Fluid Dynamics

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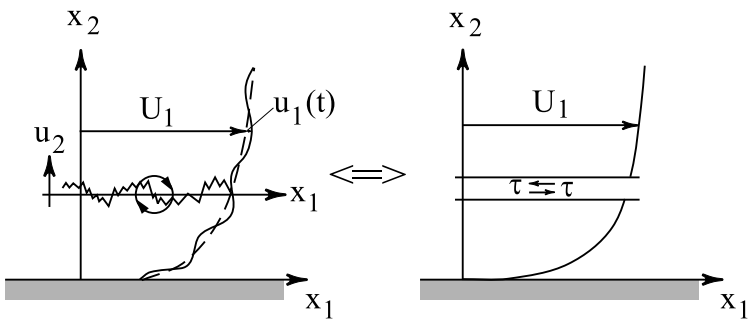

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AALBORG UNIVERSITY
Department of Civil Engineering

Aalborg University

Department of Civil Engineering

Water and Soil

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FLUID DYNAMICS

by

Michael Brorsen

March 2008

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Preface

These lecture notes are intended mainly for the 7th semester course "Fluid Dynamics" offered by the Study Committee on Civil Engineering, Aalborg University.

The notes join the Danish tradition of education laid down by the textbooks written by F. A. Engelund and J. Fredsøe, but also the transport of thermal energy is considered.

It is attempted to focus on the physics of flows and the corresponding mathematical description. Thus, many of the equations, which are solved numerically by the commercial Computational Fluid Dynamics (CFD) programs, are derived and explained. Hopefully this gives the reader a good understanding of the physics of a flow problem, making it possible to assess the quality of a numerical solution given by a CFD-program.

Aalborg, March 2008

Michael Brorsen

Contents

Chapter 1

Fluid Kinematics

1.1 Basic Assumptions

In order to find the motion of a fluid it is necessary to establish a mathematical model able to describe the conditions in the real fluid properly.

A fluid is composed of molecules moving around almost freely, and no global pattern is present in the placement of the molecules. Shear deformation of the molecules only demands weak forces, but a compression of the molecules creates very large repulsive forces.

If a resting fluid is exposed to shear forces, the molecules are always moving in such a way that the fluid takes a new shape without any shear at rest. Think e.g. of a glass of water being tilted. This is possible due to the lack of global structure of the molecules and the very small forces necessary to move the molecules in a way, which does not cause any compression.

The opposite contrast to fluids is solids, where a global structure of the molecules (a lattice) is present in most cases. Shear deformation is only possible by a deformation of this lattice, and this deformation of the lattice necessitates considerable forces, because all molecules have to be moved simultaneously. Consequently, a given shear force on a solid only creates a specific, limited deformation of the solid.

A water drop with a diameter of $\frac{1}{1000}$ mm contains approximately $N = 1.8 \cdot 10^{10}$ molecules. In practice, it is therefore impossible to describe the positions of the individual molecules in ordinary flows.

Instead we choose to consider the fluid as a *continuum*, where the mass is distributed evenly in space. In this way we may use terms like

- *density*, ρ, defined as mass per unit volume. The density, therefore, depends on the number of molecules per unit volume and the mass of the individual molecule,
- *stress*, defined as force per unit area. Stresses are in reality the sum per unit area of of all forces (including the forces caused by exchange of momentum, see below) between the molecules on opposite sides of a section.
- *fluid particle*, defined as a fluid body with negligible dimensions. The size of the fluid particle has to be small compared to the size of the flow considered, but large compared to the size of the molecules. However, these demands are not difficult to fulfill, see e.g. the example with the water drop.

Consequently, the velocity of a fluid particle, $\vec{v} = (v_1, v_2, v_3)$, is defined as the average of the velocities of the individual molecules

$$\vec{v} = \frac{1}{N} \sum_{i=1}^{N} (\vec{v})_i$$

A large computational reduction is obtained by assuming the fluid to be a continuum, but a price has to be paid due to the missing description of the motions of the individual molecules.

In a resting fluid the velocities of the molecules depend on the temperature of the fluid. Because the directions of the velocities are random in space the average value is zero, and the velocity of a fluid particle is zero.

In a moving fluid, the motions due to temperature cause an exchange of momentum over a section, and this is equivalent to both a normal force and a shear force on the section. This effect must be remembered, when considering the fluid a a continuum.

First it is seen that exchange of momentum is equivalent to a shear force. We consider two parallel speed walks moving at different velocities, see Figure 1.1.

For some reason the 'pedestrians' (quotation marks because they are too lazy to walk in the direction of the speedwalk) find it funny to swap positions. This results in an exchange of momentum between the speedwalks. All persons are assumed to have the same mass. Each time a person enters the fastest speedwalk, his velocity in the direction of the speedwalk is increased by Δv_1 and his momentum is increased proportionally. According to Newton's 2nd law, a force in

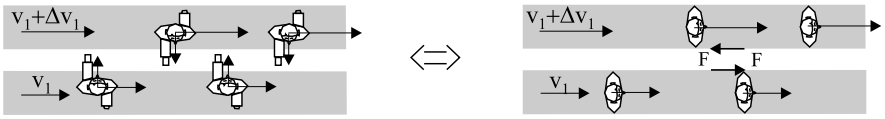

Figure 1.1: *Dynamic equivalent descriptions of persons on two speedwalks.*

the positive direction of the x_1-axis acts on him, and according to Newton's 3rd law he is acting on the speedwalk with a force of same size, but in the opposite direction.

Similarly, the reduction in momentum of the persons landing on the slowest speedwalk results in a force acting on the this speedwalk in the positive direction of the x_1-axis . The forces acting on the two speedwalks have the same size.

All in all, despite the swapping of persons, the mass of each speedwalk is constant, but the exchange of momentum is equivalent to two equal forces in opposite directions.

Consider the flow along a wall, where the fluid particles have increasing velocities in the the direction away from the wall, see Figure 1.2.

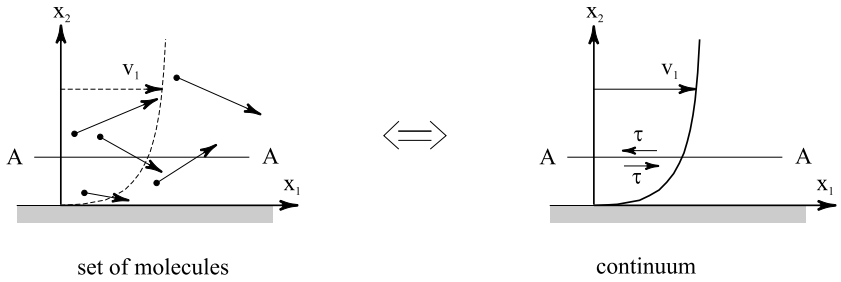

Figure 1.2: *Dynamic equivalent descriptions of a fluid.*

If the flow is described as a set of molecules (e.g. the speedwalk example), it is seen that the passage of molecules through section A-A (due the motions caused by temperature) results in an exchange of momentum equivalent to a shear force between bodies of water separated by the section. It is so, because the molecules crossing the section in downward direction on average have the largest horizontal velocity components.

If the fluid is not described as a set of molecules, it should be remembered to model the effect of the momentum exchange by introduction of the corresponding shear stresses in the continuum.

Because the shear stresses in a fluid are mainly caused by exchange of momentum, they are independent of the normal stresses. It is therefore unfortunate that shear forces in a fluid are sometimes called friction forces, because friction forces between solid bodies are proportional to the forces in the direction of the normal.

The size of the exchange of momentum depends on the temperature-dependent motions of the molecules, i.e. of the type of fluid, but also of the velocity gradient in the flow.

It is therefore natural that *experiments* have shown that the relation between shear stress and velocity gradient can be described by *Newton's formula* :

$$\boxed{\tau_{21} = \mu \cdot \frac{\partial v_1}{\partial x_2}} \tag{1.1}$$

where τ_{21} is the shear stress component in the x_1-direction on a plane perpendicular to the x_2-axis, and μ [Nm^{-2} s=Pa $\cdot$ s] is called the *dynamic* viscosity. The name is adopted, as unit of force (N) is part of the unit for this type of viscosity. Viscosity is a measure of the ability of the fluid to flow freely. Normally μ primarily depends on the type of fluid and the temperature.

Often, however, we apply the quantity *kinematic* viscosity defined as:

$$\nu = \frac{\mu}{\rho} \quad [\ \mathrm{m}^2\mathrm{s}^{-1}\ ] \tag{1.2}$$

This quantity is introduced, as the the ratio μ/ρ often appears in the equations describing flows.

In a resting fluid we have $\partial v_1/\partial x_2 = 0$ everywhere, and Newton's formula therefore correctly predicts the absence of shear stresses in a resting fluid.

Notice that in a flowing fluid we have $\tau_{12} = 0$, where the tangent to the velocity profile is 'vertical', i.e. if $\partial v_1/\partial x_2 = 0$.

1.2 Compressibility of Fluids

A fluid volume, X, subject to a small increment in pressure dp from p to $p+\Delta p$, will experience a change in volume of ΔX from X to $X + \Delta X$. Experiments have shown that the relation between Δp and ΔX reads:

$$\Delta X = -\frac{X}{K} \cdot \Delta p \tag{1.3}$$

where K is a constant, called *bulk modulus* of the fluid $[\ \mathrm{N/m^2}\ ]$. The minus-sign is present, because the volume is decreased, when the pressure is increased. For water e have $K_{water} = 22 \cdot 10^8\ \ \mathrm{N/m^2}$.

For small values of Δp equation (1.3) can be rewritten to:

$$dp = -K \cdot \frac{dX}{X} \tag{1.4}$$

Integration on both sides of the equation yields:

$$\int dp = -K \int \frac{dX}{X}$$

or

$$p = -K \ln X + C \tag{1.5}$$

The constant C is determined by adopting the initial conditions, where the pressure p_o corresponds to the volume X_o. This gives:

$$p_o = -K \ln X_o + C \quad \Rightarrow \quad C = p_o + K \ln X_o$$

Substitution of C into equation (1.5) yields:

$$p - p_o = -K \ln \frac{X}{X_o}$$

Finally this expression is rewritten to:

$$\boxed{\frac{X}{X_o} = e^{-\frac{p - p_o}{K}}} \tag{1.6}$$

If the increase in pressure is 1 atm= 1.013 bar= $1.013 \cdot 10^5\ \mathrm{N/m^2}$, the relative change in volume for water reads:

$$\frac{X}{X_o} = e^{-\frac{1.013 \cdot 10^5}{22 \cdot 10^8}} = 0.999954$$

This compression is so small that we normally may consider water incompressible.

A bit surprisingly this is also valid if the fluid is air, but only in air flows, where the flow velocity is small compared to the speed of sound (pressure waves). This can be seen by adopting a few, simple considerations.

In the first run one has to take the compressibility into account. As the mass of a fluid volume, m, is constant, the corresponding change in density reads:

$$\Delta m = \Delta(\rho\, X) = X\, \Delta\rho + \rho\, \Delta X = 0 \tag{1.7}$$

or

$$\frac{\Delta X}{X} = -\frac{\Delta \rho}{\rho} \tag{1.8}$$

Substitution of this into equation (1.3) gives:

$$\Delta p = -K \cdot \frac{\Delta X}{X} = K \cdot \frac{\Delta \rho}{\rho} \tag{1.9}$$

From experiments we know that in the flow direction the change of pressure and velocity from one point to the adjacent point approximately reads:

$$\Delta p = -\frac{1}{2} \rho \, \Delta(v^2) \tag{1.10}$$

i.e. an increase in pressure corresponds to a decrease in velocity. The expression follows from Bernoulli's equation (see chapter 3).

As shown in most textbooks on basic fluid mechanics, the propagation velocity of a change in pressure (the speed of sound), c, in a fluid reads:

$$K = \rho \, c^2 \tag{1.11}$$

Substitution of the expressions for Δp and K into equation (1.9) gives:

$$\frac{\Delta \rho}{\rho} = \frac{\Delta p}{K} = \frac{-1/2 \rho \, \Delta(v^2)}{\rho \, c^2} = \frac{-1/2 \Delta(v^2)}{c^2} \tag{1.12}$$

The order of magnitude of the relative change in density is overestimated if it is assumed that

$$|\Delta(v^2)| = |\Delta(v^2)|_{max} = v_{max}^2 - 0^2 = v_{max}^2$$

where v_{max} is the maximum velocity in the flow, and in practice one may adopt the expression

$$\frac{|\Delta \ \rho|}{\rho} < \frac{v_{max}^2}{c^2} \tag{1.13}$$

The speed of sound in air is $c_{air} \approx 330$ m/s, and for flow velocities up to 30 m/s the change in density is less than 1 %. In practice flowing air is assumed to be incompressible in civil engineering problems.

However, notice that some dynamic phenomena like underwater explosions and water hammers in pipes are impossible to describe at all, unless the compressibility is taken into account.

1.3 Basic Kinematic Concepts

Below we define some concepts, which are often adopted to describe the kinematics of a flow.

- *particle path*

  The path of a fluid particle is the curve in space along which the particle is moving. This curve can in principle be given as $x_i = x_i(t)$, where t is the time. Marking a particle with some dye and taking a photo with long exposure can visualize the path. It is also a possibility to place small, coloured spheres with the same density as the fluid, and then record the motion with a video camera. A subsequent digitizing of the position of the sphere on each picture (i.e. Δt sec apart) makes it possible to draw the path.

- *velocity field*

  The velocity field of a domain is defined as the set of velocities (vectors) for the particles at a given time.

  If the path of the particle, $x_k(t)$, is known, the velocity, v_i, for a specific particle reads:

  $$v_i = \frac{dx_i}{dt} = v_i(t) \tag{1.14}$$

  The velocity field is therefore found by determining the position and velocity for all particles at a given time t.

  However, in fluid dynamics it is normally easier to determine the velocity field first, i.e. determine the velocity of the particles, $v_i(x_k, t)$, when they are at given points x_k, and determine the paths later on by integration of equation (1.14).

  A velocity field can be visualized by marking a set of particles and take a photo with short exposure. Every particle is shown on the photo as a small line segment proportional to the velocity at that position.

- *two-dimensional flow (plane flow)*

  If the total velocity field can be found by displacing the velocity field of a single plane perpendicular to that plane, the flow is two-dimensional. If the flow takes place in e.g the $x_1\,x_2$-plane, we have $v_3 \equiv 0$. Moreover $v_1 = v_1(x_1, x_2, t)$ and $v_2 = v_2(x_1, x_2, t)$.

- *steady flow*

  If the velocity field is independent of time, i.e. $v_i = v_i(x_k)$, the flow is called *steady*. Notice that the velocity field only gives information about the velocity of particles, when the particle passes through the fixed point

x_k. If the same particle has different velocities at two points, acceleration must be present even though the velocity field is steady! See also Section 1.4.

- *streamline*

  A streamline is defined as a line, which at all points has the local velocity vector as the tangent at a given time t_o. See Fig. 1.3. If x_i denotes the

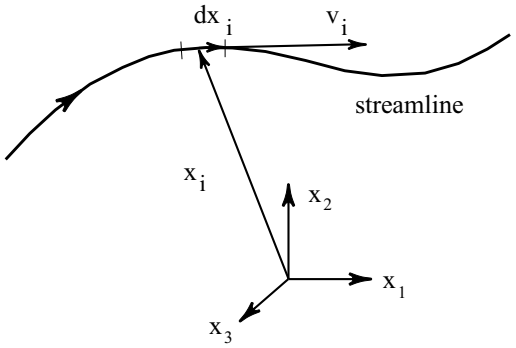

Figure 1.3: *Definition of a streamline.*

position vector to a point on the streamline, and dx_i denotes a vector which is tangent to the streamline at x_i, the coincidence of this tangent and the velocity vector at x_k may be expressed mathematically as $v_i = k \cdot dx_i$, where k is a constant. This gives:

$$k = \frac{v_1}{dx_1} = \frac{v_2}{dx_2} = \frac{v_3}{dx_3} \tag{1.15}$$

which in fact are two coupled differential equations. Sometimes these equations can be solved analytically, but a numerical solution is always possible, if we know the velocity field, $v_i = v_i(x_k, t)$.

It is directly seen that in a steady flow both streamlines and paths are firm curves. Because a particle that all the time moves along the tangent to a curve will stay on this curve, it is finally seen that in a steady flow streamlines and particle paths coincide.

Notice, that this is *not* the case in *unsteady* flows.

- *Uniform flow*

  In areas where the streamlines are parallel, and the velocity is constant along a streamline, the flow is called *uniform* or *unvarying*. Such a flow in a pipe is also called *fully developed* flow. However, the flow is called *non-uniform* or *varying* in areas, where the mutual distance between the streamlines varies, and the velocity varies along a stream line, see Fig. 1.4.

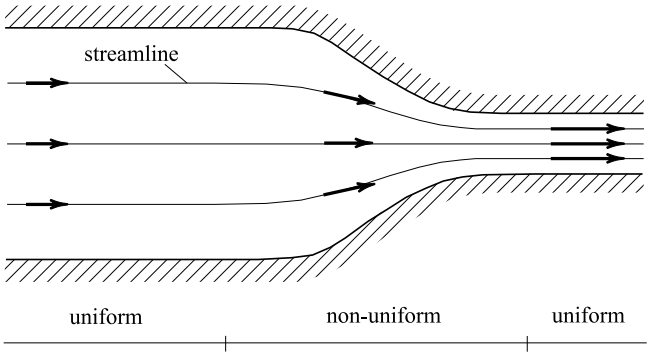

Figure 1.4: *Uniform and non-uniform flow.*

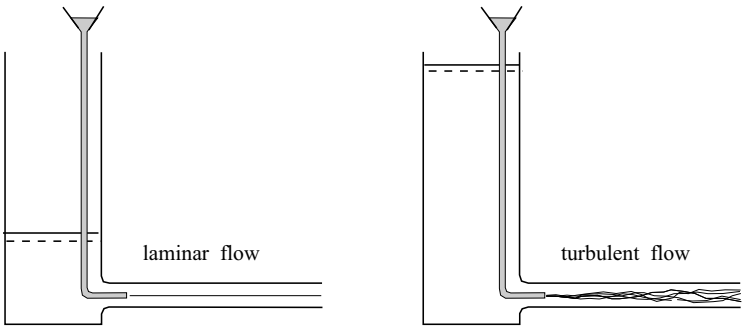

Figure 1.5: *Laminar and turbulent flow.*

- *laminar flow*

  If we consider e.g. the flow in a pipe, so-called laminar flow is present at rather small velocities. This flow is characterized by a smooth variation of the velocity field and if the paths (=streamlines in steady flow) are visualized by e.g. smoke or dye, it looks like the flow consists of layers (laminae in Latin). If dye is injected into the flow through a thin pipe, the colored line will appear sharply over a fairly long distance, see Fig. 1.5.

  If the rate of volume supplied to the pipe is constant, the velocity at every point is constant with time, and the flow is steady.

  If the rate of supplied volume is varying with time, the flow is unsteady and the velocity at every point is varying with time, but in a laminar flow this variation is smooth.

- *turbulent flow*

  When the velocity in a pipe is raised above a specific level the type of flow changes from laminar to turbulent. The latter flow type looks highly

restless, because the main flow is full of eddies of different sizes. The presence of these eddies causes a strong mixing of dye from a thin pipe, and the coloured line quickly goes blurred in a turbulent flow, see Fig. 1.5.

A time series of the velocity at a point shows strong variations (fluctuations) even though the rate of supplied volume is constant with time. Turbulent flows are in principle always unsteady.

Normally, it is impossible to describe all the details of the short periodic fluctuations, and in practice one has to be satisfied by a description of velocities, which are *smoothed* with time. All frequencies above a specific limit are filtered away by use of a so-called *low-pass filter*. The most primitive way of doing this is by calculation of the mean value of the velocity over a given time interval. The flow corresponding to these smoothed velocities is often called the *mean flow*.

A constant rate of supplied volume leads to a time invariant mean flow, and the corresponding flow is often called *steady turbulent flow*, even though it should be called steady mean flow.

The transition from laminar to turbulent flow always takes place if the velocity is exceeding a specific value. An observation that is supported by theoretical considerations, see chapter 5.1. Observations and theory have shown that the type of flow depends on the so-called *Reynolds' number:*

$$\mathrm{Re} = \frac{V \cdot L}{\nu} \quad [-]$$

where

V	is a characteristic velocity [m/s]
L	is a characteristic length [m]
ν	is the kinematic viscosity of the fluid [m^2/s]
$[-]$	means that Re is a non-dimensional parameter

1.4 Rate of Change

We consider a scalar, φ, which may depend on both position and time, i.e.

$$\varphi = \varphi(x_k, t) \tag{1.16}$$

Typical examples of such a scalar are pressure, p, temperature, T, or a component of the particle velocity like v_1.

We will focus on the changes of φ that are observed by an observer moving in space with the velocity :

$$U_i^{obs} = \frac{dx_i^{obs}}{dt} \tag{1.17}$$

If the observer moves the distance dx_i away from the initial position during dt seconds, we know from mathematics that the change of φ reads:

$$d\varphi = \frac{\partial \varphi}{\partial t}dt + \frac{\partial \varphi}{\partial x_1}dx_1^{obs} + \frac{\partial \varphi}{\partial x_2}dx_2^{obs} + \frac{\partial \varphi}{\partial x_3}dx_3^{obs} \tag{1.18}$$

Division by dt on both sides of the equation gives:

$$\begin{aligned} \frac{d\varphi}{dt} &= \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x_1}\frac{dx_1^{obs}}{dt} + \frac{\partial \varphi}{\partial x_2}\frac{dx_2^{obs}}{dt} + \frac{\partial \varphi}{\partial x_3}\frac{dx_3^{obs}}{dt} \\ &= \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x_i} \cdot U_i^{obs} \end{aligned} \tag{1.19}$$

after equation (1.17) has been substituted. As the quantity $d\varphi/dt$ indicates how quickly the scalar is changing, it feels natural to call it the *rate of change* of the scalar. Notice that the rate of change is directly proportional to the velocity of the observer.

If the observer remains on the spot, i.e. $U_i^{obs} = 0$, giving

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t}$$

it is natural to name $\frac{\partial \varphi}{\partial t}$ the *local* rate of change.

Example: Rate of change of pressure.

Around a depression (low pressure) the isobars (lines with constant pressure) are situated as shown in Fig. 1.6. The depression and the isobars are moving in the negative direction of the x_1-axis at the constant velocity 20 km/hour.

A train is moving at the constant velocity $V_{train} = 75$ km/hour in the direction of the x_1-axis.

What rate of change of the pressure (mb/hour) is observed on a barometer in the train ?

In this case the equation for the rate of change reads:

$$\left(\frac{dp}{dt}\right)_{train} = \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x_1} \cdot V_{train}$$

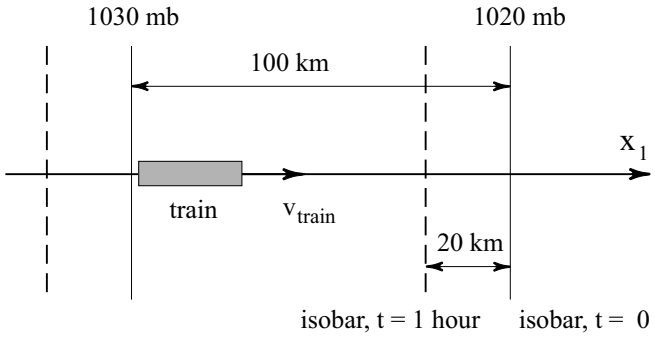

Figure 1.6: *Rate of change of pressure in a train.*

Because the isobars are moving 100 km in 5 hours, a decline in pressure of 10 mb in 5 hours is observed at a fixed point. This gives:

$$\frac{\partial p}{\partial t} \approx \frac{\Delta p}{\Delta t} = \frac{-10}{5} = -2 \text{ mb/hour}$$

Furthermore, the pressure declines 10 mb per 100 km in the direction of the x_1-axis, giving:

$$\frac{\partial p}{\partial x_1} \approx \frac{\Delta p}{\Delta x_1} = \frac{-10}{100} = -0.1 \text{ mb/km}$$

The total rate of change of pressure in the train therefore reads:

$$\left(\frac{dp}{dt}\right)_{train} = -2.0 + (-0.1 \cdot 75) = -2.0 - 7.5 = \underline{-9.5 \text{ mb/hour}}$$

Often we are interested in the rate of change corresponding to an observer following a specific fluid particle, i.e. we have $U_i^{obs} = v_i$. This rate of change is often called the *substantial* rate of change or the *particle* rate of change. Both names try to invoke the feeling that we are following a specific particle. Sometimes the substantial derivative is given the special symbol D/Dt, purely a mnemonic device, not intended to frighten the readers! The part of the rate of change caused by the motion of the observer (= the motion of the particles) is called the *convective* rate of change.

The rate of change of the velocity component (i.e a scalar) v_1 reads:

$$\frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_i} \cdot U_i^{obs}$$

Because we have similar expressions for the two other velocity components, it is seen that the expression for the rate of change of a scalar can be generalized also to be valid for vectors, i.e.

$$\frac{dv_k}{dt} = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_i} \cdot U_i^{obs} \tag{1.20}$$

If the observer follows a specific fluid particle, i.e. $U_i^{obs} = v_i$, the substantial rate of change for the particle velocity reads:

$$\frac{dv_k}{dt} \left(= \frac{Dv_k}{Dt} \right) = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_i} \cdot v_i \tag{1.21}$$

Because this substantial rate of change is the change of velocity per unit time for the actual particle, it is in fact the *acceleration* of the fluid particle.

Equation (1.21) can also be adopted to explain why acceleration may be present in steady flow. Steady flow means that all velocities do not change with time at fixed points, or we have the local rate of change of the velocity equal to zero everywhere. But the convective part of the rate of change caused by the motion of the observer (= the motion of the particles) is different from zero if the velocity gradient

$$\text{grad}\, \vec{v} = \frac{\partial v_k}{\partial x_i} \neq 0$$

This is the case for e.g. the flow in a channel of varying width.

1.5 The Continuity Equation

Conservation of mass is the basis for one of the most important equations in fluid dynamics.

The flow through a *fixed, closed* surface of arbitrary shape, A, is considered. The surface A bounds the volume X. If the outward unit normal to the surface is denoted $\vec{n}$, the definition of the area vector, $d\vec{A}$, corresponding to the area dA, reads:

$$d\vec{A} = \vec{n} \cdot dA \tag{1.22}$$

The length of the area vector is therefore equal to the corresponding area, because $|\vec{n}| = 1$. The increase in mass inside the surface is equal to the resultant *inflow* of mass through the surface. In vector notation this can be expressed as:

$$\int_X \frac{\partial \rho}{\partial t}\, dX = \int_A -\rho\, \vec{v} \cdot d\vec{A} \tag{1.23}$$

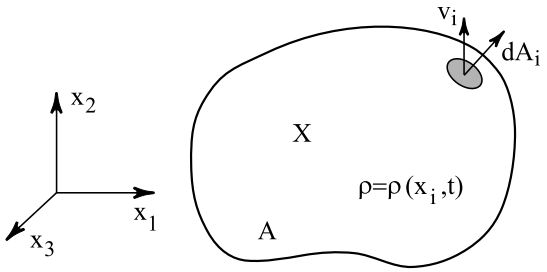

Figure 1.7: *The equation of continuity. Definition sketch.*

and in tensor notation as:

$$\int_X \frac{\partial \rho}{\partial t}\, dX = \int_A -\rho\, v_i\, dA_i \tag{1.24}$$

This equation can be rewritten by adopting *Gauss' divergence theorem*, which in vector notation reads:

$$\int_A \vec{l} \cdot \vec{dA} = \int_X \operatorname{div} \vec{l}\, dX \tag{1.25}$$

and in tensor notation reads:

$$\int_A l_i\, dA_i = \int_X \frac{\partial\, l_i}{\partial x_i}\, dX \tag{1.26}$$

Here *divergence* of a vector field, $\vec{l}$, is defined as

$$\operatorname{div} \vec{l} = \frac{\partial l_1}{\partial x_1} + \frac{\partial l_2}{\partial x_2} + \frac{\partial l_3}{\partial x_3} = \frac{\partial\, l_i}{\partial x_i} \tag{1.27}$$

If the vector field, $l_i = -\rho\, v_i$, is considered, Gauss's divergence theorem reads:

$$\int_A -\rho\, v_i\, dA_i = -\int_X \frac{\partial(\rho\, v_i)}{\partial x_i}\, dX$$

which substituted into equation (1.24) gives

$$\int_X \frac{\partial \rho}{\partial t}\, dX = -\int_X \frac{\partial(\rho\, v_i)}{\partial x_i}\, dX \quad \Rightarrow$$

$$\int_X \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\, v_i)}{\partial x_i} \right) dX = 0$$

This equation is only fulfilled for an arbitrary choice of X, if the expression in the parenthesis fulfills:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\, v_i)}{\partial x_i} = 0} \tag{1.28}$$

which is called the *continuity equation*.

Differentiation with respect to x_i gives:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} v_i + \rho \frac{\partial v_i}{\partial x_i} = 0 \tag{1.29}$$

Following a specific particle the rate of change of the density, ρ, reads:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} v_i$$

Equation (1.28) can therefore be rewritten to:

$$\boxed{\frac{d\rho}{dt} + \rho \frac{\partial v_i}{\partial x_i} = 0} \tag{1.30}$$

As shown earlier, it is a good approximation in many situations to consider a fluid incompressible, i.e. $\rho = constant \quad \Leftrightarrow \quad d\rho = 0$ for a particle moving with the flow. Consequently, the substantial rate of change reads:

$$\frac{d\rho}{dt} = 0$$

which substituted into equation (1.30) gives the continuity equation for an *incompressible* fluid:

$$\boxed{\frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0} \tag{1.31}$$

or

$$\boxed{\text{div } \tilde{\mathrm{v}} = 0} \tag{1.32}$$

1.6 Vorticity

A very important vector quantity in fluid dynamic is defined by the cross-product between the gradient vector $\partial(\ )/\partial x_i\ = \text{grad}() = \vec{\nabla}(\ )$ and the velocity vector $v_i = \vec{v}$. The new vector is called the *vorticity* of the velocity field and is normally denoted curl $\vec{v}$ or rot $\vec{v}$. In the rest of this book the notation rot $\vec{v}$ is adopted. Because the author finds it rather difficult to understand equations where cross-products are expressed in tensor notation, all equations with cross-products are expressed in vector notation below. Notice also that in the following the vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ denote unit vectors in the directions of the x_1-, x_2- and x_3-axis, respectively.

$$\text{rot}\,\vec{v} \equiv \vec{\nabla}\ \times\ \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{bmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{bmatrix} \tag{1.33}$$

In the descriptions of flowing fluids rot $\vec{v}$ frequently appears, and in practice it is important to determine, whether rot $\vec{v} = \vec{0}$ or not. Fortunately, it is possible to give a physical interpretation of rot $\vec{v}$. To do this we consider the motions of the fluid particles placed on two lines originally perpendicular to each other. The placement of the two line segments is found before and after the time dt has elapsed, see Fig. 1.8. The line segment AB turns the angle $d\alpha$ $[rad]$ during

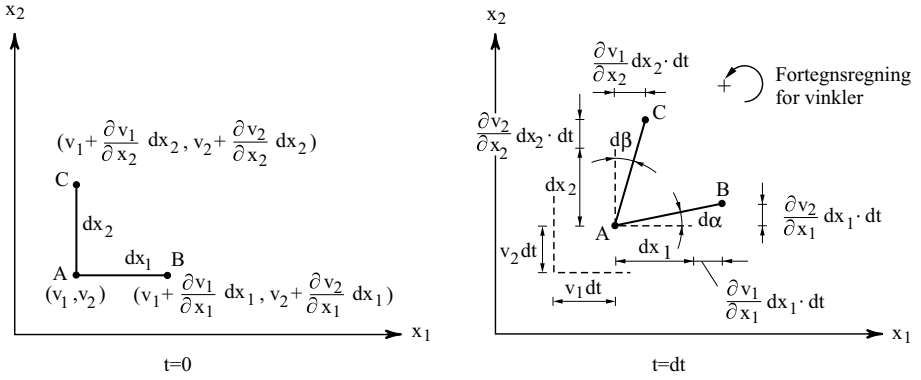

Figure 1.8: *Angular motions of line segments in a velocity field.*

dt $[secs]$, and the expression for $d\alpha$ reads:

$$d\alpha = \frac{\frac{\partial v_2}{\partial x_1}\ dx_1\ dt}{dx_1 + \frac{\partial v_1}{\partial x_1}\ dx_1\ dt}$$

This gives the angular velocity:

$$\omega_{AB} = \frac{d\alpha}{dt} = \frac{\frac{\partial v_2}{\partial x_1}\, dx_1}{dx_1 + \frac{\partial v_1}{\partial x_1}\, dx_1\, dt}$$

For $dt \to 0$ we find:

$$\omega_{AB} = \frac{\frac{\partial v_2}{\partial x_1}\, dx_1}{dx_1} = \frac{\partial v_2}{\partial x_1} \tag{1.34}$$

In the same way we find for the segment AC :

$$d\beta = \frac{-\frac{\partial v_1}{\partial x_2}\, dx_2\, dt}{dx_2 + \frac{\partial v_2}{\partial x_2}\, dx_2\, dt}$$

The negative sign is caused by a negative angular motion for $\frac{\partial v_1}{\partial x_2} > 0$. The angular velocity of AC reads:

$$\omega_{AC} = \frac{-\frac{\partial v_1}{\partial x_2}\, dx_2}{dx_2} = -\frac{\partial v_1}{\partial x_2} \tag{1.35}$$

Hereafter we find the *mean* angular velocity for the two segments to be:

$$\omega_{mean} = \frac{1}{2}\,(\omega_{AB} + \omega_{AC}) = \frac{1}{2}\,(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}) \tag{1.36}$$

If this expression is compared to the 3-component of rot $\vec{v}$, it is seen that

$$\omega_{mean} = \frac{1}{2}\,(\mathrm{rot}\,\vec{v})_3$$

or

$$(\mathrm{rot}\,\vec{v})_3 = 2 \cdot \omega_{mean} \tag{1.37}$$

Physically it is therefore possible to determine if $(\mathrm{rot}\,\vec{v})_3 = 0$ or not, if we can visualize ω_{mean}. The easiest way to do this is by insertion of a so-called "vorticity meter" into the fluid. The vorticity meter consists of two plates fixed at a right angle to each other. The plates may rotate around the line of intersection between the two plates. Each plate tries to move with the surrounding fluid. If the meter is not rotating, the two plates are therefore trying to rotate at equal, but opposite angular velocities. This corresponds to $\omega_{mean} = 0$ and also $(\mathrm{rot}\,\vec{v})_3 = 0$. By subsequent changes of the axis of rotation, it is possible to find the two other components of rot $\vec{v}$ in a similar way.

Of course rot $\vec{v}$ can be found directly from the definition, equation (1.33), if the velocity field is known. Sometimes it might be easier to determine rot $\vec{v}$ indirectly by adopting *Stokes'* theorem:

$$\int_L \vec{v} \cdot \vec{ds} = \int_A \mathrm{rot}\,\vec{v} \cdot \vec{n}\; dA \tag{1.38}$$

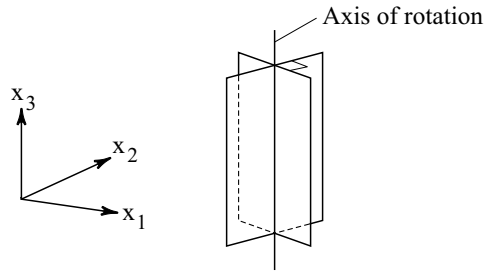

Figure 1.9: *Device ("Vorticity meter") for "measurement" of the x_3-component of* rot $\vec{v}$.

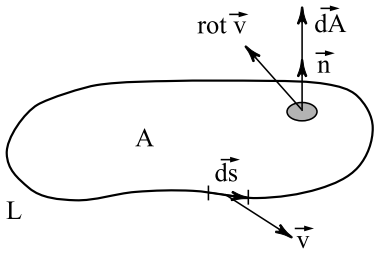

Figure 1.10: *Stokes's theorem, definition sketch.*

where $\vec{ds}$ is a vector on the closed curve L bounding the surface A and $\vec{n}$ is the unit normal to A. The left-hand side of Stokes's theorem is often called the *circulation* defined by $\Gamma = \int_L \vec{v} \cdot \vec{ds}$.

We consider a plane surface with an area A, so *small* that rot $\vec{v} \approx$ *constant* and $\vec{n} \approx$ *constant* can be assumed over the area, and the velocity is varying linearly along the boundary. In such a case Stokes's theorem reads:

$$\Gamma = \sum_L \vec{v} \cdot \vec{\Delta s} = \text{rot } \vec{v} \cdot \vec{n}\, A \tag{1.39}$$

If A is placed in the x_1 x_2-plane, we have rot $\vec{v} \cdot \vec{n} = (\text{rot } \vec{v})_3$ and from equation (1.39) one finds:

$$(\text{rot } \vec{v})_3 = \frac{\sum_L \vec{v} \cdot \vec{\Delta s}}{A}$$

It should also be stressed that in principle the size of rot $\vec{v}$ has nothing to do with the shape of the particle path. To demonstrate this we consider two plane flows both having concentric circles as paths.

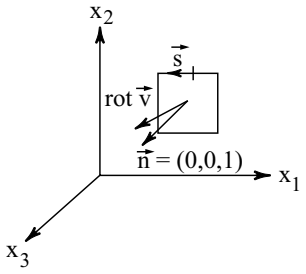

Figure 1.11: *Usage of Stokes's theorem to calculate* $(\mathrm{rot}\,\vec{v})_3$.

Example 1: irrotational flow ($\mathrm{rot}\,\vec{v} = \vec{0}$)

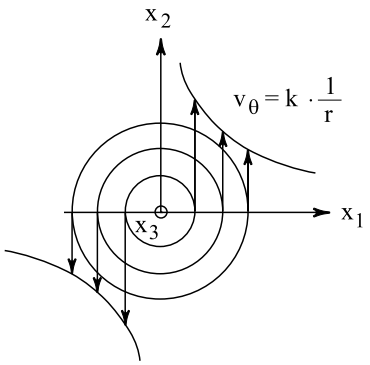

Figure 1.12: *Irrotational,* $\mathrm{rot}\,\vec{v} = \vec{0}$ *for* $r > 0$.

The velocity field in this example expressed in polar coordinates reads:

$$\vec{v} = (v_r, v_\theta) = (0, \frac{k}{r})$$

where k is a constant. The tangential velocity component is inversely proportional to the distance to the origin, and the flow is two-dimensional. In this flow the two plates of a vorticity meter will be affected to turn in opposite directions. A plate directed in the radial direction will turn *clockwise* and a plate directed in the tangential direction will turn *counter-clockwise*. If a physical experiment is conducted it can be seen that the meter is not turning. We therefore know that $(\mathrm{rot}\,\vec{v})_3 = 0$. This can of course also be shown mathematically, and in this case it is very easily done by adopting Stokes's theorem.

Example 2: rotational flow ($\mathrm{rot}\,\vec{v} \neq \vec{0}$)

The velocity field in this example expressed in polar coordinates reads:

$$\vec{v} = (v_r, v_\theta) = (0, \omega \cdot r)$$

where ω is a constant. The tangential velocity component is thus proportional to the distance to the origin. It is seen that the flow is two-dimensional, and the motion corresponds to a rigid body rotating at the angular velocity ω. The axis of rotation is perpendicular to the $x_1\,x_2$-plane.

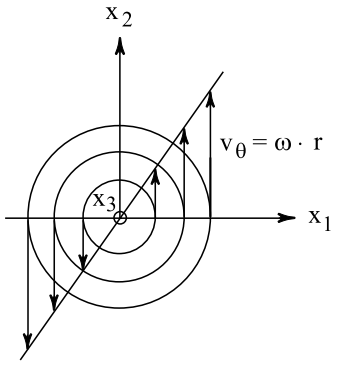

Figure 1.13: *Rotational flow,* $\mathrm{rot}\,\vec{v} \neq \vec{0}$.

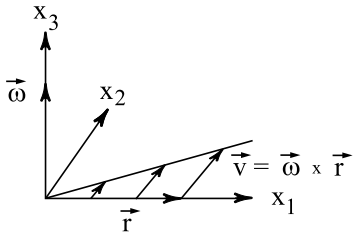

Figure 1.14: *Velocity field corresponding to rotation of a rigid body about the* x_3*-axis.*

In this case both plates of a vorticity meter are affected to turn into the same direction (counter-clockwise). Consequently, the meter will turn *counter-clockwise*, and we have $(\mathrm{rot}\,\vec{v})_3 \neq 0$.

Because both plates will rotate at the angular velocity ω, we have $\omega_{mean} = \frac{1}{2}(\omega + \omega) = \omega$. From equation (1.37) it is therefore seen that $(\mathrm{rot}\,\vec{v})_3 = 2\,\omega_{mean} = 2\,\omega$.

If this two-dimensional flow is described in an $x_1\,x_2\,x_3$-coordinate system, it can be seen that the motion corresponds to the rotation of a rigid body about the x_3-axis at the angular velocity $\vec{\omega}$, see Fig. 1.14.

The velocity field may be expressed as:

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 0 & \omega \\ x_1 & x_2 & x_3 \end{vmatrix} = (-\omega\, x_2, \omega\, x_1, 0)$$

where $\vec{\omega} = (0, 0, \omega)$ is the angular velocity and $\vec{r} = (x_1, x_2, x_3)$ is the position vector to the point at which the velocity is calculated.

This gives:

$$\operatorname{rot} \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ -\omega\, x_2 & \omega\, x_1 & 0 \end{vmatrix} = (0, 0, \omega - (-\omega)) = (0, 0, 2\,\omega)$$

If a fluid rotates about an axis like a rigid body, it is seen that both methods to determine $\operatorname{rot} \vec{v}$ result in the equation:

$$\boxed{\operatorname{rot} \vec{v} = 2\,\vec{\omega}} \tag{1.40}$$

Thus all fluid particles rotate with a rotation vector, which is half of $\operatorname{rot} \vec{v}$.

1.7 Instantaneous Velocity Field

In general the velocity field can be expressed as:

$$v_i = v_i(x_k, t) \tag{1.41}$$

where x_k is the coordinate to a fixed point and t is the time.

We now want to consider the flow at a specific moment, t_0. Time is therefore not an independent variable of the considered velocity field, which we name the *instantaneous velocity field* . We denote this field

$$v_i^{inst} = v_i(x_k, t_0) \tag{1.42}$$

In the following the superscript ^{inst} and t_0 are omitted to improve the overview.

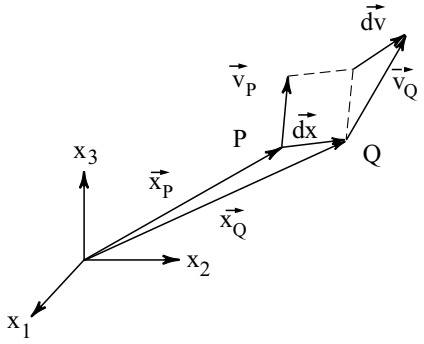

Figure 1.15: *Definition sketch, instantaneous velocity field.*

Furthermore, we only consider the instantaneous velocity field at points close to point P situated at $\vec{x}_P$. The particle velocity at this point is denoted $\vec{v}_P$. Similarly, position and velocity at the adjacent point Q are denoted $\vec{x}_Q$ and $\vec{v}_Q$, respectively. The following relations are valid:

$$\vec{x}_Q - \vec{x}_P = d\vec{x} \quad \text{and} \quad \vec{v}_Q - \vec{v}_P = d\vec{v} = (dv_1, dv_2, dv_3)$$

where

$$\begin{aligned} dv_1 &= \left.\frac{\partial v_1}{\partial x_1}\right|_P dx_1 + \left.\frac{\partial v_1}{\partial x_2}\right|_P dx_2 + \left.\frac{\partial v_1}{\partial x_3}\right|_P dx_3 \\ &= \left.\frac{\partial v_1}{\partial x_j}\right|_P dx_j = \text{grad}(v_1)|_P \cdot \vec{dx} \end{aligned} \tag{1.43}$$

where $|_P$ means that the quantity is calculated at point P.

Equation (1.43) can be generalized to

$$dv_i = \left.\frac{\partial v_i}{\partial x_j}\right|_P dx_j = \text{grad}(v_i)|_P \cdot \vec{dx} \tag{1.44}$$

where the tensor $\frac{\partial v_i}{\partial x_j}$ is called the *velocity gradient-tensor*.

Again, to improve the overview $|_P$ is omitted below. However, it is essential for the physical interpretations to remember that all gradients are calculated at point P.

The velocity gradient tensor can be divided into a *symmetric* part and an *anti-symmetric* part, which gives

$$dv_i = \frac{\partial v_i}{\partial x_j}\, dx_j = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx_j + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) dx_j \tag{1.45}$$

This equation can be rewritten to

$$dv_i = e_{ij}\, dx_j + w_{ij}\, dx_j \tag{1.46}$$

where the *deformation tensor* e_{ij} is defined as

$$\boxed{e_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})} \tag{1.47}$$

and the *spin tensor* w_{ij} is defined as

$$\boxed{w_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})} \tag{1.48}$$

The two tensors written fully read:

$$e_{ij} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}) & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \tag{1.49}$$

and

$$w_{ij} = \begin{bmatrix} 0 & \frac{1}{2}(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}) & 0 & \frac{1}{2}(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}) & 0 \end{bmatrix} \tag{1.50}$$

From the equations (1.49) and (1.50) it is clearly seen that the deformation tensor is symmetric and the spin tensor is antisymmetric.

By substitution of the definitions:

$$dv_i^{def} = e_{ij}\, dx_j \tag{1.51}$$

$$dv_i{}^{spin} = w_{ij}\, dx_j \tag{1.52}$$

the increment in velocity corresponding to the instantaneous velocity field reads:

$$dv_i = dv_i^{def} + dv_i^{spin} \tag{1.53}$$

1.7.1 Deformation Tensor

First we shall show that the motions caused by the velocities corresponding to the deformation tensor, e_{ij} causes a deformation of the body of fluid around point

P. This is done by placing a coordinate system with the axes coinciding with the *principal directions* or *eigenvectors* for e_{ij}. Per definition, the eigenvectors for e_{ij} ($= \mathbf{e}$ in matrix-notation) are the solutions to the equation

$$\mathbf{e}\, d\vec{x} = \lambda\, d\vec{x} \tag{1.54}$$

or, after substitution of equation (1.51)

$$d\vec{v}^{\,def} = \lambda\, d\vec{x} \tag{1.55}$$

To solve this equation means to determine the values of λ, called *eigenvalues*, for which equation (1.55) has a non-trivial solution $d\vec{x}$, called an *eigenvector*.

Based on equation (1.54) we might also say that if a step is taken in the direction of an eigenvector, the observed change in velocity is pointing in the same direction as the step vector.

From mathematics it is known that any real, symmetric 3x3 tensor always has 3 real values of λ and 3 corresponding eigenvectors mutually orthogonal.

First we find the eigenvectors corresponding to equation (1.55), which in tensor notation reads:

$$e_{ij}\, dx_j = \lambda\, dx_i \quad \Leftrightarrow \quad e_{ij}\, dx_j - \lambda\, dx_i = 0 \tag{1.56}$$

Introduction of Kronecker's tensor, δ_{ij}, defined as

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

gives $dx_i = \delta_{ij}\, dx_j$, and equation (1.56) can be rewritten to

$$(e_{ij} - \lambda\delta_{ij})\, dx_j = 0 \tag{1.57}$$

In order to find non-trivial solutions to this homogeneous system of equations, the determinant of the coefficients has to be zero, i.e. $det(e_{ij} - \lambda\, \delta_{ij}) = 0$. This gives a 3rd order equation of λ, and the three solutions are denoted λ_I, λ_{II} and λ_{III}, respectively. Hereafter equation (1.57) is solved three times, where the values of λ are substituted in turn. The three solutions, the eigenvectors, are denoted $(dx_I)_i$, $(dx_{II})_i$ and $(dx_{III})_i$, respectively.

