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Metastable states when the Fermi Golden Rule constant vanishes*

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Abstract

Resonances appearing by perturbation of embedded non-degenerate eigenvalues are studied in the case when the Fermi Golden Rule constant vanishes. Under appropriate smoothness properties for the resolvent of the unperturbed Hamiltonian, it is proved that the first order Rayleigh-Schrödinger expansion exists. The corresponding metastable states are constructed using this truncated expansion. We show that their exponential decay law has both the decay rate and the error term of order ε^4, where ε is the perturbation strength.

1 Introduction

In this paper we continue our study of the decay laws for resonances produced by perturbation of eigenvalues embedded in the continuous spectrum. More precisely, one considers an unperturbed Hamiltonian H having a non-degenerate eigenvalue E_0 embedded in its continuous spectrum. The degenerate case is by far more complicated and will be not discussed in this paper; we send the reader to [21, 36, 17, 30] and references therein for the results known in this case.

Our problem is to study the fate of the unperturbed state Ψ_0 corresponding to E_0 when adding a perturbation W of strength $\varepsilon \ll 1$ so that the Hamiltonian becomes $H_\varepsilon = H + \varepsilon W$. The answers are quite different depending on whether the unperturbed eigenvalue is situated near an energetic threshold or far away inside the continuous spectrum. While the previous papers in this series (see [9] and references therein) were mainly concerned with the threshold case for which there were no rigorous results available (see [21, 22] and Section 4 in [19]), in the present paper we revisit the case of properly embedded eigenvalues.

The problem of the decay laws for resonances in general and, in particular, for resonances produced by perturbation of eigenvalues embedded in the continuous spectrum, has a distinguished and ramified history ranging from experimental to rigorous levels, see e.g. [6, 12, 15, 28, 31, 7, 8, 10, 33, 34, 5, 17, 36, 19, 30, 38, 24], and references given there. As is well known, the notion of 'resonance' occurs often. It has many definitions and its meaning depends upon the context. For example, in spectral and scattering theory a resonance is a complex number which may be a pole in the analytic continuation of the resolvent of the corresponding Hamiltonian, or an eigenvalue of the dilated Hamiltonian. There is a huge literature about the subject, both at the mathematical level and at the physical level.

The scope of our paper is limited. We restrict ourselves to the perturbative setting described above and we are only interested in dynamical aspects in a Hilbert space $\mathcal{H}$. In what follows, by a resonance (probably 'metastable state' is a better name) we shall understand a *pair* $(\Psi_\varepsilon, E_\varepsilon)$ such that $\Psi_\varepsilon \in \mathcal{H}$, $