All quantities expressed in the coordinate system having the axes coinciding with the eigenvectors and origin at point P, are now marked by an asterisk (*).

If e_{ij} is transformed into the eigenvector system, denoted $x_1^*\, x_2^*\, x_3^*$, it is known

that e^*_{ij} becomes a diagonal tensor with the eigenvalues in the diagonal, i.e.

$$e^*_{ij} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}^* = \begin{bmatrix} e^*_{11} & 0 & 0 \\ 0 & e^*_{22} & 0 \\ 0 & 0 & e^*_{33} \end{bmatrix} = \begin{bmatrix} \lambda_I & 0 & 0 \\ 0 & \lambda_{II} & 0 \\ 0 & 0 & \lambda_{III} \end{bmatrix}$$

If the vector dv_i^{def} is transformed into the $x_1^* x_2^* x_3^*$-system, the expression for $(dv_i^{def})^*$ reads:

$$(dv_i^{def})^* = \begin{bmatrix} dv_1^{def} \\ dv_2^{def} \\ dv_3^{def} \end{bmatrix}^* = \begin{bmatrix} e^*_{11} & 0 & 0 \\ 0 & e^*_{22} & 0 \\ 0 & 0 & e^*_{33} \end{bmatrix}_P \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}^* \tag{1.58}$$

$$= \begin{bmatrix} e^*_{11}|_P \cdot dx^*_1 \\ e^*_{22}|_P \cdot dx^*_2 \\ e^*_{33}|_P \cdot dx^*_3 \end{bmatrix} \tag{1.59}$$

From this expression is seen that the velocity changes due to the deformation tensor are proportional to the distance to point P. These motions are only

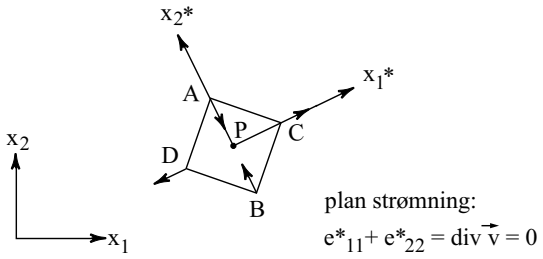

Figure 1.16: *Motions caused by* e_{ij}.

possible, if the fluid around P is deformed. This is very easy to see, if the motions of the points A, B, C and D are considered, see Fig. 1.16.

1.7.2 The Spin Tensor

Consider the motions caused by the term from the spin tensor. Due to the frequent appearance of the components of rot $\vec{v}$ the definition of this vector is

recapitulated:

$$\text{rot}\,\vec{v} = \begin{bmatrix} (\text{rot}\,\vec{v})_1 \\ (\text{rot}\,\vec{v})_2 \\ (\text{rot}\,\vec{v})_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{bmatrix} \tag{1.60}$$

It is seen that the velocity change dv_i^{spin} can be written as

$$\begin{aligned} dv_i^{spin} &= \begin{bmatrix} 0 & \frac{1}{2}(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}) & 0 & \frac{1}{2}(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}) & 0 \end{bmatrix}_P \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 0 & -(\text{rot}\,\vec{v})_3 & (\text{rot}\,\vec{v})_2 \\ (\text{rot}\,\vec{v})_3 & 0 & -(\text{rot}\,\vec{v})_1 \\ -(\text{rot}\,\vec{v})_2 & (\text{rot}\,\vec{v})_1 & 0 \end{bmatrix}_P \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \end{aligned} \tag{1.61}$$

This expression is compared to the expression:

$$\begin{aligned} (\text{rot}\,\vec{v})_P \times \vec{dx} &= \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ (\text{rot}\,\vec{v})_1 & (\text{rot}\,\vec{v})_2 & (\text{rot}\,\vec{v})_3 \\ dx_1 & dx_2 & dx_3 \end{vmatrix} \\ &= \begin{bmatrix} (\text{rot}\,\vec{v})_2\,dx_3 - (\text{rot}\,\vec{v})_3\,dx_2 \\ (\text{rot}\,\vec{v})_3\,dx_1 - (\text{rot}\,\vec{v})_1\,dx_3 \\ (\text{rot}\,\vec{v})_1\,dx_2 - (\text{rot}\,\vec{v})_2\,dx_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -(\text{rot}\,\vec{v})_3 & (\text{rot}\,\vec{v})_2 \\ (\text{rot}\,\vec{v})_3 & 0 & -(\text{rot}\,\vec{v})_1 \\ -(\text{rot}\,\vec{v})_2 & (\text{rot}\,\vec{v})_1 & 0 \end{bmatrix}_P \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \end{aligned} \tag{1.62}$$

The comparison of equation (1.61) and equation (1.62) gives:

$$d\vec{v}^{\,spin} = \frac{1}{2}\,(\text{rot}\,\vec{v})_P \times \vec{dx} \tag{1.63}$$

Because the velocity field caused by the spin tensor is described by the cross-product between a constant vector and the position vector, the velocity field,

therefore, (see example 2) corresponds to a rotation of the fluid as a rigid body around point P with the rotation vector:

$$\vec{\omega}_P = \frac{1}{2}(\text{rot } \vec{v})_P \tag{1.64}$$

Once more we see that rot $\vec{v} \neq \vec{0}$ at a point means that a fluid particle is rotating about its own axis.

1.7.3 Resulting Motion

Based on the experience obtained from the subsections 1.7.1 and 1.7.2, it is concluded that close to point P the instantaneous velocity field is caused by the following motions of the fluid:

1) a *translation* with velocity $\vec{v}_P$

2) a *deformation* due to the deformation tensor e_{ij}

3) a *rotation* with rotational vector $\vec{\omega} = \frac{1}{2}\,(\text{rot } \vec{v})_P$ due to the spin tensor w_{ij}

This is illustrated by the following important example.

Example 3: Couette-flow

Deformations and rotations are considered in a steady, two-dimensional flow with the velocity field:

$$\begin{aligned} v_1 &= 2\,k\,x_2 \quad \text{, where } k \text{ is a positive constant} \\ v_2 &= 0 \\ v_3 &= 0 \end{aligned}$$

The velocity profile is shown in Fig. 1.17. In practice this flow is seen between two parallel, plane plates, if the upper plate is moved at a constant velocity in its own plane and the lower plate is fixed.

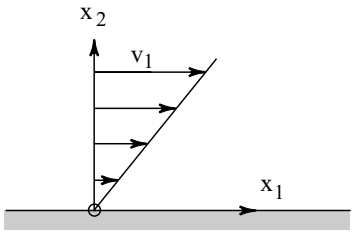

Figure 1.17: *Couette-flow: velocity profile.*

Because $\partial v_1/\partial x_1 = 0$, $\partial v_1/\partial x_2 = 2\,k$ and $\partial v_1/\partial x_3 = 0$ and all other derivatives are zero, the velocity gradient tensor reads:

$$\frac{\partial v_i}{\partial x_j} = \begin{bmatrix} 0 & 2\,k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The deformation tensor reads:

$$\begin{aligned} e_{ij} &= \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}) & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{2}(2\,k+0) & \frac{1}{2}(0+0) \\ \frac{1}{2}(0+2\,k) & 0 & \frac{1}{2}(0+0) \\ \frac{1}{2}(0+0) & \frac{1}{2}(0+0) & 0 \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

and the spin tensor reads:

$$w_{ij} = \begin{bmatrix} 0 & \frac{1}{2}(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}) & 0 & \frac{1}{2}(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2}(2\,k+0) & \frac{1}{2}(0+0) \\ \frac{1}{2}(0-2\,k) & 0 & \frac{1}{2}(0+0) \\ \frac{1}{2}(0+0) & \frac{1}{2}(0+0) & 0 \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of e_{ij} are found from the equation

$$det(e_{ij} - \lambda\,\delta_{ij}) = 0 \quad \Leftrightarrow$$

$$\begin{vmatrix} 0-\lambda & k & 0 \\ k & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0 \quad \Leftrightarrow$$

$$(0-\lambda)^3 - (0-\lambda)\cdot k \cdot k = 0$$

which after solution gives these eigenvalues:

$$\begin{aligned} \lambda_I &= -k \\ \lambda_{II} &= k \\ \lambda_{III} &= 0 \end{aligned}$$

The corresponding principal directions (eigenvectors) read:

$$\begin{aligned} (dx_I)_i &= (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0) \\ (dx_{II})_i &= (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) \\ (dx_{III})_i &= (0, 0, 1) \end{aligned}$$

These directions are shown in Fig. 1.18. Note that all three vectors are prescribed to have unit length. Transformed into the $x_1^*\,x_2^*\,x_3^*$-system the component from the deformation reads:

$$(dv_i^{def})^* = \begin{bmatrix} -k & 0 & 0 \\ 0 & +k & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1^* \\ dx_2^* \\ dx_3^* \end{bmatrix} = \begin{bmatrix} -k\,dx_1^* \\ k\,dx_2^* \\ 0 \end{bmatrix}$$

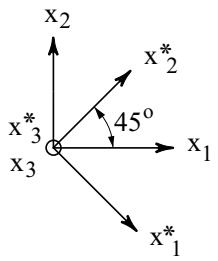

Figure 1.18: *Principal directions (eigenvectors) in Couette-flow.*

In Fig. 1.19 it is sketched, how this velocity field will deform an originally square element (broken line) around P into a rhombus (full line), because the element is shortened in the x_1^*-direction and extended in the x_2^*-direction. Thus the velocity field corresponding to the spin tensor rotates the rhombus clockwise (as $k > 0$), see Fig. 1.20, because

$$\vec{\omega} = \frac{1}{2}\text{rot } \vec{v} = \frac{1}{2}(0, 0, -2k) = (0, 0, -k)$$

Note that even though the correct size of the motions of the 4 corners has not been calculated, it is seen from Fig. 1.21 that added up, the three different motions, translation, deformation and rotation, might be able to produce the correct motion of the originally square element, i.e. a translation and a horizontal motion of the upper side of the square.

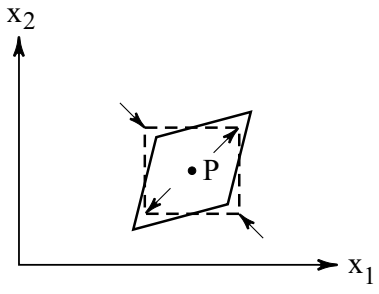

Figure 1.19: *Deformation of the fluid element at P.*

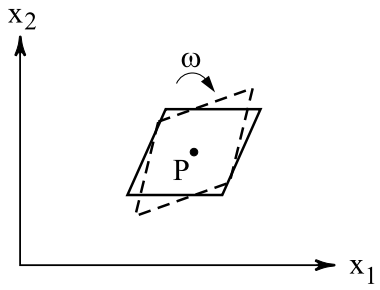

Figure 1.20: *Rotation of the fluid element at P.*

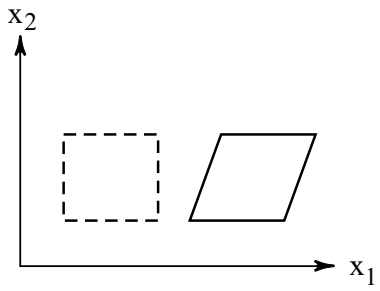

Figure 1.21: *Resulting motion of a fluid element at P.*

1.8 Velocity Potential

In this section we will consider irrotational flows, i.e. flows with velocity fields fulfilling the condition

$$\operatorname{rot} \vec{v} = \vec{0}$$

This fulfillment causes substantial computational savings, when the velocity field is calculated, as the velocity vector flow can be calculated by

$$\vec{v} = \operatorname{grad} \varphi$$

or

$$\boxed{v_i = \frac{\partial \varphi}{\partial x_i}} \tag{1.65}$$

where the scalar φ is called the *velocity potential*. It is thus only necessary to determine one variable, φ, within the flow domain, because v_1, v_2 og v_3 can subsequently be calculated by equation (1.65). Due to the existence of the velocity potential, irrotational flow is also often called *potential flow*.

First it is shown that a single-valued function, φ, exists, if the condition $\operatorname{rot} \vec{v} = \vec{0}$ is fulfilled in the flow domain.

To do that we adopt Stokes's theorem:

$$\Gamma = \int_L \vec{v} \cdot d\vec{s} = \int_A \operatorname{rot} \vec{v} \cdot d\vec{A} \tag{1.66}$$

If $\operatorname{rot} \vec{v} = \vec{0}$ everywhere on A, we find $\Gamma = 0$ along the closed boundary curve, L, to A. In the following we adopt line integrals of the type

$$\int_P^Q \vec{v} \cdot d\vec{s}$$

where P and Q are two points on the line, and the integration increment, $d\vec{s}$, is defined as positive from the start point P towards the final point Q. Adopting equation (1.66) on the curve shown in Fig. 1.22 yields:

$$\Gamma = \int_P^Q \vec{v} \cdot d\vec{s}_I + \int_Q^P \vec{v} \cdot d\vec{s}_{II} = 0 \tag{1.67}$$

where index I and II indicate the two parts, which form the total path of the line integral.

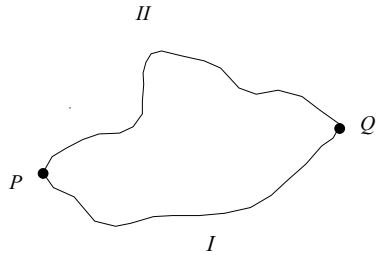

Figure 1.22: *Paths of line integrals between P and Q.*

Remembering the sign definition of $d\vec{s}$, we find:

$$\int_Q^P \vec{v} \cdot d\vec{s}_{II} = -\int_P^Q \vec{v} \cdot d\vec{s}_{II}$$

which substituted into equation (1.67) yields:

$$\int_P^Q \vec{v} \cdot d\vec{s}_I = \int_P^Q \vec{v} \cdot d\vec{s}_{II} \tag{1.68}$$

If we choose P as a *fixed* point and Q as a movable point, the definition:

$$\varphi_Q = \int_P^Q \vec{v} \cdot d\vec{s} \tag{1.69}$$

therefore gives a single-valued function φ, because the line integral is independent of the chosen path.

Next it is shown that this definition of φ also gives the correct velocity at Q, if the velocity is calculated by equation (1.65).

To do so, the two points Q and Q^* are placed as shown in Fig. 1.23. According to the definition of φ, equation (1.69), the difference in potential at the two

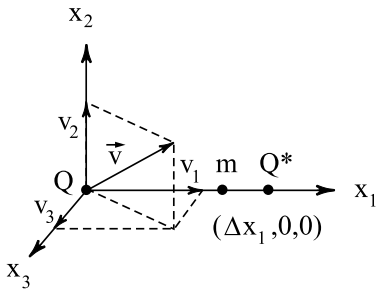

Figure 1.23: *Calculation of the velocity at point Q.*

points reads:

$$\begin{aligned}
\Delta\varphi &= \varphi_{Q^*} - \varphi_Q \\
&= \int_P^{Q^*} \vec{v}\cdot d\vec{s} - \int_P^{Q} \vec{v}\cdot d\vec{s} \\
&= \int_P^{Q} \vec{v}\cdot d\vec{s} + \int_Q^{Q*} \vec{v}\cdot d\vec{s} - \int_P^{Q} \vec{v}\cdot d\vec{s} \\
&= \int_Q^{Q^*} (v_1, v_2, v_3)\cdot(dx_1, 0, 0) = \int_0^{\Delta x_1} v_1\, dx_1 \\
&= (v_1)_m\, \Delta x_1
\end{aligned}$$

or

$$(v_1)_m = \frac{\Delta\varphi}{\Delta x_1} \tag{1.70}$$

where $(v_1)_m$ is the velocity component in the x_1-direction at the point m. According to the mean-value theorem for integrals, we only know that m is positioned somewhere between Q and Q^*, but the exact position is unknown. However, this is not a problem, if we let Q^* move arbitrarily close to Q. This gives:

$$(v_1)_m \to (v_1)_Q \quad \text{for} \quad \Delta x_1 \to 0 \qquad \text{and} \qquad \left.\frac{\Delta\varphi}{\Delta x_1}\right|_{\Delta x_1 \to 0} = \frac{\partial\varphi}{\partial x_1}$$

and therefore

$$(v_1)_Q = \frac{\partial\varphi}{\partial x_1} \tag{1.71}$$

As similar expressions are found in the x_2- and x_3-directions, it has been shown that $\vec{v} = \text{grad}\varphi$, provided φ is defined by equation (1.69).

Note that the above-mentioned proof is valid only, if the region, where rot $\vec{v} = \vec{0}$, is a so-called *simply connected* region. The formal definition of a simply connected region is a region, where any closed curve lying within the area can

be shrunk to a point without leaving the area. Roughly speaking, a simply connected region is a region without holes.

If we consider the flow in Example 1, where $(v_r, v_\theta) = (0, k/r)$, we have $\mathrm{rot}\,\vec{v} = \vec{0}$ at all points except at $r = 0$. A region containing the point $r = 0$ is therefore *not* simply connected.

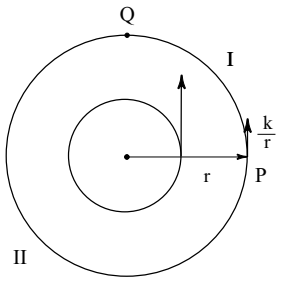

Figure 1.24: *Region with* $\mathrm{rot}\,\vec{v} = \vec{0}$ *for* $r > 0$.

If the points P and Q are placed as shown in Fig. 1.24, adoption of the two different integration paths, denoted I and II gives

$$(\varphi_Q)_I = \int_P^Q \vec{v} \cdot d\vec{s}_I = +\frac{k}{r}\frac{1}{2}\pi\, r = +\frac{1}{2}\pi\, k \tag{1.72}$$

and

$$(\varphi_Q)_{II} = \int_P^Q \vec{v} \cdot d\vec{s}_{II} = -\frac{k}{r}\frac{3}{2}\pi\, r = -\frac{3}{2}\pi\, k \tag{1.73}$$

It is seen that the potential Q is *not* single-valued, nor is the the gradient of the potential. However, the region may be transformed into a simply connected region by making a slot along a radius through point P, see Fig. 1.25. Along the two boundaries on each side of the slot velocities and pressures must be equal. With this slot it is no longer possible to have the origin placed within a region defined by a closed integration path and the potential Q becomes single-valued.

Substitution of $v_i = \partial\varphi/\partial x_i$ into the continuity equation $\partial v_i/\partial x_i = 0$ yields:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial x_i}\right) = 0$$

or

$$\frac{\partial^2 \varphi}{\partial x_i\, \partial x_i} = 0$$

which also can be written as

$$\boxed{\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial x_3^2} = 0} \tag{1.74}$$

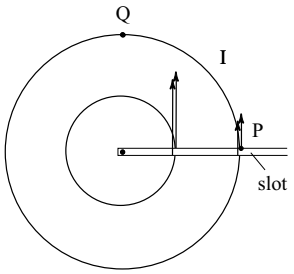

Figure 1.25: *Simply connected (slotted) region.*

Thus the velocity potential φ fulfills the *Laplace* equation.

Example: Velocity potential in uniform flow

We want to determine the velocity potential corresponding to *two-dimensional, uniform* flow, i.e. a flow having the same velocity, $v_i = (v_o, 0)$, at all points giving a rectangular velocity profile. See Fig. 1.26.

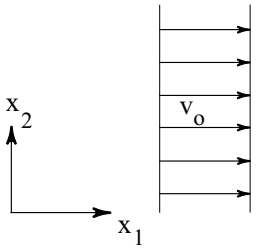

Figure 1.26: *Velocity profile in uniform flow.*

A vorticity meter placed in this flow does not rotate, as none of the two plates are affected to turning by the fluid, giving $(\mathrm{rot}\,\vec{v})_3 = 0$. This may also be seen by calculating $(\mathrm{rot}\,\vec{v})_3$ directly. This gives:

$$(\mathrm{rot}\,\vec{v})_3 \quad = \quad \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0 - 0 = 0$$

As the flow is two-dimensional, this gives $\mathrm{rot}\,\vec{v} = \vec{0}$. Thus uniform flow is a potential flow.

The velocity potential is calculated by usage of the definition equations, i.e.

$$v_1 = \frac{\partial \varphi}{\partial x_1} = v_o \qquad \Leftrightarrow \qquad \varphi = v_o\, x_1 + f(x_2)$$

which substituted into

$$v_2 = \frac{\partial \varphi}{\partial x_2} = 0$$

gives

$$0 + \frac{\partial f(x_2)}{\partial x_2} = 0 \qquad \Leftrightarrow \qquad 0 + \frac{df(x_2)}{dx_2} = 0$$

or

$$f(x_2) = K$$

where K is a constant. Finally, this gives:

$$\varphi = v_o\, x_1 + K$$

The *equipotential* lines, i.e. lines where φ is constant along the line, are straight lines parallel to the x_2-axis.

1.9 Stream Function

For a *two-dimensional* flow of an incompressible fluid it is possible to define a scalar, called the *stream function*, which has *three* convenient properties. We denote the stream function $\psi = \psi(x_1, x_2, t)$, and let the following equations define ψ:

$$\boxed{v_1 = -\frac{\partial \psi}{\partial x_2}} \tag{1.75}$$

$$\boxed{v_2 = +\frac{\partial \psi}{\partial x_1}} \tag{1.76}$$

Substitution of these definition equations into the expression for divergence yields:

$$\operatorname{div} \vec{v} = \frac{\partial}{\partial x_1}\left(-\frac{\partial \psi}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial \psi}{\partial x_1}\right) = -\frac{\partial^{\,2}\psi}{\partial x_1\, \partial x_2} + \frac{\partial^{\,2}\psi}{\partial x_1\, \partial x_2} \equiv 0$$

giving that the continuity equation

$$\operatorname{div} \vec{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \tag{1.77}$$

is always fulfilled. This is the *first* convenient property of the stream function. Notice that it is *not* necessary to assume $\operatorname{rot} \vec{v} = \vec{0}$ in order to define a stream function, but the fluid has to be incompressible.

The equation defining a streamline reads:

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2}$$

or

$$v_2\, dx_1 - v_1\, dx_2 = 0$$

where dx_i is a vector along the streamline. Substitution of equation (1.75) and equation (1.76) into this equation gives:

$$\frac{\partial \psi}{\partial x_1} dx_1 + \frac{\partial \psi}{\partial x_2} dx_2 = 0 \qquad \text{along a streamline} \tag{1.78}$$

If the flow is considered at a given time $t = t_o$, we have in general:

$$d\psi = \frac{\partial \psi}{\partial x_1} dx_1 + \frac{\partial \psi}{\partial x_2} dx_2 \tag{1.79}$$

Thus for dx_i orientated *along* the streamline, equation (1.78) gives:

$$d\psi = 0$$

or

$$\boxed{\psi = constant} \qquad \text{along a streamline} \tag{1.80}$$

This is the *second* convenient property of the stream function. A *third* convenient property is found by considering the difference between values of the stream functions on two streamlines, see Fig. 1.27.

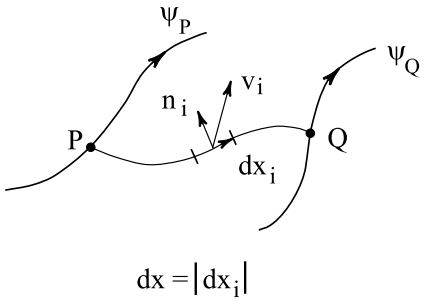

Figure 1.27: *Integration path between two streamlines.*

The two points P and Q are placed arbitrarily on each of the two streamlines, and a curve of arbitrary shape is drawn between point P and point Q. This curve might be considered to be the intersection between a curved surface perpendicular to the flow plane and the flow plane itself. We want to calculate the *volume*

flux (sometimes also called *discharge*) per metre through this curved surface. If the volume flux per metre is denoted q, the expression for q reads:

$$q = \int_P^Q v_i \, n_i |dx_i| = \int_P^Q v_i \, n_i \, dx \tag{1.81}$$

where n_i is the unit normal vector pointing in the main flow direction and dx_i is placed along the curve between P and Q. The expression for n_i reads:

$$n_i = (-dx_2, dx_1)/dx$$

as this gives $n_i \, dx_i = n_1 \, dx_1 + n_2 \, dx_2 = (-dx_2 \, dx_1 + dx_1 \, dx_2)/dx = 0$ and $n = |n_i| = 1$. Thus the expression for q can be rewritten to:

$$\begin{aligned} q &= \int_P^Q v_i \, n_i \, dx = \int_P^Q \left(v_1 \, (-dx_2) + v_2 \, dx_1 \right) \\ &= \int_P^Q \left((-\frac{\partial \psi}{\partial x_2}) \, (-dx_2) + \frac{\partial \psi}{\partial x_1} \, dx_1 \right) = \int_P^Q d\psi \\ &= [\, \psi \,]_P^Q = \psi_Q - \psi_P \end{aligned} \tag{1.82}$$

The volume flux between two streamlines is therefore equal to the *difference between stream function values on the two streamlines.* It is also seen that stream function is increasing to the *right-* hand side, when looking in the flow direction.

Example : Stream function in uniform flow

We want to determine the stream function corresponding to *two-dimensional, uniform* flow, i.e. a flow having the same velocity, $v_i = (v_o, 0)$, at all points giving a rectangular velocity profile, see Fig. 1.26. From equations 1.75 we get

$$v_1 = -\frac{\partial \psi}{\partial x_2} = v_o \qquad \Leftrightarrow \qquad \psi = -v_o \, x_2 + f(x_1)$$

which substituted into

$$v_2 = \frac{\partial \psi}{\partial x_1} = 0$$

gives

$$0 + \frac{\partial f(x_1)}{\partial x_1} = 0 \qquad \Leftrightarrow \qquad 0 + \frac{df(x_1)}{dx_1} = 0$$

or

$$f(x_1) = C$$

where C is a constant. Finally, this gives:

$$\psi = -v_o \, x_2 + C$$

The streamlines, i.e. the curves on which ψ is a constant, are therefore straight lines parallel to the x_1-axis. It is also seen that ψ is increasing to the right-hand side, when looking in the flow direction.

If the equations (1.75) and (1.76) are substituted into

$$(\text{rot}\,\vec{v})_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$$

the result reads:

$$(\text{rot}\,\vec{v})_3 = \frac{\partial}{\partial x_1}\left(\frac{\partial \psi}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(-\frac{\partial \psi}{\partial x_2}\right)$$

or

$$\boxed{\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = (\text{rot}\,\vec{v})_3} \tag{1.83}$$

It is seen that the governing equation for the stream function is a *Poisson equation.*

If the field of $(\text{rot}\,\vec{v})_3$ is known at a given time, solution of equation (1.83) gives the ψ-field, which again makes it possible to find streamlines and the velocity field.

Finally, notice finally that in general it is impossible to define a stream function for 3-dimensional flow. It is, however, possible to do it for flows symmetrical about an axis.

Chapter 2

Fluid Dynamics

2.1 State of Stress at a Point in a Fluid

In this section it is shown that in general we can adopt a stress concept for a fluid, which is quite analogous to the stress concept normally adopted for solids. In both cases the medium is considered a continuum, but the causes to the stresses in fluids and solids are quite different.

In the description of the motion of a fluid body it is necessary to consider the following two classes of forces: forces acting throughout the mass of the body (gravitational forces) and forces acting on the surface of the body (pressure forces and shear forces) from the surrounding fluid.

We consider a surface with area ΔA and unit normal vector $\vec{n}$, and we imagine that the fluid is removed on the side in the direction of the normal, but the forces from the removed fluid on the area are kept. Generally these forces are equivalent with the force $\Delta\vec{F}$ and a moment $\Delta\vec{M}$ from a pair of forces. If we divide $\Delta\vec{F}$ by ΔA and let $\Delta A \to 0$, the result approaches a limit value called *stress vector*, $\vec{\sigma}^N$, i.e.

$$\left(\frac{\Delta\vec{F}}{\Delta A}\right)_{\Delta A\to 0} = \frac{d\vec{F}}{dA} = \vec{\sigma}^N \tag{2.1}$$

However, $\Delta\vec{M}/\Delta A \to 0$ because both the pair of forces *and* the distance between the forces approach zero.

In general $\vec{\sigma}^N$ varies from point to point and it also depends on the direction of the normal $\vec{n}$. If instead we had considered the forces on the other side of the surface, the normal vector should then be expressed as $-\vec{n}$ and the stress vector

on that side is denoted $\vec{\sigma}^{-N}$. According to Newton's 3rd law (action=re-action) we get

$$\vec{\sigma}^{N} = -\vec{\sigma}^{-N} \tag{2.2}$$

In order to investigate how the stress vector $\vec{\sigma}^{N}$ at a *fixed* point depends on the direction of the normal, we shall consider at a tetrahedron of fluid cut free from the rest of the fluid, see Fig. 2.1. Stress vectors on surfaces perpendicular to the

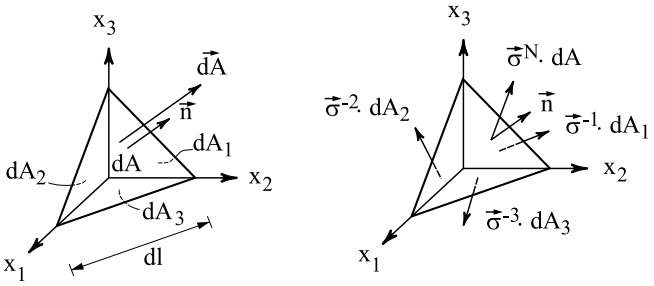

Figure 2.1: *Definition sketch, forces on surface of thetrahedron.*

coordinate axis x_i are here denoted $\vec{\sigma}^{i}$, where

$$\vec{\sigma}^{i} = (\sigma_1^i, \sigma_2^i, \sigma_3^i) \tag{2.3}$$

is acting on the side of the surface, where the normal points in the same direction as x_i. Otherwise the stress vector is denoted $\vec{\sigma}^{-i}$. Thus, the stress vector on the surface in the $x_2\, x_3$-plane depicted in Fig. 2.1 is denoted $\vec{\sigma}^{-1}$.

The expression for the unit normal $\vec{n}$ reads:

$$\vec{n} = n_1\, \vec{e}_1 + n_2\, \vec{e}_2 + n_3\, \vec{e}_3 = (n_1, n_2, n_3) = n_i \tag{2.4}$$

If θ denotes the angle between $\vec{n}$ and $\vec{e}_1$, we have $\vec{n} \cdot \vec{e}_1 = n_1 = 1 \cdot 1 \cdot \cos\theta$ or

$$n_1 = \cos\theta$$

Therefore, n_1 is often called the direction cosine between the normal and the unit vector $\vec{e}_1$. If the expression for n_1 is substituted into

$$\vec{dA} = dA\, \vec{n} \qquad \text{or} \qquad dA_i = dA\, n_i \tag{2.5}$$

the expression for the component in the x_1-direction reads:

$$dA_1 = dA \cdot n_1 = dA \cdot \cos\theta$$

As θ is also a the angle between the two planes perpendicular to $\vec{n}$ and $\vec{e}_1$, it is seen that dA_1 is the projection of the area dA on the $x_2\, x_3$-plane, see Fig. 2.1.

In the same way it is seen that n_2 and n_3 are the direction cosines between the normal and the unit vectors $\vec{e}_2$ og $\vec{e}_3$, respectively. Also dA_2 and dA_3 may be interpreted as projected areas, see Fig. 2.1.

For the tetrahedron the sum of the surface forces reads:

$$\begin{aligned}\sum \vec{F}_{surface} &= \vec{\sigma}^N\, dA + \vec{\sigma}^{-1}\, dA_1 + \vec{\sigma}^{-2}\, dA_2 + \vec{\sigma}^{-3}\, dA_3 \\ &= \vec{\sigma}^N\, dA - \vec{\sigma}^1\, dA_1 - \vec{\sigma}^2\, dA_2 - \vec{\sigma}^3\, dA_3\end{aligned}$$

after adoption of equation (2.2). If gravitation is assumed to be the only body force, Newton's 2nd law for the tetrahedron reads:

$$\rho\, dX\, \frac{\vec{dv}}{dt} = \rho\, \vec{g}\, dX + \vec{\sigma}^N\, dA - \vec{\sigma}^1\, dA_1 - \vec{\sigma}^2\, dA_2 - \vec{\sigma}^3\, dA_3$$

Shrinking of the tetrahedron, while keeping $\vec{n}$ constant, gives:

$$\frac{dX}{dA} = \frac{k\, dA\, dl}{dA} = k\, dl \to 0 \quad \text{for} \quad dl \to 0$$

where dl is a characteristic length of the tetrahedron. In this way Newton's 2nd law reads:

$$0 = 0 + \vec{\sigma}^N - \vec{\sigma}^1\, \frac{dA_1}{dA} - \vec{\sigma}^2\, \frac{dA_2}{dA} - \vec{\sigma}^3\, \frac{dA_3}{dA}$$

or

$$\vec{\sigma}^N = \vec{\sigma}^1\, \frac{dA_1}{dA} + \vec{\sigma}^2\, \frac{dA_2}{dA} + \vec{\sigma}^3\, \frac{dA_3}{dA}$$

Here substitution of equation (2.5) gives:

$$\vec{\sigma}^N = \vec{\sigma}^1\, n_1 + \vec{\sigma}^2\, n_2 + \vec{\sigma}^3\, n_3 \tag{2.6}$$

According to equation (2.3) σ_1^1, σ_1^2 and σ_1^3 are the components in the x_1-direction of $\vec{\sigma}^1$, $\vec{\sigma}^2$ and $\vec{\sigma}^3$, respectively. We see that the component in the x_1-direction of $\vec{\sigma}^N$ reads:

$$\begin{aligned}\sigma_1^N &= n_1\, \sigma_1^1 + n_2\, \sigma_1^2 + n_3\, \sigma_1^3 \\ &= \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_1^1 \\ \sigma_1^2 \\ \sigma_1^3 \end{bmatrix}\end{aligned} \tag{2.7}$$

Similarly it is found that

$$\sigma_2^N = n_1\, \sigma_2^1 + n_2\, \sigma_2^2 + n_3\, \sigma_2^3$$

and

$$\sigma_3^N = n_1\, \sigma_3^1 + n_2\, \sigma_3^2 + n_3\, \sigma_3^3$$

These three equations may be compiled to:

$$\vec{\sigma}^N = \left[\begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \right] \left[\begin{array}{ccc} \sigma_1^1 & \sigma_2^1 & \sigma_3^1 \\ \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_1^3 & \sigma_2^3 & \sigma_3^3 \end{array} \right] \tag{2.8}$$

In tensor notation this equation reads:

$$\sigma_i^N = n_j \, \sigma_i^j \tag{2.9}$$

In the literature it is often seen that both 'i' og 'j' are written as indices, i.e.

$$\sigma_{ji} = \sigma_i^j \tag{2.10}$$

and this notation will be adopted in the following. With this notation equation (2.8) reads:

$$\vec{\sigma}^N = \left[\begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \right] \left[\begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array} \right] \tag{2.11}$$

and in tensor notation this equation reads:

$$\sigma_i^N = n_j \, \sigma_{ji} \tag{2.12}$$

where σ_{ji} is called the *stress tensor*.

For a solid this tensor is symmetric, and it will be shown below that this is also the case for the stress tensor of a fluid.

A fluid cube with edge length dx is depicted in Fig. 2.2. The figure are also shows the stresses creating a moment about an axis through the centre of gravity of the cube and parallel to the x_1-axis.

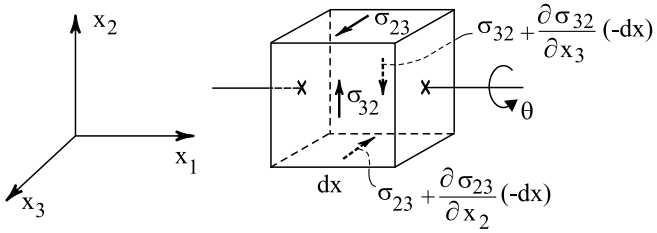

Figure 2.2: *Definition sketch, stresses on a cube.*

The general equation of motion with respect to the angular motion about the axis reads:

$$\sum M_{ext} = I\,\ddot{\theta} \tag{2.13}$$

where $\sum M_{ext}$ is the moment from the external forces about the axis, I is the mass moment of inertia about the axis, and θ is the turning angle. As the external forces are solely stresses on the surface, equation (2.13) reads:

$$\begin{aligned}\sum M_{ext} \quad = \quad & +(\sigma_{23} + (\sigma_{23} - \frac{\partial \sigma_{23}}{\partial x_2}\,dx))\,dx^2\,\frac{dx}{2} \\ & -(\sigma_{32} + (\sigma_{32} - \frac{\partial \sigma_{32}}{\partial x_3}\,dx))\,dx^2\frac{dx}{2}\end{aligned} \tag{2.14}$$

Radius of inertia of a cube is $i = dx/\sqrt{6}$ giving:

$$I = m\,i^2 = \rho\,dx^3 \cdot \frac{1}{6}\,dx^2 = \rho\,\frac{1}{6}\,dx^5 \tag{2.15}$$

Substitution of the equations (2.14) and (2.15) into equation (2.13) followed by a division by dx^3 gives:

$$\frac{1}{2}(\sigma_{23} + (\sigma_{23} - \frac{\partial \sigma_{23}}{\partial x_2}\,dx)) - \frac{1}{2}(\sigma_{32} + (\sigma_{32} - \frac{\partial \sigma_{32}}{\partial x_3}\,dx)) = \rho\,\frac{1}{6}\,dx^2\,\ddot{\theta} \tag{2.16}$$

Letting $dx \to 0$ gives:

$$\frac{1}{2}(\sigma_{23} + \sigma_{23}) - \frac{1}{2}(\sigma_{32} + \sigma_{32}) \to 0$$

unless $\ddot{\theta} \to \infty$, but this is not possible from a physical point of view. We therefore end up with

$$\sigma_{23} = \sigma_{32}$$

Considering the motions about the x_2- and x_3- axis, respectively, we similarly obtain:

$$\sigma_{13} = \sigma_{31} \qquad \text{and} \qquad \sigma_{12} = \sigma_{21}$$

From these three expressions of the shear stresses, we see that the stress tensor in a fluid is *symmetric*.

From calculus it is known that the stress field corresponding to a symmetric stress tensor has 3 principal stress directions with principal stress values here denoted σ_I, σ_{II} and σ_{III}. It is also known that the trace of the stress tensor is invariant to a coordinate transformation, i.e.

$$\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_I + \sigma_{II} + \sigma_{III} = constant \tag{2.17}$$

Finally the symmetry yields that the expression for the stress vector acting on a surface, equation (2.14), may be rewritten as:

$$\sigma_i^N = n_j\,\sigma_{ji} = n_j\,\sigma_{ij}$$

or

$$\boxed{\sigma_i^N = \sigma_{ij}\, n_j} \tag{2.18}$$

2.1.1 Stresses in a Resting Fluid

Per definition, the shear stresses in a resting fluid are zero. Consequently the stress vector on a plane is perpendicular to the plane despite the direction of the normal to the plane, i.e. all directions are principal directions and all planes are principal planes. This causes the size of normal stress at a point to be independent of the direction of the normal to the plane, which is shown by considering the fluid body depicted in Fig. 2.3.

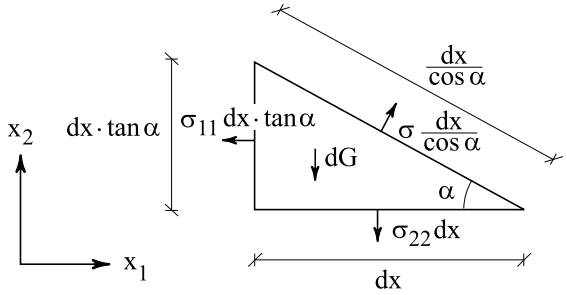

Figure 2.3: *Definition sketch, stresses on a fluid body in a resting fluid.*

As the considered body is at rest, the equations of motion are reduced to equilibrium equations.

Vertical equilibrium:

$$\sigma\, \frac{dx}{\cos\alpha}\, \cos\alpha = \sigma_{22}\, dx + \rho\, g\, \frac{1}{2}\, dx^2\, \tan\alpha$$

After division by dx, the expression for the stress on the inclined plane reads:

$$\sigma = \sigma_{22} + \rho\, g\, \frac{1}{2}\, dx\, \tan\alpha$$

Letting $dx \to 0$, while keeping α constant (less than 90°) gives:

$$\sigma = \sigma_{22}$$

Horizontal equilibrium:

$$\sigma \frac{dx}{\cos\alpha} \sin\alpha = \sigma_{11}\, dx \, \tan\alpha$$

or

$$\sigma = \sigma_{11}$$

Similarly, turning of the body 90° about a vertical axis gives:

$$\sigma = \sigma_{33}$$

Therefore, the normal stress at a point is the same on all planes through the point, and the expression for the stress tensor at a point reads:

$$\boxed{\sigma_{ij} = \sigma\, \delta_{ij}} \tag{2.19}$$

where δ_{ij} is Kronecker's tensor.

2.1.2 Stresses in a Moving Fluid

In general the shear stresses in a moving fluid are *not* zero, but normally they are very small compared to the normal stresses.

Example : Stresses in Couette-flow

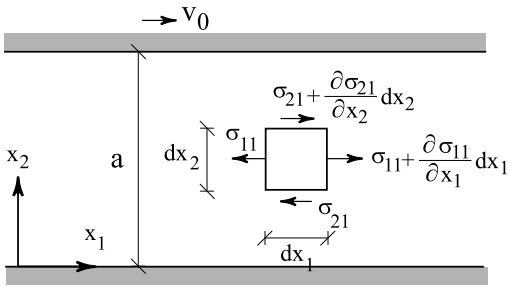

Figure 2.4: *Definition sketch, Couette-flow.*

We consider the plane flow between two horizontal, parallel plates, see Fig.2.4. The upper plate is forced to move at the constant velocity v_o. If we assume that

$$\frac{\partial\, \sigma_{11}}{\partial\, x_1} = 0 \tag{2.20}$$

the flow is therefore solely driven by the motion of the upper plate. Due to the parallel plates the flow is *uniform*, and due to the constant velocity of the upper plate, the flow is also *steady*. The velocity field may be expressed as:

$$v_i = (v_1(x_2), 0)$$

Consequently, the acceleration of a fluid particle is zero in the x_1- and the x_2-direction, and the horizontal component of Newton's 2nd law for the fluid particle reads:

$$0 = \frac{\partial \sigma_{21}}{\partial x_2} dx_2 \cdot dx_1 + \frac{\partial \sigma_{11}}{\partial x_1} dx_1 \cdot dx_2$$

Substitution of equation (2.20) and division by $dx_1 \cdot dx_2$ gives:

$$\frac{\partial \sigma_{21}}{\partial x_2} = 0$$

or

$$\sigma_{21} = K(x_1)$$

However, K cannot depend on x_1 in a uniform flow, and σ_{21} is therefore constant. As the stress tensor has to be symmetric, we find:

$$\boxed{\sigma_{21} = \sigma_{12} = K} \tag{2.21}$$

at all points in the flow region.

The velocity field is calculated by adoption of Newton's formula. This gives

$$\sigma_{21} = \mu \frac{\partial v_1}{\partial x_2} = \mu \frac{dv_1}{dx_2} \quad \text{(as } v_1 \text{ only depends on } x_2\text{)}.$$

After substitution of equation (2.21) this expression reads:

$$\mu \frac{dv_1}{dx_2} = K$$

which after integration gives:

$$v_1 = \frac{K}{\mu} x_2 + K_1$$

i.e. the velocity profile is a straight line. Due to adhesion between the fluid and the plates, the boundary conditions read:

$$\begin{aligned} v_1 &= 0 \qquad \text{for} \quad x_2 = 0 \\ v_1 &= v_o \qquad \text{for} \quad x_2 = a \end{aligned}$$

The first boundary condition gives $K_1 = 0$ and the second leads to $K = \mu\, v_o/a$. Therefore, the expression for the velocity profile reads:

$$\boxed{v_1 = \frac{v_0}{a}\, x_2}$$

This flow is called a *Couette*-flow, and it should be noticed that results obtained from considering this flow are important, when it comes to setting up the constitutive equations for a fluid.

2.2 Equations of Motion

The equations of motion for a fluid body are set up by a summation of Newton's 2nd law for all the fluid particles, which make up the body considered. As all forces between the particles, two and two, are equal and opposite, the resultant force on the body only consists of body forces on the volume of the body, X, and forces acting on the surface of the body, A.

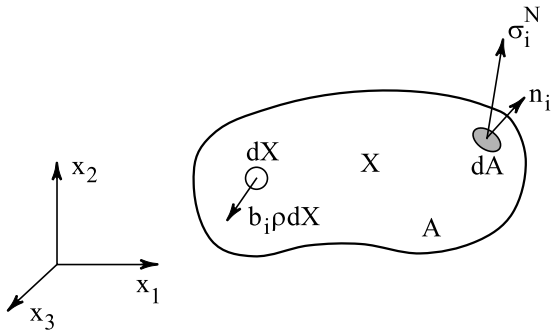

Figure 2.5: *Definition sketch, equations of motion.*

The body forces (defined as force per unit mass) considered in fluid dynamics are normally gravity forces, inertial forces ($-\frac{d\vec{v}}{dt}$) and the Coriolis force. Inclusion of the latter is only necessary, if we want to describe the motion in a coordinate system fixed to the Earth, i.e. a non-inertial system rotating with the angular velocity of the Earth, $\vec{\Omega}$.

In general the expression for the body force , $\vec{b} = b_i$, reads:

$$\vec{b} = \vec{g} - \frac{d\vec{v}}{dt} - 2\,\vec{\Omega} \times \vec{v} \tag{2.22}$$

Notice that the horizontal component of the Coriolis force can also be expressed as

$$2\,\Omega\,\sin\left(lat\right)\,\hat{\vec{v}} \tag{2.23}$$

where lat is the latitude of the position and $\hat{\vec{v}}$ is $\vec{v}$ rotated 90°clockwise. On the northern hemisphere ($lat > 0$) the Coriolis force is thus orientated to the right, if one looks in the direction of motion. Normally, the Coriolis force is unimportant as $\Omega = (2\,\pi)/(24 \cdot 60 \cdot 60) = 7.27 \cdot 10^{-5}$, but for flows with a large time scale or for flows, where the horizontal components of the other forces are small (flows in the atmosphere), it is necessary to include the Coriolis force.

Inclusion of the inertia forces into the body force makes it possible to interpret Newton's 2nd law as an equilibrium equation. For the body depicted in Fig. 2.5 this equation reads:

$$\int_A \sigma_i^N\,dA + \int_X b_i\,\rho\,dX = 0 \quad , \quad i = 1,2,3 \tag{2.24}$$

where $\sigma_i^N = \vec{\sigma}^N$ is the stress vector on dA and $\vec{n} = n_i$ is defined as the outward, unit normal to the plane. The general expression for the stress vector reads:

$$\sigma_i^N = \sigma_{ij}\,n_j \tag{2.25}$$

giving this x_1-component:

$$\sigma_1^N = \sigma_{1j}\,n_j = \begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \vec{\sigma}^1 \cdot \vec{n}$$

Hereby the x_1-component of the surface integral in equation (2.24) can be written:

$$\begin{aligned}\int_A \sigma_1^N\,dA &= \int_A \vec{\sigma}^1 \cdot \vec{n}\,dA \\ &= \int_X \mathrm{div}(\tilde{\sigma}^1)\;\mathrm{dX} \\ &= \int_X \frac{\partial \sigma_j^1}{\partial x_j}\,dX = \int_X \frac{\partial \sigma_{1j}}{\partial x_j}\,dX\end{aligned}$$

after adoption of Gauss's divergence theorem and $\sigma_{1j} = \sigma_j^1$. Similar equations are valid for the two other components of the surface integral in equation (2.24), and in general we get

$$\int_A \sigma_i^N\,dA = \int_X \frac{\partial \sigma_{ij}}{\partial x_j}\,dX \qquad , \; i = 1,2,3$$

Substitution of this expression into equation (2.24) gives:

$$\int_X \left(\frac{\partial \sigma_{ij}}{\partial x_j} + \rho\,b_i\right)\,dX = 0 \qquad , \; i = 1,2,3 \tag{2.26}$$

Equation (2.26) is seen to be satisfied for an arbitrary volume X if

$$\boxed{\frac{\partial \sigma_{ij}}{\partial x_j} + \rho\, b_i = 0} \tag{2.27}$$

which is the general equation of motion for a fluid. This equation is identical with the equilibrium equation for a solid, but even though stresses are described by equation (2.25) in both fluid and solids, the cause of stresses is different for the two types of substance. Also remember also that inertia force and Coriolis's force are "hidden" in the body force for a fluid. If only gravity force and inertia force are taken into account in the body force, we get:

$$b_i = g_i - \frac{dv_i}{dt}$$

and the equation of motion reads:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho\,(g_i - \frac{dv_i}{dt}) = 0$$

or

$$\boxed{\rho\,\frac{dv_i}{dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho\, g_i} \tag{2.28}$$

It is clearly seen that in order to calculate the velocity field a relation between σ_{ij} and v_i is necessary, i.e. a so-called *constitutive* equation for the fluid has to be established.

2.3 Constitutive Equation for a Newtonian Fluid

In order to establish the constitutive equation for a fluid we attempt to generalize the knowledge obtained from the description of some simple flows as e.g. resting fluid, pure translation, rotation of the fluid as a rigid body and the Couette-flow.

Simple static/dynamic considerations for the first three cases show that shear stresses are absent in these flows. It is also clear that in these three cases, the deformation tensor is zero, $e_{ij} = 0$.

On the other hand, it was seen that for the Couette-flow e_{ij} was constant in the entire flow domain, and σ_{ij} was also constant, but only for $i \neq j$. The motions corresponding to e_{ij} of a originally square body are shown in Fig. 2.6, and next to it the shear stresses on the same body are shown.

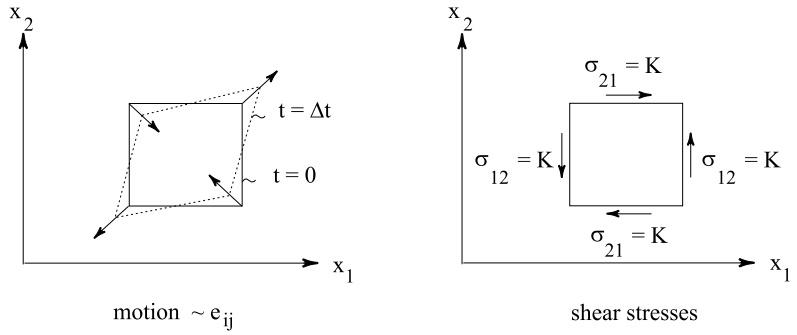

Figure 2.6: *Deformations and shear stresses in Couette-flow.*

As the shear stresses tend to deform the body in the way actually done by e_{ij}, it is obvious to consider constitutive equations of the type:

$$\sigma_{ij} = f(e_{ij}) \tag{2.29}$$

where so far the function f is unknown.

This equation cannot, however, describe the conditions correctly in a resting fluid, where the stress state is *isentropic*

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} = \sigma\,\delta_{ij} \tag{2.30}$$

and all velocity gradients are zero giving:

$$e_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Instead of σ_{ij} we consider the tensor σ^d_{ij}, which describes the deviations from the isentropic stress state, i.e.

$$\sigma_{ij} = \bar{\sigma}\ \delta_{ij} + \sigma^d_{ij} \tag{2.31}$$

or

$$\sigma^d_{ij} = \sigma_{ij} - \bar{\sigma}\ \delta_{ij} = \begin{bmatrix} \sigma_{11} - \bar{\sigma} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \bar{\sigma} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \bar{\sigma} \end{bmatrix} \tag{2.32}$$

where $\bar{\sigma}$ is defined as

$$\bar{\sigma} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \tag{2.33}$$

This definition is adopted, as the sum of the elements in the diagonal of σ_{ij} does not depend on the actual coordinate system. In this way the definition of $\bar{\sigma}$ is unique at all points.

In a resting fluid without shear stresses we have $\sigma_{ij}^d = 0$, and this fact makes it possible to relate σ_{ij}^d and e_{ij}.

In order to keep things simple we first consider the simple, linear equation:

$$\sigma_{ij}^d = C \cdot e_{ij} \tag{2.34}$$

where C is a constant.

Written fully expanded, equation (2.34) reads:

$$\begin{bmatrix} \sigma_{11} - \bar{\sigma} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \bar{\sigma} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \bar{\sigma} \end{bmatrix}$$

$$= C \cdot \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}) & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

However, the constitutive equation (2.34) considered is still only a *hypothesis*, and at least it has to be shown that adoption of the hypothesis gives correct results for the four simple flows. Otherwise the hypothesis must be rejected.

The requirements to be fulfilled are:

1) a symmetrical stress tensor, i.e. $\sigma_{ij} = \sigma_{ji}$

2) Newton's formula for Couette-flow, i.e. $\sigma_{21} = \mu \, \frac{\partial v_1}{\partial x_2}$

3) $\sigma_{ij} = 0$ in a resting flow for $i \neq j$ and $\sigma_{11} = \sigma_{22} = \sigma_{33}$

As shown below the hypothesis $\sigma_{ij}^d = C \, e_{ij}$ fulfills these requirements:

ad 1) As $e_{ij} = e_{ji}$, equation (2.34) directly gives $\sigma_{ij}^d = \sigma_{ji}^d$. According to equation (2.31) this gives $\sigma_{ij} = \sigma_{ji}$. The stress tensor is therefore symmetrical.

ad 2) In a Couette-flow with $v_i = (2\,k\,x_2, 0, 0)$ we have

$$e_{21} = \frac{1}{2}(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}) = \frac{1}{2}\,\frac{\partial v_1}{\partial x_2}$$

Substitution into equation (2.34) gives:

$$\sigma^d_{21} = C\,\frac{1}{2}\frac{\partial v_1}{\partial x_2}$$

However, according to equation (2.32) we have $\sigma^d_{21} = \sigma_{21}$, giving:

$$\sigma_{21} = C\,\frac{1}{2}\frac{\partial v_1}{\partial x_2}$$

So far C is an arbitrary constant, but with $C = 2\,\mu$ it is seen that the stress predicted by the hypothesis is actually the same as Newton's formula.

ad 3) In a resting fluid we have $e_{ij} = 0$, and according to equation (2.34), we get $\sigma^d_{ij} = 0$. From equation (2.32) is seen that $\sigma_{ij} = \bar{\sigma}\;\delta_{ij}$ and therefore $\sigma_{ij} = 0$ for $i \neq j$. Furthermore we see that $\sigma_{11} = \sigma_{22} = \sigma_{33}$ in a resting fluid.

Consequently, we can accept the hypothesis with $C = 2\,\mu$, which gives:

$$\sigma^d_{ij} = 2\,\mu\,e_{ij} \tag{2.35}$$

After substitution of this expression into equation (2.31) the constitutive equation reads:

$$\boxed{\sigma_{ij} = \bar{\sigma}\,\delta_{ij} + 2\,\mu\,e_{ij}} \tag{2.36}$$

Notice that equation (2.36) is still a hypothesis and should only be finally accepted, if it also gives satisfactory predictions for flows in general. However, this is actually the case for many fluids as e.g. water, air and oil. All fluids having this linear relation between the stress and the deformation tensor are called *newtonian fluids*.

On the other hand, many fluids do not have a linear relation between the stress and the deformation tensor. Typical examples are honey, asphalt, sludge from a sewage disposal plant and non-drip paint, but the description of such fluids is beyond the framework of this book.

Equation (2.36) shows that in a *moving* newtonian fluid the normal stresses depend on the orientation of the normal to the plane, but in practice the deviations from an isotropic stress state are insignificant. The x_1-component of equation (2.36) reads:

$$\sigma_{11} = \bar{\sigma} + 2\,\mu\,\frac{1}{2}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_1}\right) = \bar{\sigma} + 2\,\mu\,\frac{\partial v_1}{\partial x_1}$$

Generally shear stresses are small compared to normal stresses in a fluid, i.e.

$$\sigma_{21} = \mu \frac{\partial v_1}{\partial x_2} \ll \bar{\sigma}$$

If the main flow direction coincides with the x_1-axis, we have

$$\frac{\partial v_1}{\partial x_1} \ll \frac{\partial v_1}{\partial x_2}$$

Therefore,

$$2\mu \frac{\partial v_1}{\partial x_1} \ll \bar{\sigma}$$

and it is normally assumed that

$$\sigma_{11} \approx \bar{\sigma}$$

In a uniform flow directed along the x_1-axis we have $\frac{\partial v_1}{\partial x_1} = 0$ and consequently $\sigma_{11} = \bar{\sigma}$.

Similarly we find:

$$\sigma_{22} = \bar{\sigma} + 2\,\mu \frac{\partial v_2}{\partial x_2} \approx \bar{\sigma}$$

and

$$\sigma_{33} = \bar{\sigma} + 2\,\mu \frac{\partial v_3}{\partial x_3} \approx \bar{\sigma}$$

Due to the fact that a fluid cannot withstand tension stresses (if the pressure is lower than the saturated vapour pressure, the fluid boils instead), it is practical to adopt the following unique definition of *pressure*, denoted p, in a *moving* fluid:

$$\boxed{p = -\bar{\sigma} = -\frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)} \tag{2.37}$$

With this definition p is always positive, and the corresponding stress state, $-p\,\delta_{ij}$, is isentropic, just like the stress state in a resting fluid. In practice it is only necessary to include $-p\,\delta_{ij}$ at the calculation of normal forces on a plane in a moving fluid.

Substitution of equation (2.37) into equation (2.36) leads to the final version of the constitutive equation for a newtonian fluid:

$$\boxed{\sigma_{ij} = -p\,\delta_{ij} + 2\,\mu\,e_{ij}} \tag{2.38}$$

2.4 Navier-Stokes' Equation

Based on the equation of motion

$$\rho \frac{dv_i}{dt} = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{2.39}$$

and the constitutive equation for a newtonian fluid

$$\sigma_{ij} = -p\,\delta_{ij} + 2\,\mu\, e_{ij} \tag{2.40}$$

it is possible to set up equations for a direct calculation of the velocity field of a newtonian fluid.

Derivation of the constitutive equation with respect to x_j gives this resultant of the surface forces on a particle:

$$\begin{aligned} \frac{\partial \sigma_{ij}}{\partial x_j} &= \frac{\partial(-p\,\delta_{ij} + 2\,\mu\, e_{ij})}{\partial x_j} \\ &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right) \end{aligned} \tag{2.41}$$

Substitution of this expression into the equation of motion (2.39) gives

$$\boxed{\rho \frac{dv_i}{dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right)} \tag{2.42}$$

which is called the *Navier-Stokes equation.*

In this version of the equation the last term is zero due to the assumption of an incompressible fluid. Consequently we can rewrite the Navier-Stokes equation to:

$$\boxed{\rho \frac{dv_i}{dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j\, \partial x_j}} \tag{2.43}$$

This equation is always valid for flows of a newtonian fluid, but normally it is adopted only for *laminar* flows. In most cases of turbulent flows a direct numerical solution of the equation is too costly, because very small vortices have to be resolved in the calculation. Instead a kind of "smoothed" version of the equation is solved. In this solution the dynamic effect (exchange of momentum) of the small vortices is taken into account by use of a so-called *turbulence model.* See chapter 7.

The left-hand side of equation (2.42) might be interpreted as mass times acceleration of a particle having the volume one unit (m^3), and the right-hand side is therefore the sum of the external forces on this volume.

The first term on the right-hand side is the gravity force and the second term is the resulting force from pressure p on the volume (see appendix A). The last term on the right-hand side is therefore the resultant of the so-called *viscous* stresses, which comprise both normal stresses and shear stresses. Normally the resultant viscous normal stress is insignificant compared to the resulting force from pressure.

The pressure distribution in a resting fluid is called *hydrostatic* pressure distribution, and this pressure is denoted p_{hyd}. It is easily determined from the Navier-Stokes equation, which for a resting fluid reads:

$$0 = \rho\, g_i - \frac{\partial p_{hyd}}{\partial x_i} \tag{2.44}$$

As g_i has a vertical component only, p_{hyd} has to be constant on horizontal planes, and therefore only varies in the vertical direction. If z denotes a *vertical* coordinate (positive in the *upward* direction), we find by integration of equation (2.44) that in a resting fluid the expression for pressure reads:

$$p_{hyd} = -\rho\, g\, z + C \tag{2.45}$$

where C is a constant.

As p_{hyd} does not create any motion of the fluid, only the part of p deviating from p_{hyd} is considered in many cases. This deviation in pressure is called either *excess* pressure or *dynamic* pressure. In these notes it is denoted p^+, but many books use the notation p_d. Thus, the definition equation for p^+ reads:

$$\boxed{p = p_{hyd} + p^+} \tag{2.46}$$

Substitution of equation (2.45) gives

$$p^+ = p + \rho\, g\, z - C \tag{2.47}$$

The expression for the pressure gradient can therefore be rewritten to:

$$\begin{aligned} \frac{\partial p}{\partial x_i} &= \frac{\partial p_{hyd}}{\partial x_i} + \frac{\partial p^+}{\partial x_i} \\ &= \rho\, g_i + \frac{\partial p^+}{\partial x_i} \end{aligned} \tag{2.48}$$

by substitution of equation (2.44). After substitution of equation (2.48) into equation (2.42) we have this alternative version of the Navier-Stokes equation:

$$\boxed{\rho \frac{dv_i}{dt} = -\frac{\partial p^+}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right)} \tag{2.49}$$

For an incompressible fluid this version of Navier-Stokes equation can be rewritten to:

$$\boxed{\rho \frac{dv_i}{dt} = -\frac{\partial p^+}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \, \partial x_j}} \tag{2.50}$$

At first glance all versions of the Navier-Stokes equation look rather simple, but they are in fact nonlinear. The expression for acceleration reads:

$$\frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} \cdot v_j$$

where the last term corresponding to convective acceleration is nonlinear! This prevents an analytical solution except for the very few flows, where the convective term is zero, i.e. when the flow is uniform.

In case of laminar flow the Navier-Stokes equation can be solved numerically in practice, but considerable computer power is necessary.

In the following some examples of analytical solution of Navier-Stokes equations are considered.

Example 1: Poiseuille-flow

A *steady* flow between 2 plane parallel plates with mutual distance a and slope β is considered, see Fig. 2.7.

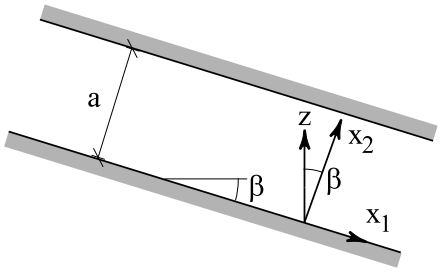

Figure 2.7: *Definition sketch, Poiseuille-flow.*

The flow is driven by a pressure gradient in the x_1-direction, i.e. $\partial p^+/\partial x_1 < 0$, and this gradient is so small that the flow remains *laminar*. The flow is two-dimensional and because the parallel plates give parallel streamlines the flow is also *uniform*. The only non-zero velocity component is therefore v_1.

The pressure distribution across the flow is calculated by the x_2-component of the Navier-Stokes equation:

$$\rho \frac{dv_2}{dt} = -\frac{\partial p^+}{\partial x_2} + \mu \left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} \right) \tag{2.51}$$

As $v_2 = 0$ everywhere, equation (2.51) reduces to

$$\frac{\partial p^+}{\partial x_2} = 0$$

Thus, in sections perpendicular to the streamlines, where $x_1 = \textit{constant}$, we get:

$$p^+ = K(x_1)$$

If this expression is substituted into $p = p_{hyd} + p^+ = -\rho\, g\, z + p^+$, we find:

$$p = -\rho\, g\, z + K(x_1) = -\rho\, g\, x_2\, \cos\beta + K(x_1) \tag{2.52}$$

i.e. the pressure distribution is linear, and it only deviates from the pressure distribution in a resting fluid by the constant factor $\cos\beta$. Despite this factor we also call this distribution a *hydrostatic* pressure distribution. Also, notice that equation (2.52) is valid for all flows, where the streamlines are straight and parallel at the section considered.

The variation (or profile) of v_1 is calculated by the x_1-component of the Navier-Stokes equation, which reads:

$$\rho \frac{dv_1}{dt} = -\frac{\partial p^+}{\partial x_1} + \mu \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} \right)$$

Substitution of the rate of change for v_1 gives:

$$\rho \left[\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} \right] = -\frac{\partial p^+}{\partial x_1} + \mu \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} \right)$$

Due to uniform flow we have

$$\frac{\partial v_1}{\partial x_1} = 0 \quad \text{og} \quad \frac{\partial^2 v_1}{\partial x_1^2} = 0$$

and due to steady flow we have

$$\frac{\partial v_1}{\partial t} = 0$$

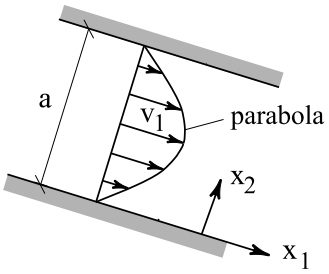

Figure 2.8: *Velocity profile in Poiseuille-flow.*

As also $v_2 = 0$, the left-hand side of the Navier-Stokes equation is zero, i.e. the acceleration is zero. This gives:

$$\mu \frac{\partial^2 v_1}{\partial x_2^2} = \frac{\partial p^+}{\partial x_1}$$

This partial differential equation can be changed to an ordinary differential equation, because $v_1 = v_1(x_2)$ and the pressure gradient is constant. The Navier-Stokes equation now reads:

$$\mu \frac{d^2 v_1}{d x_2^2} = \frac{\partial p^+}{\partial x_1}$$

Integration twice gives:

$$v_1 = \frac{1}{2\,\mu} \frac{\partial p^+}{\partial x_1}\, x_2^2 + C_1\, x_2 + C_2$$

The constants C_1 and C_2 are determined by adoption of the boundary conditions:

$$v_1 = 0 \;\text{ for }\; x_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$v_1 = 0 \;\text{ for }\; x_2 = a \quad \Rightarrow \quad C_1 = -\frac{1}{2\,\mu} \frac{\partial p^+}{\partial x_1}\, a$$

After substitution of C_1 and C_2, the expression for v_1 reads:

$$v_1 = \frac{1}{2\,\mu} \frac{\partial p^+}{\partial x_1}\, \left(x_2^2 - a\, x_2\right)$$

or

$$\boxed{v_1 = \frac{1}{2\,\mu} \frac{\partial p^+}{\partial x_1}\, (x_2 - a)\; x_2}$$

The velocity profile in the slot is therefore parabolic as seen in Fig. 2.8.

Example 2: unsteady flow above an oscillating plate

We consider a flow caused by a plate oscillating in its own plane at the velocity:

$$v_1 = v_o \cos(\frac{2\pi}{T} t) = v_o \cos(\omega t) \tag{2.53}$$

Here T is the period and $\omega = 2\pi/T$ is the cyclic frequency.

As the motions of the plate in its own plane are unable to produce any pressure gradients in the x_1-direction, we have $\partial p^+/\partial x_1 = 0$. It is also evident that $\partial v_1/\partial x_1 = 0$ and $v_2 = 0$, i.e. the flow is *uniform*. Therefore, the x_1-component of the the Navier-Stokes equation reads:

$$\rho \frac{\partial v_1}{\partial t} = \mu \frac{\partial^2 v_1}{\partial x_2^2} \tag{2.54}$$

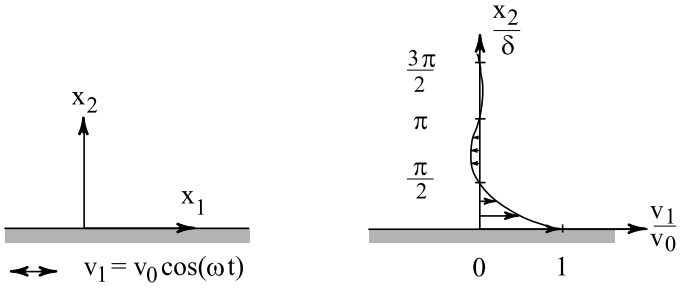

Figure 2.9: *Definition sketch and velocity profile at* $t = 0$.

As the motion of the plate is periodic with time, the fluid at an arbitrary distance from the plate also has to move periodically with time with period T, but a lag in the phase will be present. At the plate the fluid and the plate will move together due to adhesion, and it is expected that the motions of the fluid will decrease with increasing distance to the plate. In principle the fluid velocity can be expressed as:

$$v_1(x_2, t) = v_o \, f(x_2) \cos(\omega t - \theta(x_2)) \tag{2.55}$$

The solution to equation (2.54) reads:

$$v_1(x_2, t) = v_o \, e^{-\frac{x_2}{\delta}} \cos(\omega t - \frac{x_2}{\delta}) \tag{2.56}$$

where

$$\delta = \sqrt{\frac{2\nu}{\omega}}$$

The solution is built correctly according to equation (2.55), but it is left to the reader to show that it fulfills both equation (2.54) and the boundary conditions. The variable δ is often called the *Stokes distance*, and it is seen to be a measure of how far away the motions of the plate can be felt. As $\nu \approx 1.00 \cdot 10^{-6}$ m^2/s for pure water at 20° C, a period of $T = 5$ seconds gives $\delta = 1.3$ mm, i.e. a very small quantity.

In Fig. 2.9 the velocity profile at $t = n \cdot 2\pi/T$ $(n = 0, 1, 2....)$ is shown.

2.4.1 Vorticity Transport Equation

A *two-dimensional* flow of an *incompressible* fluid is considered. Adoption of the operator:

$$\nabla^2( ) = \frac{\partial^2( )}{\partial x_1^2} + \frac{\partial^2( )}{\partial x_2^2} \tag{2.57}$$

makes it possible to write the Navier-Stokes equation in the more compact form:

$$\rho \frac{dv_1}{dt} = -\frac{\partial p^+}{\partial x_1} + \mu \nabla^2 v_1 \tag{2.58}$$

$$\rho \frac{dv_2}{dt} = -\frac{\partial p^+}{\partial x_2} + \mu \nabla^2 v_2 \tag{2.59}$$

If equation (2.58) is derived with respect to x_2 and equation (2.59) is derived with respect to x_1, both equations contain the term:

$$-\frac{\partial^2 p^+}{\partial x_1 \, \partial x_2}$$

This term disappears, if equation (2.59) is subtracted from equation (2.58). The result reads:

$$\rho \frac{\partial}{\partial x_2}\left(\frac{dv_1}{dt}\right) - \rho \frac{\partial}{\partial x_1}\left(\frac{dv_2}{dt}\right) = \mu \frac{\partial}{\partial x_2}\left(\nabla^2 v_1\right) - \mu \frac{\partial}{\partial x_1}\left(\nabla^2 v_2\right) \tag{2.60}$$

After substitution of the operator for the rate of change:

$$\frac{d( )}{dt} = \frac{\partial( )}{\partial t} + v_1 \frac{\partial( )}{\partial x_1} + v_2 \frac{\partial( )}{\partial x_2} \tag{2.61}$$

and the ∇^2-operator, equation (2.60) reads:

$$\begin{aligned} &\frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2}\right) \\ &\quad = \frac{\mu}{\rho}\left[\frac{\partial}{\partial x_2}\left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2}\right)\right] \end{aligned} \tag{2.62}$$

This equation is rewritten to:

$$\frac{\partial}{\partial t}\left(\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}\right)+v_1\frac{\partial}{\partial x_1}\left(\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}\right)+v_2\frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}\right)$$
$$+\frac{\partial v_1}{\partial x_2}\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\right)-\frac{\partial v_2}{\partial x_1}\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\right)$$
$$=\nu\left[\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}\right)+\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}\right)\right] \tag{2.63}$$

Substitution of the continuity equation

$$\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=0 \tag{2.64}$$

and

$$\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}=-(\mathrm{rot}\,\vec{v})_3$$

gives:

$$\frac{\partial(\mathrm{rot}\,\vec{v})_3}{\partial t}+v_1\frac{\partial(\mathrm{rot}\,\vec{v})_3}{\partial x_1}+v_2\frac{\partial(\mathrm{rot}\,\vec{v})_3}{\partial x_2}=\nu\left(\frac{\partial^2(\mathrm{rot}\,\vec{v})_3}{\partial x_1^2}+\frac{\partial^2(\mathrm{rot}\,\vec{v})_3}{\partial x_2^2}\right) \tag{2.65}$$

Use of the operator for the rate of change and the ∇^2-operator finally gives:

$$\boxed{\frac{d(\mathrm{rot}\,\vec{v})_3}{dt}=\nu\,\nabla^2(\mathrm{rot}\,\vec{v})_3} \tag{2.66}$$

This equation is called the *vorticity transport equation* for a *two-dimensional* flow, because the equation describes how rot $\vec{v}$ is dispersed in a two-dimensional flow. This is shown by the following example.

Example: Dispersion of a dissolved substance in a two-dimensional flow

The dispersion of a dissolved substance is considered. Concentration of the dissolved substance may be stated in many different ways, but here concentration is defined as kg of dissolved substance per m^3 of the mixture of fluid and substance, and it is denoted c [kg/m^3]. By means of continuity considerations it is possible to derive an equation that describes the dispersion of the substance. First of all it should be noted that the dispersion of a substance is caused by two different transport mechanisms: *diffusion* and *convection*.

We consider a resting fluid, where the concentration profile at a given moment is depicted in Fig. 2.10. If the flux of substance through a section

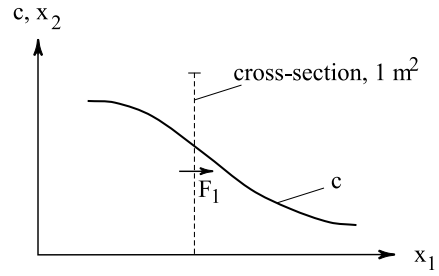

Figure 2.10: *Definition sketch, diffusion.*

perpendicular to the x_1-axis is denoted F_1, experiments show that the expression for this flux reads:

$$F_1 = -D\,\frac{\partial c}{\partial x_1} \quad [\mathrm{kg/s/m^2}]$$

where D is called the *diffusion coefficient* or the *diffusivity* $[\mathrm{m^2/s}]$. This equation is called *Fick's 1st law*. Notice that this spreading of substance is caused by the heat-dependent motions of the molecules, and it will always create a net transport away from the domains with most molecules of the substance. In the following this spreading mechanism is called *diffusion*. It is seen that omitting the description of the motions of the individual molecules make it necessary to introduce a mechanism (diffusion), which has the same effect as the details omitted. We have seen this earlier, where it was necessary to introduce shear stresses, because the individual motions of the molecules were omitted.

Hereafter we consider convection, which is the transport of dissolved substance caused by the motion of the fluid particles. The volume flux of the mixture due to the velocity $v_1 \neq 0$ is $v_1 \cdot 1$ $[\mathrm{m^3/s/m^2}]$ through a section perpendicular to the x_1-axis, and the flux of substance is therefore $c\,v_1$ $[\mathrm{kg/s/m^2}]$.

The expression for the total flux of substance (diffusion and convection) reads:

$$F_1 = -D\frac{\partial c}{\partial x_1} + v_1\,c$$

In a similar way we get in the x_2-direction:

$$F_2 = -D\frac{\partial c}{\partial x_2} + v_2\,c$$

If the control surface shown in Fig. 2.11 is considered, the resulting flux of dissolved substance *inwards* is equal to the increase in substance stored

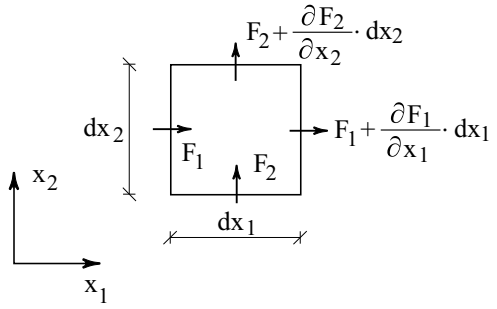

Figure 2.11: *Flux of dissolved substance through a control surface.*

within the control surface. This can be expressed as:

$$\frac{\partial c}{\partial t}\,dx_1\,dx_2 = F_1\,dx_2 - (F_1 + \frac{\partial F_1}{\partial x_1}\,dx_1)\,dx_2 + F_2\,dx_1 - (F_2 + \frac{\partial F_2}{\partial x_2}\,dx_2)\,dx_1$$

or

$$\frac{\partial c}{\partial t}\,dx_1\,dx_2 = -\frac{\partial F_1}{\partial x_1}\,dx_1\,dx_2 - \frac{\partial F_2}{\partial x_2}\,dx_2\,dx_1$$

Division by $d_1\ dx_2$ gives:

$$\frac{\partial c}{\partial t} + \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0$$

Substitution of the expressions for F_1 og F_2 gives:

$$\frac{\partial c}{\partial t} - D\,\frac{\partial^2 c}{\partial x_1^2} + \frac{\partial (v_1\,c)}{\partial x_1} - D\,\frac{\partial^2 c}{\partial x_2^2} + \frac{\partial (v_2\,c)}{\partial x_2} = 0$$

After derivation of the products and substitution of the continuity equation (2.64), we get this equation:

$$\frac{\partial c}{\partial t} + v_1\,\frac{\partial c}{\partial x_1} + v_2\,\frac{\partial c}{\partial x_2} = D\,\left(\frac{\partial^2 c}{\partial x_1^2} + \frac{\partial^2 c}{\partial x_2^2}\right)$$

The left-hand side is seen to be the rate of change of c that is observed by following a particle, and the equation can be re-written to:

$$\boxed{\frac{dc}{dt} = D\,\nabla^2 c} \tag{2.67}$$

Now we can interpret the vorticity transport equation (2.66) by comparing it to equation (2.67). It is seen that the two equations are identically

set up, and it can be concluded that the vorticity $(\mathrm{rot}\,\vec{v})_3$ is dispersed in the fluid by convection and by diffusion with the diffusion coefficient ν [m^2/s]. In practice all vorticity is created near solid boundaries. It is therefore possible to get an impression of the dispersion of the vorticity, if one imagines how dye, discharged at the boundaries, will disperse in the actual flow. This is a useful tool, when we want to decide whether a certain domain of the flow is irrotational or not.

2.5 Dissipation

The macroscopic (organized) kinetic energy of fluid particles is constantly transformed into (unorganized) kinetic energy at molecular level in a flow. This transformation corresponds to an increasing temperature of the fluid, and is normally called *dissipation*.

As it is impossible to transform the unorganized molecular motions back into organized motions of particles, the transformation to unorganized motions is often called *loss of mechanical energy*, where mechanical energy is defined as the sum of kinetic and potential energy of fluid particles. It is seen that the concept of dissipation has to be introduced, because we have assumed the fluid to be a continuum.

The actual dissipation can be determined, if we first calculate the amount of work produced by all forces acting on the particle, and from this work subtract the increase of the kinetic energy of the particle.

We consider a body of fluid having the volume X bounded by the closed surface A, see Fig. 2.5. The work produced by the external forces during the time dt is given by:

$$A_{external} = \int_X v_i\,dt\;(\rho\,g_i)\;dX + \int_A v_i\,dt\,\sigma_i^N\,dA \tag{2.68}$$

Substitution of $\sigma_i^N = \sigma_{ij}\,n_j$, and use of the divergence theorem on the vector field $l_j = v_i\,\sigma_{ij}$ gives:

$$A_{external} = (\int_X v_i(\rho\,g_i)\,dX + \int_X \frac{\partial(v_i\,\sigma_{ij})}{\partial x_j}\,dX)\,dt \tag{2.69}$$

Per unit *volume* of the fluid the work of the external forces is:

$$\begin{aligned} a_{external} &= \frac{A_{external}}{X} \\ &= (v_i\,\rho\,g_i + \frac{\partial(v_i\,\sigma_{ij})}{\partial x_j})\,dt \\ &= (v_i\,\rho\,g_i + v_i\,\frac{\partial \sigma_{ij}}{\partial x_j} + \sigma_{ij}\,\frac{\partial v_i}{\partial x_j})\,dt \end{aligned} \tag{2.70}$$

Substitution of the equation of flow derived in section 2.2

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho\,(g_i - \frac{d\,v_i}{dt}) = 0 \tag{2.71}$$

gives

$$\begin{aligned} a_{external} &= \left(v_i\,\rho\,g_i - v_i\,\rho\,(g_i - \frac{d\,v_i}{dt}) + \sigma_{ij}\,\frac{\partial v_i}{\partial x_j}\right)\,dt \\ &= \rho\,v_i\,d\,v_i + \sigma_{ij}\,\frac{\partial v_i}{\partial x_j}\,dt \\ &= d\,(\frac{1}{2}\,\rho\,v^2) + \sigma_{ij}\,\frac{\partial v_i}{\partial x_j}\,dt \end{aligned} \tag{2.72}$$

If the constitutive equation (2.38)

$$\sigma_{ij} = -p\,\delta_{ij} + 2\,\mu\,e_{ij}$$

is substituted, we get:

$$a_{external} = d\,(\frac{1}{2}\,\rho\,v^2) - p\,\delta_{ij}\,\frac{\partial v_i}{\partial x_j}\,dt + \mu\,(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})\,\frac{\partial v_i}{\partial x_j}\,dt \tag{2.73}$$

However, for an incompressible fluid the middle term is zero as:

$$\delta_{ij}\,\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_j} = 0$$

Thus equation (2.73) can be written as:

$$a_{external} = d\,(\frac{1}{2}\,\rho\,v^2) + \mu\,(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})\,\frac{\partial v_i}{\partial x_j}\,dt \tag{2.74}$$

The first term is seen to be the increase of the kinetic energy (per unit volume) during the time dt, and the second term is therefore the dissipation per unit volume, i.e. the part of the work, which is transformed into kinetic energy at molecular level (heat).

In many cases the dissipation per *unit mass* per *unit time* is considered, and it is normally denoted ϵ. It is found by division of equation (2.74) by $\rho\, dt$ giving

$$\boxed{\epsilon = \nu\,(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})\,\frac{\partial v_i}{\partial x_j}} \tag{2.75}$$

As

$$\begin{aligned}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})^2 &= \frac{\partial v_i}{\partial x_j}\cdot\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\cdot\frac{\partial v_j}{\partial x_i} + 2\,\frac{\partial v_i}{\partial x_j}\cdot\frac{\partial v_j}{\partial x_i} \\ &= 2\,(\frac{\partial v_i}{\partial x_j})^2 + 2\,\frac{\partial v_i}{\partial x_j}\cdot\frac{\partial v_j}{\partial x_i}\end{aligned} \tag{2.76}$$

equation (2.75) can be rewritten to:

$$\epsilon = \frac{1}{2}\,\nu\,(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})^2 > 0 \tag{2.77}$$

Thus the dissipation is always positive, i.e. mechanical energy is constantly transformed into heat at all points having a non-zero velocity gradient.

Chapter 3

Ideal Fluids

3.1 Basic assumptions

In most flows the shear stresses are very small compared to the normal stresses. Is is therefore natural to omit the shear stresses in the description of a flow. This is obtained by the assumption

$$\nu = 0$$

which e.g. yields:

$$\sigma_{21} = \rho\,\nu(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}) = 0$$

but notice that shear stresses and viscous normal stresses all become zero. Furthermore, if we assume that the fluid is incompressible, the fluid is called an *ideal* fluid.

The theory for ideal fluids gives excellent results in many cases, but in general the small shear stresses play a decisive role in the course of the flow, e.g. on the look of the streamlines.

The Navier-Stokes equation for an ideal fluid ($\nu = 0$) reads:

$$\rho\frac{dv_i}{dt} = -\frac{\partial p^+}{\partial x_i} \tag{3.1}$$

This equation is often called *Euler's* equation. It is easy to derive it using Newton's 2nd law as shear stresses are missing. Equation (3.1) simply states that for an ideal fluid the particle acceleration is solely caused by gradients in the dynamic pressure, p^+, i.e. the part of the pressure deviating from the

hydrostatic pressure, p_{hyd}. As

$$p = p_{hyd} + p^+$$

and

$$p_{hyd} = -\rho\, g\, z + C$$

where z denotes a *vertical* coordinate (positive in the *upward* direction) and C is a constant, we have:

$$p^+ = p + \rho\, g\, z - C$$

As only gradients of p^+ are able to create motion, the constant C is neglected in the following. After substitution of the expression for p^+ into equation (3.1) and subsequent division by $\gamma = \rho\, g$ on both sides of the equation, Euler's equation is rewritten to:

$$\frac{1}{g}\frac{dv_i}{dt} + \frac{\partial}{\partial x_i}\left(z + \frac{p}{\gamma}\right) = 0 \tag{3.2}$$

The component of this vector equation in the direction of $e_i = v_i/v$ reads:

$$\frac{1}{g}\frac{v_i}{v}\frac{dv_i}{dt} + \frac{v_i}{v}\frac{\partial}{\partial x_i}\left(z + \frac{p}{\gamma}\right) = 0 \tag{3.3}$$

where v_i is the velocity of the fluid particle.

As $v_i = dx_i/dt$, where dx_i is a vector on the particle path, equation (3.3) can be rewritten to:

$$\frac{d}{dt}\left(\frac{v^2}{2\,g}\right) + \frac{dx_i}{dt}\frac{\partial}{\partial x_i}\left(z + \frac{p}{\gamma}\right) = 0 \tag{3.4}$$

Furthermore, if the flow is assumed to be *steady*, we have:

$$d\left(z + \frac{p}{\gamma}\right) = \frac{\partial}{\partial x_i}\left(z + \frac{p}{\gamma}\right) \cdot dx_i$$

Substitution of this expression into equation (3.4) gives:

$$\frac{d}{dt}\left(\frac{v^2}{2\,g}\right) + \frac{d}{dt}\left(z + \frac{p}{\gamma}\right) = 0 \tag{3.5}$$

or

$$\frac{d}{dt}\left(z + \frac{p}{\gamma} + \frac{v^2}{2\,g}\right) = 0 \tag{3.6}$$

This equation states that the rate of change of the quantity within the parenthesis is zero if observed from a fluid particle. This corresponds to

$$\boxed{z + \frac{p}{\gamma} + \frac{v^2}{2\,g} = constant} \qquad \text{along a } \textit{particle path} \tag{3.7}$$

This is the famous *Bernoulli equation*. As particle paths and streamlines coincide in a steady flow, Bernoulli's equation is also valid along a *streamline*. Notice that the value of the constant normally varies from streamline to streamline.

Example: Steady flow under a sluice gate

In an open channel with horizontal bottom a sluice gate is placed. A sluice gate is a vertical plate under which the flow has to pass, see Fig. 3.1.

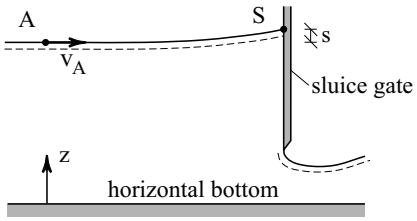

Figure 3.1: *Flow under a sluice gate.*

In front of the gate the free surface is elevated the distance s, see Fig. 3.1. At point S, where there is a break on the path line (= streamline), the particle velocity has to be zero. It is so, because a break at points, where the particle velocity is non-zero, would necessitate an infinitely large force according to Newton's 2nd law. The elevation of the surface at S is calculated applying Bernoulli's equation to the streamline at the surface between A and S:

$$z_A + \frac{p_A}{\gamma} + \frac{v_A^2}{2\,g} = z_S + \frac{p_S}{\gamma} + \frac{v_S^2}{2\,g}$$

where the level z has datum at the bottom of the channel.

Substitution of $v_S = 0$ and $p_A = p_S = p_{atm}$ yields:

$$s = z_S - z_A = \frac{v_A^2}{2\,g}$$

Thus, the increase in surface level is due to a transformation of the kinetic energy of the fluid particle at A into the extra potential energy of the particle, when the particle arrives to S.

A more general equation can be derived from Euler's equation if the flow is irrotational flow, i.e. the velocity field fulfils:

$$\begin{aligned} \operatorname{rot}\vec{v} &= \vec{0} \qquad \Leftrightarrow \\ \frac{\partial v_i}{\partial x_j} &= \frac{\partial v_j}{\partial x_i} \end{aligned} \tag{3.8}$$

In such a flow the velocities are given by

$$v_i = \frac{\partial \varphi}{\partial x_i} \tag{3.9}$$

where φ is the velocity potential. First the substantial rate of change of v_i is substituted into Euler's equation:

$$\frac{1}{g}\frac{dv_i}{dt} + \frac{\partial}{\partial x_i}\left(z + \frac{p}{\gamma}\right) = 0 \tag{3.10}$$

yielding:

$$\frac{1}{g}\left(\frac{\partial v_i}{\partial t} + v_j\,\frac{\partial v_i}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(z + \frac{p}{\gamma}\right) = 0 \tag{3.11}$$

After substitution of equation (3.9) and equation (3.8) the equation reads:

$$\frac{1}{g}\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial x_i}\right) + \frac{1}{g}\,v_j\,\frac{\partial v_j}{\partial x_i} + \frac{\partial}{\partial x_i}\left(z + \frac{p}{\gamma}\right) = 0 \tag{3.12}$$

or

$$\frac{1}{g}\frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial t}\right) + \frac{1}{g}\,\frac{\partial}{\partial x_i}\left(\frac{1}{2}\,v_j\,v_j\right) + \frac{\partial}{\partial x_i}\left(z + \frac{p}{\gamma}\right) = 0 \tag{3.13}$$

which is rewritten to:

$$\frac{\partial}{\partial x_i}\left(\frac{1}{g}\,\frac{\partial \varphi}{\partial t} + \frac{v^2}{2g} + z + \frac{p}{\gamma}\right) = 0 \tag{3.14}$$

As the gradient of the quantity in the parenthesis is zero everywhere in the flow, this quantity has to be constant in the entire flow domain at a given time, i.e.

$$\boxed{z + \frac{p}{\gamma} + \frac{v^2}{2g} + \frac{1}{g}\,\frac{\partial \varphi}{\partial t} = C(t)} \qquad \text{everywhere in the flow} \tag{3.15}$$

where $C(t)$ can only depend on the time. This equation is called *Bernoulli's generalized equation*. Substitution of $p^+ = p + \gamma z$ gives this alternative version of Bernoulli's generalized equation:

$$\boxed{\frac{p^+}{\gamma} + \frac{v^2}{2g} + \frac{1}{g}\,\frac{\partial \varphi}{\partial t} = C(t)} \qquad \text{everywhere in the flow} \tag{3.16}$$

Notice that it is *not* necessary to limit the application of this equation to a single streamline, but remember the condition of irrotational flow !

Bernoulli's generalized equation is widely adopted to calculate pressures and pressure forces for irrotational flows. First the pressure distribution on the surface of a body is calculated. Then the pressures on the body are integrated yielding the force from the fluid acting on the body.

3.2 Potential Flow

In this section we solely consider irrotational flows, i.e. flows where the condition $\mathrm{rot}\,\vec{v} = \vec{0}$ is fulfilled, and where the velocity field can consequently be calculated from a velocity potential φ. Therefore this type of flow is often called *potential flow* instead of irrotational flow. In general φ fulfills the Laplace equation:

$$\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2} = 0 \tag{3.17}$$

which for two-dimensional flows is reduced to

$$\boxed{\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} = 0} \tag{3.18}$$

The stream function for a two-dimensional flow fulfills equation (1.83), which is a Poisson equation. Because $(\mathrm{rot}\,\vec{v})_3 = 0$ in potential flow, equation (1.83) is reduced to a Laplace equation

$$\boxed{\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} = 0} \tag{3.19}$$

For a potential flow both φ and ψ fulfill the Laplace equation, and because this equation is linear, the superposition principle is valid for potential flow.

If knowledge of φ (or ψ) has been obtained for some basic potential flows, these potentials can be superposed to produce potentials corresponding to more complicated flows. It is always true that superposition of two potential flows gives a new potential flow, but it is not always possible to fulfill the physical boundary conditions with the velocities corresponding to the superposed potentials. In the following some examples of superposition of basic potential flows will be given.

In this connection it should be remembered that the boundary conditions at a wall are special for an ideal fluid. Due to the missing shear stresses we cannot demand that fluid and wall shall have the same velocity component along the wall. The only boundary condition left for an *ideal fluid* is therefore that fluid and wall shall have the *same velocity component along the normal to the wall.*

3.2.1 Line Source and Line Sink

We consider the two-dimensional flow around a line source with discharge q per unit length along the line, see Fig. 3.2. Due to conservation of volume we have:

$$q = 2\,\pi\,r \cdot 1 \cdot v_r$$

or

$$v_r = \frac{q}{2\,\pi}\frac{1}{r} = \frac{k}{r}$$

where $k = q/(2\,\pi) > 0$ is called the *source strength*. Due to symmetry we have $v_\theta = 0$, and consequently the velocity field expressed in polar coordinates reads:

$$\vec{v} = (v_r, v_\theta) = (\frac{k}{r}, 0)$$

Expressed in polar coordinates the component of rot $\vec{v}$ perpendicular to the flow

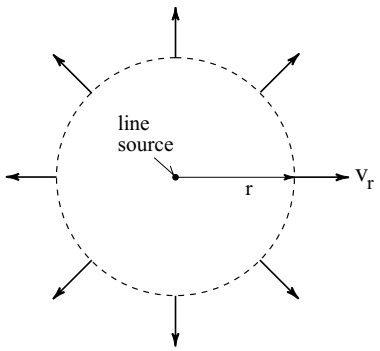

Figure 3.2: *Velocity field at a line source.*

plane reads:

$$\boxed{(\mathrm{rot}\,\vec{v})_3 = \frac{1}{r}\left(\frac{\partial(r\,v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)} \tag{3.20}$$

and it is straightforward to see that $(\mathrm{rot}\,\vec{v})_3 = 0$ for this flow. Thus we find $\mathrm{rot}\,\vec{v} = \vec{0}$, as the two components in the flow plane are always zero for a two-dimensional flow. Thus, the flow is a potential flow.

In order to calculate the potential for the source, we adopt the expression for a gradient, which expressed in polar coordinates reads:

$$\mathrm{grad}\,\varphi = \left(\frac{\partial\varphi}{\partial r}\,,\ \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) \tag{3.21}$$

giving

$$v_r = \frac{\partial\varphi}{\partial r} = \frac{k}{r} \qquad \Leftrightarrow \qquad \varphi = k\,\ln\,r + f(\theta) + C$$

where C is a constant. Substitution of this expression for φ into

$$v_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = 0$$

gives

$$0 + \frac{f'(\theta)}{d\theta} + 0 = 0 \qquad \Leftrightarrow \qquad f(\theta) = C_1$$

which substituted into the expression for φ gives:

$$\varphi = k \ln r + C_2$$

As the gradient of a constant is zero, we can omit the constant C_2. Therefore, the potential for the flow around a line source is

$$\boxed{\varphi_{source} = k \ln r} \tag{3.22}$$

As it was very easy to find the velocity field around a source expressed in polar coordinates, we shall also set up definition equations for the stream function expressed in polar coordinates.

In polar coordinates the expression for divergence reads:

$$\boxed{\operatorname{div} \vec{v} = \frac{1}{r}\frac{\partial (r\, v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} \tag{3.23}$$

By analogy with the definitions in Cartesian coordinates, we adopt these definitions in polar coordinates:

$$\boxed{v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}} \tag{3.24}$$

$$\boxed{v_\theta = \frac{\partial \psi}{\partial r}} \tag{3.25}$$

Substitution of equation (3.24) and equation (3.25) into equation (3.23), yields $\operatorname{div} \vec{v} = 0$. Accordingly the continuity equation for an incompressible fluid is always fulfilled as wanted.

If equation (3.24) is used for the flow at the source one get:

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{k}{r} \qquad \Leftrightarrow \qquad \psi = -k\,\theta + f(r) + C$$

where C is a constant. Substitution of this expression for ψ into:

$$v_\theta = \frac{\partial \psi}{\partial r} = 0$$

gives

$$0 + \frac{f(r)}{dr} + 0 = 0 \qquad \Leftrightarrow \qquad f(r) = C_1$$

Substitution of $f(r)$ into the expression for ψ gives

$$\psi = -k\,\theta + C_2$$

As the gradient of a constant is zero, we can omit the constant C_2 and the expression for ψ reads:

$$\boxed{\psi_{source} = -k\,\theta} \tag{3.26}$$

Replacement of q by $-q$ (where q is a positive quantity) in the equations causes a sign shift of the velocity, i.e. the flow is still in a radial direction, but now towards the origin corresponding to a *line sink*.

3.2.2 Two-Dimensional Dipole Flow

A two-dimensional flow is produced by superposition of a line source with discharge $+q$ and a line sink with discharge $-q$. The sink is placed at $r = 0$ and the source the distance s to the right, see Fig. 3.3. Initially the velocity potential

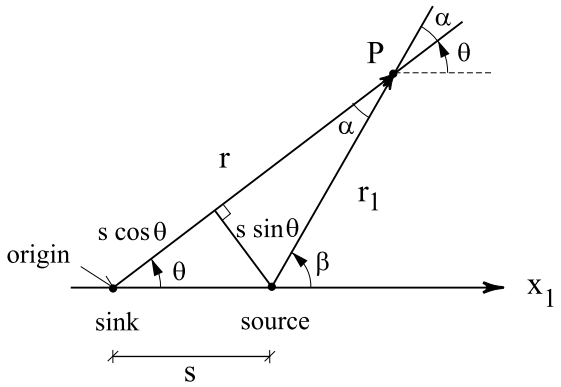

Figure 3.3: *Definition sketch, two-dimensional dipole flow.*

and velocity are calculated at point P placed at the distance r from the sink and the distance r_1 from the source.

By superposition we get

$$\varphi = \varphi_{source} + \varphi_{sink} = +k \ln r_1 - k \ln r \tag{3.27}$$

where equation (3.22) has been adopted with positive source strength $+k$ at the source and with negative source strength $-k$ at the sink.

In the following we only consider the asymptotic expressions for φ (and ψ), valid far from the sink and the drain, i.e. it is assumed that $r >> s$, but equation (3.27) is of course valid everywhere. Seen from point P the distance between the source and the sink looks very small, and the configuration is named a *dipole*.

From Fig. 3.3 it is seen that for $r >> s$, we can approximate the distance from the source to P by $r_1 \approx r - s \cos\theta$. Substitution into equation (3.27) gives:

$$\begin{aligned} \varphi_{dipole} &\approx +k \ln(r - s\cos\theta) - k \ln r \\ &= k \ln\left(\frac{r - s\cos\theta}{r}\right) \\ &= k \ln(1 - \frac{s}{r}\cos\theta) \approx k\,(-\frac{s}{r}\cos\theta) \end{aligned} \tag{3.28}$$

where $\ln(1-\epsilon) \approx -\epsilon$ for $\epsilon << 1$ has been used.

Defining *the strength of the dipole* as

$$m = \frac{q\,s}{2\,\pi} = k\,s \tag{3.29}$$

i.e. $m > 0$ is assumed, the equation for the potential around the dipole reads:

$$\boxed{\varphi_{dipole} = -m\,\frac{\cos\theta}{r}} \tag{3.30}$$

The velocity components at point P read:

$$v_r = \frac{\partial\varphi}{\partial r} = \frac{-1}{r^2}\,(-m\cos\theta) = m\,\frac{\cos\theta}{r^2} \tag{3.31}$$

and

$$v_\theta = \frac{1}{r}\,\frac{\partial\varphi}{\partial\theta} = \frac{1}{r}\,(-m)\,\frac{-\sin\theta}{r} = m\,\frac{\sin\theta}{r^2} \tag{3.32}$$

The stream function for the dipole flow is calculated by superposition of the stream functions for a sink and a source, respectively. The general expression reads:

$$\psi = \psi_{source} + \psi_{sink} = -k\,\beta + k\,\theta = -k\,(\theta + \alpha) + k\,\theta = -k\,\alpha \tag{3.33}$$

as $\beta = \theta + \alpha$.

For $r >> s$ we approximate α by

$$\alpha \approx \frac{s \sin\theta}{r - s\cos\theta} = \frac{s \sin\theta}{r(1 - \frac{s}{r}\cos\theta)} \approx \frac{s \sin\theta}{r}$$

which substituted into (3.33) gives this stream function for a dipole flow:

$$\boxed{\psi_{dipole} = -m\frac{\sin\theta}{r}} \tag{3.34}$$

The course of the streamlines can be determined by solving the differential equation for a streamline, equation (1.15). However, in many cases it is much easier to make use of the convenient property: the stream function is constant on a streamline. The course of a streamline is therefore found by locating the curve, where $\psi = C$.

In the expression for ψ we substitute:

$$r = \sqrt{x_1^2 + x_2^2} \qquad \text{and} \qquad \sin\theta = \frac{x_2}{r} = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

yielding:

$$\psi = -m\frac{x_2}{x_1^2 + x_2^2}$$

and this expression is rewritten to:

$$\begin{aligned} \Rightarrow \qquad x_1^2 + x_2^2 + \frac{m}{\psi}x_2 &= 0 \\ \Rightarrow \qquad x_1^2 + (x_2 + \frac{m}{2\,\psi})^2 - (\frac{m}{2\,\psi})^2 &= 0 \\ \Rightarrow \qquad x_1^2 + (x_2 - (-\frac{m}{2\,\psi}))^2 &= (\frac{m}{2\,\psi})^2 \end{aligned} \tag{3.35}$$

The course of the streamline with $\psi = C$ is then found by substituting $\psi = C$ into equation (3.35).

This yields:

$$x_1^2 + \left(x_2 - (-\frac{m}{2\,C})\right)^2 = (\frac{m}{2\,C})^2 \tag{3.36}$$

i.e. the equation for a *circle* having the centre at

$$(x_1, x_2) = (0, -\frac{m}{2\,C}) \tag{3.37}$$

and radius

$$R = \left|\frac{m}{2\,C}\right| \tag{3.38}$$

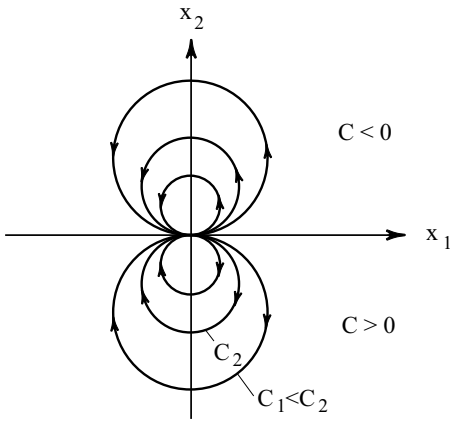

Figure 3.4: *Streamlines, two-dimensional dipole flow.*

as the equation for a circle with centre at (a, b) and radius R reads:

$$(x_1 - a)^2 + (x_2 - b)^2 = R^2$$

If $C > 0$, the streamline is placed under the x_1-axis, and R is decreasing for $|C|$ increasing. The streamlines are sketched in Fig. 3.4. It is also seen that ψ is increasing to the right at all positions on the x_2-axis, when looking into the direction of the flow.

3.2.3 Flow past a Circular Cylinder

A circular cylinder with radius R is placed in a uniform flow with the velocity field: $v_i = (-v_o, 0)$, see Fig. 3.5.

We want to describe the flow by use of the stream function ψ, which must fulfill both the differential equation (the flow equation)

$$\nabla^2 \psi = 0 \tag{3.39}$$

and the following two boundary conditions.

The first boundary condition can be expressed as:

$$\begin{aligned} & v_r = 0 \quad \text{for} \quad r = R \\ \Leftrightarrow \quad & \psi = 0 \quad \text{for} \quad r = R \end{aligned} \tag{3.40}$$

as the surface of the cylinder is a streamline. However, the value of the stream function on this boundary is arbitrarily chosen to be zero. Due to reasons of

symmetry the x_1-axis is a streamline. This streamline splits into two streamlines at the stagnation point S, where the velocity is zero.

The second boundary condition is

$$(v_1, v_2) \to (-v_o, 0) \quad \text{for} \quad r \to \infty \tag{3.41}$$

corresponding to undisturbed flow far away from the cylinder.

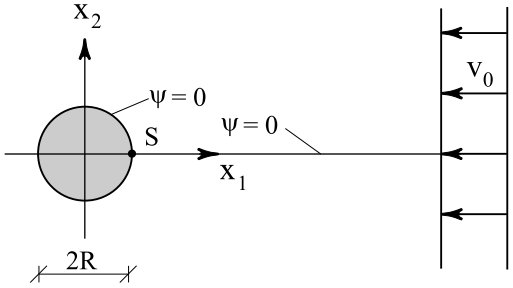

Figure 3.5: *Definition sketch. Two-dimensional flow around a cylinder.*

The problem is to find out if the potential flow formed by superposition of a dipole flow (1 source + 1 sink) and a uniform flow with velocity field $(v_1, v_2) = (-v_o, 0)$ fulfills the boundary conditions stated above.

First we notice that the velocity components around a dipole are:

$$(v_r, v_\theta) = (m\,\frac{\cos\,\theta}{r^2}\,,\, m\,\frac{\sin\,\theta}{r^2}) \to (0, 0) \tag{3.42}$$

for $r \to \infty$. Consequently the superposed velocity field approaches uniform flow far away from the cylinder. Therefore, the second boundary condition is fulfilled.

Superposition gives:

$$\begin{aligned} \psi &= \psi_{par} + \psi_{dipole} = -(-v_o)\,x_2 + (-m\frac{\sin\,\theta}{r}) \\ &= v_o\,r\,\sin\,\theta - m\frac{\sin\,\theta}{r} \\ &= v_o\,\frac{\sin\,\theta}{r}(r^2 - \frac{m}{v_o}) \end{aligned} \tag{3.43}$$

After substitution of R_* , defined as:

$$R_* \equiv \sqrt{\frac{m}{v_o}} \tag{3.44}$$

the expression for the stream function reads:

$$\psi = v_o \frac{\sin\theta}{r}(r^2 - R_*^2) \tag{3.45}$$

For $r = R_*$ we have $\psi = 0$, and the circle with radius R_* is consequently a streamline. By scaling the dipole strength m, and hereby scaling of R_*, it is also possible to 'move' the streamline having $\psi = 0$, until it coincides with the surface of the cylinder.

The demand $R_* = R$ is fulfilled if

$$\sqrt{\frac{m}{v_o}} = R$$

or

$$m = v_o R^2 \tag{3.46}$$

With this dipole strength both boundary conditions are fulfilled, and the corresponding stream function reads:

$$\boxed{\psi = v_o \frac{\sin\theta}{r}(r^2 - R^2)} \quad \text{for} \quad r \geq R \tag{3.47}$$

If the potential flow is symmetric with respect to both coordinate axes, so is the velocity distribution along the surface of the cylinder. The streamlines are depicted in Fig. 3.6.

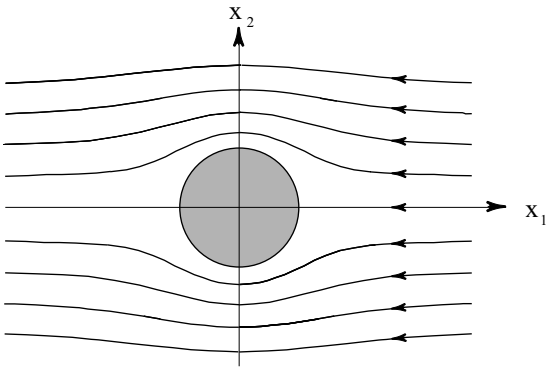

Figure 3.6: *Two-dimensional potential flow around a circular cylinder.*

The pressure distribution on the cylinder surface is calculated by use of Bernoulli's equation along the streamline on the surface of the cylinder. This gives

$$z + \frac{p}{\rho g} + \frac{v^2}{2g} = C \quad \text{for} \quad r = R$$

Substitution of the dynamic pressure, defined as the deviation from hydrostatic pressure, i.e.

$$p^+ = p + \rho\, g\, z$$

where z denotes a *vertical* coordinate (positive in the *upward* direction) gives:

$$z + \frac{p^+ - \rho\, g\, z}{\rho\, g} + \frac{v^2}{2\, g} = C$$

or

$$\frac{p^+}{\rho\, g} + \frac{v^2}{2\, g} = C \tag{3.48}$$

As v^2 is symmetric about both axes, Bernoulli's equation tells that also the distribution of p^+ is symmetric about both axes.

Therefore, the resulting force from p^+ is zero, which means that the potential flow *does not create any pressure force* on the cylinder.

This result is strikingly against all practical experience, and the reason for the misleading result simply is that the requirement for potential flow, rot $\vec{v} = \vec{0}$, is not fulfilled everywhere.

Even though the uniform flow is irrotational, vorticity is evidently dispersed into the flow from the surface of the cylinder. In real fluids vorticity is constantly created at a wall due to the presence of the shear stresses, and although the shear stresses are very small, they are able to change the velocity profile near the wall significantly, see Fig. 3.7.

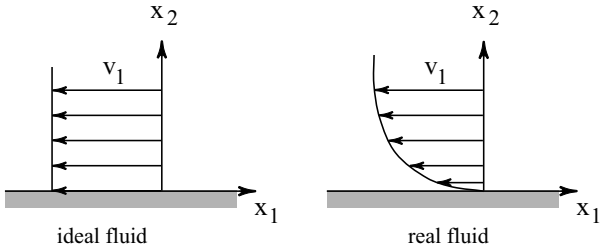

Figure 3.7: *Velocity profiles near a wall in ideal and real fluid.*

If the pressure p^+ is increasing in the flow direction, the pressure force on fluid particles will create a negative acceleration. The particles closest to the wall have the lowest velocities, and consequently they will stop sooner than the particles farther away from the wall. This means that at some section the velocity profile has a tangent perpendicular to the wall, and in practice the flow is no longer following the wall. This phenomenon is called *separation*.

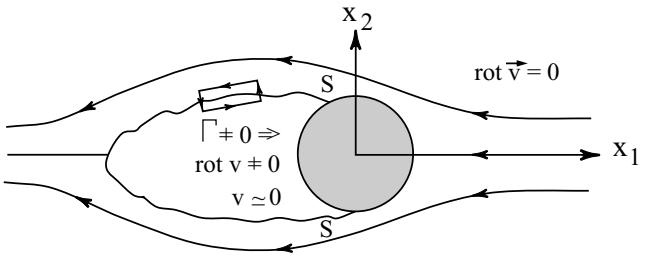

Figure 3.8: *Flow of a real fluid around a circular cylinder.*

In case of an un-separated flow the vorticity close to the wall is dispersed very slowly into the rest of the fluid domain, because the vorticity is transported by diffusion only. In separated flow the vorticity is normally dispersed very quickly into the fluid domain, as both diffusion and convection transport the vorticity.

As p^+ is increasing in the flow direction on the rear side of the cylinder, separation takes place there, and a lot of vorticity is dispersed into the flow behind the cylinder. In this flow domain the potential theory is bound to produce false results, as we have solved the differential equation $\nabla^2 \psi = 0$, and we should have solved $\nabla^2 \psi = (\mathrm{rot}\, \vec{v})_3$.

An example of the course of the real flow is depicted in Fig. 3.8, but notice that the course depends on the magnitude of the Reynolds number for the flow.

From this example one should *not* conclude that potential theory is useless. However, potential theory shall only be adopted for such parts of the flow, where the vorticity is confined in a rather thin layer close to the wall. In fact potential theory gives fine results for the upstream part of the flow around the cylinder!

3.2.4 Hydrodynamic Mass (Added Mass)

The concept of hydrodynamic mass is illustrated by the example given below.

Consider an air bubble with volume X, see Fig. 3.9. At the time $t = 0$ the bubble starts to rise, i.e. the velocity of the bubble is $U = 0$ and the acceleration $dU/dt \neq 0$.

The initial acceleration, dU/dt, is caused by the external forces on the bubble. In this case we have the gravity force, $F_g = \rho_L\, X\, g$, and the buoyancy, $F_o = \rho_w\, X\, g$, where ρ_L is the air density and ρ_w is the water density. Notice that buoyancy is defined as the resultant of the pressure forces on a body placed in a resting fluid, see Appendix A.

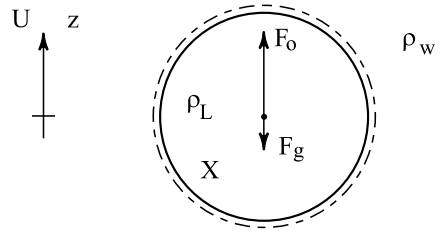

Figure 3.9: *Forces on an air bubble at rest.*

Newton's 2nd law reads:

$$\rho_L \cdot X \cdot \frac{dU}{dt} = \sum F = \rho_w \, X \, g - \rho_L \, X \, g = gX \, (\rho_w - \rho_L) \tag{3.49}$$

and division by $\rho_L X$ gives:

$$\frac{dU}{dt} = \frac{gX \, (\rho_w - \rho_L)}{\rho_L X} \simeq g \, \frac{800\rho_L - \rho_L}{\rho_L} = 799 \; g \;\; !! \tag{3.50}$$

This initial acceleration looks incredibly large, which is confirmed by measurements.

The reason for the miscalculation of dU/dt is of course that it is impossible to accelerate the bubble without accelerating some water, and this fact has not been taken into account by the calculation.

The correct size of the force from the water F acting on a body having the acceleration dU/dt is found below. For simplicity it assumed that the flow around the body is a *potential flow* and that the fluid is *ideal*. In practice, they are excellent approximations until boundary layers are developed and separation occurs.

Due to the assumption of ideal fluid the force $\vec{F}_p$ is solely caused by pressure on the surface of the body. If the area vector $d\vec{A}$ is defined positive from water towards the body, the expression for $\vec{F}_p$ reads:

$$\vec{F}_p = \int_A p \, d\vec{A} = \int_X -\text{grad}\, p \, dX \tag{3.51}$$

according to gradient theorem, see Appendix A. Notice that a uniform pressure distribution ($\Rightarrow$ grad $p = 0$) *does not* create any resulting pressure force on the body. The total force component in the $\vec{U}-$direction can be expressed as:

$$F_U \;\; = \;\; \vec{F}_p \cdot \vec{e}_U = \int_A p \, d\vec{A} \cdot \vec{e}_U \tag{3.52}$$

where $\vec{e}_U$ is a unit vector in the direction of the acceleration, see Fig. 3.10.

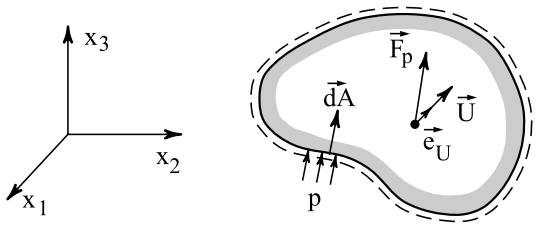

Figure 3.10: *Definition sketch.*

As we are interested only in the part of the force caused by the acceleration, it is necessary to subtract the buoyancy, $\vec{F}_o$, caused by the hydrostatic pressure, p_{hyd} .

The total pressure reads:

$$p = p_{hyd} + p^+ \tag{3.53}$$

where p^+ is defined as the deviation from hydrostatic pressure, p_{hyd}.

Substitution of equation (3.53) into the expression for the total force gives:

$$\vec{F}_p = \int_A (p_{hyd} + p^+) d\vec{A} = \int_A p_{hyd}\, d\vec{A} + \int_A p^+ d\vec{A} = \vec{F}_o + \vec{F_{p^+}} \tag{3.54}$$

The force component in the $\vec{U}$-direction caused by acceleration therefore is:

$$F_U^+ = \vec{F_{p^+}} \cdot \vec{e}_U = \int_A p^+\, d\vec{A} \cdot \vec{e}_U \tag{3.55}$$

The distribution of p^+ on the surface of the body is found by Bernoulli's generalized equation:

$$p^+ + \frac{1}{2}\,\rho_w v^2 + \rho_w\,\frac{\partial \varphi}{\partial t} = C(t) \tag{3.56}$$

where $\vec{v}$ is the fluid particle velocity. The term $C(t)$ is constant in space, i.e. it does not contribute to the resulting pressure force, so it is assumed that $C(t) = 0$ in the following. This gives:

$$p^+ = -\rho_w\,\frac{\partial \varphi}{\partial t} - \frac{1}{2}\,\rho_w v^2 \tag{3.57}$$

Immediately after the start of the motion $v^2 \simeq 0$ everywhere, and p^+ can be

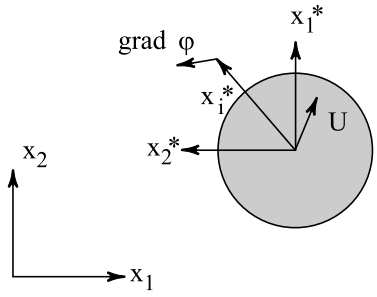

Figure 3.11: *Definition sketch. Velocity potential around a moving body.*

approximated by:

$$p^+ \simeq -\rho_w \frac{\partial \varphi}{\partial t} \tag{3.58}$$

It is now postulated that the velocity potential expressed in the $x_1^* \, x_2^* \, x_3^*$-system fixed to the body reads:

$$\boxed{\varphi(x_i^*, t) = U(t)\varphi_1(x_i^*)} \tag{3.59}$$

where

$U(t)$ is the velocity of the body with respect to the $x_1 \, x_2 \, x_3$-system fixed in space

x_i^* is the position vector in the $x_1^* \, x_2^* \, x_3^*$-system fixed to the body

φ_1 is the potential around the body corresponding to $U = 1$

and this expression is verified below for the flow around a moving circular cylinder.

As $\partial\varphi(x_i^*, t)/\partial t$ is the change in potential observed from a fixed position, i.e. for x_i^* = constant, it is seen from the equations (3.58) and (3.59), that

$$\begin{aligned} p^+ &= -\rho_w \frac{\partial \varphi(x_i^*, t)}{\partial t} \\ &= -\rho_w \, \varphi_1(x_i^*) \frac{\partial U}{\partial t} \end{aligned} \tag{3.60}$$

The velocity of the body is only depending on time, giving

$$\frac{dU}{dt} = \frac{\partial U}{\partial t}$$

and the acceleration of the body therefore creates the pressure:

$$p^+ = -\rho_w \cdot \varphi_1 \frac{dU}{dt} \tag{3.61}$$

Substitution of this expression into equation (3.55) gives:

$$F_U^+ = -\int_A \rho_w \, \varphi_1 \frac{dU}{dt} \, d\vec{A} \cdot \vec{e}_U$$

or

$$\boxed{F_U^+ = -\left(\rho_w \int_A \varphi_1 d\vec{A} \cdot \vec{e}_U\right) \frac{dU}{dt}} \tag{3.62}$$

Notice that the force from the fluid is directly proportional to the acceleration dU/dt.

Hereafter Newton's 2nd law reads:

$$\begin{aligned} m \cdot \frac{dU}{dt} &= F_{water} + F_g \\ &= F_o + F_U^+ + F_g \end{aligned} \tag{3.63}$$

which is rewritten to:

$$\begin{aligned} m \frac{dU}{dt} - F_U^+ &= F_o + F_g \\ \left(m + \rho_w \int_A \varphi_1 d\vec{A} \cdot \vec{e}_U\right) \frac{dU}{dt} &= F_o + F_g \\ (m + m') \frac{dU}{dt} &= F_o + F_g \end{aligned} \tag{3.64}$$

where m' is called hydrodynamic mass (or added mass). From equation (3.64) is seen that

$$\boxed{m' = \rho_w \int_A \varphi_1 \, d\vec{A} \cdot \vec{e}_U} \tag{3.65}$$

Sometimes it is possible to calculate φ_1 analytically, and subsequently calculate m' by equation (3.65). In general, however, m' is calculated numerically or by by use of model experiments.

The advantage of the introduction of hydrodynamic mass is of course that for a given body being accelerated in a given direction, we only need to calculate a

hydrodynamic mass once. Hereafter we can calculate the dynamic forces from the water by simply adding m_1 to the body mass in Newton's 2nd law, and the external forces from the water shall only comprise the known buoyancy.

Example: hydrodynamic mass of a circular cylinder

Consider the potential flow around a cylinder moving in the direction of the z-axis (upwards) at the velocity U in stagnant water. Due to the motion of the body a water volume is pushed from the upper side of the cylinder to the lower sider. This flow is very similar to the flow at a dipole having the source placed at $\left(\frac{s}{2}\ ,\ 0\right)$ and the sink placed at $\left(-\frac{s}{2}\ ,\ 0\right)$ in the $x_1^*\,x_2^*$-coordinate system, see Fig. 3.12.

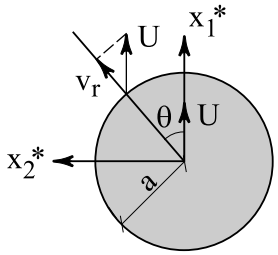

Figure 3.12: *Boundary conditions for the flow around a moving cylinder.*

Is it possible to fulfill the boundary conditions for this flow simply by scaling the dipole strength to a suitable value? The answer is a bit surprising: "yes".

Expressed in the $x_1^*\,x_2^*$-system fixed to the cylinder, the velocity potential reads:

$$\varphi_{dipol} = -m\,\frac{\cos\theta}{r}$$

where m is the dipole strength:

$$m = \frac{qs}{2\pi}$$

The boundary conditions are:

1) no flow through the surface of the cylinder $\Leftrightarrow\ v_r = (v_r)_{cyl} = U\ \cos\theta$ for $r = a$

2) no fluid motion far away from the cylinder $\Leftrightarrow$ $v \to 0$ for $r \to \infty$

Fulfillment of boundary condition 2) only requires that the dipole strength is limited.

As

$$v_r = \frac{\partial \varphi_{dipol}}{\partial r}$$

gives:

$$v_r = -m \left(-\frac{1}{r^2} \right) \cos\theta = m \frac{1}{r^2} \cos\theta$$

boundary condition 1) can be rewritten to:

$$v_r = m \frac{1}{a^2} \cos\theta = U \cos\theta$$

Boundary condition 1) is therefore fulfilled for all θ, if $m = U\, a^2$.

Thus both boundary conditions are fulfilled by this dipole strength. The velocity potential expressed in the coordinate system fixed to the cylinder reads:

$$\varphi_{dipol} = -U\, a^2 \frac{\cos\theta}{r} \tag{3.66}$$

This expression has the form:

$$\varphi = U(t)\varphi_1$$

and the unit potential for the flow around a circular cylinder therefore reads:

$$\varphi_1 = -a^2 \frac{\cos\theta}{r} \tag{3.67}$$

Notice that the flow is unsteady seen from a coordinate system fixed in space, when the dipole is moving, see Fig 3.13.

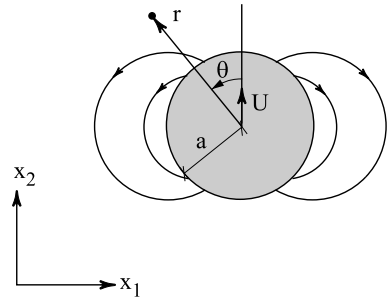

Figure 3.13: *Streamlines seen from a coordinate system fixed in space.*

Substitution of $\varphi_1 = -a^2(\cos\theta/r)$ into the expression for m' gives:

$$\begin{aligned} m' &= \rho_w \int_A \varphi_1 d\vec{A} \cdot \vec{e}_U \\ &= \rho_w \int_0^{2\pi} (-a^2 \frac{\cos\theta}{a})(-a\, d\theta \cdot \cos\theta) \\ &= \rho_w a^2 \int_0^{2\pi} \cos^2 d\theta \\ &= \rho_w a^2 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} \\ &= \rho_w a^2 \cdot \pi \end{aligned}$$

or the hydrodynamic mass for a circular cylinder is equal to the displaced mass of fluid per unit length !

From this we can conclude that in order to produce the acceleration dU/dt for a cylinder with diameter D, we need the force:

$$F = (m + m')\,\frac{dU}{dt} = (m + \rho_w\,\frac{\pi}{4}\,D^2)\,\frac{dU}{dt} \quad N/m$$

After this example we shall return to the calculation of the initial acceleration of an air bubble.

For a sphere with volume X it is in a similar way (by superposition of a point source and a point sink) found that $m' = \frac{1}{2}\,\rho_w X$. Newton's 2nd law for the air bubble therefore reads:

$$\begin{aligned} (m + m')\,\frac{dU}{dt} &= \sum F \\ (\rho_L X + \frac{1}{2}\,\rho_w X)\,\frac{dU}{dt} &= \rho_w\, g\, X - \rho_L\, g\, X \end{aligned}$$

giving the initial acceleration:

$$\begin{aligned} \frac{dU}{dt} &= \frac{gX(\rho_w - \rho_L)}{X(\rho_L + \frac{1}{2}\rho_w)} \\ &\simeq g \cdot \frac{\rho_w}{\frac{1}{2}\rho_w} \\ &= 2\,g \quad ! \end{aligned}$$

This magnitude of the acceleration is confirmed by measurements, but it should be remembered that the basic assumptions are valid only a short time after the bubble is generated.

Chapter 4

Boundary Layers

In a *real fluid* adhesion causes the particle velocity at the wall and the velocity of a wall to be equal (often zero). If *ideal fluid* is assumed, only the components along the normal to the wall have to be equal, and calculations (potential theory or solution of Euler's equations) give different values of the tangential velocity component at the wall. However, measurements often show that the calculated velocities are in excellent agreement with velocities measured not far away from the wall. In a domain close to a wall the tangential velocity therefore increases from zero to the value calculated by the theory for an ideal fluid. See Fig. 4.1. This domain is called the *boundary layer*.

Based on this observation, Prandtl (1904) suggested that the flow domain should be divided into two sub domains:

1) *the boundary layer*, where the fluid is assumed to be *real*. The effects from shear stresses are taken into account, and the Navier-Stokes equation is used.

2) *the domain outside* the boundary layer, where the fluid is assumed to be *ideal*. In this domain the shear stresses are unimportant, and the flow can be approximated by potential flow.

This approach is not a guarantee for good results, but it has proven its capability in the following cases:

- flow immediately after a sudden start of the motion of a body
- the flow caused by relatively high frequent oscillations of a body, if the amplitude of the motion is small compared to a characteristic dimension of the body
- the flow around very slender bodies like e.g. wings of an aeroplane.

4.1 Flow Equations for Boundary Layers

Due to the assumption of real fluid in the boundary layer we must use Navier-Stokes equations. In order to keep things simple we consider a two-dimensional flow along a thin plate, where velocity profiles are observed as depicted in Fig. 4.1.

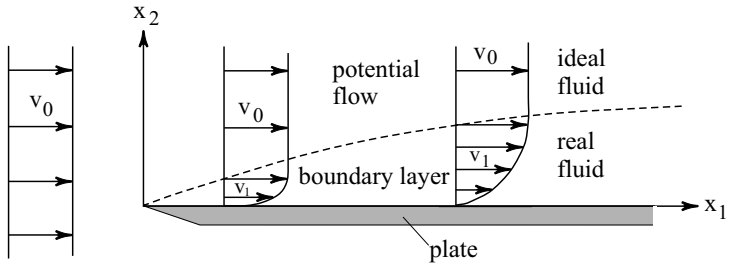

Figure 4.1: *Definition sketch, boundary layer.*

Notice that the thickness of the boundary layer is increasing down the plate, and that velocities and streamlines are approximately parallel to the plate, i.e. $v_2 \ll v_1$.

The component of Navier-Stokes equation in the x_2-direction reads:

$$\rho \frac{dv_2}{dt} = -\frac{\partial p^+}{\partial x_2} + \mu \left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} \right) \tag{4.1}$$

As $v_2 \approx 0$ everywhere, equation (4.1) is reduced to

$$\frac{\partial p^+}{\partial x_2} \approx 0$$

This pressure distribution was also found in the flow between to parallel plates, and it is in fact always found in parts of a flow, where the streamlines locally are parallel.

In a section perpendicular to the streamlines we have approximately:

$$\boxed{p^+ = K(x_1)} \tag{4.2}$$

Thus, in the boundary layer the dynamic pressure is controlled by the dynamic pressure p_o^+ in the potential flow just outside the boundary layer, as they are equal at the top of the boundary layer, i.e. $p^+ = p_o^+$.

The Navier-Stokes equation in the flow direction reads:

$$\rho\left(\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}\right) = -\frac{\partial p^+}{\partial x_1}+\mu\left(\frac{\partial^2 v_1}{\partial x_1^2}+\frac{\partial^2 v_1}{\partial x_2^2}\right) \tag{4.3}$$

As the variations in velocity across the main flow direction are much larger than the variations in the flow direction, we have:

$$\frac{\partial^2 v_1}{\partial x_1^2} \ll \frac{\partial^2 v_1}{\partial x_2^2}$$

and the equation of motion is therefore reduced to:

$$\boxed{\rho\left(\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}\right)=-\frac{\partial p^+}{\partial x_1}+\mu\frac{\partial^2 v_1}{\partial x_2^2}} \tag{4.4}$$

Notice that normally we cannot neglect the last term on the left-hand side of the equation. The component v_2 is indeed a very small quantity, but $\partial v_1/\partial x_2$ is on the other hand a very large quantity within the boundary layer. Finally it is seen that due to equation (4.2), the pressure gradient in equation (4.4) is equal to the gradient in the potential flow, and this gradient is calculated by adopting Bernoullis's equation. Equation (4.4) is often named the *boundary layer equation.*

In reality there is no well defined transition between boundary layer and potential flow as the influence of the shear stresses is gradually decreasing. It is therefore not possible to define the thickness of the boundary layer uniquely.

4.1.1 Boundary Layer on a Flat Plate in Uniform Flow

We consider the flow along a flat plate placed in a steady, uniform flow. In the potential flow Bernoulli's generalized equation reads:

$$\frac{p^+}{\gamma}+\frac{v^2}{2g}=constant \quad \text{as} \quad \frac{\partial\varphi}{\partial t}=0 \tag{4.5}$$

In the potential flow we have $v \simeq v_o = constant$, and thus $p^+ = constant$ everywhere in the potential flow. According to equation (4.2) this means that $p^+ = constant$ everywhere in the boundary layer, and consequently we have $\partial p^+/\partial x_1 = 0$ in the boundary layer.

The boundary layer equation now reads:

$$v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}=\nu\frac{\partial^2 v_1}{\partial x_2^2} \tag{4.6}$$

Notice that for continuity reasons v_2 is not zero, even though $v_2 \ll v_1$. We need two equations to determine v_1 and v_2, and the second equation is the continuity equation:

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \tag{4.7}$$

The boundary conditions are:

$$\begin{aligned} x_2 = 0 \quad &: \quad v_1 = v_2 = 0 \\ x_2 \to \infty \quad &: \quad v_1 \to v_o \end{aligned}$$

Measurements of velocity profiles along the plate show similar profiles, see Fig. 4.2. This was utilized in the solution presented by Blasius (1908).

In Fig. 4.2 the non-dimensional coordinate perpendicular to the flow direction reads:

$$y = \frac{x_2}{\sqrt{\frac{\nu x_1}{v_o}}} \tag{4.8}$$

Similarity of the velocity profiles can be expressed by

$$\boxed{\frac{v_1}{v_o} = f'(y)} \tag{4.9}$$

where $f'(y)$ is a function depending of y only. The stream function can be expressed as

$$\psi = -\sqrt{\nu x_1 v_o} \cdot f(y) \tag{4.10}$$

and the correctness of expression is seen by substitution into the first definition equation for ψ, which gives:

$$v_1 = -\frac{\partial \psi}{\partial x_2} = \sqrt{\nu x_1 v_o}\, \frac{\partial f}{\partial y}\, \frac{\partial y}{\partial x_2} = \sqrt{\nu x_1 v_o}\, \frac{1}{\sqrt{\frac{\nu x_1}{v_o}}} \cdot f'(y) = v_o \cdot f'(y)$$

Substitution into the second definition equation gives:

$$v_2 = \frac{\partial \psi}{\partial x_1} = \frac{1}{2} \sqrt{\frac{\nu v_o}{x_1}}\, (y f'(y) - f(y)) \tag{4.11}$$

If these expressions for v_1 and v_2 are substituted into the boundary layer equation (4.6), it is possible to reduce this equation to the ordinary differential equation:

$$\boxed{f\, f'' + 2\, f''' = 0} \tag{4.12}$$

with the boundary conditions:

$$\begin{aligned} y = 0 \quad &: \quad f = f' = 0 \\ y \to \infty \quad &: \quad f' \to 1 \end{aligned}$$

It is not possible to solve equation (4.12) analytically, but it is easy to get a numerical solution of the equation. The only problem is the boundary condition at infinite distance from the plate, but in practice excellent results are obtained if the boundary condition is fulfilled for $y \geq 8$. The boundary condition $f'(8) = 1$ has been used to calculate the velocity profile shown with a broken line in Fig. 4.2 .

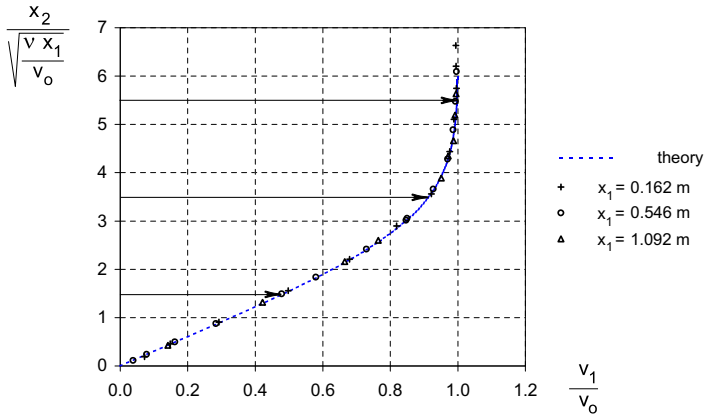

Figure 4.2: *Velocity profiles in air flow along a plate corresponding to* $v_o = 10$ *m/s and* $\nu = 15 \cdot 10^{-5}$ m^2/s.

In practice the thickness of the boundary layer is infinite, but based on Fig. 4.2 a visual assessment of the boundary layer thickness would be:

$$y = \frac{x_2}{\sqrt{\frac{\nu x_1}{v_o}}} \approx 5 - 6 \tag{4.13}$$

From the numerical solution it is seen that $v_1 = 0.99\, v_o$ for $y = 4.9$ and $v_1 = 0.999\, v_o$ for $y = 6.0$. It is therefore not evident how to define a thickness that is of practical use.

Often the so-called *displacement thickness,* δ^*, is adopted as an objective measure for the boundary layer thickness. The basic idea is that the discharge corresponding to the actual profile in the boundary layer shall be equal to the discharge corresponding to an imaginary velocity profile with $v_1 = 0$ for $x_2 \leq \delta^*$ and $v_1 = v_o$ for $x_2 > \delta^*$, see Fig. 4.3. This is expressed as:

$$q = \int_0^\infty v_1 dx_2 = \int_{\delta^*}^\infty v_o dx_2 = \int_0^\infty v_o dx_2 - \int_0^{\delta^*} v_o dx_2$$

or

$$\int_0^\infty v_1 dx_2 = \int_0^\infty v_o dx_2 - v_o\, \delta^*$$

which gives:

$$\boxed{\delta^* = \int_0^\infty (1 - \frac{v_1}{v_o})\, dx_2} \tag{4.14}$$

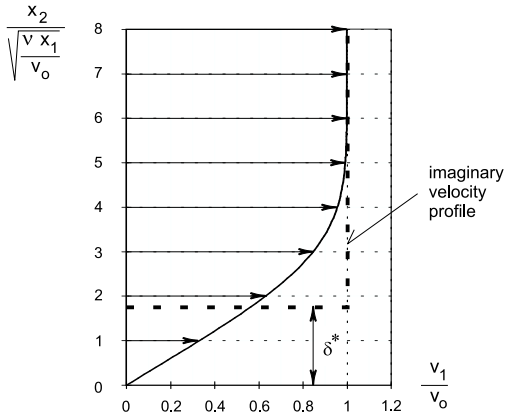

Figure 4.3: *Imaginary velocity profile defining* δ^*.

The displacement thickness is therefore a measure of the distance that the potential flow has been displaced from the plate due to the velocity reduction in the boundary layer. Substitution of $v_1/v_o = f'(y)$ into this expression gives:

$$\delta^* = 1.72 \sqrt{\frac{\nu x_1}{v_o}}$$

This value is shown in Fig. 4.3.

4.2 The Momentum Equation

Sometimes it is easier to consider a fluid body of finite volume instead of fluid particle. Like in chapter 2.2 a fluid body with volume X and surface area A is considered, see Fig. 2.5, but this time the results are not rewritten to be valid for a fluid particle. Newton's 2nd law for the fluid body reads:

$$\int_X \rho\, dX\, \frac{dv_i}{dt} = \int_X \rho\, dX\, g_i + \int_A \sigma_i^N\, dA \quad , \quad i = 1, 2, 3 \tag{4.15}$$

where $\sigma_i^N = \vec{\sigma}^N$ is the stress vector on the area dA. This stress vector is equivalent to a normal force, $-p\, dA_i$, and a shear force, $\tau_i\, dA$, on dA, and

equation (4.15) can be rewritten to:

$$\int_X \rho\, dX\, \frac{dv_i}{dt} = \int_X \rho\, dX\, g_i - \int_A p\, dA_i + \int_A \tau_i\, dA \quad , \quad i = 1,2,3 \tag{4.16}$$

As the resultant of the hydrostatic pressure, p_{hyd}, on A is equal to the gravity force on X, equation (4.16) can be reduced to:

$$\int_X \rho\, dX\, \frac{dv_i}{dt} = -\int_A p^+\, dA_i + \int_A \tau_i\, dA \quad , \quad i = 1,2,3 \tag{4.17}$$

where p^+ is the deviation in pressure from hydrostatic pressure. Substitution of

$$\frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_k}\, v_k \tag{4.18}$$

gives

$$\int_X \rho\, dX \left[\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_k}\, v_k\right] = -\int_A p^+\, dA_i + \int_A \tau_i\, dA \quad , \quad i = 1,2,3 \tag{4.19}$$

The second term on the left-hand side is rewritten by use of the divergence theorem for the vector fields $v_1 v_k$, $v_2 v_k$ and $v_3 v_k$ giving:

$$\int_A v_i\, v_k\, dA_k = \int_X \frac{\partial (v_i\, v_k)}{\partial x_k}\, dX = \int_X \left[\frac{\partial v_i}{\partial x_k}\, v_k + v_i \frac{\partial v_k}{\partial x_k}\right] dX = \int_X \frac{\partial v_i}{\partial x_k}\, v_k\, dX$$

for $i = 1,2,3$, where $\partial\, v_k / \partial\, x_k = 0$ due to the assumption of incompressible fluid. Hereafter equation (4.19) reads:

$$\boxed{\int_X \rho\, \frac{\partial v_i}{\partial t}\, dX + \int_A \rho\, v_i\, v_k dA_k = -\int_A p^+\, dA_i + \int_A \tau_i\, dA} \tag{4.20}$$

This equation is called the *general momentum equation.* Notice that it is valid for any volume X (called the *control volume*) of an incompressible fluid bounded by a *closed* surface A (called the *control surface*). If the control surface cuts solid structures, the stresses on such sections should be included into the surface forces on the right-hand side of equation (4.20). Remember also that $d\vec{A}$ is defined positive in the *outward* direction.

Often equation (4.20) is written in vector notation giving:

$$\boxed{\int_X \rho\, \frac{\partial \vec{v}}{\partial t}\, dX + \int_A \rho\, \vec{v}\, (\vec{v} \cdot d\vec{A}) = -\int_A p^+\, d\vec{A} + \int_A \vec{\tau}\, dA} \tag{4.21}$$

4.3 Momentum Equation for Boundary Layers

The general momentum equation is now applied to a boundary layer on a flat plate. It is assumed that the the boundary layer has a *finite* thickness, δ, even though the transition from boundary layer to potential flow is not well defined.

The control surface is depicted in Fig. 4.4, and it is extended by 1 m perpendicular to the flow plane.

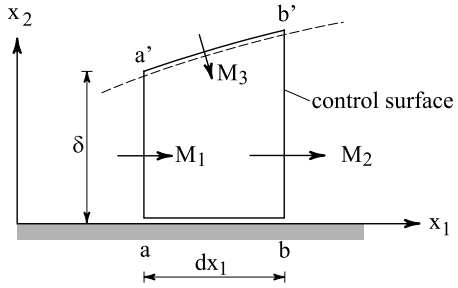

Figure 4.4: *Control surface for boundary layer flow.*

Consider the x_1 component of the momentum equation. The 1st term on the left-hand side (change of momentum within the control surface) reads:

$$\int_X \rho \frac{\partial v_1}{\partial t}\, dX = dx_1 \cdot 1 \int_0^\delta \rho \frac{\partial v_1}{\partial t}\, dx_2$$

and the 2nd term (change of momentum due to different inflow and outflow of momentum trough the control surface) reads:

$$\int_A \rho\, v_1\, (\vec{v} \cdot d\vec{A})$$

Here $\vec{v} \cdot d\vec{A}$ is the discharge (volume flux) through dA, and $\rho\, v_1\, (\vec{v} \cdot d\vec{A})$ is thus the momentum flux through dA. Before this integral is determined, it is practical to consider the fluxes of mass through the control surface. For an incompressible fluid we have:

$$M_2 = M_1 + M_3 \tag{4.22}$$

where M_1, M_2 and M_3 are shown in Fig. 4.4. The mass flux through the surface aa' is:

$$M_1 = \int_0^\delta \rho\, v_1\, dx_2 \cdot 1$$

and the mass flux through bb' can be expressed as:

$$M_2 = M_1 + \frac{\partial M_1}{\partial x_1}\, dx_1 = M_1 + dx_1 \frac{\partial}{\partial x_1} \int_0^\delta \rho\, v_1\, dx_2$$

After substitution of M_1 and M_2 into equation (4.22) the mass flux *inwards* through the top side of the boundary layer reads:

$$M_3 = dx_1 \frac{\partial}{\partial x_1} \int_0^\delta \rho\, v_1\, dx_2 \tag{4.23}$$

According to the definition of mass flux we might also express M_3 as:

$$M_3 = -\int_{a'}^{b'} \rho\, (\vec{v} \cdot d\vec{A}) \tag{4.24}$$

and comparison of the two expressions for M_3 :

$$\int_{a'}^{b'} \rho\, (\vec{v} \cdot d\vec{A}) = -dx_1 \frac{\partial}{\partial x_1} \int_0^\delta \rho\, v_1\, dx_2 \tag{4.25}$$

The next step is to find the momentum flux through the control surface. Through the surface aa' the flux reads:

$$B_1 = \int_0^\delta \rho\, v_1\, (\vec{v} \cdot d\vec{A}) = \int_0^\delta \rho\, v_1\, (-v_1\, dx_2) = -\int_0^\delta \rho\, v_1^2\, dx_2 \tag{4.26}$$

and through the surface bb' it is:

$$B_2 = \int_0^\delta \rho\, v_1\, (v_1\, dx_2) + dx_1 \frac{\partial}{\partial x_1} \int_0^\delta \rho\, v_1\, (v_1\, dx_2) \tag{4.27}$$

At the top side of the boundary layer (the surface $a'b'$) the expression for the momentum flux reads:

$$\begin{aligned} B_3 &= \int_{a'}^{b'} \rho\, v_1\, (\vec{v} \cdot d\vec{A}) = v_o \int_{a'}^{b'} \rho\, \vec{v} \cdot d\vec{A} \\ &= -v_o\, dx_1 \frac{\partial}{\partial x_1} \int_0^\delta \rho\, v_1\, dx_2 \end{aligned} \tag{4.28}$$

as $v_1 = v_o$ at the top side of the boundary layer, and equation (4.25) has been substituted.

On the right-hand side of the momentum equation the resultant of the pressure forces is rewritten to:

$$P_i^+ = -\int_A p^+\, dA_i = -\int_X \frac{\partial p^+}{\partial x_i} dX \tag{4.29}$$

by use of the *gradient theorem* derived in Appendix A. As the extent of the control surface in the x_1-direction is infinitesimal and p^+ does not vary in the x_2-direction, we know that $\partial p^+/\partial x_i = constant$ within the control surface. Consequently we can rewrite the pressure force on A to:

$$P_i^+ = -\int_X \frac{\partial p^+}{\partial x_i} dX = -\frac{\partial p^+}{\partial x_i} \int_X dX = -\frac{\partial p^+}{\partial x_i} X \tag{4.30}$$

and the component in the x_1-direction reads:

$$P_1^+ = -\frac{\partial p^+}{\partial x_1} X = -\frac{\partial p^+}{\partial x_1} \cdot dx_1 \cdot \delta \cdot 1 \tag{4.31}$$

Shear stresses are only present at the bottom of the control surface. Being directed opposite to the flow direction, the x_1-component can expressed as:

$$T_1 = -\tau_o \cdot dx_1 \cdot 1 \tag{4.32}$$

where τ_o (> 0) is the shear stress at the plate. Substitution of these expressions into the momentum equation and division by dx_1 on both sides gives:

$$\int_0^\delta \rho \frac{\partial v_1}{\partial t} dx_2 + \frac{\partial}{\partial x_1} \int_0^\delta \rho v_1^2 dx_2 - v_o \frac{\partial}{\partial x_1} \int_0^\delta \rho v_1 dx_2 = -\delta \frac{\partial p^+}{\partial x_1} - \tau_o \tag{4.33}$$

or

$$\boxed{\int_0^\delta \rho \frac{\partial v_1}{\partial t} dx_2 + \frac{\partial}{\partial x_1} \int_0^\delta \rho v_1 (v_1 - v_o) dx_2 = -\delta \frac{\partial p^+}{\partial x_1} - \tau_o} \tag{4.34}$$

This equation is called the *momentum equation for boundary layers*. In practice it is used to calculate approximate values for the thickness of the boundary layer and the wall shear stress τ_o. The pressure gradient is determined by the pressure gradient in potential flow outside the boundary layer, and if similar velocity profiles along the plate are assumed, only a single velocity profile has to be assessed in order to find δ from equation (4.34).

Example: Calculation of the thickness of boundary layer on a flat plate

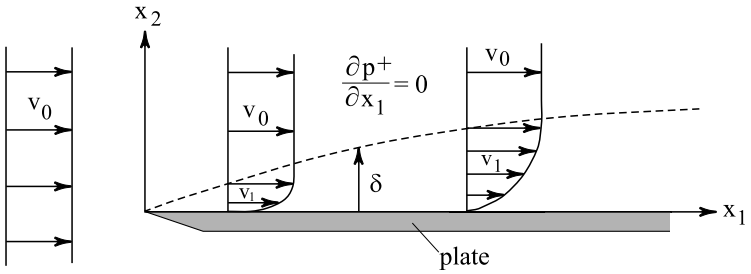

Figure 4.5: *Boundary layer on a flat plate.*

The flow depicted in Fig. 4.5 is steady assuming $v_o = constant$. For the potential flow Bernoulli's generalized equation reads:

$$\frac{p^+}{\gamma} + \frac{v^2}{2g} = constant \quad \text{as} \quad \frac{\partial \varphi}{\partial t} = 0 \tag{4.35}$$

In the potential flow we have $v \simeq v_o = constant$, and thus $p^+ = constant$. According to equation (4.2) this means that $p^+ = constant$ everywhere in the boundary layer, and consequently we have $\partial p^+/\partial x_1 = 0$ in the boundary layer.

Thus the momentum equation for the boundary layer reads:

$$0 + \frac{\partial}{\partial x_1} \int_0^\delta \rho\, v_1 (v_1 - v_o)\, dx_2 = -0 - \tau_o \tag{4.36}$$

The assessed velocity profile has to fulfill the following boundary conditions:

1) $v_1 = 0$ for $x_2 = 0$
2) $v_1 = v_o$ for $x_2 = \delta$
3) $\frac{\partial v_1}{\partial x_2} = 0$ for $x_2 = \delta$ (corresponding to no shear stress at the top layer)

These conditions are fulfilled by the profile:

$$v_1 = v_o \sin \frac{\pi x_2}{2\delta} \tag{4.37}$$

but they can also be fulfilled by an appropriate polynomial.

The shear stress at the plate is calculated by Newton's formula, which after substitution of the assessed profile reads:

$$\tau_o = \mu \left(\frac{\partial v_1}{\partial x_2} \right)_{x_2=0} = \mu \frac{\pi v_o}{2\delta} \left(\cos \frac{\pi x_2}{2\delta} \right)_{x_2=0} = \mu \frac{\pi v_o}{2\delta} \cdot 1 \tag{4.38}$$

Substitution of v_1 gives:

$$\begin{aligned}\int_0^\delta \rho\, v_1(v_1 - v_o)\, dx_2 &= \int_0^\delta \rho\, v_o \sin\frac{\pi x_2}{2\delta}\left(v_o \sin\frac{\pi x_2}{2\delta} - v_o\right) dx_2 \\ &= \rho\, v_o^2 \frac{2\delta}{\pi}\int_0^\delta \sin\frac{\pi x_2}{2\delta}\left(\sin\frac{\pi x_2}{2\delta} - 1\right) d\frac{\pi x_2}{2\delta} \\ &= \rho\, v_o^2 \frac{2\delta}{\pi}\left(\frac{\pi}{4} - 1\right) \qquad (4.39)\end{aligned}$$

As δ depends on x_1 only, the left-hand side of the momentum equation reads:

$$\frac{\partial}{\partial x_1}\int_0^\delta \rho\, v_1(v_1 - v_o)\, dx_2 = \rho\, v_o^2 \frac{2}{\pi}\left(\frac{\pi}{4} - 1\right)\frac{d\,\delta}{d\,x_1} \qquad (4.40)$$

Substitution of the equations (4.38) and (4.40) into equation (4.36) yields:

$$\rho\, v_o^2 \frac{2}{\pi}\left(\frac{\pi}{4} - 1\right)\frac{d\,\delta}{d\,x_1} = -\frac{\mu\pi v_o}{2\delta} \qquad (4.41)$$

or

$$\frac{d\delta^2}{dx_1} = \frac{\mu\pi^2}{2\rho v_o(1-\frac{\pi}{4})} = \frac{\nu\pi^2}{2v_o(1-\frac{\pi}{4})} \qquad (4.42)$$

where $\mu = \nu\,\rho$ has been substituted. Integration gives:

$$\delta^2 = \frac{\nu\pi^2}{2v_o(1-\frac{\pi}{4})}\, x_1 + C_1 \qquad (4.43)$$

The constant C_1 is determined by using the boundary condition: $\delta = 0$ for $x_1 = 0$, which gives $C_1 = 0$. The approximate expression for the boundary layer thickness therefore reads:

$$\delta = \sqrt{\frac{\nu\pi^2 x_1}{2v_o(1-\frac{\pi}{4})}} = 4.80\sqrt{\frac{\nu\, x_1}{v_o}} \qquad (4.44)$$

The corresponding displacement thickness reads:

$$\delta^* = \int_0^\delta (1 - \sin\frac{\pi x_2}{2\delta})\, dx_2 = 0.363\,\delta = 1.74\sqrt{\frac{\nu x_1}{v_o}}$$

The choice of a velocity profile could be crucial for the accuracy of the estimates. That's why estimates obtained by two different assessed profiles are shown in the table below. Note that $\eta = x_2/\delta$ in the table.

velocity profile	$\delta^*\sqrt{\frac{v_o}{\nu x_1}}$	$\frac{\tau_o}{\rho v_o^2}\sqrt{\frac{v_o x_1}{\nu}}$
$f(\eta) = 1.5\eta - 0.5\eta^3$	1.74	0.32
$f(\eta) = \sin(\frac{\pi}{2}\eta)$	1.74	0.33
correct profile	1.72	0.33

It is seen that both the displacement thickness and the shear at the plate are estimated well by the momentum equation for boundary layers, and apparently the choice of velocity profile is not important.

4.4 Separation

In general the velocity profiles in a boundary layer are *not* similar. Such layers can be analyzed by a method proposed by Pohlhausen (1921).

It is assumed that the flow is *steady*, and that the velocity profiles can be described by a 4th order polynomial, i.e.

$$\frac{v_1}{v_o} = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4 \tag{4.45}$$

where $\eta = \frac{x_2}{\delta}$, i.e. it is assumed that the thickness of the layer is finite. This polynomial is seen to fulfill the boundary condition $v_1 = 0$ for $x_2 = 0$.

The four coefficients a, b, c and d are determined by adopting the following boundary conditions:

1) Continuous velocity profile:

$$f(1) = 1, \text{ corresponding to } \quad \frac{v_1}{v_o} = 1 \ \text{ for } \ \frac{x_2}{\delta} = 1$$

2) No shear stress at the top of the boundary layer:

$$f'(1) = 0, \text{ corresponding to } \quad \frac{\partial v_1}{\partial x_2} = 0 \ \text{ for } \ \frac{x_2}{\delta} = 1$$

3) No curvature of the velocity profile at the top of the boundary layer:

$$f''(1) = 0, \text{ corresponding to } \quad \frac{\partial^2 v_1}{\partial x_2^2} = 0 \ \text{ for } \ \frac{x_2}{\delta} = 1$$

4) Very close to a wall the velocities remain low. Consequently also accelerations are small close to a wall, and the equation of motion (4.4) is approximated by:

$$0 \approx -\frac{\partial p^+}{\partial x_1} + \mu \frac{\partial^2 v_1}{\partial x_2^2}$$

or

$$\frac{\partial^2 v_1}{\partial x_2^2} = \frac{1}{\mu}\frac{\partial p^+}{\partial x_1} \quad \text{for} \quad x_2 = 0 \tag{4.46}$$

Here $\partial p^+/\partial x_1$ is controlled by the flow outside the boundary layer and by use of Bernoulli's generalized equation i.e. $p^+ + 1/2\rho v_o^2 = constant$ derived with respect to x_1 , the gradient reads:

$$\frac{\partial p^+}{\partial x_1} = -\rho v_o \frac{\partial v_o}{\partial x_1} \tag{4.47}$$

As

$$\frac{\partial^2 v_1}{\partial x_2^2} = \frac{v_o}{\delta^2} f''(\eta)$$

equation (4.46) can be rewritten to:

$$v_o \, f''(0) \frac{1}{\delta^2} = \frac{1}{\mu} \frac{\partial p^+}{\partial x_1}$$

or

$$f''(0) = \frac{\delta^2}{\mu \, v_o} \frac{\partial p^+}{\partial x_1} \tag{4.48}$$

Notice that close to a wall the curvature of the profile depends on the pressure gradient outside the boundary layer.

Defining the parameter Λ as:

$$\boxed{\Lambda = -\frac{\delta^2}{\mu v_o} \frac{\partial p^+}{\partial x_1}} \tag{4.49}$$

and making use of the four boundary conditions for $f(\eta)$ gives (after some algebra) expressions for a, b, c og d. Without going into details the result can be expressed as:

$$\boxed{\frac{v_1}{v_o} = F(\eta) + \Lambda G(\eta)} \tag{4.50}$$

where

$$\begin{aligned} F(\eta) &= 2\eta - 2\eta^3 + \eta^4 \\ G(\eta) &= \frac{1}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4) \end{aligned}$$

Thus, the velocity profiles are *not* similar, but they are part of a one-parameter profile family with form parameter Λ.

Substitution of this expression into the momentum equation makes it possible to calculate the boundary layer thickness δ and wall shear stress τ_o the same way it was done for similar profiles.

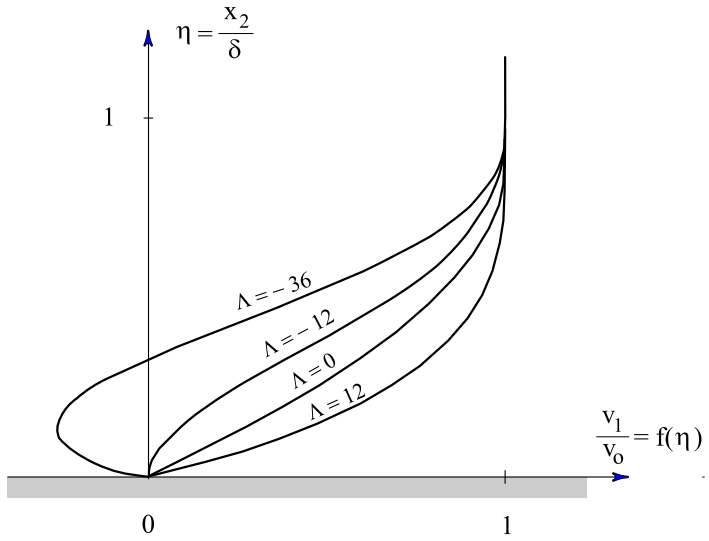

Figure 4.6: *Non-similar velocity profiles in boundary layers.*

The sign and magnitude of Λ are, however, decisive for the shape of the velocity profile, see Fig. 4.6.

If the fluid particles are flowing towards decreasing pressure, we have $\Lambda > 0$, as the pressure gradient is negative. From Fig. 4.6 it is seen that the velocity profiles are close to being similar for $\Lambda > 0$.

On the other hand, if the fluid particles flow towards increasing pressure, we have $\Lambda < 0$, as the pressure gradient is positive. In this case the shape of the profile strongly depends on Λ. From Fig. 4.6 it is seen that for $\Lambda = -12$, the tangent to the velocity profile at the wall is perpendicular to the wall, and for $\Lambda < -12$, the flow is reversed in the area closest to the wall. The spot in the boundary layer, where $\Lambda = -12$ appears, is called the *point of separation*. On the length after this point, we say that the flow is separated from the wall.

If we consider the flow past a circular cylinder, see Fig. 3.8, the pressure is decreasing from the upstream stagnation point up to the point, where the x_2-axis intersects the cylinder. Therefore, separation does not occur on this length. On the downstream side of the cylinder the pressure is increasing in the flow direction, and separation occurs as depicted in the figure. In practice, the position of the point of separation and the course of the flow behind the cylinder strongly depend on Reynolds' number, and the appearance in Fig. 3.8 only occurs for $10 < Re < 40$.

Chapter 5

Turbulent Flows

A transition from laminar to turbulent flow always takes place, when either the characteristic velocity or the characteristic length in Reynolds' number, $Re = V\,L\,/\,\nu$, exceeds a certain value. For the flow in a circular pipe with diameter D and discharge (volume flux) Q, the diameter D and the average velocity $V = Q/A$ are often selected as characteristic length and characteristic velocity, respectively. Experiments show that the transition to turbulent flow takes place at

$$Re = \frac{V\,D}{\nu} \approx 2300 \tag{5.1}$$

As for pure water $\nu \approx 10^{-6}$ m^2/s, the transition takes place at $V = 0.046$ m/s for a flow in a pipe with diameter $D = 0.05$ m. Consequently nearly all pipe flows are turbulent in practice.

Turbulence is caused by instability of a flow. If a disturbance is introduced into a flow, it has been shown that an increase of the energy corresponding to the disturbance always takes place if the Reynolds number is large enough. The disturbance is spread into the entire flow, and the result is the very chaotic type of flow, which we call turbulent flow.

The velocity field corresponding to smoothed (with time, i.e. low-pass filtered) values of the velocities is called the *mean flow*, and a turbulent flow may be considered as the mean flow superposed by eddies causing the velocity fluctuations of stochastic nature.

Navier-Stokes equation is still valid for the fluid particles in a turbulent flow, but an analytical solution is excluded. Also, a direct numerical solution is excluded in most cases, as it is necessary to describe (resolve) all the small eddies in the flow. In practice this demands so many grid points in the calculation that both

memory and computational speed are inadequate except for the so-called super computers. However, some direct numerical solutions exist, and they are used as benchmarks for the approximate solutions obtainable with ordinary commercial numerical solvers and computers. These solvers do not try to resolve the small eddies, but they attempt to take the effect of the eddies into account by use of so-called turbulence models.

5.1 Stability of Laminar Flow

In the following it is shown that for sufficiently large values of Reynolds' number, any disturbance in a laminar flow will grow with time, and transition to turbulent flow takes place.

The stability analysis is based on *the vorticity transport equation* for two-dimensional flow, i.e.

$$\frac{d(\mathrm{rot}\,\vec{v})_3}{dt} = \nu\,\nabla^2(\mathrm{rot}\,\vec{v})_3 \tag{5.2}$$

and a flow close to a wall is considered, see Fig. 5.1.

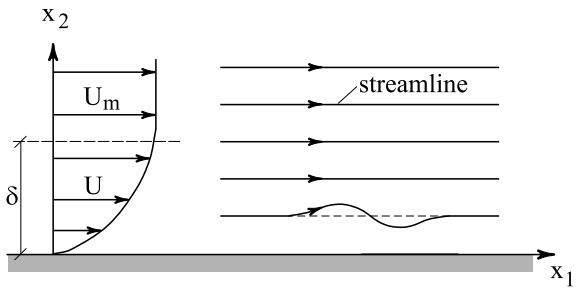

Figure 5.1: *Disturbance of an initially steady, laminar flow close to a wall.*

In this section variables corresponding to the initially, steady flow are denoted with capital letters and the disturbances with small letters.
The velocity field in the *undisturbed* laminar flow is:

$$\begin{aligned} v_1 &= U(x_2) \\ v_2 &= 0 \end{aligned}$$

and the angular velocity of the fluid particles, Ω, in the undisturbed flow therefore

reads:

$$\begin{aligned}\Omega &= \frac{1}{2}(\mathrm{rot}\,\vec{v})_3 \\ &= \frac{1}{2}(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}) \\ &= -\frac{1}{2}\frac{\partial U}{\partial x_2} \end{aligned} \tag{5.3}$$

Substitution of $\Omega = 1/2(\mathrm{rot}\,\vec{v})_3$ and the expression for rate of change into equation (5.2) gives:

$$\frac{\partial \Omega}{\partial t} + v_1\frac{\partial \Omega}{\partial x_1} + v_2\frac{\partial \Omega}{\partial x_2} = \nu\,\nabla^2\Omega \tag{5.4}$$

Here $\partial\Omega/\partial t = 0$ due to steady laminar flow, and $v_2 = 0$ gives:

$$U\frac{\partial \Omega}{\partial x_1} = \nu\,\nabla^2\Omega \tag{5.5}$$

After introduction of the disturbance, the velocity field is:

$$\begin{aligned} v_1 &= U(x_2) + u_1(t) \\ v_2 &= 0 + u_2(t) \end{aligned}$$

and equation (5.2) reads:

$$\frac{\partial(\Omega+\omega)}{\partial t} + (U+u_1)\frac{\partial(\Omega+\omega)}{\partial x_1} + (0+u_2)\frac{\partial(\Omega+\omega)}{\partial x_2} = \nu\,\nabla^2(\Omega+\omega) \tag{5.6}$$

where ω is the angular velocity corresponding to the disturbance. As the changes of the undisturbed flow are much smaller in the flow direction than in the direction perpendicular to the flow direction, it is assumed that:

$$\frac{\partial \Omega}{\partial x_1} \ll \frac{\partial \Omega}{\partial x_2} \tag{5.7}$$

Again $\partial\Omega/\partial t = 0$, and substitution of equation (5.5) into equation (5.6) gives:

$$\frac{\partial \omega}{\partial t} + (U+u_1)\frac{\partial \omega}{\partial x_1} + (0+u_2)\frac{\partial \omega}{\partial x_2} = -u_2\,\frac{\partial \Omega}{\partial x_2} + \nu\,\nabla^2\omega \tag{5.8}$$

According to the expression for rate of change the left-hand side of this equation is the substantial rate of change for ω, and equation (5.8) is rewritten to:

$$\frac{d\omega}{dt} = -u_2\,\frac{\partial \Omega}{\partial x_2} + \nu\,\nabla^2\omega \tag{5.9}$$

After multiplication by ω on both sides, the equation reads:

$$\frac{d}{dt}(\frac{1}{2}\omega^2) = -\frac{\partial \Omega}{\partial x_2}\,u_2\,\omega + \nu\,\omega\,(\frac{\partial^2\omega}{\partial x_1^2} + \frac{\partial^2\omega}{\partial x_2^2}) \tag{5.10}$$

Next the individual terms of this equation will be interpreted on a physical basis.

The term on the left-hand side is proportional to the rate of change of the rotational energy of a fluid particle. If this rate of change is positive, more energy is bound to the disturbance, i.e. the disturbance will grow.

The first term on the right-hand side of equation (5.10) describes the production of rotational energy in the disturbance, as shown below. The fundamental variation of Ω is depicted in Fig. 5.2. Notice that $\Omega < 0$, as the particles are turning clockwise. Consider a fluid particle, which at the time $t = 0$ has the angular

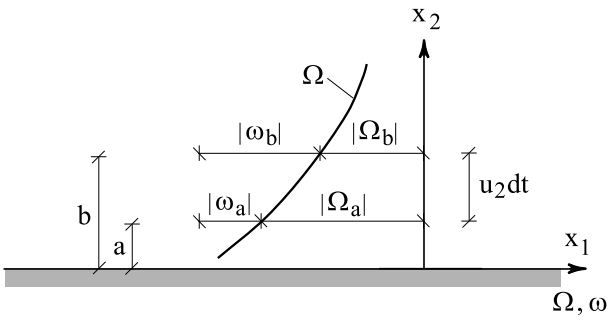

Figure 5.2: *Production of rotational energy.*

velocity $\Omega_a + \omega_a$ at the distance a from the wall. At the time $t = 0 + dt$ the particle's distance to the wall is $b = a + u_2\, dt$, and due to the limited influence of viscosity during dt, the angular velocity of the particle remains nearly constant, i.e.

$$\Omega_a + \omega_a \approx \Omega_b + \omega_b$$

Substitution of

$$\Omega_b = \Omega_a + \frac{\partial \Omega}{\partial x_2}\, u_2\, dt$$

gives

$$\omega_a \approx \frac{\partial \Omega}{\partial x_2}\, u_2\, dt + \omega_b$$

or

$$\frac{d\omega}{dt} = \frac{\omega_b - \omega_a}{dt} \approx -\frac{\partial \Omega}{\partial x_2}\, u_2$$

Multiplication by ω on both sides of the equation gives

$$\frac{d}{dt}(\frac{1}{2}\omega^2) \approx -\frac{\partial \Omega}{\partial x_2}\, u_2\, \omega$$

When the right-hand side is positive, there is an increase of the rotational energy of the particle corresponding to the disturbance, and consequently

$$-\frac{\partial \Omega}{\partial x_2}\, u_2\, \omega$$

may be interpreted as a term that represents the production of rotational energy corresponding to the disturbance.

The last term on the right-hand side of equation (5.10) represents the work done by the viscous forces, i.e. the transformation of mechanical energy to heat, also called dissipation. This term has to be negative, as a disturbance will always fade away unless the flow is constantly supplied with energy.

The ratio between production and dissipation can be assessed from equation (5.10) on non-dimensional form. Below, non-dimensional variables are marked with an apostrophe, and to obtain non-dimensional variables we make use of the variables:

δ	the thickness of the boundary layer near the wall, where $\Omega \neq 0$
Ω_o	the angular velocity at the wall
U_m	the velocity outside the boundary layer.

The equations defining the non-dimensional variables are chosen to be:

$$t = \delta/U_m\, t' \quad , \quad \Omega = \Omega_o\, \Omega' \quad , \quad \omega = \Omega_o\, \omega'$$
$$u_2 = U_m\, u_2' \quad , \quad x_1 = \delta\, x_1' \quad , \quad x_2 = \delta\, x_2'$$

Integration of equation (5.10) over the thickness of the boundary layer and subsequent substitution of the non-dimensional variables yield:

$$\frac{d}{dt'} \int_0^1 \frac{1}{2} (\omega')^2\, dx_2' = \int_0^1 (-\frac{\partial \Omega'}{\partial x_2'}\, u_2'\, \omega')\, dx_2' + \frac{1}{Re} \int_0^1 \omega'\, (\frac{\partial^2 \omega'}{\partial {x_1'}^2} + \frac{\partial^2 \omega'}{\partial {x_2'}^2})\, dx_2' \quad (5.11)$$

where

$$Re = \frac{U_m\, \delta}{\nu}$$

It is seen that Re can be interpreted as a Reynolds number for the flow. It is also seen that for increasing values of Re, the importance of the last term of equation (5.11) decreases.

If Re is small enough, the dissipative term becomes large compared to the production term, and all disturbances introduced to the laminar flow will fade away.

On the other hand, if Re is sufficiently large, the dissipative term becomes small compared to the production term, and consequently any disturbance will grow, until the flow ends up as a fully turbulent flow.

5.2 Description of Turbulent Flow

Often turbulent flow is considered as the mean flow superposed by eddies causing velocity fluctuations of stochastic nature.

Even though all turbulent flows are per definition unsteady, we talk about *steady turbulent flow*, if the mean flow is steady. If we consider a steady mean flow in a pipe, time series of the three velocity components will in principle look as depicted in Fig. 5.3. In the rest of this book the velocity field of the mean flow

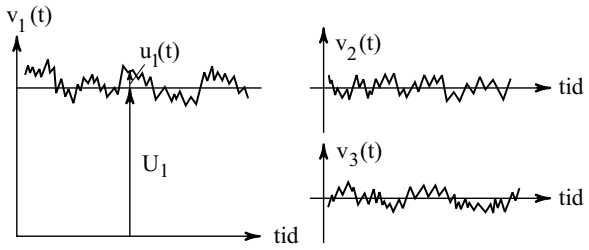

Figure 5.3: *Time series of velocity components in steady mean flow*

is denoted U_i and the turbulent fluctuations are denoted u_i.

For an unsteady mean flow the general velocity field $v_i(x_k, t)$ reads:

$$v_i(x_k, t) = U_i(x_k, t) + u_i(x_k, t) \tag{5.12}$$

Fig. 5.4 shows a time series of fluctuations in a pipe flow, where the discharge (volume flux) is varying periodically with time. Notice that the discharge is varying much slower than the turbulent fluctuations do.

The fluctuations can be considered as stochastic variables, so we are in practice satisfied, if it is possible to determine the mean flow $U_i(x_k, t)$ after, in some way or another, the effect of the fluctuations has been taken into account.

If Δt denotes a sufficiently large time span (compared to the time scale of the fluctuations), a smoothed value of the variable $a(t)$ is defined as:

$$\overline{a(t)} = \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} a(t + t')dt' \tag{5.13}$$

where $\overline{\ }$ denotes a value smoothed over time (low-pass filtered).

Thus velocity components in the mean flow reads:

$$U_i(t) = \overline{v_i(t)} = \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} v_i(t + t')dt' \tag{5.14}$$

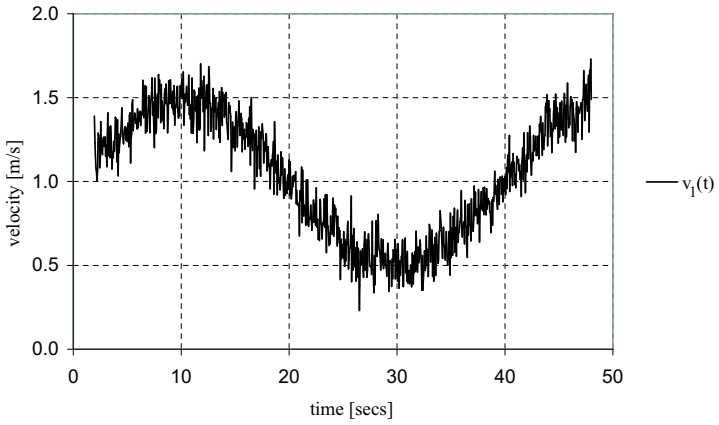

Figure 5.4: *Time series of velocity component in pipe flow with slowly varying discharge.*

If the x_1-axis coincides with the pipe axis, we see that $U_1(x_k, t) \neq 0$ and the two other components are zero.

Application of $\Delta t = 4$ secs when smoothing the time series in Fig. 5.4, leads to the mean flow and fluctuations depicted in Fig. 5.5. Notice that the breakdown into mean flow and fluctuations by smoothing with time only makes sense if $T_f \ll \Delta t \ll T_m$, where T_f and T_m are typical periods in fluctuations and mean flow, respectively.

Substitution of equation (5.12) into equation (5.14) yields:

$$U_i(t) = \frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} U_i(t+t')dt' + \frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} u_i(t+t')dt'$$

If the variation with time of the mean flow is so slow that the velocity varies approximately linearly over Δt, we get:

$$U_i(t) = U_i(t) + \overline{u_i(t)}$$

and therefore:

$$\boxed{\overline{u_i} = 0} \tag{5.15}$$

In the following it is necessary to calculate smoothed values of derivatives like e.g $\partial a(t)/\partial x_i$. In such a case we find:

$$\overline{\frac{\partial a}{\partial x_i}} = \frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} \frac{\partial a(t+t')}{\partial x_i}dt' = \frac{\partial}{\partial x_i}(\frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} a(t+t')\,dt') = \frac{\partial \overline{a}}{\partial x_i} \tag{5.16}$$

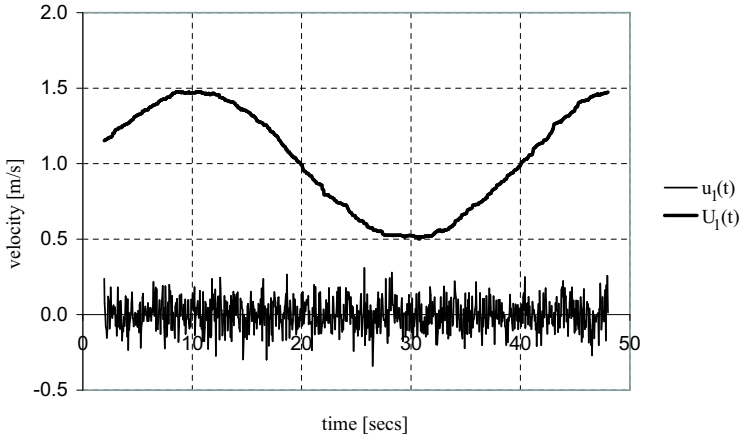

Figure 5.5: *Time series mean flow and fluctuations in pipe flow with slowly varying discharge.*

The smoothed version of the continuity equation reads:

$$\overline{\frac{\partial v_i}{\partial x_i}} = \overline{\frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}} = \frac{\partial U_i}{\partial x_i} + \overline{\frac{\partial u_i}{\partial x_i}} = 0 \tag{5.17}$$

and application of equation (5.16) to the last term gives:

$$\frac{\partial U_i}{\partial x_i} + \frac{\partial \overline{u_i}}{\partial x_i} = 0 \tag{5.18}$$

As $\partial\overline{u_i}/\partial x_i = 0$, when $\overline{u_i} = 0$ everywhere, the *continuity equation for the mean flow* reads:

$$\boxed{\frac{\partial U_i}{\partial x_i} = 0} \tag{5.19}$$

Consider the Navier-Stokes equation:

$$\rho \frac{dv_i}{dt} = -\frac{\partial p^+}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right) \tag{5.20}$$

Below a smoothed version of this Navier-Stokes equation is derived. The acceleration term is rewritten to:

$$\frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} \cdot v_j = \frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j}(v_i\, v_j) - v_i \frac{\partial v_j}{\partial x_j} \tag{5.21}$$

where the last term according to the continuity equation is zero. Substitution of equation (5.21) into equation (5.20) gives:

$$\rho \frac{\partial v_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(v_i\, v_j) = -\frac{\partial p^+}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right) \tag{5.22}$$

Like the velocities, the dynamic pressure p^+ is , divided into a smoothed part $\overline{p^+}$ and a fluctuation $(p^+)'$, i.e.

$$p^+ = \overline{p^+} + (p^+)' \tag{5.23}$$

where $\overline{(p^+)'} = 0$ per definition.
Substitution of equation (5.12) and (5.23) into equation (5.22), and subsequent smoothing give:

$$\rho \frac{\partial U_i}{\partial t} + \rho \overline{\frac{\partial}{\partial x_j}\left(\left(U_i + u_i\right)\left(U_j + u_j\right)\right)} = -\frac{\partial(\overline{p^+})}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right) \tag{5.24}$$

where $\overline{U_i} = U_i$, $\overline{u_i} = 0$ and $\overline{(p^+)'} = 0$ has been used. The second term on the left-hand side is rewritten as follows:

$$\begin{aligned}
\overline{\frac{\partial}{\partial x_j}(U_i\, U_j + U_i\, u_j + u_i\, U_j + u_i\, u_j)} &= \\
\frac{\partial}{\partial x_j}(\overline{U_i\, U_j} + \overline{U_i\, u_j} + \overline{u_i\, U_j} + \overline{u_i\, u_j}) &= \\
\frac{\partial(U_i\, U_j)}{\partial x_j} + \frac{\partial(U_i\, \overline{u_j})}{\partial x_j} + \frac{\partial(\overline{u_i}\, U_j)}{\partial x_j} + \frac{\partial(\overline{u_i\, u_j})}{\partial x_j} &= \frac{\partial(U_i\, U_j)}{\partial x_j} + \frac{\partial(\overline{u_i\, u_j})}{\partial x_j}
\end{aligned} \tag{5.25}$$

where $\overline{u_i} = 0$ is applied.

The left-hand side of the smoothed Navier-Stokes equation (5.24) is then rewritten to:

$$\begin{aligned}
\rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial(U_i\, U_j)}{\partial x_j} + \rho \frac{\partial(\overline{u_i\, u_j})}{\partial x_j} &= \rho \frac{\partial U_i}{\partial t} + \rho\, U_j \frac{\partial U_i}{\partial x_j} + \rho\, U_i \frac{\partial U_j}{\partial x_j} + \rho \frac{\partial(\overline{u_i\, u_j})}{\partial x_j} \\
&= \rho \frac{\partial U_i}{\partial t} + \rho\, U_j \frac{\partial U_i}{\partial x_j} + \rho \frac{\partial(\overline{u_i\, u_j})}{\partial x_j} \\
&= \rho \frac{dU_i}{dt} + \rho \frac{\partial(\overline{u_i\, u_j})}{\partial x_j}
\end{aligned}$$

where the expression for rate of change and the continuity equation for the mean flow (5.19) have been used. In this way the smoothed Navier-Stokes equation

reads:

$$\rho \frac{dU_i}{dt} = -\frac{\partial \overline{p^+}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right) - \rho \frac{\partial(\overline{u_i\, u_j})}{\partial x_j} \tag{5.26}$$

or

$$\boxed{\rho \frac{dU_i}{dt} = -\frac{\partial \overline{p^+}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \rho\, \overline{u_i\, u_j}\right)} \tag{5.27}$$

This version is also named *the Navier-Stokes equation for turbulent flow.*

Notice that the last term on the right-hand side appeared, when the acceleration term in the Navier-Stokes equation was smoothed. Therefore, this term describes an acceleration (and change in momentum), which cannot be described by use of the velocities in the mean flow. As shown below, this term describes the effect of the exchange of momentum caused by the fluctuations. The term is therefore quite similar to the viscous term (the next-to-last term on the right-hand side), which describes the effect of the exchange of momentum caused by the motion of the molecules.

Notice that if the continuity equation for the mean flow is used, we can rewrite equation (5.27) to:

$$\boxed{\rho \frac{dU_i}{dt} = -\frac{\partial \overline{p^+}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial U_i}{\partial x_j} - \rho\, \overline{u_i\, u_j}\right)} \tag{5.28}$$

Example: *The effect of exchange of momentum between two trains*

Two trains, denoted $L(eft)$ and $R(ight)$, are moving in the same direction at different velocities. Train L has the velocity $U_1 + \Delta U_1$ and train R has the velocity U_1. Suddenly, the staff on the trains start to spray water on each other. Both teams use a discharge of Q m^3/s, see Fig. 5.6.

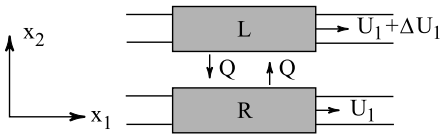

Figure 5.6: *Exchange of momentum between two trains.*

Even though the mass of each train is constant with time, the engine driver on train L has to increase the driving force from the engine in order to keep the velocity unchanged. The opposite happens for train R. Why ?

The conditions in the direction of the x_1-axis are considered. $Q\,\Delta t$ m^3 of water is transferred from train R to train L during Δt secs. As the velocity of this water is increased from U_1 to $U_1 + \Delta U_1$, the momentum of the water is also increased. This requires the impulse

$$F^L_{water,1} \cdot \Delta t = m\,(U_1 + \Delta U_1) - m\,U_1 = m\,\Delta U_1 = \rho\,Q\,\Delta t\,\Delta U_1$$

where $F^L_{water,1}$ is the force from the train on the water. After division by Δt the expression for this force reads:

$$F^L_{water,1} = \rho\,Q\,\Delta U_1 > 0$$

As action = re-action (Newton's third law) the force from the water on train L becomes:

$$F^L_{train,1} = -F^L_{water,1} = -\rho\,Q\,\Delta U_1 < 0$$

Similarly we find for the water transferred from train L to train R:

$$F^R_{water,1} = \rho\,Q\,(-\Delta U_1) < 0$$

and therefore

$$F^R_{train,1} = -F^R_{water,1} = -\rho\,Q\,(-\Delta U_1) = \rho\,Q\,\Delta U_1 > 0$$

The exchange of momentum is therefore equivalent to the shear for $F = \rho\,Q\,\Delta U_1$ between the trains.

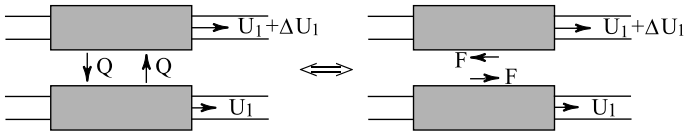

Figure 5.7: *Dynamically equivalent descriptions of two trains.*

Notice that you need not describe the discharges between the trains, if you remember to include their effect, which is the shearing force between the two trains, see Fig. 5.7.

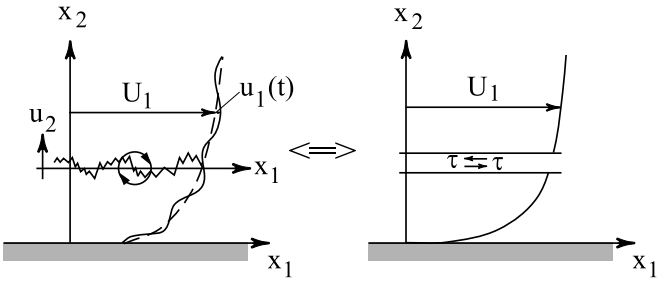

Figure 5.8: *Dynamically equivalent descriptions of turbulent flow.*

In many cases, the eddies, causing the fluctuations in a turbulent flow, will also cause an exchange of momentum across a section dividing two parts of the flow. If we do not want to describe these eddies, i.e. we only want to calculate the mean flow, we must remember to include their dynamic effect in the calculation. This is done by introducing shear stress to the mean flow, see Fig. 5.8.

The x_1-component of equation (5.27) for a two-dimensional flow reads:

$$\begin{aligned}\rho\,\frac{dU_1}{dt} &= -\frac{\partial\overline{p^+}}{\partial x_1}+\frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial U_1}{\partial x_1}+\frac{\partial U_1}{\partial x_1}\right)-\rho\,\overline{u_1\,u_1}\right)\\ &\quad+\frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial U_1}{\partial x_2}+\frac{\partial U_2}{\partial x_1}\right)-\rho\,\overline{u_1\,u_2}\right)\end{aligned}$$

In case of *uniform* flow in the direction of the x_1-axis, we have $\partial U_1/\partial x_1 = 0$ and $\partial\,(\overline{u_1\,u_1})/\partial\,x_1 = 0$, which gives this version of the Navier-Stokes equation:

$$\rho\,\frac{dU_1}{dt} = -\frac{\partial\,\overline{p^+}}{\partial\,x_1}+\frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial U_1}{\partial x_2}+\frac{\partial U_2}{\partial x_1}\right)-\rho\,\overline{u_1\,u_2}\right) \tag{5.29}$$

Because most literature on fluid mechanics denotes shear stresses τ, this is also done in the following.

As $\mu\,(\partial U_1/\partial x_2 + \partial U_2/\partial x_1)$ is a smoothed value of the shear stress caused by viscosity, we introduce the name *viscous shear stress*, and we have:

$$\overline{\tau_{21}^{visc}} = \mu\left(\frac{\partial U_1}{\partial x_2}+\frac{\partial U_2}{\partial x_1}\right)$$

Equation (5.29) can now be written as:

$$\rho\,\frac{dU_1}{dt} = -\frac{\partial\overline{p^+}}{\partial x_1}+\frac{\partial}{\partial x_2}(\overline{\tau_{21}^{visc}}-\rho\,\overline{u_1\,u_2}) \tag{5.30}$$

and it is seen that the term $-\rho\,\overline{u_1\,u_2}$ can be interpreted as an extra shear stress caused by the turbulence.

In turbulent flows we thus have two contributions to shear stress:

$$\boxed{\overline{\tau_{ji}} = \overline{\tau_{ji}^{visc}} + \tau_{ji}^{turb}} \tag{5.31}$$

where

$$\boxed{\overline{\tau_{ji}^{visc}} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)} \quad \text{for} \quad i \neq j \tag{5.32}$$

and

$$\boxed{\tau_{ji}^{turb} = -\rho \, \overline{u_i \, u_j}} \quad \text{for} \quad i \neq j \tag{5.33}$$

Except in certain domains, where large velocity gradients are present (usually in very thin layers close to fixed boundaries), we find $\overline{\tau_{ji}^{visc}} \ll \tau_{ji}^{turb}$.

In general the term $-\rho \, \overline{u_i \, u_j}$ can be interpreted as extra stresses normally named *Reynolds' stresses*.

Unfortunately, equation (5.27) cannot be applied to calculate the mean flow, because a relation between $\tau_{ji}^{turb} = -\rho \, \overline{u_i \, u_j}$ and U_i is missing. Despite considerable research within the area, no general theory exists, and in practice we are forced to apply approximate relations describing the relation between turbulence and mean flow. These approximate relations are normally named *turbulence models*, and in the following the oldest and most simple turbulence model is described.

5.3 Mixing Length Theory

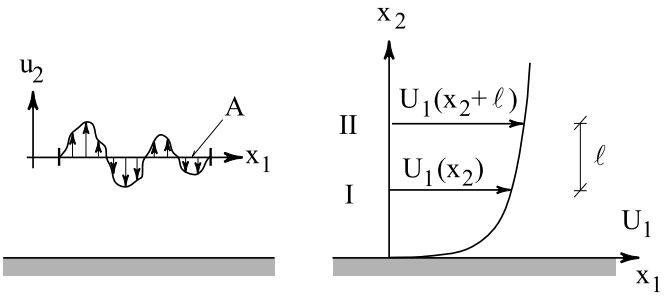

Figure 5.9: *Exchange of momentum across a section in the flow direction.*

A *uniform* turbulent flow may be interpreted as the mean flow superposed with eddies of different size. These eddies will cause an exchange of fluid and momentum across a section parallel to the wall. At a given section it is assumed that the size of the eddies is comparable to a length scale ℓ, but the size of this scale is unknown for the time being. It is assumed that an eddy of size ℓ can transport fluid from level I up to level II, if the distance between these two levels is of the order of magnitude ℓ. Therefore, ℓ is often named the *mixing length*. Due to continuity the same amount of fluid has to be transported downwards.

Thus we have two equal, oppositely directed discharges (volume fluxes) through the area A. The magnitude of these is denoted $Q(> 0)$.

Prandtl (1925) assumed that fluid particles were able to move a finite distance crosswise before they adopt the velocity of the surrounding fluid. Thus he assumed that fluid particles moving upwards from level I will reach level II, before they are moving in the x_1-direction at the same velocity as the surrounding fluid. Thus the fluid particle increases its velocity by

$$\Delta U_1 = U_1(x_2 + \ell) - U_1(x_2) = \ell \frac{\partial U_1}{\partial x_2} \tag{5.34}$$

As shown in the previous example, this exchange of momentum is equivalent to the shear force

$$F = \rho\, Q\, \Delta U_1$$

and substitution of $F = \tau\, A$ and the expression for ΔU_1 gives:

$$\tau = \rho \frac{Q}{A} \cdot \ell \frac{\partial U_1}{\partial x_2} \tag{5.35}$$

Furthermore, Prandtl assumed that the order of magnitude of the fluctuations in the x_1-direction was proportional to the change of velocity in the mean flow over

the distance ℓ. If the standard deviation is used to assess the order of magnitude of u_1, we have:

$$\sigma_{u_1} = k_1 \left|\Delta U_1\right| = k_1 \, \ell \left|\frac{\partial U_1}{\partial x_2}\right| \tag{5.36}$$

but k_1 is so far unknown. An eddy causes fluctuations in both directions, wherefore the fluctuations must have the same order of magnitude in both directions, i.e.

$$\sigma_{u_2} \approx \sigma_{u_1} \tag{5.37}$$

The exchange of fluid across A is caused by the eddies, wherefore Q is proportional to the size of the crosswise fluctuations. This gives:

$$Q = k_2 \, \sigma_{u_2} \, A \approx k_2 \, \sigma_{u_1} \, A \tag{5.38}$$

or

$$Q = k_2 \cdot k_1 \, \ell \left|\frac{\partial U_1}{\partial x_2}\right| \cdot A \tag{5.39}$$

Substitution of this expression into equation (5.35) yields:

$$\tau = \rho \cdot k_2 \, k_1 \, \ell \left|\frac{\partial U_1}{\partial x_2}\right| \, A \frac{1}{A} \cdot \ell \frac{\partial U_1}{\partial x_2} = \rho \left(k_1 \, k_2 \, \ell^2\right) \left|\frac{\partial U_1}{\partial x_2}\right| \, \frac{\partial U_1}{\partial x_2}$$

Shear stress in a turbulent flow, can thus be expressed as:

$$\boxed{\tau = \rho \, \ell_b^2 \left|\frac{\partial U_1}{\partial x_2}\right| \, \frac{\partial U_1}{\partial x_2}} \tag{5.40}$$

where $\ell_b^2 = k_1 \, k_2 \, \ell^2$. It can be a bit confusing that ℓ_b is also named the *mixing length*, but in practice this causes no misunderstandings. We can determine ℓ_b only by measuring corresponding values of τ and $\frac{\partial U_1}{\partial x_2}$. Therefore, the magnitudes of k_1 k_2 and ℓ remain unknown.
Equation (5.40) is of no use, unless we can determine the variation of the mixing length ℓ_b. In order to do so, a few general considerations are relevant.

At the wall no turbulence is present, and, consequently, $\ell_b = 0$ at the wall. As the size of the eddies is increasing with the distance to the wall, we also expect ℓ_b to be is increasing with the distance from the wall. Thus, the simplest and dimensionally correct assumption is that ℓ_b is proportional to the distance from the wall x_2 giving

$$\boxed{\ell_b = \kappa \, x_2} \tag{5.41}$$

The constant κ is named von Kármán's universal constant. Experiments have shown that for fully developed turbulent flows close to a wall, $\kappa \approx 0.41$, but it

should be remembered that $\ell_b = \kappa\, x_2$ is valid only within a limited distance from the wall, see Fig 7.1.

The exchange of momentum caused by the eddies in a turbulent flow is quite analog to the exchange of momentum caused by the heat dependent motions of the molecules. As we do not want to describe the individual molecular motions, i.e. we assume the fluid to be a continuum, it is necessary to take their effect into account. This is done by the introduction of equivalent shear stresses, named viscous shear stresses, given by Newton's formula:

$$\overline{\tau_{21}^{visc}} = \rho\,\nu\,(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1})$$

In the *technical turbulence theory* we do not want to describe the fluctuations, but their effect (exchange of momentum) is taken into account by the introduction of equivalent shear stresses, named turbulent shear stresses. Often they are modelled by Boussinesq's approximation, which is an equation similar to Newton's formula:

$$\tau_{21}^{turb} = \rho\,\nu_T\,(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}) \tag{5.42}$$

where ν_T is named the *eddy viscosity*. If the equations (5.40) and (5.42) are compared for $U_2 = 0$, it is seen that

$$\nu_T = \ell_b^2 \left|\frac{\partial U_1}{\partial x_2}\right| \tag{5.43}$$

Notice that ν_T varies in space in contrast to ν, which is a constant depending of fluid type and temperature only. Therefore, in technical turbulence theory the expression for the *total* shear stress reads:

$$\tau_{21} = \overline{\tau_{21}^{visc}} + \tau_{21}^{turb} = \rho(\nu + \nu_T)\,(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}) \tag{5.44}$$

In general $\nu_T >> \nu$, and often $\overline{\tau^{visc}}$ is ignored. However, the assumption $\tau \approx \tau^{turb}$ is *not* valid in general. For example it would lead to $\tau^{wall} \approx 0$ at a smooth wall, even though the largest shear stresses are found at the wall in reality.

Notice that according to equation (5.43) we find $\nu_T = 0$ at places, where $|\partial U_1/\partial x_2| = 0$. This is fairly unfortunate seen from a physical point of view.

Consider the flux F of a dissolved substance through a section parallel to the direction of the mean flow, see Fig. 5.10. Concentration is denoted c and has the unit kg (substance) per unit volume of the mixture (substance/fluid). A gradient of c into direction of the x_2-axis is present.

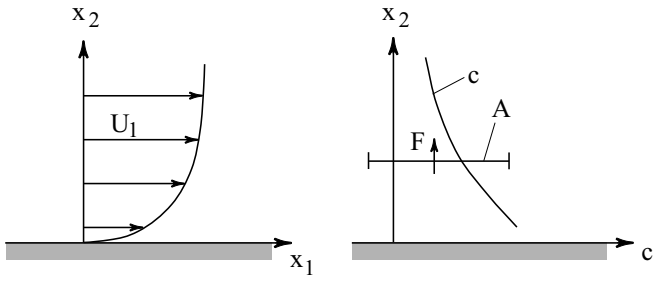

Figure 5.10: *Flux of dissolved substance across the flow direction.*

As the mean flow has no velocity component perpendicular to the section, the only cause to the flux is diffusion, i.e.

$$F = -(D + D_T)\,\frac{\partial c}{\partial x_2} \approx -D_T\,\frac{\partial c}{\partial x_2} \tag{5.45}$$

where D is the ordinary diffusion coefficient (corresponding to the thermal motions of the molecules) and D_T is a diffusion coefficient (m^2/s), whose magnitude depends on the strength and size of the eddies. Unless the section is very close to the wall, we normally have $D \ll D_T$.

Next equation (5.42) is rewritten to

$$\tau_{21}^{turb} = \rho\,\nu_T\,\frac{\partial U_1}{\partial x_2} = \nu_T\,\frac{\partial(\rho\,U_1)}{\partial x_2} \tag{5.46}$$

where $\rho\,U_1$ is momentum per unit volume.

Except from different signs, equation (5.46) and equation (5.45) are built in the same way. Thus, we can interpret τ_{21}^{turb} as the flux of momentum through the section, and interpret ν_T as the diffusion coefficient of the process.

In technical turbulence theory it is often assumed that $D_T \approx \nu_T$, i.e the dispersion mechanism for matter is the same as the dispersion mechanism for momentum.

The presence of eddies will always create a mixing of matter across the direction of the mean flow, and, consequently, the turbulent diffusion coefficient used to describe this mixing should be different from zero. It is therefore unfortunate, that equation (5.43) predicts $\nu_T = 0$, at places where $|\partial U_1/\partial x_2| = 0$. In practice eddies are present there, and calculations based on $D_T = \nu_T$ will underestimate the flux of matter.

Thus more advanced models of the connection between turbulence and shear stress are normally applied. The minimum requirement is usually a model, which always predicts $\nu_T > 0$ in domains with eddies.

5.4 Velocity Profiles in Turbulent Flows

Consider the velocity fields in a turbulent flow. For the sake of simplicity only two-dimensional, steady and uniform flows in channels are considered. However, the results are valid in nearly unchanged form for flows in completely full pipes.

We consider the flow in a very wide channel with uniform bottom slope β (the angle between the bottom and horizontal). In steady, uniform flow the use of simple dynamic considerations gives that the shear stress at the bottom has to be in equilibrium with the gravity force component parallel to the bottom, if we ignore the very small shear stress between fluid and air at the surface. It is also easy to show that the shear stress varies linearly over a section perpendicular to the bottom. Both results are valid for both laminar and turbulent flows.

Thus the shear stress at the bottom reads:

$$\tau_o = \rho\, g\, D\, \sin\beta \tag{5.47}$$

where D is the constant fluid depth, and the distribution of the shear stress is

$$\tau = \tau_o(1 - \frac{x_2}{D}) \tag{5.48}$$

Furthermore, it turns out to be practical to introduce the quantity named *friction velocity*, which is defined as:

$$\boxed{U_F \equiv \sqrt{\frac{\tau_o}{\rho}}} \tag{5.49}$$

5.4.1 Smooth Bottom

For flows with a smooth bottom the velocity profile near the wall can be calculated by use of the mixing length model. Equation (5.44) with $U_2 = 0$ reads:

$$\frac{\tau}{\rho} = (\nu + \nu_T)\,\frac{\partial U_1}{\partial x_2} \tag{5.50}$$

Substitution of equation (5.48), the expression for U_F and equation (5.43) give:

$$\begin{aligned} U_F^2\,(1 - \frac{x_2}{D}) &= (\nu + \nu_T)\,\frac{\partial U_1}{\partial x_2} \\ &= \left(\nu + \ell_b^2 \left|\frac{\partial U_1}{\partial x_2}\right|\right)\,\frac{\partial U_1}{\partial x_2} \\ &= \nu\,\frac{\partial U_1}{\partial x_2} + \ell_b^2 \left|\frac{\partial U_1}{\partial x_2}\right|\,\frac{\partial U_1}{\partial x_2} \end{aligned} \tag{5.51}$$

Close to the bottom we have $\ell_b = \kappa\, x_2$ and $x_2/D \ll 1$, which after substitution into equation (5.51) gives:

$$U_F^2 \approx \nu \frac{\partial U_1}{\partial x_2} + (\kappa\, x_2)^2 \left(\frac{\partial U_1}{\partial x_2}\right)^2 \tag{5.52}$$

Even though ν is a very small quantity, the first term on the right–hand side will always be dominant if we are close enough to the bottom. This corresponds to $\tau^{visc} \gg \tau^{turb}$ very close to the bottom. If we ignore the turbulent stresses, equation (5.52) is reduced to:

$$U_F^2 = \nu \frac{\partial U_1}{\partial x_2} \tag{5.53}$$

which, after usage of the boundary condition $U_1 = 0$ for $x_2 = 0$, gives:

$$\boxed{\frac{U_1}{U_F} = \frac{U_F}{\nu}\, x_2} \tag{5.54}$$

The velocity profile of the mean flow close to the bottom is seen to be *linear*, and the flow turns out to be mainly laminar. Fluctuations might occur, but as the viscous stresses are dominant, such fluctuations do not grow to real turbulence. This flow domain is often named the *viscous sub-layer*.

However, we need not go particularly far away from the bottom before $\nu_T \gg \nu$, and the turbulent stresses become dominant. In that case equation (5.51) is reduced to:

$$U_F^2 = (\kappa\, x_2)^2 \left(\frac{\partial U_1}{\partial x_2}\right)^2$$

as we still have $x_2/D \ll 1$. Because $U_1 = U_1(x_2)$, the equation can be rewritten to:

$$dU_1 = U_F \frac{1}{\kappa} \frac{dx_2}{x_2} \tag{5.55}$$

which, after integration, reads:

$$\frac{U_1}{U_F} = \frac{1}{\kappa} \ln x_2 + K_1 \tag{5.56}$$

The velocity profile is seen to be logarithmic, but we are not able to determine the constant K_1 by use of mixing length theory. To do so, experiments or more advanced turbulence modelling is necessary. In Fig. 5.11, measured data from experiments with full pipes are shown. The drawing is based on the non-dimensional distance $(U_F\, x_2)/\nu$ from the wall. As the data points in the semi-logarithmic coordinate system fit a straight line excellently, except close to the wall, the theoretically logarithmic velocity profile is confirmed.

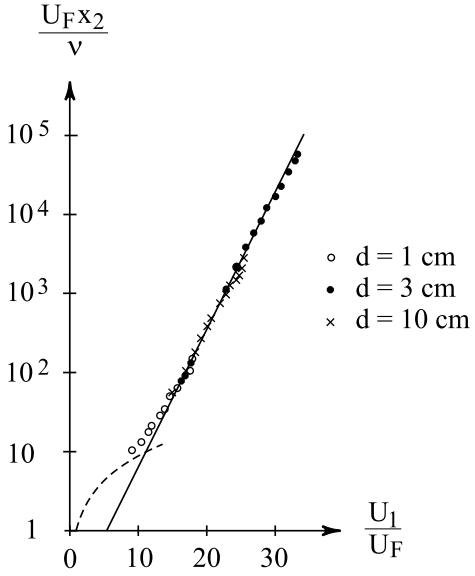

Figure 5.11: *Smooth wall. Velocity profiles in pipes measured outside the viscous sub-layer.*

A fit of the data (for $U_F x_2/\nu > 50$) to a straight line in the semi-logarithmic coordinate system gives:

$$\boxed{\frac{U_1}{U_F} = 5.7 + 2.45 \ln(\frac{U_F}{\nu} x_2)} \tag{5.57}$$

According to equation (5.56) the factor of 2.45 has to be equal to $1/\kappa$, which gives $\kappa \approx 0.41$.

The deviation of the points from the straight line close to the wall in Fig. 5.11 is caused by the transition from a logarithmic profile to a linear profile within the viscous sub-layer. The linear profile, shown with the broken line in Fig. 5.11, is curved in a semi-logarithmic coordinate system. In practice we use the linear profile, equation (5.54), up to the intersection point with the logarithmic profile, equation (5.57), and the logarithmic profile above that point. See Fig. 5.12.

The thickness of the viscous sub-layer, δ, is thus defined as the distance from the wall to the intersection point of the two types of profile, i.e.

$$\frac{U_F}{\nu} \delta = 5.7 + 2.45 \ln(\frac{U_F}{\nu} \delta) \tag{5.58}$$

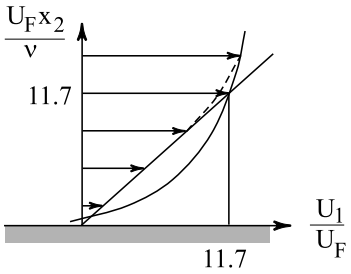

Figure 5.12: *Smooth wall. Velocity profile close to wall.*

which solved with respect to δ gives:

$$\boxed{\delta = \frac{11.7\,\nu}{U_F}} \tag{5.59}$$

In most cases δ is a very small quantity, and it should be noticed that δ decreases with increasing values of U_F.

Example : Thickness of a viscous sub-layer

Consider a channel with constant water depth $D = 1$ m and slope $\beta = 10^{-3}$. According to equation (5.47) we have $\tau_o = 10^3 \cdot 9.81 \cdot 1 \cdot \sin 10^{-3} = 9.81$ N/m^2. This gives $U_F = \sqrt{9.81\,/\,10^3} = 0.099$ m/s, and, accordingly the thickness of the viscous sub-layer is:

$$\delta = (11.7 \cdot 10^{-6})/0.099 = 1.2 \cdot 10^{-4}\ \text{m} = 0.12\ \text{mm}$$

5.4.2 Rough Bottom

In practice it is impossible to obtain a completely smooth wall from a physical point of view. Even if the surface looks smooth to the naked eye, a look on the surface with a microscope will reveal roughness elements on the surface, and in the following the characteristic size of these roughness elements is denoted k. While $\delta \gg k$, i.e. the roughness elements are small compared to the thickness of the viscous sub-layer, the roughness has no influence and the velocity distribution corresponding to smooth wall is valid.

However, from equation (5.59) it is seen that δ is inversely proportional to U_F, and if U_F becomes large enough, δ becomes comparable to k, i.e. the largest roughness elements are no longer covered by the sub-layer.

In that case the shear stress on the wall, τ_o , is gradually changing from viscous shear stress to drag forces on the individual roughness elements, where the drag force on an element is due to excess pressure on the upstream side of the element and low pressure on the downstream side. Formally it can be expressed as a shear stress in this way:

$$\tau_o = \frac{F_{wall}}{A_{wall}} = \frac{\sum F_{element}}{A_{wall}} = \frac{\sum c_D \cdot \frac{1}{2}\,\rho\, v^2\, A_{element}}{A_{wall}}$$

where c_D is a drag coefficient, $A_{element}$ is the area of an element projected on a plane perpendicular to the flow direction and v is a reference velocity like e.g. the velocity immediately above the top of the roughness elements.

The size of the gradient $\partial U_1/\partial x_2$ is normally so large immediately above the top of the roughness elements, that τ^{visc} is a significant part of the total shear stress τ there. However, the importance of the viscous stresses is only significant at points very close to the wall, and normally it is assumed that $\tau \approx \tau^{turb}$, when the distance to the top of the elements is of the same order of magnitude as k.

In the domain (approximately 1/10 of the depth or radius), where $\tau \approx \tau^{turb}$ and the mixing length is given by $\ell_b = \kappa\, x_2$, calculations similar to the calculations made for a smooth wall give:

$$\frac{U_1}{U_F} = \frac{1}{\kappa} \ln x_2 + K_2 \tag{5.60}$$

Again the velocity profile is logarithmic, and again we must rely on experiments in order to determine the integration constant K_2.

In Fig. 5.13 measured data from experiments with full-flowing pipes are shown. The value of k for the roughness elements is the so-called Nikuradse sand roughness (or equivalent sand roughness), which is the diameter of sand grains (glued to the wall) that gives the same shear stress on the wall as the actual roughness do. Some typical values are given in the table below, but notice that coatings building up over time on plastic or steel surfaces can increase k by a factor 3 to 5!

Wall material	equivalent sand roughness, k mm
Plastic	0.02
Steel	0.1
Concrete	1
Soil (channel)	10

Table 5.1: *Typical values of equivalent sand roughness.*

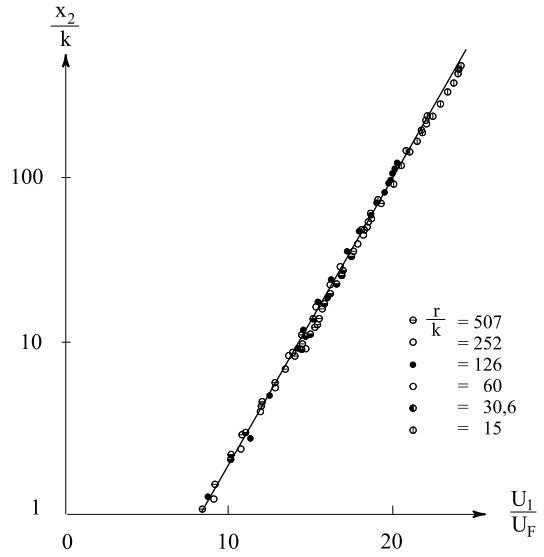

Figure 5.13: *Rough wall. Measured velocity profiles.*

The drawing is based on the non-dimensional distance x_2/k from the wall. In case of a rough wall it is of course difficult to define the location of $x_2 = 0$. In practice it is done during the fit of the points to a straight line. The location of $x_2 = 0$ is simply varied systematically until the best possible fit to a straight line is obtained. In most cases is $x_2 = 0$ situated close to the peaks of the roughness elements, but fortunately the exact placement is not very important in practice. As the data points in the semi-logarithmic coordinate system fit a straight line excellently, the theoretically logarithmic velocity profile is confirmed. The best fit of the the data to a straight line in the semi-logarithmic coordinate system reads:

$$\boxed{\frac{U_1}{U_F} = 8.6 + 2.45 \ln \frac{x_2}{k}} \tag{5.61}$$

This equation can also be expressed as:

$$\boxed{\frac{U_1}{U_F} = 2.45 \ln \frac{x_2}{k/33}} \tag{5.62}$$

The expression predicts $U_1 = 0$ at $x_2 = k/33$, i.e. close to the top of the roughness elements. The equation is in principle only valid in the domain, where $\ell_b = \kappa\, x_2$. In practice the deviations between the actual velocity profile and the profile predicted by equation (5.62) are so small that equation (5.62) is often

used in the whole range from the bottom to the surface in a channel or to the centre of the pipe in case of pipe flow.

5.5 Considerations on energy in turbulent flow

The turbulence in a uniform, steady turbulent flow, e.g. a pipe flow, is named *preserved* turbulence, as the three standard deviations of the turbulent fluctuations at a given point keep their size and distribution constant with time. This is not always the case. For example, if turbulence is created by the motion of a grid through a stagnant fluid, this turbulence will fade away as time goes by.

In pipe flow the turbulence at a given point can be preserved, because energy is steadily transferred from the mean flow to turbulence (in order to keep the main flow steady, we must therefore submit energy to the main flow from e.g. a pump).

Below it is attempted to explain how turbulent energy is produced in uniform, steady main flow.

The definition of turbulent kinetic energy, k, per *unit mass* reads:

$$\boxed{k = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) = \frac{1}{2}\, u_i\, u_i} \tag{5.63}$$

i.e. k is the kinetic energy due to the fluctuations.

The total kinetic energy per *unit mass* of the fluid is $1/2\, v^2$, and smoothing yields:

$$\begin{aligned} \frac{1}{2}\overline{v^2} &= \frac{1}{2}\overline{v_i\, v_i} = \frac{1}{2}\overline{(U_i + u_i)\,(U_i + u_i)} = \frac{1}{2}\overline{U_i\, U_i + u_i\, u_i + 2\, U_i\, u_i} \\ &= \frac{1}{2}\, U_i^2 + \frac{1}{2}\overline{u_i\, u_i} = \frac{1}{2}\, U_i^2 + \overline{k} \end{aligned} \tag{5.64}$$

It is seen that the total kinetic energy after smoothing is the sum of the kinetic energy of the mean flow and the smoothed turbulent kinetic energy.

Consider the change of k, when, due to velocity fluctuations, a fluid particle is moving across the streamlines of a mean flow.

Due to u_2 the fluid particle moves from the level $x_2 = a$ to the level $x_2 = b$ during the time Δt. Thus, the distance between the two positions is:

$$\Delta x_2 = b - a = u_2\, \Delta t$$

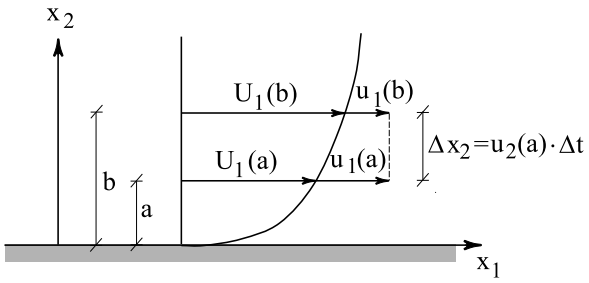

Figure 5.14: *Production of turbulent kinetic energy.*

See Fig. 5.14. If the action of the external forces is neglected, this change in position is seen to correspond to a change in k. We have:

$$\begin{aligned} v_1(a) &\approx v_1(b) \qquad \Rightarrow \\ U_1(a)+u_1(a) &= U_1(b)+u_1(b) \qquad \Rightarrow \\ U_1(a)+u_1(a) &= U_1(a)+\frac{\partial U_1}{\partial x_2}\cdot\Delta x_2 \; + u_1(b) \qquad \Rightarrow \\ u_1(a) &= \frac{\partial U_1}{\partial x_2}\cdot u_2\,\Delta t + u_1(b) \end{aligned}$$

Thus at $x_2 = b$ the size of the x_1-component of u_1 is:

$$u_1(b) = u_1(a) - \frac{\partial U_1}{\partial x_2}\cdot u_2\,\Delta t$$

The two other components of fluctuations are not changed, because the corresponding two components of the mean flow are zero at both levels.

The corresponding change of the of the turbulent kinetic energy of the particle per *unit volume* during the time Δt becomes:

$$\begin{aligned} \rho\,(k_b-k_a) &= \frac{\rho}{2}[\,u_1(b)^2+u_2(a)^2+u_3(a)^2\,]-\frac{\rho}{2}[\,u_1(a)^2+u_2(a)^2+u_3(a)^2\,] \\ &= \frac{\rho}{2}[\,u_1(a)-\frac{\partial U_1}{\partial x_2}\cdot u_2\,\Delta t\,]^2-\frac{\rho}{2}\,u_1(a)^2 \\ &= \frac{\rho}{2}[\,u_1(a)^2+(\frac{\partial U_1}{\partial x_2}\cdot u_2\,\Delta t)^2-2\,u_1(a)\,\frac{\partial U_1}{\partial x_2}\cdot u_2\,\Delta t\,]-\frac{\rho}{2}u_1(a)^2 \\ &= \frac{\rho}{2}[\,(\frac{\partial U_1}{\partial x_2}\cdot u_2\,\Delta t)^2-2\,u_1(a)\,\frac{\partial U_1}{\partial x_2}\cdot u_2\,\Delta t\,] \end{aligned}$$

or, after division by Δt :

$$\rho \frac{k_b - k_a}{\Delta t} = \rho \frac{\Delta k}{\Delta t} = \frac{\rho}{2} (\frac{\partial U_1}{\partial x_2} \cdot u_2)^2 \, \Delta t - \rho \, u_1(a) \frac{\partial U_1}{\partial x_2} \cdot u_2$$

For $\Delta t \to 0$ this expression reads:

$$\rho \frac{dk}{dt} = -\rho \, u_1 \, u_2 \frac{\partial U_1}{\partial x_2}$$

Therefore, the smoothed rate of change of the turbulent kinetic energy per *unit volume* is:

$$\rho \frac{d\overline{k}}{dt} = -\rho \, \overline{u_1 \, u_2} \frac{\partial U_1}{\partial x_2} \tag{5.65}$$

In fully developed turbulence the shear stress is identical to the Reynolds stress, i.e. $\tau_{21} = \tau_{21}^{turb} = -\rho \, \overline{u_1 \, u_2}$, and the right-hand side of equation (5.65) can be written:

$$\tau_{21} \frac{\partial U_1}{\partial x_2} \tag{5.66}$$

As both terms in this expression are positive, we say the expression corresponds to *production* of turbulent energy, as $\overline{k}$ increases during the shift in position. The expression can also be interpreted as the work done on a unit cube, i.e. per unit volume of the fluid. In turbulent flow the work of the shear stresses is initially transformed into turbulent energy in contrast to laminar flow, where the work done on a particle is initially transformed into heat. The turbulent kinetic energy is not transformed into heat in the large eddies, as closer investigations have shown that the turbulent energy is transformed to still smaller eddies, and the main transformation into heat, i.e. dissipation, takes place in the smallest eddies. The dissipation per *unit mass* is again denoted ϵ.

From equation (5.65) it is seen that the production of turbulent energy is largest close to the wall, as both τ and the velocity gradient are largest there. It is also seen that the production is 0, where τ or the velocity gradient is 0 as e.g. at the pipe axis or the free surface in channel flow.

From experiments and dimensional analysis it is known that the smoothed dissipation per *unit mass* can be expressed as:

$$\boxed{\overline{\epsilon} = A \frac{\overline{k}^{\frac{3}{2}}}{\ell_d}} \tag{5.67}$$

where A is a non-dimensional factor, and ℓ_d is a measure for the size of the eddies. Names as *dissipation length* or *length scale of turbulence* are often used for ℓ_d.

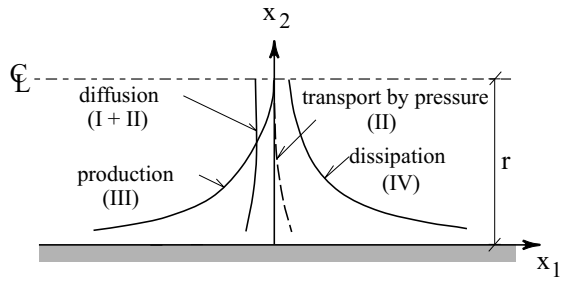

Figure 5.15: *Energy balance of turbulent energy, $\overline{k}$, in a pipe flow. Roman numerals refer to the terms of equation (7.14).*

It is possible to show that $A \approx 0.09$, and that ℓ_d has the same order of magnitude as the mixing length ℓ_b, see chapter 7.

From equation (5.67) it is seen that dissipation takes place at all points, where $\overline{k} > 0$. As $\overline{k} > 0$ at the centerline of a pipe, where the production is zero, a transport of energy from the wall towards the centerline must take place, and because this direction is perpendicular to the streamlines of the mean flow, it is diffusive transport only. Finally, a closer examination shows that the work done by the fluctuations of the pressure also creates a transport of energy towards the centerline. However, this contribution is included in the total diffusion, so the transport by pressure is shown only by a broken line in Fig. 5.15.

In Fig. 5.15 size and distribution of the above-mentioned contributions in a pipe flow are depicted, and the equation describing these contributions is discussed in chapter 7.

Chapter 6

Turbulent Boundary Layers

All boundary layers are turbulent if the Reynolds number is sufficiently large.

The turbulent eddies cause the dividing surface between potential flow and boundary layer to move irregularly, and these motions have an effect on the potential flow. Even though the potential flow upstream of the boundary layer is steady, the motion of the dividing surface causes the potential flow to be weakly unsteady near the boundary layer.

If the mean flow in a turbulent boundary layer is considered, the same characteristics are found for laminar and turbulent boundary layers, i.e. gradients of velocity and pressure are much larger perpendicular to the flow direction than in the flow direction. Consequently, part of the terms in the equations of motion can be discarded.

Even though the thickness of the turbulent boundary layer increases along the plate (in the x_1-direction), and continuity therefore gives $U_2 \neq 0$, we normally find $U_2 \ll U_1$. Thus, the streamlines of the mean flow are also nearly parallel.

6.1 Flow Equations for Turbulent Boundary Layers

The x_2-component of the Navier-Stokes equation for turbulent flow, equation (5.27), reads:

$$\begin{aligned}\rho\frac{dU_2}{dt} &= -\frac{\partial\overline{p^+}}{\partial x_2}+\frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial U_2}{\partial x_1}+\frac{\partial U_1}{\partial x_2}\right)-\rho\,\overline{u_2\,u_1}\right)\\ &\quad+\frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial U_2}{\partial x_2}+\frac{\partial U_2}{\partial x_2}\right)-\rho\,\overline{u_2\,u_2}\right)\end{aligned}\tag{6.1}$$

As $U_2 \approx 0$ everywhere, the accelerations of the mean flow in the x_2-direction are also small, and because the gradients in the x_1-direction are small compared to the gradients in the x_2-direction, equation (6.1) can be reduced to:

$$\frac{\partial}{\partial x_2}\left(\overline{-p^+}-\rho\,\overline{u_2\,u_2}\right)\approx 0\tag{6.2}$$

From equation (6.2) we find for a section perpendicular to the plate:

$$\overline{p^+}=-\rho\,\overline{u_2\,u_2}+K(x_1)\tag{6.3}$$

If the dynamic pressure in the potential flow is denoted p_o^+, equation (6.3) reads:

$$\overline{p^+}=p_o^+-\rho\,\overline{u_2\,u_2}\tag{6.4}$$

as $K(x_1)=p_o^+ + \rho\,\overline{u_2\,u_2}|_{x_2=\delta}=p_o^+$, because $\overline{u_2\,u_2}=0$ at the transition to potential flow.

Normally we have $-\rho\,\overline{u_2\,u_2} \ll \overline{p^+}$, and this is the case also for turbulent boundary layers. The dynamic pressure within the boundary layer is therefore determined by the dynamic pressure in the potential flow, i.e.

$$\boxed{\overline{p^+}=p_o^+}\tag{6.5}$$

The component of the Navier-Stokes equation in the flow direction reads:

$$\begin{aligned}\rho\frac{dU_1}{dt} &= -\frac{\partial\overline{p^+}}{\partial x_1}+\frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial U_1}{\partial x_1}+\frac{\partial U_1}{\partial x_1}\right)-\rho\,\overline{u_1\,u_1}\right)\\ &\quad+\frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial U_1}{\partial x_2}+\frac{\partial U_2}{\partial x_1}\right)-\rho\,\overline{u_1\,u_2}\right)\end{aligned}\tag{6.6}$$

Again it is used that gradients in the x_1-direction are much smaller than gradients in the x_2-direction, so equation (6.6) can be reduced to:

$$\boxed{\rho \frac{dU_1}{dt} = -\frac{\partial p_o^+}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial U_1}{\partial x_2} - \frac{\partial U_2}{\partial x_1}\right) - \rho\,\overline{u_1\,u_2}\right)} \tag{6.7}$$

As shown in section 5.2, the quantity in the last parenthesis is equal to the shear stress $\bar{\tau}_{21}$, and the flow equation can also be written as:

$$\rho \frac{dU_1}{dt} = -\frac{\partial p_o^+}{\partial x_1} + \frac{\partial \bar{\tau}_{21}}{\partial x_2} \tag{6.8}$$

If a particle with unit volume is considered in the mean flow, equation (6.8) can be interpreted as an ordinary Newton's 2nd law for the particle, where the Reynolds stresses are part of the shear stresses.

6.2 Momentum Equation for Turbulent Boundary layers

After smoothing of the ordinary momentum equation for boundary layers, equation (4.34), reads:

$$\boxed{\int_0^\delta \rho \frac{\partial U_1}{\partial t}\,dx_2 + \frac{\partial}{\partial x_1}\int_0^\delta \rho\,U_1(U_1 - v_o)\,dx_2 = -\delta\,\frac{\partial p_o^+}{\partial x_1} - \tau_o} \tag{6.9}$$

Notice that a *finite* value of the boundary layer thickness is assumed, and for convenience the smoothed value of the wall shear stress is simply denoted τ_o in the following. In the second integral a small term has been neglected, as we normally find $\overline{u_1 u_1} \leq 0.01 \cdot U_1^2$ for flows having the same character as boundary layer flow.

Equation (6.9) is named the *momentum equation for turbulent boundary layers*. Usage of this equation makes it rather simple to find approximate values of the thickness of the boundary layer after assessment of a velocity profile and a relation between the velocity outside the layer and the wall shear stress. It is, however, not possible to express directly the smoothed wall shear stress by the assessed velocity profile, as shear stress in a turbulent flow consists of both smoothed viscous shear stresses and Reynolds' stresses.

Measurements of velocity profiles in turbulent boundary layers have shown the existence of a domain close to the wall, where the velocity profiles with good approximation can be described as shown in section 5.4.

Consider e.g. the velocity profile close to a *rough* wall:

$$\frac{U_1(x_2)}{U_F} = 2.45 \ln \frac{x_2}{k/33} \tag{6.10}$$

where the friction velocity U_F is defined as:

$$U_F = \sqrt{\frac{\tau_o}{\rho}} \tag{6.11}$$

If we consider a point at the top side of the smoothed boundary layer, equation (6.10) gives this relation between the friction velocity and the velocity outside the boundary layer, v_o:

$$\frac{v_o}{U_F} = 2.45 \ln \frac{\delta}{k/33} \tag{6.12}$$

After substitution of equation (6.11) the relation between v_o and τ_o for a *rough* wall reads:

$$\frac{v_o}{\sqrt{\frac{\tau_o}{\rho}}} = 2.45 \ln \frac{\delta}{k/33} \tag{6.13}$$

If this expression and equation (6.10) are substituted into the momentum equation, we get a differential equation. Unfortunately no analytical solution exists, as δ appears on the right-hand side of equation (6.13).

A numerical solution of the momentum equation based on equation (6.13) is described in Fredsøe (1990). Below a simpler method of solution is described, which is based on the same assumptions, i.e. steady, uniform mean flow.

If we consider the boundary layer on a pipe wall or on the bottom of a channel, the thickness of such a boundary layer increases until the top of the boundary layer reaches the axis of the pipe or the free surface in the channel. Downstream of this section the flow becomes an ordinary turbulent flow. Uniform, turbulent flow in pipes or channels may therefore be considered as boundary layer flow which fills up the entire cross-section.

From dimensional analysis of uniform flow in pipes or channels it is known that the relation between the average velocity (in the mean flow) over the cross-sectional area, A, and the wall shear stress reads:

$$\tau_o = f \cdot \frac{1}{2} \rho V^2 \tag{6.14}$$

where f is named the *friction factor*, and V is defined as:

$$V = \frac{1}{A} \int_A U_1 \, dA \tag{6.15}$$

A is the area through which the fluid is flowing. As shown in basic books on fluid mechanics it is practical to introduce the quantity *hydraulic radius,* R, defined as:

$$R = \frac{A}{P} \tag{6.16}$$

where P is the length of the wetted part of the perimeter. If R is chosen as the characteristic size of the cross-section, measurements show that the friction factor is given by Colebrook & White's formula:

$$\sqrt{\frac{2}{f}} = 6.4 - 2.45 \ln\left(\frac{k}{R} + \frac{4.7}{Re\sqrt{f}}\right) \tag{6.17}$$

where k is the equivalent sand roughness, and Re is the Reynolds number defined as:

$$Re = \frac{V\,R}{\nu} \tag{6.18}$$

Equation (6.17) is valid for all steady, uniform turbulent flows, but it is implicit. In order to avoid an iterative solution of this implicit equation, f can be approximated by:

$$f = \frac{0.341}{\left[\ln\left(\frac{k}{14.8\,R} + \frac{1.65}{Re^{0.9}}\right)\right]^2} \tag{6.19}$$

Notice that this expression is only valid for $4 \cdot 10^{-5} < k/R < 0.08$ and $1000 < Re < 2.5 \cdot 10^7$.

For a *smooth* wall equation (6.17) is approximated by:

$$f = \frac{0.056}{\sqrt[4]{Re}} \quad \text{for} \quad Re < 31000 \tag{6.20}$$

and for a *rough* wall the approximation reads:

$$\sqrt{\frac{2}{f}} = 8.1\left(\frac{R}{k}\right)^{\frac{1}{6}} \quad \text{for} \quad 0.0033 < k/R < 0.21 \tag{6.21}$$

With respect to the assessment of the velocity profile in turbulent boundary layers, an often adopted expression reads:

$$\frac{U_1(x_2)}{v_o} = \left(\frac{x_2}{\delta}\right)^{\frac{1}{n}} \tag{6.22}$$

where the magnitude of n slightly depends on the type of flow. In many cases $n = 7$ is a good approximation, and this value is also used here. It has been shown earlier that the values of δ obtained by the momentum equation are rather

insensitive to the shape of the velocity profile assessed. Thus, the mean velocity becomes:

$$V = \frac{1}{\delta}\int_0^\delta v_o \left(\frac{x_2}{\delta}\right)^{\frac{1}{7}}\, dx_2 = \frac{\delta}{\delta}\int_0^\delta v_o \left(\frac{x_2}{\delta}\right)^{\frac{1}{7}}\, d\left(\frac{x_2}{\delta}\right) = \frac{7}{8}v_o \tag{6.23}$$

With these assumptions an analytically solution of the momentum equation for turbulent boundary layers becomes possible.

6.2.1 Turbulent Boundary Layer on a Smooth Plate in Uniform Flow

Assume that $v_o = constant$ in a uniform flow, i.e. the potential flow is *steady*. Bernoulli's generalized equation gives $p^+ = constant$ in the potential flow, and thus the smoothed momentum equation for the boundary layer reads:

$$\frac{\partial}{\partial x_1}\int_0^\delta \rho\, U_1(U_1 - v_o)\, dx_2 = -\tau_o \tag{6.24}$$

Furthermore similar velocity profiles are expected, based on the experience obtained from laminar boundary layers. Substitution of $U_1(x_2)$, equation (6.22), into the momentum equation, gives:

$$\rho v_o^2 \frac{\partial}{\partial x_1}\int_0^\delta \left[\left(\frac{x_2}{\delta}\right)^{\frac{2}{7}} - \left(\frac{x_2}{\delta}\right)^{\frac{1}{7}})\right] dx_2 = -\rho v_o^2 \frac{\partial}{\partial x_1}\left(\frac{7}{72}\delta\right) = -\tau_o$$

or

$$\rho v_o^2 \frac{7}{72}\frac{\partial \delta}{\partial x_1} = \tau_o \tag{6.25}$$

As a smooth wall is assumed, f is given by equation (6.20):

$$f = \frac{0.056}{\sqrt[4]{Re}} = \frac{0.056}{\sqrt[4]{\frac{V\,R}{\nu}}} = \frac{0.056}{\sqrt[4]{\frac{\frac{7}{8}v_o\,\delta}{\nu}}}$$

as $R = \delta$ for the actual boundary layer. Substitution of this expression for f into equation (6.14), where $V = 7/8\, v_o$, gives:

$$\tau_o = \frac{0.056}{\sqrt[4]{\frac{\frac{7}{8}v_o\,\delta}{\nu}}} \cdot \frac{1}{2}\rho(\frac{7}{8}v_o)^2 = 0.022\rho v_o^2 \left(\frac{v_o\delta}{\nu}\right)^{-\frac{1}{4}} \tag{6.26}$$

As δ only depends on x_1 , equation (6.25) is an ordinary differential equation, which after substitution of equation (6.26) reads:

$$\frac{7}{72}\frac{d\delta}{dx_1} = 0.022\left(\frac{v_o\delta}{\nu}\right)^{-\frac{1}{4}} \tag{6.27}$$

Integration gives

$$\frac{\delta^{\frac{5}{4}}}{\frac{5}{4}} = 0.022 \cdot \frac{72}{7} \left(\frac{v_o}{\nu}\right)^{-\frac{1}{4}} \cdot x_1 + C \tag{6.28}$$

where C is a constant.

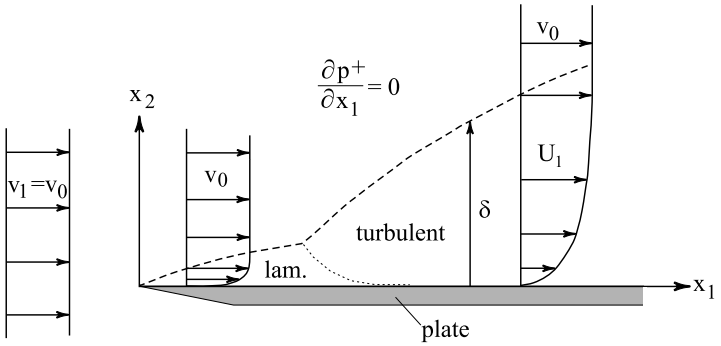

Figure 6.1: *Sketch of the transition between laminar and turbulent boundary layer.*

To determine C is not trivial, as the boundary layer is laminar close to the edge of the plate, see Fig. 6.1. At a certain distance from the edge the boundary layer becomes turbulent, and we will assume that the thickness of the laminar layer is equal to the thickness of the turbulent layer at a point of transition, even though the transition in fact takes place over a finite reach. In literature it is often stated that the transition takes place for

$$Re_{x_1} \approx \frac{v_o x_1}{\nu} \approx 3.5 \cdot 10^5 \tag{6.29}$$

but this value is only valid if the potential flow is nearly perfectly uniform and the plate is very smooth.

In practice even small disturbances in the potential flow or roughness elements on the plate will lead to a transition to turbulent boundary flow for

$$Re = \frac{VR}{\nu} \approx \frac{v_o \delta}{\nu} \approx 600$$

as it is also seen for ordinary pipe or channel flows. If we use the approximate expression for δ in a laminar boundary layer, i.e.

$$\delta_{lam} = 4.8\sqrt{\frac{\nu x_1}{v_o}}$$

it is seen that transition takes place at $x_1 \approx \frac{0.020}{v_o}$, if $\nu = 1.3 \cdot 10^{-6}$ m^2/s ($\sim$ water). For $v_o = 1$ m/s the transition takes place a few centimeters from the leading edge, and the thickness of the boundary layer is only

$$\delta_{lam} = 4.8\sqrt{1.3 \cdot 10^{-6} \cdot 0.020/1} = 0.00078 \text{ m} \approx 1\text{mm}$$

Consequently, it is often assumed that the boundary condition for the turbulent layer is $\delta = 0$ for $x_1 = 0$, giving $C = 0$ in equation (6.28). Thus the thickness of a turbulent boundary layer on a smooth plate reads:

$$\boxed{\delta = 0.37 \left(\frac{\nu}{v_o}\right)^{\frac{1}{5}} \cdot x_1^{\frac{4}{5}}} \tag{6.30}$$

Notice that the turbulent boundary layer grows approximately linearly with the distance from the leading edge, whereas the growth of the laminar boundary layer is parabolic only.

If v_o is sufficiently increased, any plate surface will change from smooth to rough. For a rough plate the thickness of the turbulent boundary layer also grows almost linearly with the distance from the leading edge.

6.3 Free Turbulence

So far we have only considered turbulent flows, where the presence of solid boundaries has been of vital importance for the course of the flow in the entire domain.

As a contrast to that kind of flows, we have flows dominated by so-called *free turbulence*. This name is used, if the turbulence is due solely to the interaction of fluids with different velocities, and such flows have substantial shear stresses even at large distances from the solid boundaries that created the velocity difference.

In Fig. 6.2 the three most common types of free turbulence are depicted. All three types have in common that the pressure gradients are very small, as pressure gradients are normally due to convection caused by solid boundaries. Furthermore, we only consider the flow domains situated far downstream of the solid boundaries that initiated the turbulence. Within such domains the flows have the same character as boundary layer flow, i.e. gradients across the flow direction are much larger than gradients in the flow direction, and it is also a good approximation to assume similar velocity profiles.

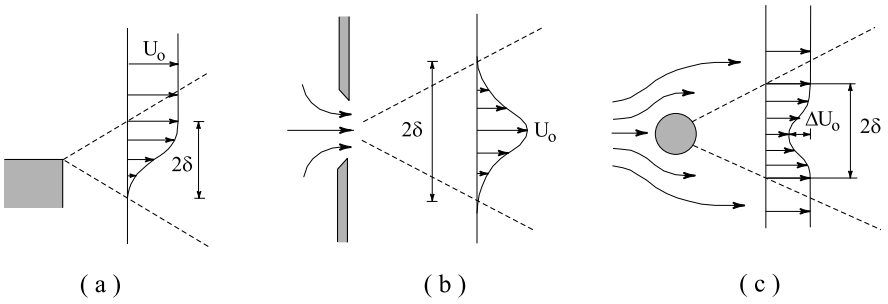

Figure 6.2: *Three types of free turbulence: (a) mixing layer, (b) submerged jet, (c) wake behind a body*

6.3.1 Turbulent two-dimensional, submerged jet

Consider the fluid flow through a slot into a half-space filled with stagnant fluid. The fluid of the jet is identical to the stagnant fluid. If the volume flux through the slot exceeds a certain value, the downstream submerged jet becomes turbulent. As the flow is two-dimensional, the jet is named a two-dimensional jet. In

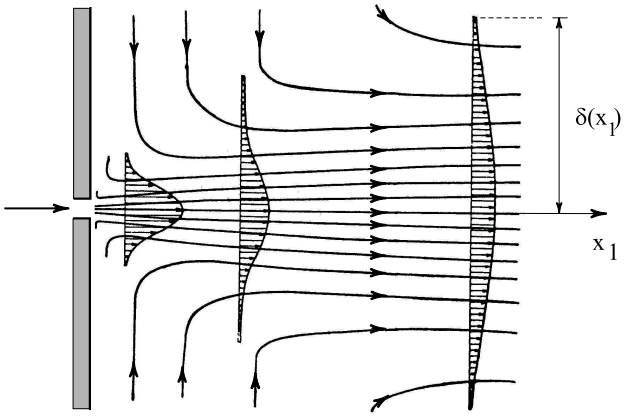

Figure 6.3: *Sketch of flow in two-dimensional jet.*

Fig. 6.3 it is depicted, how the velocity profiles of the jet are changing in the flow direction, but notice that it is only a sketch with unprecise dimensions in the two directions. Where the jet meets the stagnant fluid turbulent mixing layers are formed, and the thickness of these layers is growing quickly. At a distance of the same order of magnitude as the width of the slot, the jet is turbulent over the entire cross-section. After that point the velocity profiles are gradually approaching a shape similar to a Gaussian density distribution. At distances

greater than 20-50 times the width of the slot, the velocities are approximately similar, i.e

$$U_1 = U_o \cdot f(\frac{x_2}{\delta}) \tag{6.31}$$

where $U_o(x_1)$ is the maximum velocity in a velocity profile, and $\delta(x_1)$ is half of the total width of the jet. In the following only the domain with similar profiles is considered. Measurements show that the volume flux of the jet is slightly increasing in the flow direction, and thus surrounding fluid is drawn into the jet.

Some insight can be obtained by applying the general momentum equation (4.20), to a *closed* control surface, where the sections a and b are both perpendicular to the x_1-axis, see Fig. 6.4. The hight of the control surface perpendicular to the flow plane is 1, and the width of the control surface is $2\,\delta_b$. The x_1-component of the momentum equation is considered, and we also assume that the volume flux in the slot is constant with time. We note that no momentum in

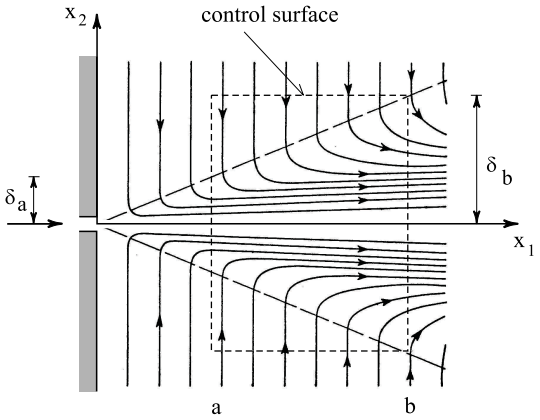

Figure 6.4: *Sketch of control surface.*

the x_1-direction enters the control surface trough the parts of the surface parallel to the x_1-axis. On the two sections a and b we have pressure forces of equal magnitude, as the pressure is constant in the x_1-direction due to the absent pressure gradient outside the the jet. Finally, we have no shear stresses outside the jet.

After smoothing of the x_1-component of the momentum equation, it reads:

$$-\int_a \rho\, U_1^2\, dA + \int_b \rho\, U_1^2\, dA = 0 \tag{6.32}$$

which, due to symmetry with respect to the x_1-axis and because $U_1 = 0$ for $x_2 > \delta_a$ at section a, can be written:

$$-2\int_0^{\delta_a} \rho\, U_1^2\, dx_2 + 2\int_0^{\delta_b} \rho\, U_1^2\, dx_2 = 0 \tag{6.33}$$

From this is concluded that:

$$\int_0^\delta U_1^2 \, dx_2 = constant \tag{6.34}$$

for all sections perpendicular to the jet. Substitution of equation (6.31) yields:

$$\int_0^\delta U_o^2 \, f^2(\frac{x_2}{\delta}) dx_2 = constant \tag{6.35}$$

or

$$\int_0^\delta U_o^2 \, \delta \, f^2(\frac{x_2}{\delta}) d(\frac{x_2}{\delta}) = constant \tag{6.36}$$

With introduction of the substitution:

$$\zeta = \frac{x_2}{\delta} \tag{6.37}$$

equation (6.36) reads:

$$U_o^2 \, \delta \int_0^1 f^2(\zeta) d\zeta = constant \tag{6.38}$$

as both U_o and δ only depend on x_1. As $f(\zeta)$ is the same at all sections, it can be concluded that for all cross-sections of a *two-dimensional* jet we have:

$$\boxed{U_o^2 \, \delta = constant} \tag{6.39}$$

The way δ depends on x_1 can be investigated by use of several methods. The most simple method is dimensional analysis. The dominant forces in the considered part of the jet are solely inertial forces, as the turbulent shear stresses are equivalent to inertial forces. In the domain with similar velocity profiles the width is assumed to depend on 1) the distance from the slot, 2) the density of fluid and 3) a characteristic velocity, i.e.

$$\delta = g(x_1, \rho, U_o) \tag{6.40}$$

As the mass-dimension is present for ρ only and the time-dimension for U_o only, the only possible, dimensionally correct expression for the width reads:

$$\boxed{\delta = K_1 \, x_1 + K_2} \tag{6.41}$$

where K_1 and K_2 are unknown constants.

By the way, this expression can also be derived under the assumption of similar shear stresses in the cross-sections, i.e.

$$\bar{\tau}_{21} = \rho \, U_o^2 \, f_1(\frac{x_2}{\delta}) \tag{6.42}$$

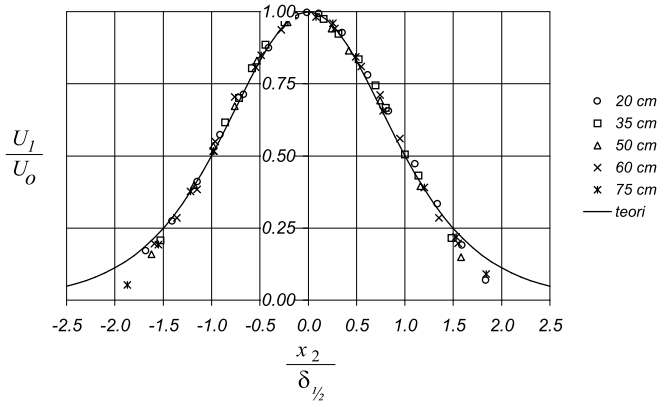

Figure 6.5: *Two-dimensional jet. The distance from the slot to a velocity profile is given in the legend. The theoretical curve corresponds to equation (6.46).*

and a subsequent substitution of this expression into the flow equations.

Combination of equation (6.39) and equation (6.41) shows that the velocity at the centre of the jet varies principally as:

$$\boxed{U_0 \sim \frac{1}{\sqrt{x_1}}} \tag{6.43}$$

If more detailed information about velocity profile and shear stress distribution is required, one has to use technical turbulence theory. In a flow along a solid wall, it was seen that the size of the eddies, and thereby ν_T, was linearly increasing with the distance to the wall. In the present jet we have no solid boundaries at which the turbulent fluctuation are prevented. Consequently, it is often assumed that ν_T is constant over the entire cross-section of the jet, and a rational and dimensionally correct expression reads:

$$\nu_T \sim \delta U_o \tag{6.44}$$

The size of δ is not well defined, as the velocities near the boundary only asymptotically go towards zero, which is a problem also faced at the determination of the thickness of ordinary boundary layers. For jets, the width of the jet is often estimated by the quantity $\delta_{1/2}$, which is defined as the distance from the axis to the point in the cross-section, where $U_1 = 1/2\, U_o$. Measurements have shown that in general we rewrite equation (6.44) to:

$$\nu_T = 0.037 \delta_{1/2} U_o \tag{6.45}$$

If this expression is adopted at the solution of the flow equations and the continuity equation, the result reads:

$$\boxed{\frac{U_1}{U_o} = 1 - \tanh^2(\sigma \frac{x_2}{\delta_{1/2}})} \tag{6.46}$$

where σ is a constant, whose magnitude is unknown, but measurements show that $\sigma \simeq 0.88$. In Fig. 6.5 the measured velocity profiles at different distances from the slot and the theoretical expression (6.46) are depicted. The agreement is seen to be excellent except close to the edges of the jet.

Chapter 7

Turbulence Models

In most cases it is sufficient to determine the mean flow . However, we cannot solve the Navier-Stokes equation for the mean flow without a so-called *turbulence model*, which is a relation between U_i and $\tau_{ji}^{turb} = -\rho\overline{u_i\,u_j}$. In technical turbulence theory the Reynolds stresses are modelled by Boussinesq's approximation:

$$\tau_{ji}^{turb} = \rho\,\nu_T\,(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}) \qquad \text{for} \qquad i \neq j \tag{7.1}$$

where the *eddy viscosity*, ν_T, is calculated by a suitable turbulence model. Notice that ν_T can be interpreted as a diffusion coefficient, so it is preferable only to apply turbulence models that predict $\nu_T > 0$ in a domain with turbulent fluctuations.

The eddy viscosity must depend on both the magnitude of the turbulent fluctuations and the size of the eddies, see e.g. equation (5.35). As $\bar{k}$ denotes the smoothed turbulent kinetic energy per unit mass, i.e.

$$\bar{k} = \frac{1}{2}(\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}) = \frac{1}{2}\overline{u_i\,u_i} \tag{7.2}$$

$\sqrt{\bar{k}}$ can be interpreted as a characteristic velocity fluctuation. In order to follow the tradition in the literature on the subject, we shall in the following denote smoothed turbulent kinetic energy by k and smoothed dissipation by ϵ, i.e. we omit the over bar for these quantities.

As the characteristic size of the eddies we shall use the length scale, ℓ_d, which was used at the calculation of the dissipation per unit mass, see equation (5.67). Instead of ℓ_d we could have adopted the mixing length ℓ_b, but the choice is not very important, as a relation between ℓ_d and ℓ_b exists, see section 7.2.

A dimensionally correct expression for ν_T reads:

$$\boxed{\nu_T = \ell_d \sqrt{k}} \tag{7.3}$$

This expression predicts increasing values of ν_T for increasing values of ℓ_d and k, as it is expected, and the expression fulfills the requirement $\nu_T > 0$ at all points, where $k > 0$.

In principle, equation (7.3) can be considered as the equation that defines ℓ_d, because both ν_T and k can be measured in an experiment.

It is seen that if equation (7.3) shall be of any practical value, two extra equations determining ℓ_d and k are necessary.

7.1 Mixing Length Model (Algebraic Model or Zero-Equation Model)

Algebraic (or zero equation) turbulence models are *not* based on equation (7.3). Instead the calculation of ν_T is based on the *mixing length* ℓ_b like e.g. in the simple turbulence model set up in chapter 5.3:

$$\nu_T = \ell_b^2 \left| \frac{\partial U_1}{\partial x_2} \right| \tag{7.4}$$

This model is not based on a characteristic velocity fluctuation, and therefore the mixing length theory erroneously predicts $\nu_T = 0$, where $\partial U_i / \partial x_j = 0$. Furthermore, it is impossible to calculate the variation ℓ_b analytically except in the layer close to a wall. In practice the variation af ℓ_b is found by substitution of measured shear stresses and velocity profiles into equation (5.40), and we hope that the expression obtained has relevance for flows of the same character. Finally, mixing length theory cannot describe convection and diffusion of the turbulent kinetic energy, phenomena which are important in many cases.

Despite these shortcomings, mixing length theory has shown good results at least in flows, where a reasonable estimate for the variation of ℓ_b exists. Below some examples of flows are given, where the variation of ℓ_b can be predicted with reasonable accuracy.

- Boundary layer on a wall

  In most cases the thickness of the viscous sub-layer can be neglected. The

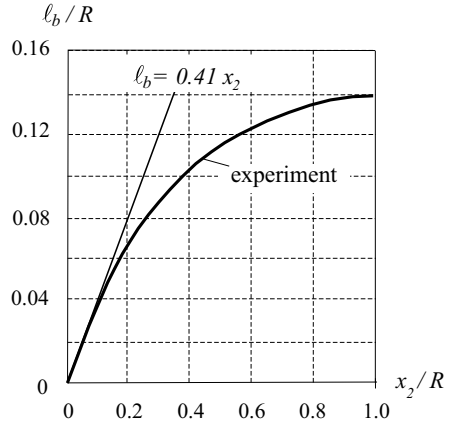

Figure 7.1: *Variation of mixing length in a pipe flow.*

simplest and dimensionally correct assumption about ℓ_b, i.e. a variation proportional to the distance to the wall, is adopted for the quarter part of the boundary layer closest to the wall. In the rest of the boundary layer ℓ_b is assumed constant. Thus, the expression for the mixing length reads:

$$\begin{aligned} \ell_b &= \kappa\, x_2 \qquad \text{for} \qquad x_2 < 0.25\,\delta \\ \ell_b &= \kappa\, 0.25\,\delta \qquad \text{for} \qquad 0.25\,\delta < x_2 < \delta \end{aligned} \tag{7.5}$$

where $\kappa = 0.41$ is the universal Kármán constant and $\delta = \delta(x_1)$ is the local thickness of the boundary layer. The thickness is here defined as the distance from the wall up to the point in the velocity profile, where $U_1 = 0.99\, U_o$ and U_o is the velocity outside the boundary layer.

- Fully developed turbulence in pipes and channels

  From experiments it is known that the variation of ℓ_b can be expressed as:

$$\frac{\ell_b}{R} = 0.14 - 0.08(1 - \frac{x_2}{R})^2 - 0.06(1 - \frac{x_2}{R})^4 \tag{7.6}$$

  where R is radius of the pipe or one half of the distance between the two walls in a channel, and x_2 is the distance from the wall. See Fig. 7.1. For channel flow with a free surface, R is the depth of the channel, and x_2 is the distance from the bottom. Notice that close to a wall, equation (7.6) correctly predicts $\ell_b = 0.41\, x_2$, see Fig. 7.1.

- Free turbulence

  Experiments have shown that ℓ_b can be assumed to be constant within the turbulent domain and proportional to the thickness of the turbulent layer

type of flow	mixing length ℓ_b
2-D mixing layer	$0.07\,\delta$
2-D wake	$0.08\,\delta$
2-D jet (into stagnant fluid)	$0.045\,\delta$
3-D jet (into stagnant fluid)	$0.038\,\delta$

Table 7.1: *Expressions for mixing length, free turbulence.*

here denoted δ. Some characteristic values for δ are given in the table below. In the first case, δ is the distance between the two points, where $U_1 = 0.01\,U_o$ and $U_1 = 0.99\,U_o$, respectively, and U_o is the velocity of the driving layer. In the last three flows, δ is the distance between the points where $U_1 = 0.01\,U_o$, and U_o is the velocity of the flow upstream to the wake or the velocity at the line of symmetry of the jet.

7.2 The *k* Model (One-Equation Model)

In order to avoid predictions like $\nu_T = 0$, we need to base the calculation of the eddy viscosity on $\nu_T = \ell_d \sqrt{k}$.

We would also like to calculate flows, where the local level of turbulence depends on the flow conditions either upstream or downstream.

If it is possible to establish a reasonable assessment of the variation ℓ_d (based on e.g. ℓ_b), only *one* extra equation is necessary. The equation that determines k is named the *transport equation* for k. This transport equation describes:

1) how much turbulent kinetic energy is produced

2) how much turbulent kinetic energy is transformed into heat (dissipation)

3) how is the turbulent kinetic energy transported by the mean flow (convection and diffusion)

This transport equation is set up by a multiplication of the three components of the Navier-Stokes equation, equation (5.20), by the corresponding velocity fluctuation followed by adding the three equations into one equation. Expressed by index notation the result reads:

$$\rho\, u_i \,\frac{d(U_i + u_i)}{dt} = -u_i \,\frac{\partial p^+}{\partial x_i} + u_i\, \mu \frac{\partial^2 (U_i + u_i)}{\partial x_j\, \partial x_j} \tag{7.7}$$

This equation is then smoothed. First the term on the left-hand side is considered.

$$\begin{aligned}
\overline{u_i \frac{d(U_i+u_i)}{dt}} &= \overline{u_i \frac{\partial(U_i+u_i)}{\partial t}} + \overline{u_i (U_j+u_j)\frac{\partial(U_i+u_i)}{\partial x_j}} \\
&= \overline{u_i \frac{\partial u_i}{\partial t}} + \overline{u_i\, u_j}\frac{\partial U_i}{\partial x_j} + U_j \overline{u_i \frac{\partial u_i}{\partial x_j}} + \overline{u_i\, u_j \frac{\partial u_i}{\partial x_j}} \\
&= \frac{\partial k}{\partial t} + \overline{u_i\, u_j}\frac{\partial U_i}{\partial x_j} + U_j \frac{\partial k}{\partial x_j} + \overline{u_j \frac{\partial k}{\partial x_j}} \\
&= \frac{\partial k}{\partial t} + \overline{u_i\, u_j}\frac{\partial U_i}{\partial x_j} + U_j \frac{\partial k}{\partial x_j} + \overline{\frac{\partial (u_j\, k)}{\partial x_j}} \qquad (7.8)
\end{aligned}$$

The continuity equation for the fluctuations:

$$\boxed{\frac{\partial u_j}{\partial x_j} = 0} \qquad (7.9)$$

has been used to rewrite the last term of equation (7.8). The continuity equation for fluctuations is derived by substitution of the continuity equation for the mean flow, $\partial U_j/\partial x_j = 0$, into the continuity equation for the total velocity, $\partial v_j/\partial x_j = \partial(U_j + u_j)/\partial x_j = 0$.

Smoothing of equation (7.7) and substitution of equation (7.8) yield:

$$\rho \frac{\partial k}{\partial t} + \rho\, U_j \frac{\partial k}{\partial x_j} + \rho \overline{\frac{\partial (u_j\, k)}{\partial x_j}} + \rho\, \overline{u_i\, u_j}\frac{\partial U_i}{\partial x_j} = -\overline{u_i \frac{\partial p^+}{\partial x_i}} + \overline{u_i\, \mu \frac{\partial^2 (u_i)}{\partial x_j\, \partial x_j}} \qquad (7.10)$$

Equation (7.9) makes it possible to rewrite the first term on the right-hand side to:

$$\overline{u_i \frac{\partial p^+}{\partial x_i}} = \overline{u_i \frac{\partial p^+}{\partial x_i}} + \overline{p^+ \frac{\partial u_i}{\partial x_i}} = \overline{\frac{\partial (u_i\, p^+)}{\partial x_i}} = \overline{\frac{\partial (u_j\, p^+)}{\partial x_j}} \qquad (7.11)$$

and equation (7.10) then reads:

$$\rho \frac{\partial k}{\partial t} + \rho\, U_j \frac{\partial k}{\partial x_j} + \rho \overline{\frac{\partial (u_j\, k)}{\partial x_j}} + \rho\, \overline{u_i\, u_j}\frac{\partial U_i}{\partial x_j} = -\overline{\frac{\partial (u_j\, p^+)}{\partial x_j}} + \overline{u_i\, \mu \frac{\partial^2 (u_i)}{\partial x_j\, \partial x_j}} \qquad (7.12)$$

After using the expression for the rate of change, equation (1.19), the definition equation $p^+ = \overline{p^+} + p^{+'}$, and $\overline{u_i} \cdot \overline{p^+} = 0$, equation (7.12) finally reads:

$$\boxed{\rho \frac{dk}{dt} = -\rho \overline{\frac{\partial (u_j\, k)}{\partial x_j}} - \overline{\frac{\partial (u_j\, p^{+'})}{\partial x_j}} - \rho\, \overline{u_i\, u_j}\frac{\partial U_i}{\partial x_j} + \mu\, \overline{u_i \frac{\partial^2 (u_i)}{\partial x_j\, \partial x_j}}} \qquad (7.13)$$

In order to give a physical interpretation of the individual terms in equation (7.13), this equation is applied to a turbulent boundary layer on a plate. It is assumed that the boundary layer is caused by a uniform flow free of turbulence, and we again use that gradients in the direction perpendicular to the flow are dominant. After using these assumptions, equation (7.13) is reduced to:

$$\rho \frac{dk}{dt} = \underbrace{-\rho \frac{\partial \overline{(u_2\, k)}}{\partial x_2}}_{\text{I}} \underbrace{- \frac{\partial \overline{(u_2\, p^{+\prime})}}{\partial x_2}}_{\text{II}} \underbrace{- \rho\, \overline{u_1\, u_2} \frac{\partial U_1}{\partial x_2}}_{\text{III}} \underbrace{+ \mu\, \overline{u_i \frac{\partial^2 (u_i)}{\partial x_j\, \partial x_j}}}_{\text{IV}} \tag{7.14}$$

The left-hand side of the equation is the increase of turbulent kinetic energy per unit volume for a fluid particle.

The terms I and II can be interpreted as net diffusion of kinetic energy across the boundary layer. Term I is directly seen to be the net transport of kinetic energy due to the turbulent fluctuations. It is a bit more difficult to interpret term II, but it can be considered as the increase of energy due to work done on the particle by the pressure fluctuations. Both terms are diffusive terms, which is seen by integrating the terms from the plate up to the top of the boundary layer. Outside the boundary layer no turbulent fluctuations are present, and at the plate the velocity fluctuations are zero causing both integrals to be zero. Consequently, these terms do not create any change of the total kinetic energy in the boundary layer, i.e. they only redistribute the energy.

As pointed out in section 5.5, the production of kinetic energy is described by term III.

Consequently, term IV has to be a sink of turbulent kinetic energy, i.e. it must describe the dissipation.

A similar equation can be derived for a uniform pipe flow, and the corresponding variations of the terms (I+II), III and IV are shown in Fig. 5.15. Term II is shown as a broken line.

Equation (7.14) is of no immediate use, as prescriptions of how to calculate the fluctuations are missing. Instead we make the following assumptions:

- Diffusive terms (I and II)
  These terms are lumped together into one term, and it is assumed that the diffusion coefficient can be approximated by the diffusion coefficient for momentum, i.e. by ν_T, as the diffusion of energy and momentum are caused by the same turbulent fluctuations. Thus, the flux of turbulent energy is expressed in the same way as the flux of momentum, i.e.

$$-\rho\, \overline{u_2\, k} - \overline{u_2\, p^{+\prime}} = \rho\, \nu_T \frac{\partial k}{\partial x_2} \tag{7.15}$$

- Production term (III)

$$-\rho\,\overline{u_1\,u_2}\frac{\partial U_1}{\partial x_2} = \tau_{21}^{turb}\cdot\frac{\partial U_1}{\partial x_2} = \rho\nu_T\left(\frac{\partial U_1}{\partial x_2}+\frac{\partial U_2}{\partial x_1}\right)\frac{\partial U_1}{\partial x_2} \tag{7.16}$$

- Dissipation term (IV)
  The term describes the dissipation per unit volume. This dissipation is equal to the dissipation per unit mass, equation (5.67), multiplied by ρ, i.e.

$$\overline{\mu\,u_i\frac{\partial^2(u_i)}{\partial x_j\,\partial x_j}} = -\rho\,A\frac{k^{\frac{3}{2}}}{\ell_d} \tag{7.17}$$

  In the following example it is shown that $A \approx 0.09$.

Substitution of these expressions into equation (7.14) gives:

$$\rho\,\frac{dk}{dt} = \frac{\partial(\rho\nu_T\,\frac{\partial k}{\partial x_2})}{\partial x_2} + \rho\nu_T\left(\frac{\partial U_1}{\partial x_2}+\frac{\partial U_2}{\partial x_1}\right)\frac{\partial U_1}{\partial x_2} - \rho\,A\frac{k^{\frac{3}{2}}}{\ell_d} \tag{7.18}$$

Finally

$$\nu_T = \ell_d\,\sqrt{k} \tag{7.19}$$

is substituted into equation (7.18), giving the *transport equation* for k in a boundary layer:

$$\boxed{\rho\,\frac{dk}{dt} = \frac{\partial(\rho\,\ell_d\,\sqrt{k}\,\frac{\partial k}{\partial x_2})}{\partial x_2} + \rho\,\ell_d\,\sqrt{k}\left(\frac{\partial U_1}{\partial x_2}+\frac{\partial U_2}{\partial x_1}\right)\frac{\partial U_1}{\partial x_2} - \rho\,A\frac{k^{\frac{3}{2}}}{\ell_d}} \tag{7.20}$$

It is seen that knowledge of the length scale for the turbulent eddies, ℓ_d, is necessary in order to determine k. In a one-equation turbulence model an equation for the determination of ℓ_d is missing, and one has to rely on the best possible guess of the variation of ℓ_d at the solution of the flow problem. Exactly as we had to do with the mixing length model.

The only improvement obtained by solving an extra equation seems to be that a k-model does not predict $\nu_T = 0$.

In the next example it is shown that a relation between ℓ_b and ℓ_d exists for wall turbulence, and that we can determine the value of the constant A by looking at wall turbulence.

Example: The k-equation in the equilibrium layer

In section 5.5 it was shown that for a turbulent pipe- or channel flow, we have almost equilibrium between production and dissipation of

turbulent kinetic energy within a layer extending from the wall up to approximately 15 % of the radius or depth, see Fig. 5.15. Naturally, this layer is named *the equilibrium layer.*

In Fig 7.2 the measured distribution of k/U_F^2 over the cross-section of a pipe is shown. U_F is the friction velocity i.e. $U_F = (\tau_o/\rho)^{0.5}$, and τ_o is the shear stress at the wall. In the figure also the linear distribution of the shear stress is shown.

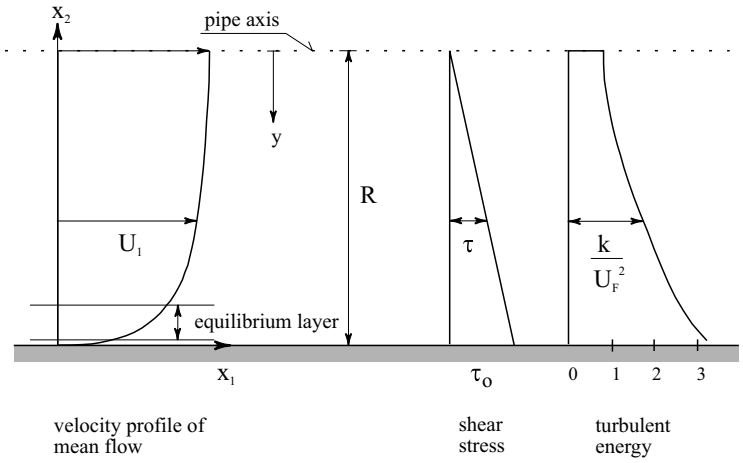

Figure 7.2: *Variation of τ and k in turbulent pipe flow.*

Within the equilibrium layer it is assumed that production is equal to dissipation:

$$\rho\,\nu_T\,(\frac{\partial U_1}{\partial x_2})^2 = \rho\,A\frac{k^{\frac{3}{2}}}{\ell_d} \tag{7.21}$$

Multiplication by ν_T on both sides and substitution of $\tau/\rho = \nu_T(\partial U_1/\partial x_2)$ gives:

$$(\frac{\tau}{\rho})^2 = \nu_T\,A\frac{k^{\frac{3}{2}}}{\ell_d} = A\,k^2 \tag{7.22}$$

where the right-hand side has been rewritten by use of $\nu_T = \ell_d\,\sqrt{k}$. The expression for shear stress therefore reads:

$$\tau = \rho\sqrt{A}\,k \tag{7.23}$$

From Fig. 7.2 it is seen that k is varying approximately linearly except close to the axis of the pipe. The linear variation of k can be expressed as:

$$k \approx \frac{y}{R}\cdot 3.3\,U_F^2 \qquad \Leftrightarrow \qquad \rho\,U_F^2 \approx \frac{R}{y}\,0.3\,\rho \tag{7.24}$$

where y is the distance from the pipe axis, and R is the radius of the pipe. The linear variation of the shear stress can be expressed as:

$$\tau = \frac{y}{R} \cdot \tau_o = \frac{y}{R} \cdot \rho \, U_F^2 \tag{7.25}$$

and by combining equation (7.24) and equation (7.25), we find:

$$\tau \approx 0.3 \, \rho \, k \tag{7.26}$$

This expression is valid, where k is varying approximately linearly, i.e. it is also valid in the equilibrium layer.

Comparison of equation (7.23) and equation (7.26) gives:

$$\sqrt{A} \approx 0.3$$

or

$$\boxed{A \approx 0.09} \tag{7.27}$$

Notice that the expression for dissipation, equation (5.67), is valid in general for a fully developed turbulent flow. The value of A is therefore also valid outside the equilibrium layer.

Within the equilibrium layer it is possible to find a relation between ℓ_b and ℓ_d.

Consider the two expressions:

$$\frac{\tau}{\rho} = \ell_b^2 \, (\frac{\partial U_1}{\partial x_2})^2 \tag{7.28}$$

and

$$\begin{aligned} \frac{\tau}{\rho} &= \nu_T \, \frac{\partial U_1}{\partial x_2} \\ &= \ell_d \, \sqrt{k} \, \frac{\partial U_1}{\partial x_2} \end{aligned} \tag{7.29}$$

The latter is rewritten to:

$$(\frac{\tau}{\rho})^2 = \ell_d^2 \, \frac{\tau}{\rho \, \sqrt{A}} \, (\frac{\partial U_1}{\partial x_2})^2 \quad \Leftrightarrow \quad \frac{\tau}{\rho} = \ell_d^2 \, \frac{1}{\sqrt{A}} \, (\frac{\partial U_1}{\partial x_2})^2 \tag{7.30}$$

Comparison of equation (7.28) and equation (7.30) yields:

$$\ell_b^2 = \ell_d^2 \, \frac{1}{\sqrt{A}} \tag{7.31}$$

Substitution of $A \approx 0.09$ means that in the equilibrium layer we have:

$$\boxed{\ell_d = A^{1/4} \, \ell_b \approx 0.55 \, \ell_b} \tag{7.32}$$

7.3 Two-Equation Models

One of the drawbacks of the k-model is the lacking knowledge of ℓ_d in the general flow situation. In many cases a k-model will not predict better results than those obtainable with a mixing-length model, even though the solution of the k-transport equation in the entire domain increases the computational work considerably.

It is therefore natural to set up a transport equation for ℓ_d also, and thus eliminate any guesses or prior knowledge of ℓ_d . If ℓ_d and k can be calculated by transport equations, we can find the eddy viscosity by use of:

$$\nu_T = \ell_d \sqrt{k} \tag{7.33}$$

A turbulence model including a second transport equation is named a *two-equation model.* In the course of time many different versions of a second transport equation have been proposed. The most frequently applied turbulence models do not have a transport equation specifically for ℓ_d, as we shall se in the following.

7.3.1 The k-ϵ Model

Instead of a transport equation for ℓ_d, the k-ϵ model uses a transport equation for the dissipation i.e. for:

$$\epsilon = A\frac{k^{\frac{3}{2}}}{\ell_d} \tag{7.34}$$

where $A \approx 0.09$ is valid everywhere as mentioned above.

If k and ϵ are calculated by use of transport equations, we can subsequently calculate ℓ_d by equation (7.34) and ν_T by equation (7.33). In principle, the transport equation for ϵ and k are set up in the same way. Without going into details, after smoothing the ϵ-transport equation contains some terms depending on turbulent fluctuations, and these terms are rewritten in the same way as the 'fluctuation terms' were rewritten in the k-equation. The rewriting causes introduction of some extra coefficients like the coefficient A. All of these coefficients are named *closure coefficients*. The resulting transport equation for a fully turbulent flow is given below.

As shown in e.g. Wilcox (1994), Rodi (1984) or Versteeg (1995), adoption of the k-ϵ model means that the calculations of the mean flow are based on these equations:

Continuity equation:

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{7.35}$$

Navier-Stokes' equation:

$$\rho \frac{dU_i}{dt} = -\frac{\partial \overline{p^+}}{\partial x_i} + \frac{\partial \tau_{ji}^{turb}}{\partial x_j} \tag{7.36}$$

Shear stress:

$$\tau_{ji}^{turb} = \rho \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \tag{7.37}$$

Eddy viscosity:

$$\nu_T = \ell_d \sqrt{k} \tag{7.38}$$

Dissipation:

$$\epsilon = A \frac{k^{3/2}}{\ell_d} \tag{7.39}$$

k-equation:

$$\rho \frac{dk}{dt} = \rho \frac{\partial}{\partial x_j} \left(\frac{\nu_T}{\sigma_k} \frac{\partial k}{\partial x_j} \right) + \tau_{ji}^{turb} \frac{\partial U_i}{\partial x_j} - \rho \epsilon \tag{7.40}$$

ϵ-equation:

$$\rho \frac{d\epsilon}{dt} = \rho \frac{\partial}{\partial x_j} \left(\frac{\nu_T}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right) + C_1 \frac{\epsilon}{k} \tau_{ji}^{turb} \frac{\partial U_i}{\partial x_j} - C_2 \rho \frac{\epsilon^2}{k} \tag{7.41}$$

The coefficients σ_k and σ_ϵ are introduced in order to have a possibility to compensate for the errors introduced by the assuming that the diffusion coefficients for momentum, turbulent energy and dissipation are equal.

After substitution of τ_{ji}^{turb} into the Navier-Stokes equation, the k-equation and the ϵ-equation, knowledge of the closure coefficients A, σ_k, σ_ϵ, C_1 and C_2 means that we have 8 equations with these 8 unknowns:

U_1, U_2, U_3, $\overline{p^+}$, ν_T, ℓ_d, k og ϵ

i.e. with proper boundary conditions we are able to solve the system of equations and determine the mean flow.

Often applied values of the closure coefficients are:

$A = 0.09$, $\sigma_k = 1.0$, $\sigma_\epsilon = 1.3$, $C_1 = 1.44$ and $C_2 = 1.92$

as these values have resulted in accurate prediction of many flows. However, notice for special flow problems that it can be necessary to adjust the values for some of the coefficients. In the literature it is often claimed that the k-ϵ model might give inaccurate results for flow with adverse pressure gradients, i.e. if the flow takes place in the direction of increasing pressure.

7.3.2 The k-ω Model

In order to improve the predictions for flow with adverse pressure gradients, the literature often recommends application of the $k-\omega$ model, which is a turbulence model, where the second transport equation is set up for the so-called *specific dissipation rate*, ω, defined as:

$$\boxed{\omega = \frac{\epsilon}{A\,k}} \tag{7.42}$$

Thus, no transport equation is set up for ℓ_d directly, but knowledge of ω makes it possible to calculate ℓ_d by the expression:

$$\ell_d = \frac{\sqrt{k}}{\omega} \tag{7.43}$$

This expression is derived by substitution of :

$$\epsilon = A\frac{k^{\frac{3}{2}}}{\ell_d} \tag{7.44}$$

into equation (7.42), which gives:

$$\omega = \frac{\epsilon}{A\,k} = \frac{A\,k^{\frac{3}{2}}/\ell_d}{A\,k} = \frac{\sqrt{k}}{\ell_d} \tag{7.45}$$

or

$$\ell_d = \frac{\sqrt{k}}{\omega} \tag{7.46}$$

If this expression for ℓ_d is substituted into $\nu_T = \ell_d\,\sqrt{k}$, the result reads:

$$\boxed{\nu_T = \frac{k}{\omega}} \tag{7.47}$$

which is the equation for the eddy viscosity in the k-ω model. The setup of the transport equation for ω is rather spectacular, see e.q. Wilcox (1994). The most simple version of the transport equation for ω (for fully turbulent flow) is given below.

As shown in Wilcox (1994) adoption of the k-ω model means that the calculations of the mean flow are based on these equations:

Continuity equation:

$$\frac{\partial\,U_i}{\partial\,x_i} = 0 \tag{7.48}$$

Navier-Stokes equation:

$$\rho \frac{dU_i}{dt} = -\frac{\partial \overline{p^+}}{\partial x_i} + \frac{\partial \tau_{ji}^{turb}}{\partial x_j} \tag{7.49}$$

Shear stress:

$$\tau_{ji}^{turb} = \rho \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \tag{7.50}$$

Eddy viscosity:

$$\nu_T = \frac{k}{\omega} \tag{7.51}$$

Dissipation:

$$\epsilon = A \frac{k^{3/2}}{\ell_d} \tag{7.52}$$

k-equation:

$$\rho \frac{dk}{dt} = \rho \frac{\partial}{\partial x_j} \left(\frac{\nu_T}{\sigma_k} \frac{\partial k}{\partial x_j} \right) + \tau_{ji}^{turb} \frac{\partial U_i}{\partial x_j} - \rho \epsilon \tag{7.53}$$

ω-equation:

$$\rho \frac{d\omega}{dt} = \rho \frac{\partial}{\partial x_j} \left(\sigma \nu_T \frac{\partial \omega}{\partial x_j} \right) + \alpha \frac{\omega}{k} \tau_{ji}^{turb} \frac{\partial U_i}{\partial x_j} - \beta \rho \omega^2 \tag{7.54}$$

After substitution of τ_{ji}^{turb} into the Navier-Stokes equation, the k-equation and the ω-equation, knowledge of the closure coefficients A, σ_k, σ, α and β means that we have 8 equations with these 8 unknowns:

U_1, U_2, U_3, $\overline{p^+}$, ν_T, k, ω and ϵ

i.e. with proper boundary conditions, we are able to solve the system of equations and determine the mean flow.

Often applied values of the closure coefficients are:

$A = 0.09$, $\sigma_k = 1.0$, $\sigma = 0.5$, $\alpha = 0.56$ and $\beta = 0.025$

as these values have resulted in accurate prediction of many flows. However, notice for special flow problems that it can be necessary to adjust the values for some of the coefficients, see e.g Wilcox (2006), where a modified version of the transport equation for ω is presented. For flows with adverse pressure gradients, i.e. if the flow takes place in the direction of increasing pressure and separation is likely to occur, application of the k-ω model is normally recommended.

Chapter 8

Transport Equation for Dissolved Substance

In a flow the transport of a dissolved substance is caused by two transport mechanisms named *diffusion* and *convection*.

The flux through a section due to diffusion reads:

$$\vec{F}_D = -D \operatorname{grad} c \tag{8.1}$$

or

$$\boxed{(F_D)_i = -D \frac{\partial c}{\partial x_i}} \tag{8.2}$$

where c is the concentration of the dissolved substance (kg/m^3), grad c is the gradient of c, $|\vec{F}_D|$ is the flux of substance (kg/s/m^2) through a surface perpendicular to grad c and D is the *diffusivity* (often named the diffusion coefficient) (m^2/s). As mentioned in section 2.4.1, diffusion has to be introduced, because we not intend to describe the motions of the individual molecules. Normally D is of the same order of magnitude as the kinematic viscosity ν of the fluid.

Consider a fixed, closed surface A of arbitrary shape, bounding the volume X, see Fig. 8.1. The *inflow* of substance through dA is the sum of convection and diffusion, i.e.

$$F\,dA = -c\,v_i\,dA_i - (F_D)_i\,dA_i = (-c\,v_i + D\,\frac{\partial c}{\partial x_i})\,dA_i$$

where dA_i is the outward area vector. The increase of substance within the volume X has to be the net inflow of substance through A plus the substance

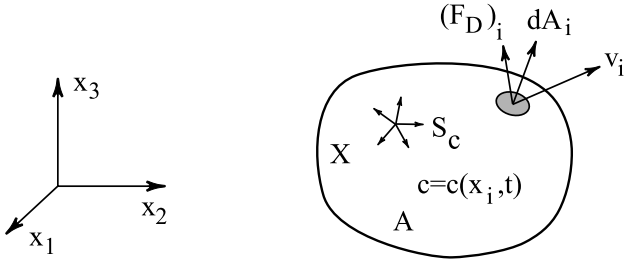

Figure 8.1: *Definition sketch, inflow of substance.*

added to X from any internal sources. Expressed mathematically this reads:

$$\int_X \frac{\partial c}{\partial t}\, dX = \int_A (-c\, v_i + D\, \frac{\partial c}{\partial x_i})\, dA_i + \int_X S_c\, dX \tag{8.3}$$

where S_c is the source strength (kg/m^3/s). If the surface integral is rewritten by use of Gauss' divergence theorem:

$$\int_A l_i\, dA_i = \int_X \frac{\partial l_i}{\partial x_i}\, dX$$

equation (8.3) reads:

$$\int_X \left[\frac{\partial c}{\partial t} + \frac{\partial (c\, v_i)}{\partial x_i} - \frac{\partial}{\partial x_i}(D\, \frac{\partial c}{\partial x_i}) - S_c \right] dX = 0 \tag{8.4}$$

Equation (8.4) is, however, only fulfilled for an arbitrary volume if:

$$\frac{\partial c}{\partial t} + \frac{\partial (c\, v_i)}{\partial x_i} - \frac{\partial}{\partial x_i}(D\, \frac{\partial c}{\partial x_i}) - S_c = 0 \tag{8.5}$$

or

$$\frac{\partial c}{\partial t} + c\, \frac{\partial v_i}{\partial x_i} + v_i\, \frac{\partial c}{\partial x_i} = \frac{\partial}{\partial x_i}(D\, \frac{\partial c}{\partial x_i}) + S_c \tag{8.6}$$

For an incompressible fluid, $\partial v_i / \partial x_i = 0$, equation (8.6) is reduced to:

$$\frac{\partial c}{\partial t} + v_i\, \frac{\partial c}{\partial x_i} = \frac{\partial}{\partial x_i}(D\, \frac{\partial c}{\partial x_i}) + S_c \tag{8.7}$$

If we finally use the expression for the rate of change, equation (1.19), we can write equation (8.7) as:

$$\boxed{\frac{dc}{dt} = \frac{\partial}{\partial x_i}(D\, \frac{\partial c}{\partial x_i}) + S_c} \tag{8.8}$$

This equation is named the *transport equation for substance.*

8.1 Turbulent Flow

In a turbulent flow the instantaneous value of concentration reads:

$$c = \overline{c} + c' \tag{8.9}$$

where $\overline{c}$ is the smoothed concentration and c' is the fluctuation, i.e. $\overline{c'} = 0$.

Substitution of equation (8.9) into equation (8.5) and a subsequent smoothing give:

$$\frac{\partial \overline{c}}{\partial t} + \frac{\partial \overline{(c\, v_i)}}{\partial x_i} = \frac{\partial}{\partial x_i}(D \frac{\partial \overline{c}}{\partial x_i}) + S_c \tag{8.10}$$

The second term on the left-hand side is rewritten as follows:

$$\begin{aligned} \frac{\partial \overline{(\overline{c} + c')\,(U_i + u_i)}}{\partial x_i} &= \frac{\partial \,(\overline{c}\, U_i + \overline{c'}\, U_i + \overline{c}\,\overline{u_i} + \overline{c'\, u_i})}{\partial x_i} \\ &= \frac{\partial \,(\overline{c}\, U_i + \overline{c'\, u_i})}{\partial x_i} \\ &= \frac{\partial \,(\overline{c}\, U_i)}{\partial x_i} + \frac{\partial \overline{c'\, u_i}}{\partial x_i} \\ &= U_i \frac{\partial \overline{c}}{\partial x_i} + \frac{\partial \overline{c'\, u_i}}{\partial x_i} \end{aligned} \tag{8.11}$$

where the continuity equation for the mean flow:

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{8.12}$$

has been used. Substitution of equation (8.11) into equation (8.10) gives:

$$\frac{\partial \overline{c}}{\partial t} + U_i \frac{\partial \overline{c}}{\partial x_i} + \frac{\partial \overline{c'\, u_i}}{\partial x_i} = \frac{\partial}{\partial x_i}(D \frac{\partial \overline{c}}{\partial x_i}) + S_c \tag{8.13}$$

which after use of the expression for the rate of change, equation (1.19), reads:

$$\frac{d\overline{c}}{dt} + \frac{\partial \overline{c'\, u_i}}{\partial x_i} = \frac{\partial}{\partial x_i}(D \frac{\partial \overline{c}}{\partial x_i}) + S_c \tag{8.14}$$

The term $\overline{c'\, u_i}$ can be interpreted as a convective transport of substance through a surface, but equation (8.13) is of no use in practice unless a relation between the mean flow and the term $\overline{c'\, u_i}$ can be established.

Consider the area A of a section parallel to the wall in a uniform mean flow along the flat wall, see Fig. 8.2. The smoothed value of the convective flux of

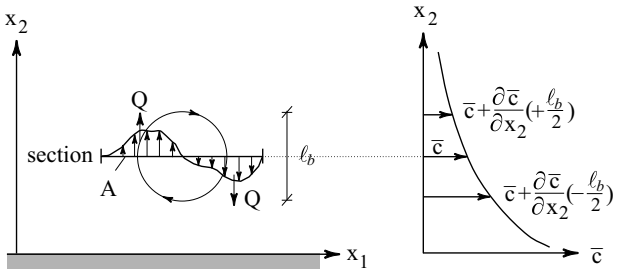

Figure 8.2: *Definition sketch, convective transport caused by fluctuations.*

substance through the section can be expressed as $\bar{F} = \overline{c' u_2}$. However, if the way of thinking from the mixing length theory is applied, another expression describing the same convective transport can be set up.

The fluid crossing the section in the upward direction is on average coming from a domain positioned $0.5\,\ell_b$ under the section. Here ℓ_b is the mixing length. The concentration of this fluid is denoted $\overline{c_u}$, and it can be expressed as:

$$\overline{c_u} = \overline{c} + \frac{\partial\,\overline{c}}{\partial\,x_2} \cdot \Delta x_2 = \overline{c} + \frac{\partial\,\overline{c}}{\partial\,x_2} \cdot \frac{-\ell_b}{2}$$

The average concentration of the water crossing the section in the downward direction, denoted $\overline{c_d}$, is similarly expressed as:

$$\overline{c_d} = \overline{c} + \frac{\partial\,\overline{c}}{\partial\,x_2} \cdot \Delta x_2 = \overline{c} + \frac{\partial\,\overline{c}}{\partial\,x_2} \cdot \frac{\ell_b}{2}$$

Based on continuity the upward discharge (volume flux) has to be equal to the downward discharge, and the magnitude of these is denoted Q. The average transport of substance (positive in the direction of the x_2-axis) caused by this swapping of volume reads:

$$\begin{aligned}
\overline{F}\,A &= Q\,\overline{c_u} - Q\,\overline{c_d} \\
&= Q\,(\overline{c} + \frac{\partial\,\overline{c}}{\partial\,x_2} \cdot \frac{-\ell_b}{2}) - Q\,(\overline{c} + \frac{\partial\,\overline{c}}{\partial\,x_2} \cdot \frac{\ell_b}{2}) \\
&= \overline{c}\,(Q - Q) - Q\,\ell_b\,\frac{\partial\,\overline{c}}{\partial\,x_2} = -Q\,\ell_b\,\frac{\partial\,\overline{c}}{\partial\,x_2}
\end{aligned} \tag{8.15}$$

Thus the smoothed flux of transported substance reads:

$$\overline{F} = -\frac{Q\,\ell_b}{A}\,\frac{\partial\,\overline{c}}{\partial\,x_2} \tag{8.16}$$

We can interpret $\overline{F}$ as the result of a diffusion process, because equation (8.16) can also be written as:

$$\overline{F} = -D_T\,\frac{\partial\,\overline{c}}{\partial\,x_2} \tag{8.17}$$

where the turbulent diffusion coefficient, D_T , reads:

$$D_T = \frac{Q\,\ell_b}{A} \tag{8.18}$$

As both Q and ℓ_b vary as the distance from the wall changes, it is seen that also the turbulent diffusion coefficient will vary over the flow field.

As $\overline{c'\,u_2} = \bar{F}$ (both describe the convective transport caused by the fluctuations), we have:

$$\overline{c'\,u_2} = -D_T\,\frac{\partial\,\bar{c}}{\partial\,x_2} \tag{8.19}$$

and in this way a relation between fluctuations and the mean flow has been set up. As turbulence normally causes a very strong dispersion, we have $D_T \gg D$ in most cases.

In section 5.3 it was shown that turbulent shear stress is caused by exchange of momentum, and can be expressed as:

$$\tau_{21}^{turb} = \nu_T\,\frac{\partial\,(\rho\,U_1)}{\partial\,x_2} \tag{8.20}$$

However, this expression also shows that turbulent shear stress can be interpreted as diffusion of momentum with diffusion coefficient ν_T. As the exchange of both momentum and substance is caused by the same eddies, it is often assumed that the two processes have approximately equal diffusion coefficients. If the non-dimensional *Schmidt* number is defined as:

$$\boxed{\sigma_c = \frac{\nu_T}{D_T}} \tag{8.21}$$

it is therefore expected that we have $\sigma_c \approx 1$. An often applied value is $\sigma_c = 0.9$. With this assumption we can find D_T from equation (8.21), when the eddy viscosity ν_T has been calculated by a suitable turbulence model.

The one-dimensional equation (8.19) can be generalized to:

$$\overline{c'\,u_i} = -D_T\,\frac{\partial\,\bar{c}}{\partial\,x_i} \tag{8.22}$$

and after substitution into equation (8.14), we have the *transport equation for substance* in a turbulent flow:

$$\boxed{\frac{d\bar{c}}{dt} = \frac{\partial}{\partial\,x_i}\left((D + \frac{\nu_T}{\sigma_c})\,\frac{\partial\,\bar{c}}{\partial\,x_i}\right) + S_c} \tag{8.23}$$

Chapter 9

Transport Equation for Thermal Energy

The sum of kinetic and potential energy for a fluid particle is often called the *mechanical energy*. Besides this energy a fluid particle also has the so-called *internal energy* e, which for an incompressible fluid simply is *thermal energy* ($\mathrm{Nm/m^3}$ or $\mathrm{J/m^3}$). The increase in thermal energy for an incompressible fluid is found by $de = \rho\, c\, dT$, where c is the specific heat ($\mathrm{J/kg/K}$) and T is the temperature (degree Kelvin, K). The flows considered in civil engineering normally have so small flow velocities that even air can be considered incompressible, and only rather small pressure variations are present. It is therefore a good approximation to estimate the increase in thermal energy by $de = \rho\, c_p\, dT$, where c_p is the specific heat at constant pressure.

The concept of thermal energy has to be introduced, because we want describe the fluid as a continuum instead of a set of molecules. The thermal energy is thus the kinetic energy corresponding to velocities that cancel out, when the average velocity of the fluid particle is calculated.

In the following it is assumed that thermal energy can only propagate by *conduction* (exchange of energy on molecular level) and by *convection*, but in practical heat transfer problems also propagation by *radiation* is important.

Thermal energy propagating by conduction is described by *Fourier's* law:

$$\vec{q} = -\lambda \operatorname{grad} T \tag{9.1}$$

or

$$\boxed{q_i = -\lambda \frac{\partial T}{\partial x_i}} \tag{9.2}$$

where $|q_i|$ is the thermal flux (J/s/m^2) through a surface perpendicular to the temperatur gradient, λ is the *thermal conductivity* (J/m/grad/s) and T is the temperature.

As equation (9.2) can be rewritten to:

$$q_i = -\frac{\lambda}{\rho\, c_p} \frac{\partial\,(\rho\, c_p\, T)}{\partial\, x_i} \tag{9.3}$$

i.e. the thermal flux is proportional to the gradient of the thermal energy, it is natural that the quantity:

$$\boxed{D_J = \frac{\lambda}{\rho\, c_p}} \tag{9.4}$$

is named the *thermal diffusivity* (m^2/s), or thermal diffusion coefficient. Notice that often a is used to denote the thermal diffusivity.

The non-dimensional ratio between the kinematic viscosity and the thermal diffusivity is denoted Pr and is named *Prandtl's* number:

$$\boxed{Pr = \frac{\nu}{D_J}} \tag{9.5}$$

Thus Prandtl's number is a constant for a given substance. Prandtl's number is for water $Pr \approx 7$ and for air $Pr \approx 0.7$.

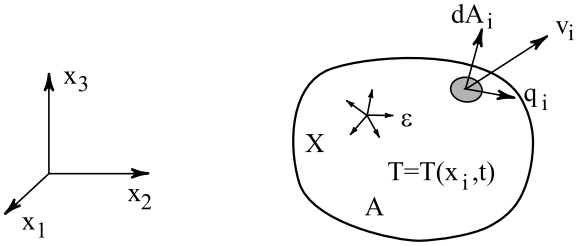

Figure 9.1: *Definition sketch, inflow of thermal energy.*

Consider a fixed, closed surface A of arbitrary shape, bounding the volume X, see Fig. 9.1. The *inflow* of thermal energy through dA is the sum of convection

and conduction, i.e.

$$F\,dA = -c_p\,T\,\rho\,v_i\,dA_i - q_i\,dA_i = (-c_p\,T\,\rho\,v_i + \lambda\,\frac{\partial\,T}{\partial\,x_i})\,dA_i$$

where dA_i is the outward area vector. The increase of thermal energy within the volume X has to be the net inflow of thermal energy through A plus the thermal energy added to X from internal heat sources as e.q. the dissipation ϵ. Expressed mathematically, this reads:

$$\int_X \frac{\partial\,T}{\partial\,t}\,c_p\,\rho\,dX = \int_A (-c_p\,T\,\rho\,v_i + \lambda\,\frac{\partial\,T}{\partial\,x_i})\,dA_i + \int_X \epsilon\,\rho\,dX \tag{9.6}$$

If the surface integral is rewritten by use of Gauss' divergence theorem:

$$\int_A l_i\,dA_i = \int_X \frac{\partial\,l_i}{\partial\,x_i}\,dX$$

equation (9.6) can be written as:

$$\int_X \left[c_p\,\rho\,\frac{\partial\,T}{\partial\,t} + c_p\,\rho\,\frac{\partial\,(v_i\,T)}{\partial\,x_i} - \frac{\partial}{\partial\,x_i}(\lambda\,\frac{\partial\,T}{\partial\,x_i}) - \epsilon\,\rho\right]\,dX = 0 \tag{9.7}$$

Equation (9.7) is, however, only fulfilled for an arbitrary volume if :

$$c_p\,\rho\,\frac{\partial\,T}{\partial\,t} + c_p\,\rho\,\frac{\partial\,(v_i\,T)}{\partial\,x_i} - \frac{\partial}{\partial\,x_i}(\lambda\,\frac{\partial\,T}{\partial\,x_i}) - \epsilon\,\rho = 0 \tag{9.8}$$

or

$$c_p\,\rho\,\left(\frac{\partial\,T}{\partial\,t} + \frac{\partial\,(v_i\,T)}{\partial\,x_i}\right) = \frac{\partial}{\partial\,x_i}(\lambda\,\frac{\partial\,T}{\partial\,x_i}) + \epsilon\,\rho \tag{9.9}$$

For an incompressible fluid, where $\partial\,v_i/\partial\,x_i = 0$, this expression can be rewritten to:

$$c_p\,\rho\,\left(\frac{\partial\,T}{\partial\,t} + v_i\,\frac{\partial\,T}{\partial\,x_i}\right) = \frac{\partial}{\partial\,x_i}(\lambda\,\frac{\partial\,T}{\partial\,x_i}) + \epsilon\,\rho \tag{9.10}$$

If the expression for the rate of change, equation (1.19), is used, we can write equation (9.10) as:

$$\boxed{c_p\,\rho\,\frac{dT}{dt} = \frac{\partial}{\partial\,x_i}(\lambda\,\frac{\partial\,T}{\partial\,x_i}) + \epsilon\,\rho} \tag{9.11}$$

This equation is named the *transport equation for thermal energy* or simply the *energy equation*.

9.1 Turbulent Flow

In a turbulent flow we have:

$$T = \overline{T} + T' \tag{9.12}$$

where $\bar{T}$ is the smoothed concentration and T' is the fluctuation, i.e. $\overline{T'} = 0$.

After substitution of (9.12) into equation (9.9), a subsequent smoothing gives:

$$c_p\,\rho\left(\frac{\partial\,\overline{T}}{\partial\,t} + \frac{\partial\,\overline{(U_i + u_i)\,(\overline{T} + T')}}{\partial\,x_i}\right) = \frac{\partial}{\partial\,x_i}(\lambda\,\frac{\partial\,\overline{T}}{\partial\,x_i}) + \overline{\epsilon}\,\rho \tag{9.13}$$

The second term on the left-hand side is rewritten as follows:

$$\begin{aligned}\frac{\partial\,\overline{(U_i + u_i)\,(\overline{T} + T')}}{\partial\,x_i} &= \frac{\partial\,(U_i\,\overline{T} + U_i\,\overline{T'} + \overline{u_i}\,\overline{T} + \overline{u_i\,T'})}{\partial\,x_i} \\ &= \frac{\partial\,(U_i\,\overline{T} + \overline{u_i\,T'})}{\partial\,x_i} \\ &= \frac{\partial\,(U_i\,\overline{T})}{\partial\,x_i} + \frac{\partial\,\overline{u_i\,T'}}{\partial\,x_i}\end{aligned} \tag{9.14}$$

and substitution into to (9.13) gives:

$$c_p\,\rho\left(\frac{\partial\,\overline{T}}{\partial\,t} + \frac{\partial\,(U_i\,\overline{T})}{\partial\,x_i} + \frac{\partial\,\overline{u_i\,T'}}{\partial\,x_i}\right) = \frac{\partial}{\partial\,x_i}(\lambda\,\frac{\partial\,\overline{T}}{\partial\,x_i}) + \overline{\epsilon}\,\rho \tag{9.15}$$

If the continuity equation of the mean flow is used

$$\frac{\partial\,U_i}{\partial\,x_i} = 0 \tag{9.16}$$

equation (9.15) can be rewritten to:

$$c_p\,\rho\left(\frac{\partial\,\overline{T}}{\partial\,t} + U_i\,\frac{\partial\,\overline{T}}{\partial\,x_i} + \frac{\partial\,\overline{u_i\,T'}}{\partial\,x_i}\right) = \frac{\partial}{\partial\,x_i}(\lambda\,\frac{\partial\,\overline{T}}{\partial\,x_i}) + \overline{\epsilon}\,\rho \tag{9.17}$$

If the expression for the rate of change, equation (1.19), is used, equation (9.17) reads:

$$c_p\,\rho\left(\frac{d\overline{T}}{dt} + \frac{\partial\,\overline{u_i\,T'}}{\partial\,x_i}\right) = \frac{\partial}{\partial\,x_i}(\lambda\,\frac{\partial\,\overline{T}}{\partial\,x_i}) + \overline{\epsilon}\,\rho \tag{9.18}$$

The term $\rho\,c_p\,\overline{u_i\,T'} = \overline{u_i\,\rho\,c_p\,T'}$ can be interpreted as a convective transport of thermal energy through a surface, but equation (9.18) is of no use in practice, unless a relations between the mean flow and the terms $\overline{u_i\,T'}$ and $\bar{\epsilon}$ can be established. The expression for the smoothed dispersion reads:

$$\bar{\epsilon} = \overline{\nu\,(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})\,\frac{\partial v_i}{\partial x_j}} \tag{9.19}$$

but as the non-linear fluctuation terms have the same character as the Reynolds stresses, we must either use the expression for $\overline{\epsilon}$, equation (5.67):

$$\overline{\epsilon} = A\,\frac{k^{3/2}}{\ell_d} \tag{9.20}$$

where $A \approx 0.09$ and ℓ_d is the length scale for the eddies, or calculate the dissipation directly, as it is done by a k-ϵ turbulence model. If a k-ω model is used, we know that $\ell_d = k/\omega$. If a k-model is applied, assessed values for ℓ_d have to be used.

The term $\rho\, c_p\, \overline{u_i\, T'} = \overline{u_i\, \rho\, c_p\, T'}$ can be interpreted as the transport of thermal energy due to the fluctuations, and we describe it as a diffusion process, i.e as a product of a diffusion coefficient and a gradient of the 'concentration' of the thermal energy $\overline{\rho\, c_p\, T}$. We did exactly the same, when the transport of a substance from fluctuations had to be described. This gives:

$$\overline{u_i\, \rho\, c_p\, T'} = -D_{J,T}\,\frac{\partial\, \overline{\rho\, c_p\, T}}{\partial\, x_i} \tag{9.21}$$

where $D_{J,T}$ is named *the turbulent thermal diffusivity (diffusion coefficient)* (m^2/s). As the exchange of both momentum and thermal energy is caused by the same eddies, it is often assumed that the two diffusion processes have approximately the same diffusion coefficient. If the non-dimensional number named the *turbulent Prandtl number* is defined as:

$$\boxed{\sigma_h = \frac{\nu_T}{D_{J,T}}} \tag{9.22}$$

the value $\sigma_h \approx 1$ is expected. Notice that the turbulent Prandtl number is sometimes also denoted Pr_T. For fluids having a Prandtl number $Pr = \nu/D_J > 0.7$, experiments show that $\sigma_h (= Pr_T) = 0.9 - 1.0$ is a good estimate except for free turbulence (jets, wakes and mixing layers). In the latter cases $\sigma_h \approx 0.7$ is a better estimate.

Substitution of equation (9.21) and equation (9.22) into equation (9.18) gives:

$$\boxed{c_p\, \rho\, \frac{d\overline{T}}{dt} = \frac{\partial}{\partial\, x_i}\left((\lambda + \frac{\rho\, c_p\, \nu_T}{\sigma_h})\,\frac{\partial\, \overline{T}}{\partial\, x_i}\right) + \overline{\epsilon}\, \rho} \tag{9.23}$$

The equation is named *the transport equation for thermal energy* in turbulent flow. Division by $c_p\, \rho$ gives:

$$\frac{d\overline{T}}{dt} = \frac{\partial}{\partial\, x_i}\left((\frac{\lambda}{c_p\, \rho} + \frac{\nu_T}{\sigma_h})\,\frac{\partial\, \overline{T}}{\partial\, x_i}\right) + \frac{\overline{\epsilon}}{c_p} \tag{9.24}$$

Substitution of the expression for the thermal diffusion coefficient, equation (9.4), gives finally:

$$\boxed{\frac{d\overline{T}}{dt} = \frac{\partial}{\partial x_i}\left((D_J + \frac{\nu_T}{\sigma_h})\frac{\partial \overline{T}}{\partial x_i}\right) + \frac{\bar{\epsilon}}{c_p}} \tag{9.25}$$

It is seen that the equation has the same structure as the transport equation for a substance in turbulent flow, i.e. the rate of change is equal to a gradient of the diffusion plus a source term. This is not surprising, as $c_p\,\rho\,T$ can be interpreted as the 'concentration' of thermal energy.

Chapter 10

Non-Homogeneous Fluids

Sometimes the density of a fluid is varying within the flow field. These variations may be caused by e.g. variations in the concentration of dissolved matter or variations of the temperature of the fluid. Variations in density create buoyancy forces, which sometimes affect the course of the flow considerably. Typically buoyancy force has to be taken into account for cases like e.g. flow in a heated room, flows caused by fire, smoke plumes from chimneys and the outlet of waste water into a recipient.

Below it is described how flows of fluid with a continuous variation of the density can be treated.

10.1 Boussinesq's approximation

The flow of a mildly non-homogenous fluid can by analyzed properly by adopting an approximation proposed first by Boussinesq.

It is assumed that density gradients are so small that the local equation of continuity can be written (as usual):

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{10.1}$$

This implies that the Navier-Stokes equation can be written (also as usual):

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right) \tag{10.2}$$

but ρ is now a variable quantity. The equation expresses that the inertial forces are balanced by pressure forces, gravity forces and viscous forces.

Solution of the Navier-Stokes equation coupled with a transport equation for dissolved substance and the energy equation for thermal energy, yields the actual variation of the density, as

$$\rho = \rho(c, T) \tag{10.3}$$

is assumed to be a known relation.

In some cases, most frequently with flows of air, it is sufficient to assume

$$\rho = \rho(T) \tag{10.4}$$

and in such cases the Navier-Stokes equation is often rewritten as shown below. The purpose of the rewriting is primarily a simplification of the computations during the solution of the equations.

The gradients of ρ are assumed to be small, which means that the effect from density variations on inertial and viscous forces can be neglected. The density in the expression for the inertial force is therefore replaced by a reference density denoted ρ_o, and the viscosity $\mu = \rho\,\nu$ is replaced by $\rho_o\,\nu$.

On the other hand, to replace ρ in the gravity term by ρ_o is not allowed. For example, a local heating of a resting fluid leads to unequal sizes of the buoyancy and gravity force on the heated particle, and this net force creates a motion of the fluid.

In order to obtain a unique definition of the dynamic pressure p^+, i.e. the difference between the actual pressure and hydrostatic pressure, it is necessary to calculate the hydrostatic pressure on the basis of the reference density. The definition equation for p^+ reads:

$$p = p^+ - \rho_o\, g\, z \tag{10.5}$$

where z is a vertical coordinate, positive upwards. This implies that $\partial\, z/\partial\, x_i$ is a vertical, upward unit vector giving $g\,\partial\, z/\partial\, x_i = -g_i$. From equation (10.5) is seen that

$$\frac{\partial\, p}{\partial\, x_i} = \frac{\partial\, p^+}{\partial\, x_i} + \rho_o\, g_i \tag{10.6}$$

Substitution of this expression into equation (10.2) and the actual density for the inertial and viscous forces replaced by the reference density gives:

$$\rho_o\, \frac{dv_i}{dt} = -\frac{\partial p^+}{\partial x_i} + (\rho - \rho_0)\, g_i + \frac{\partial}{\partial\, x_j}\left(\rho_o\, \nu \left(\frac{\partial\, v_i}{\partial\, x_j} + \frac{\partial\, v_j}{\partial\, x_i}\right)\right) \tag{10.7}$$

The deviation of the actual density from the reference density is denoted $\Delta\rho$, i.e.

$$\Delta\rho = \rho - \rho_o \tag{10.8}$$

Heating of a mass m with volume X increases the volume, and the expression for the increment reads:

$$\Delta X = \beta\, X\, \Delta T \tag{10.9}$$

where β is the *spatial coefficient of thermal expansion* and ΔT is the increment of the temperature. For an ideal gas (as e.g. air) the coefficient is $\beta = 1/T_{Kelvin} \approx 1/293 = 3.4 \cdot 10^{-3}$ deg^{-1}. The coefficient is varying from fluid to fluid and e.g. $\beta \approx 5.0 \cdot 10^{-5}$ deg^{-1} for water.

The corresponding change of the density is calculated based on the principle of conservation of mass:

$$\Delta m = \Delta(\rho\, X) = X\, \Delta\rho + \rho\, \Delta X = 0 \tag{10.10}$$

giving

$$\Delta\rho = -\rho \frac{\Delta\, X}{X} = -\rho\, \beta\, \Delta T \tag{10.11}$$

after substitution of equation (10.9). If T_o corresponds to ρ_o, the deviation in density is:

$$\rho - \rho_o = -\rho\, \beta\, (T - T_o) \approx -\rho_o\, \beta\, (T - T_o) \tag{10.12}$$

which substituted into the Navier-Stokes equation gives:

$$\rho_o \frac{dv_i}{dt} = -\frac{\partial p^+}{\partial x_i} - \rho_o\, \beta\, (T - T_o)\, g_i + \frac{\partial}{\partial\, x_j} \left(\rho_o\, \nu \left(\frac{\partial\, v_i}{\partial\, x_j} + \frac{\partial\, v_j}{\partial\, x_i} \right) \right) \tag{10.13}$$

Application of this equation makes a separate calculation of ρ unnecessary. Equation (10.13) is valid for snapshots of turbulent flow. The equation valid for the mean flow is obtained after smoothing, which gives:

$$\begin{aligned} \rho_o \frac{dU_i}{dt} \quad &= \quad -\frac{\partial \overline{p^+}}{\partial x_i} - \rho_o\, \beta\, (\overline{T} - T_o)\, g_i \\ &\quad + \frac{\partial}{\partial\, x_j} \left(\rho_o\, \nu \left(\frac{\partial\, U_i}{\partial\, x_j} + \frac{\partial\, U_j}{\partial\, x_i} \right) - \rho_o\, \overline{u_i\, u_j} \right) \end{aligned} \tag{10.14}$$

In technical turbulence theory the Reynolds' stresses $-\rho_o\, \overline{u_i\, u_j}$ are modelled as

$$-\rho_o\, \overline{u_i\, u_j} = \nu_T \left(\frac{\partial\, U_i}{\partial\, x_j} + \frac{\partial\, U_j}{\partial\, x_i} \right) \quad \text{for} \quad i \neq j \tag{10.15}$$

where ν_T is calculated by a suitable turbulence model.

All in all we have:

$$\boxed{\rho_o \frac{dU_i}{dt} = -\frac{\partial \overline{p^+}}{\partial x_i} - \rho_o\, \beta\, (\overline{T} - T_o)\, g_i + \frac{\partial}{\partial\, x_j} \left(\rho_o\, (\nu + \nu_T) \left(\frac{\partial\, U_i}{\partial\, x_j} + \frac{\partial\, U_j}{\partial\, x_i} \right) \right)} \tag{10.16}$$

Notice that we cannot expect the usual expressions for ν_T to be valid for flows having very large density gradients corresponding to a stratification of the fluid.

In order to decide whether variations in density shall be taken into account or not, we need a parameter related to the differences in density.

Consider a boundary layer flow solely driven by differences in density like the flow along a heated, vertical wall, where the velocity outside the boundary layer is zero, $v_o = 0$. The constant density *outside* the boundary layer is denoted ρ_o and the density at the heated wall is denoted ρ_w.

The forces acting on a fluid particle within the boundary layer are the vertical pressure force (buoyancy) and the gravity force.

According to the gradient-theorem derived in Appendix A, the pressure force reads: $O = |-\text{grad}\, p\,| \cdot X$, where X is the volume of the particle. As usual the pressure gradient within the boundary layer is equal to the gradient outside the layer, and outside the layer the fluid is at rest giving $|-\text{grad}\, p\,| = \rho_o\, g$. Thus the buoyancy reads: $O = |-\text{grad}\, p\,| \cdot X = \rho_o\, g\, X$.

Due to the variation of the density across the boundary layer, the gravity force depends on the position of the particle. A typical value of the gravity force is found by using the average value of the density i.e. $\rho = 1/2(\rho_o + \rho_w)$, and in this case the total vertical force on the particle reads:

$$F = O - G = \rho_o\, g\, X - \frac{\rho_o + \rho_w}{2}\, X\, g = \frac{1}{2}(\rho_o - \rho_w) g\, X \tag{10.17}$$

Introduction of the notation:

$$\Delta\rho = \rho_w - \rho_o \tag{10.18}$$

means that the total force can be expressed as:

$$F = \frac{1}{2}\, |\Delta\rho|\, g\, X \tag{10.19}$$

Being at rest at the time $t = 0$, a particle will start to move upwards due to the force F. Having moved the distance L, the velocity of the particle V can be calculated by the classical energy theorem for a particle stating that the increment of kinetic energy is equal to the work done by the external forces. If we approximate the external forces by F (i.e. friction forces are ignored) we have:

$$\frac{1}{2}\, |\Delta\rho|\, g\, X\, L \approx \frac{1}{2}\, \rho\, X\, V^2 \tag{10.20}$$

or

$$V \approx \left(g\, L \frac{|\Delta\rho|}{\rho} \right)^{\frac{1}{2}} \tag{10.21}$$

This velocity can by adopted in the following definition of a Reynolds number:

$$Re_b = \frac{V\,L}{\nu} = \frac{\left(g\,L\frac{|\Delta\rho|}{\rho}\right)^{\frac{1}{2}}\,L}{\nu} \tag{10.22}$$

However, in most literature is applied a parameter, which is the Reynolds number raised to the power 2, i.e. defined as:

$$\boxed{Gr = (Re_b)^2 = \frac{g\,L^3}{\nu^2}\frac{|\Delta\rho|}{\rho}} \tag{10.23}$$

and the parameter Gr is named *Grashof's number*.

If the differences in density are caused by temperature, Grashof's number can be expressed as:

$$Gr = \frac{g\,L^3\,\beta\,|\Delta T|}{\nu^2} \tag{10.24}$$

after substitution of equation (10.11)

$$\Delta\rho \;=\; -\rho\,\beta\,\Delta T \tag{10.25}$$

In situations where the boundary flow is driven by differences in density and by a flow outside the boundary layer at the velocity v_o, the influence of the differences in density on the course of the flow can be assessed by a comparison of Grashof's number and the usual Reynolds number squared, i.e. $Re^2 = (v_o\,L/\nu)^2$.

In this context one has to distinguish between these two completely different types of flow in the boundary layer:

1) *forced convection*, where the flow in the boundary layer is mainly driven by the flow outside the boundary layer, i.e. differences in density are unimportant. This corresponds to $Gr \ll Re^2$.

2) *free convection*, where the flow in the boundary layer is driven mainly by differences in density caused be e.g. differences in temperature or concentration. This corresponds to $Gr \gg Re^2$.

For *air* having $\beta = 3.4 \cdot 10^{-3}$, differences in temperature give this ratio between Grashof's number and the squared Reynolds number:

$$\left(\frac{Gr}{Re^2}\right)_{air} = \frac{g\,L\,\beta\,|\Delta T|}{v_o^2} = \frac{9.81 \cdot L \cdot 3.4 \cdot 10^{-3} \cdot |\Delta T|}{v_o^2} = \frac{0.03\,L\,|\Delta T|}{v_o^2}$$

Chapter 11

Transfer of Heat

In many cases the total transfer of heat between a body and the flow past the body is the main issue, and the details of the heat transport are only of minor importance.

The transfer of heat, denoted q, between the body and the flow is determined by *Fourier's* law applied at a point in the flow just outside the surface of the body. Thus

$$q(x_1) = -\lambda \left. \frac{\partial T}{\partial x_2} \right|_{x_2=0} \tag{11.1}$$

where λ is the thermal conductivity, T is the temperature. The unit for q is $\mathrm{J/m^2/sec}$.

11.1 Forced Convection in Boundary Layers

An exact solution (due to Blasius, see section 4.1.1) exists to the equations of motion for a laminar boundary layer.

No exact solutions exist for turbulent boundary layers, but use of the momentum equation for boundary layers with an assessed velocity profile can provide approximate solutions, where the accuracy of the calculated thickness of the boundary layer is approximately 5-10%.

If variation of the density is neglected, the Navier-Stokes equation reads:

$$\frac{d(\rho v_i)}{dt} = \frac{\partial}{\partial x_j}\left(\nu\left(\frac{\partial(\rho v_i)}{\partial x_j} + \frac{\partial(\rho v_j)}{\partial x_i}\right)\right) - \frac{\partial p^+}{\partial x_i} \tag{11.2}$$

If this version of the Navier-Stokes equation is compared to the transport equation for thermal energy $c_p\,\rho T$, which reads:

$$\frac{d(c_p\,\rho T)}{dt} = \frac{\partial}{\partial\, x_j}(\frac{\lambda}{c_p\,\rho}\frac{\partial\,(c_p\,\rho T)}{\partial\, x_j}) + \epsilon\,\rho \tag{11.3}$$

it is seen that the Navier-Stokes equation may be interpreted as a transport equation for momentum $B_i = \rho\, v_i$.

Thus, it makes sense to believe that the approximate methods for the calculation of flow in boundary layers can be used to calculate the heat/substance transport in boundary layers.

11.1.1 Boundary Layers on a Plate at Zero Pressure Gradient

Application of the usual boundary layer approximation:

$$\frac{\partial}{\partial\, x_1} \ll \frac{\partial}{\partial\, x_2} \quad \text{and} \quad v_2 \ll v_1 \tag{11.4}$$

gives that the equation for momentum ($B_1 = \rho v_1$) in a two-dimensional, steady flow without a pressure gradient reads:

$$v_1\frac{\partial\, v_1}{\partial\, x_1} + v_2\frac{\partial\, v_1}{\partial\, x_2} = \nu\,\frac{\partial^{\,2} v_1}{\partial\, x_2^2} \tag{11.5}$$

with corresponding boundary conditions:

1)
$$v_1 = 0 \;\text{ and }\; v_2 = 0 \quad \text{for} \quad x_2 = 0$$

2)
$$\frac{v_1}{v_o} \to 1 \quad \text{for} \quad x_2 \to \infty$$

Based on an assumption of similar profiles, equation (11.5) and the continuity can be solved numerically giving:

$$\frac{v_1}{v_o} = g_1\left(\frac{x_2}{\sqrt{\frac{\nu\, x_1}{v_o}}}\right) \quad \text{and} \quad \frac{v_2}{v_o} = g_2\left(\frac{x_2}{\sqrt{\frac{\nu\, x_1}{v_o}}}\right) \tag{11.6}$$

where the functions $g_1()$ and $g_2()$ are known functions, first derived by Blasius (1908). The function $g_1()$ is seen in Fig. 4.2.

If the boundary layer approximation is used for the transport equation for thermal energy in a two-dimensional, steady flow gives:

$$c_p \rho \left(v_1 \frac{\partial T}{\partial x_1} + v_2 \frac{\partial T}{\partial x_2} \right) = \lambda \frac{\partial^2 T}{\partial x_2^2} + \epsilon \rho \tag{11.7}$$

The dissipation ϵ is unimportant for flows with small velocity gradients or large differences in temperature between the outer flow and the wall. Therefore the dissipation term is often neglected for small values ($\ll 1$) of *the Eckart number*, E_c, defined as

$$E_c = \frac{v_o^2}{c_p (T_w - T_o)} \tag{11.8}$$

where v_o is the velocity outside the velocity boundary layer, T_o is the temperature outside the thermal boundary layer, and T_w is the temperature of the wall.

Neglecting dissipation makes it possible to rewrite the transport equation for thermal energy to:

$$v_1 \frac{\partial T}{\partial x_1} + v_2 \frac{\partial T}{\partial x_2} = \frac{\lambda}{c_p \rho} \frac{\partial^2 T}{\partial x_2^2} = D_J \frac{\partial^2 T}{\partial x_2^2} \tag{11.9}$$

where D_J is the thermal diffusivity defined as:

$$D_J = \frac{\lambda}{\rho c_p} \tag{11.10}$$

If a non-dimensional temperature, θ, is defined as:

$$\theta = \frac{T - T_w}{T_o - T_w} \tag{11.11}$$

the transport equation for thermal energy reads:

$$v_1 \frac{\partial \theta}{\partial x_1} + v_2 \frac{\partial \theta}{\partial x_2} = D_J \frac{\partial^2 \theta}{\partial x_2^2} \tag{11.12}$$

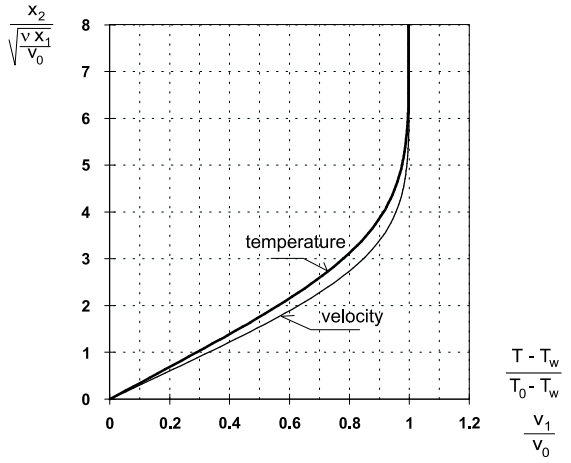

Figure 11.1: *Profiles of velocity and temperature in air,* $Pr = 0.7$, $\partial\, p^+ / \partial\, x_1 = 0$.

with boundary conditions:

1)
$$\theta = 0 \quad \text{for} \quad x_2 = 0$$

2)
$$\theta \to 1 \quad \text{for} \quad x_2 \to \infty$$

The boundary layer equations for momentum and transport of thermal energy are two coupled differential equations, which can be solved numerically to any specified accuracy.

A single solution is known already, as the equations for momentum and transport of thermal energy and corresponding boundary conditions are identical for $D_J = \nu$, i.e. if the Prandtl number is $Pr = 1$. In this case the profiles for velocity and temperature are coincident.

For $Pr \neq 1$ the deviation between the temperature profile and the velocity profile only depends on the Prandtl number. In Fig. 11.1 both profiles are shown for air flow ($Pr = 0.7$). It is seen that the extent of temperature profiles is largest. The opposite is true for fluids with $Pr > 1$. For water, $Pr = 7.0$.

After some algebra, including the use of equation (11.1), the transfer of heat from the wall to the fluids can be expressed as:

$$q(x_1) \;=\; 0.332\, \lambda\, Pr^{1/3} \sqrt{\frac{v_o}{\nu\, x_1}} (T_v - T_o) \quad \text{for} \quad 0.6 < Pr < 10 \qquad (11.13)$$

Heat transfer is often expressed by use of a non-dimensional quantity named the local *Nusselt number*. It is defined as:

$$\boxed{Nu_x = \frac{q \cdot x_1}{\lambda\,(T_v - T_o)}} \tag{11.14}$$

where x_1 is the distance from the leading edge of the plate. Comparison of the equations (11.13) and (11.14) gives:

$$\boxed{Nu_x = 0.332\,Pr^{1/3}\,\sqrt{Re_x}} \quad \text{for} \quad 0.6 < Pr < 10 \tag{11.15}$$

where Reynolds' number is defined as:

$$Re_x = \frac{v_o\,x_1}{\nu} \tag{11.16}$$

11.1.2 Boundary Layers on a Plate at Zero Pressure Gradient, Approximate Calculation

Approximate methods based on assessed velocity profiles normally predict the thickness of a velocity boundary layer, δ, with an accuracy of say 5-10%. The same approximate methods will now be adopted to calculate the thickness of thermal boundary layers , δ_T.

Consider a two-dimensional, steady, laminar flow of an incompressible fluid. Along a heated plate boundary layers for velocity and temperature are formed, see Fig. 11.2.

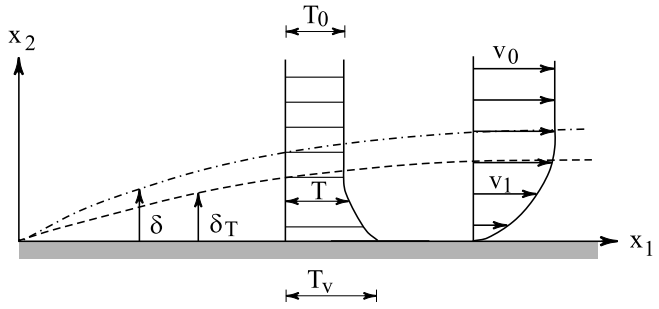

Figure 11.2: *Definition sketch. Boundary layers for velocity and temperature on a heated plate.*

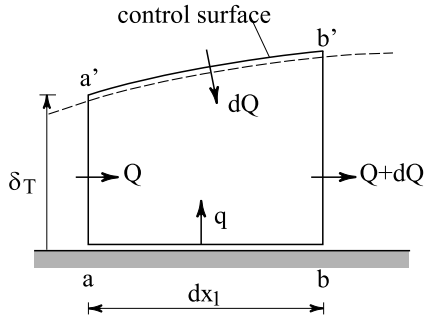

Figure 11.3: *Definition sketch. Thermal boundary layer. Q is volume flux and q is the transfer of thermal energy from plate to fluid.*

The resulting *inward* transport of thermal energy through the control surface depicted in Fig. 11.3 is considered.

Inflow of thermal energy:

a a':

$$\int_0^{\delta_T} \rho\, c_p\, T\, v_1\, dx_2 \qquad \text{(J/s/m)}$$

a' b':

$$\rho\, c_p\, T_o\, dQ$$

b b':

$$-\left(\int_0^{\delta_T} \rho\, c_p\, T\, v_1\, dx_2 + \frac{\partial}{\partial\, x_1} \int_0^{\delta_T} \rho\, c_p\, T\, v_1\, dx_2 \cdot dx_1\right)$$

a b :

$$q \cdot dx_1$$

Due to continuity the inflow of volume through the surface a' b' can be expressed as:

$$dQ \;=\; \frac{\partial}{\partial\, x_1} \int_0^{\delta_T} v_1\, dx_2 \cdot dx_1 \tag{11.17}$$

Neglecting the thermal energy due to dissipation, the resulting *inflow* of thermal energy has to be zero, as the situation is steady, i.e. the temperature is constant at any fixed point. Mathematically this is expressed as:

$$\begin{aligned} &\int_0^{\delta_T} \rho\, c_p\, T\, v_1\, dx_2 + \rho\, c_p\, T_o\, \frac{\partial}{\partial\, x_1} \int_0^{\delta_T} v_1\, dx_2 \cdot dx_1 \\ -&\int_0^{\delta_T} \rho\, c_p\, T\, v_1\, dx_2 - \frac{\partial}{\partial\, x_1} \int_0^{\delta_T} \rho\, c_p\, T\, v_1\, dx_2 \cdot dx_1 + q\, dx_1 \;=\; 0 \end{aligned} \tag{11.18}$$

or

$$\boxed{q(x_1) \;=\; \frac{\partial}{\partial\, x_1} \int_0^{\delta_T} \rho\, c_p (T - T_o)\, v_1\, dx_2} \tag{11.19}$$

This equation is the local energy equation for thermal energy on integral form for a boundary layer. The left-hand side can also be expressed as:

$$q(x_1) = -\lambda \left.\frac{\partial\, T}{\partial\, x_2}\right|_{x_2=0} \tag{11.20}$$

Substitution of assessed profiles of temperature and velocity into equation (11.19) makes it possible to calculate δ_T as shown below. The choice of profiles is not very important, and e.g. parabolic profiles may be used. The assessed velocity profile:

$$\frac{v_1}{v_o} \;=\; 2\frac{x_2}{\delta} - \frac{x_2^2}{\delta^2} \tag{11.21}$$

substituted into the momentum equation for boundary layers gives:

$$\delta \approx 5.5 \sqrt{\frac{\nu\, x_1}{v_o}} \tag{11.22}$$

Substitution of the assessed temperature profile:

$$\frac{T - T_o}{T_v - T_o} = 1 - 2\frac{x_2}{\delta_T} + \frac{x_2^2}{\delta_T^2} \tag{11.23}$$

into equation (11.20) gives:

$$q(x_1) \;=\; -\lambda\,(T_v - T_o)\,(-\frac{2}{\delta_T}) = \frac{2\,\lambda\,(T_v - T_o)}{\delta_T} \tag{11.24}$$

If $q(x_1)$ is given, this expression makes it possible to calculate δ_T directly. However, in practice we want to find $q(x_1)$.

This means that $\delta_T(x_1)$ has to be calculated by a numerical solution of equation (11.19) after substitution of equation (11.24). If the partial derivative on the right-hand side of the equation is substituted by a finite difference, equation (11.19) reads:

$$\frac{2\,\lambda\,(T_v - T_o)}{\delta_T} \;\approx\; \frac{\Delta}{\Delta x_1} \int_0^{\delta_T} \rho\, c_p (T - T_o)\, v_1\, dx_2 \tag{11.25}$$

or

$$\frac{2\,\lambda\,(T_v - T_o)}{\delta_T} \;\approx\; \frac{\int_0^{\delta_T^B} \rho\, c_p (T - T_o)\, v_1\, dx_2 - \int_0^{\delta_T^A} \rho\, c_p (T - T_o)\, v_1\, dx_2}{\Delta x_1} \tag{11.26}$$

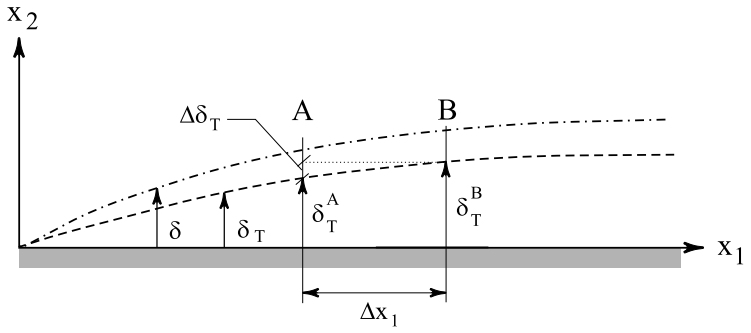

Figure 11.4: *Definition sketch. Numerical calculation of thermal boundary layer*

Assume that the conditions at section A including δ_T^A are known, see Fig. 11.4. The next step is to find the placing of the section, where the thickness of the thermal boundary layer is increased by $\Delta\delta_T$. The size of $\Delta\delta_T$ is chosen beforehand, i.e. $\delta_T^B = \delta_T^A + \Delta\delta_T$ is known, but $x_1^B = x_1^A + \Delta x_1$ is unknown. Equation (11.26) is then used to calculate the distance Δx_1 between section A and section B. Iteration is necessary, as T_v and δ (and hereby v_1) depend on x_1. This calculation is repeated from one section to the next one until $\delta_T(x_1)$ is known along the entire plate. Finally the transfer of heat $q(x_1)$ is calculated by equation (11.24).

11.2 Free Convection in Boundary Layers

11.2.1 Boundary Layer at a vertical, heated plate

It is assumed that the fluid outside the boundary layer is at rest, i.e. $v_o = 0$. Temperature and density at the wall are denoted T_w and ρ_w, respectively, and outside the thermal boundary layer they are denoted T_o and ρ_o. As $Gr \gg Re^2$ for this flow, the flow type is free convection, where density variations have to be taken into account. This is done by use of the Navier-Stokes equation for non-homogeneous fluids, equation (10.13).

Moreover, assume two-dimensional *steady* flow and the usual boundary layer approximations to be valid, which implies that the component in the direction of the x_1-axis (positive upwards) reads:

$$\rho_o \left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} \right) = -\frac{\partial p^+}{\partial x_1} - \rho_o \beta (T - T_o) g_1 + \frac{\partial}{\partial x_2} \left(\rho_o \nu \left(\frac{\partial v_1}{\partial x_2} + 0 \right) \right)$$

The pressure gradient within the boundary layer is (as usual) equal to the pres-

sure gradient outside the layer. Thus

$$\frac{\partial p^+}{\partial x_1} = 0$$

within the boundary layer, as $p^+ = 0$ in the resting fluid outside the layer. Substitution of the pressure gradient and $g_1 = -g$, yields after division by ρ_o:

$$v_1 \frac{\partial\, v_1}{\partial\, x_1} + v_2 \frac{\partial\, v_1}{\partial\, x_2} = +\beta\,(T - T_o)\, g + \frac{\partial^{\,2} v_1}{\partial\, x_2^2} \tag{11.27}$$

Having three unknowns, v_1, v_2 and T, two more equations are necessary to solve the problem. These equations are the continuity equation:

$$\frac{\partial\, v_1}{\partial\, x_1} + \frac{\partial\, v_2}{\partial\, x_2} = 0 \tag{11.28}$$

and the transport equation for thermal energy rewritten to:

$$v_1 \frac{\partial\, T}{\partial\, x_1} + v_2 \frac{\partial\, T}{\partial\, x_2} = D_J\, \frac{\partial^{\,2} T}{\partial\, x_2^2} \tag{11.29}$$

where $D_J = \lambda/(c_p\, \rho)$ is the thermal diffusivity and the effect of dissipation has been ignored.

The boundary conditions read:

1)
$$v_1 = 0 \;\text{ and }\; v_2 = 0 \quad \text{for} \quad x_2 = 0$$

2)
$$v_1 \to 0 \quad \text{for} \quad x_2 \to \infty$$

and

1)
$$T = T_v(x_1) \quad \text{for} \quad x_2 = 0$$

2)
$$T \to T_o \quad \text{for} \quad x_2 \to \infty$$

These equations can be numerically solved in the same way as the equations for forced convection of the boundary layer were solved. See e.g. Schlichting (1979).

Bibliography

Blasius, H., 1908
Grenzschichten in Flüssigkeiten mit kleiner Reibung, Zeitschrift für Mathematik und Physik, Berlin, Tyskland, vol. 56, pp. 1-37.

Engelund, F.A., 1968
Hydrodynamik, Den private Ingeniørfond, Danmarks Tekniske Universitet, Lyngby

Fredsøe, J., 1990
Hydrodynamik, Den private Ingeniørfond, Danmarks Tekniske Universitet, Lyngby

Nielsen, P. V., 1994
Computational Fluid Dynamics in Ventilation, Aalborg University.

Pohlhausen, K., 1921
Zue näherungsweisen Integration der Differentialgleichung der laminaren Reibungsschicht, Zeitschrift für angewandte Mathematik und Mechanik, Berlin, Tyskland, vol. 1, pp. 252-268.

Prandtl, L., 1904
Über Flüssigkeitsbewegung bei sehr kleiner Reibung, Proceedings of the Third International Mathematics Congress, Heidelberg, Tyskland.

Prandtl, L., 1925
Über die ausgebildete Turbulenz, Zeitschrift für angewandte Mathematik und Mechanik, Berlin, Tyskland, vol. 5, pp. 136-139.

Rodi, W., 1984
Turbulence Models and their application in Hydraulics - A State of the Art Review, IAHR, Delft, Holland.

Schlichting, H., 1979
Boundary-Layer Theory (7. edition), McGraw-Hill, Inc.

Wilcox, D. C., 1994
Turbulence Modeling for CFD, DCW Industries, Inc., California, USA.

Wilcox, D. C., 2006
Turbulence Modeling for CFD, Third Edition, DCW Industries, Inc., California, USA.

Versteeg, H. K. & Malalasekera, W., 1995
An introduction to Computational Fluid Dynamics. The Finite Volume Method, Longman Group Ltd.

Appendix A

The Gradient Theorem

The subject of this appendix is the calculation of the *total* pressure force on a *closed* surface in a fluid.

Denoting the closed surface A bounding the volume X, means that the total pressure force on A can be expressed as:

$$\vec{F}_p = \int_A p\,(-d\vec{A}) \tag{A.1}$$

where p is the pressure and $d\vec{A}$ is the *outwards* directed area vector. As it may be difficult to calculate the surface integral, we shall next derive a theorem, which can be applied in many cases with advantage.

Gauss's divergence theorem applied to the vector field $\vec{c} = -p\,\vec{k}$, where $\vec{k}$ is an arbitrary constant, reads:

$$\int_A \vec{c}\cdot d\vec{A} = \int_X \operatorname{div}\vec{c}\; dX \tag{A.2}$$

or

$$\int_A -p\,\vec{k}\cdot d\vec{A} = \int_X \operatorname{div}\left(-p\,\vec{k}\right) dX = -\int_X \operatorname{div}\left(p\,\vec{k}\right) dX \tag{A.3}$$

Here

$$\begin{aligned} \operatorname{div}(p\,\vec{k}) &= \frac{\partial\,(p\,k_1)}{\partial\,x_1} + \frac{\partial\,(p\,k_2)}{\partial\,x_2} + \frac{\partial\,(p\,k_3)}{\partial\,x_3} = k_1\frac{\partial\,p}{\partial\,x_1} + k_2\,\frac{\partial\,p}{\partial\,x_2} + k_3\,\frac{\partial\,p}{\partial\,x_3} \\ &= \vec{k}\cdot\operatorname{grad} p \end{aligned} \tag{A.4}$$

as $\vec{k}$ is a constant vector.

Substitution of this expression into equation (A.3) yields:

$$\int_A -p\,\vec{k}\cdot d\vec{A} = \int_X -\vec{k}\cdot \operatorname{grad} p\, dX \tag{A.5}$$

As $\vec{k}$ is (still) a constant vector, equation (A.5) is rewritten to:

$$\vec{k}\cdot\left(\int_A -p\,d\vec{A} + \int_X \operatorname{grad} p\, dX\right) = 0 \tag{A.6}$$

The first integral in this equation is seen to be the total pressure force on A, and the equation is rewritten to:

$$\vec{k}\cdot\left(\vec{F_p} + \int_X \operatorname{grad} p\, dX\right) = 0 \tag{A.7}$$

As the direction of $\vec{k}$ is arbitrary, the vector within the parenthesis must be the null-vector, which gives:

$$\boxed{\vec{F_p} = \int_X -\operatorname{grad} p\, dX} \tag{A.8}$$

This equation is sometimes named the *gradient theorem*, and it shows that the pressure force on an arbitrary, closed surface A can be found as the volume integral of the negative pressure gradient over the volume bounded by the surface.

Notice that *the pressure force from a constant pressure is always zero*, as the gradient of a constant is zero!

In a *resting fluid* the pressure is constant, as $p_{hyd} = -\rho\, g\, z$ gives:

$$\operatorname{grad} p_{hyd} = (0, 0, -\rho\, g) \tag{A.9}$$

when the x_3-axis is vertical and positive upwards. Substitution of this pressure gradient into equation (A.9) yields:

$$\vec{F}_{p_{hyd}} = \int_X -\operatorname{grad} p\, dX = \int_X -(0, 0, -\rho\, g)\, dX = (0, 0, \rho\, g)\, X \tag{A.10}$$

This force is the pressure force on the fluid body from the surrounding fluid. However, if X is replaced by e.g a solid body at rest, the pressure distribution on A is unchanged, resulting in the same pressure force. In this way it has been shown that the pressure force on an immersed body in a resting fluid is a force directed *upwards* and equal to the gravity force on the displaced volume of fluid. This was formulated by Archimedes back in antiquity, and today this pressure force is named the *buoyancy*.

Appendix B

On the Notation of Cartesian Tensors

The subject of this appendix is the index notation of Cartesian tensors.

The components of a *3*-dimensional vector $\vec{x}$ are denoted x_1, x_2, x_3, i.e.

$$\vec{x} = x_1\,\vec{e_1} + x_2\,\vec{e_2} + x_3\,\vec{e_3} = (x_1, x_2, x_3)$$

where $\vec{e_1}$, $\vec{e_2}$ and $\vec{e_3}$ are unit vectors in the x_1-, x_2- and x_3-directions, respectively, of an ordinary cartesian coordinate system.

Sometimes equations can be expressed in a short form by use of index notation. The first convention of index notation reads:

if an index is a lower-case letter, it shall run through its range, which is normally 1, 2 and 3.

In general the *range* of the index i of an n-dimensional vector is the set of integers from 1 to n.

The second convention of index notation reads:

a variable with one of the numbers 1, 2 or 3 as index, indicates the component of the variable in either the x_1-, x_2- or x_3-direction.

Thus, with index notation the vector $\vec{x}$ can be expressed as x_i.

The third convention of index notation (Einstein's summation convention) reads:

repetition of an index in a term denotes summation with respect to that index over its range

An index that is summed over is called a *dummy index* and one that is not summed over is called a *free index*.

Consider two vectors a_i and b_i. Due to the summation convention the scalar product (or dot product) of the vectors expressed by index notation reads:

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{B.1}$$

Since a dummy index simply indicates summation, it does not matter what symbol is being used. Thus $a_i b_i$ may be replaced by $a_j b_j$, which is obvious if the following expression (in ordinary notation) is considered:

$$\sum_{i=1}^{3} a_i b_i = \sum_{j=1}^{3} a_j b_j \tag{B.2}$$

Another example of an equation with a dummy index is the continuity equation for an incompressible fluid:

$$\operatorname{div} \vec{v} = 0 \quad \Leftrightarrow \quad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \tag{B.3}$$

which in tensor notation reads:

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{B.4}$$

Next consider an example of an equation with one free index:

$$\operatorname{grad} p = \vec{k} \quad \Leftrightarrow \quad \left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\right) = (k_1, k_2, k_3) \tag{B.5}$$

in ordinary notation. In index notation equation (B.5) reads:

$$\frac{\partial p}{\partial x_i} = k_i \tag{B.6}$$

So far only scalars and vectors have been considered. It is, however, also possible to express matrix equations in index notation. To do so, we use two indices. The first index corresponds to the row number and the second index to the column number. Consider a 3x3 matrix $[A]$ defined by:

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{B.7}$$

With index notation this matrix is expressed as A_{ij}.

If we post-multiply $[A]$ by a 3x1 column vector $\vec{x}$, the result is a 3x1 column vector $\vec{y}$, i.e.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{B.8}$$

With index notation and use of the summation convention, equation (B.8) reads:

$$y_i = A_{ij} x_j \tag{B.9}$$

where j is a dummy index and i is a free index.

Index

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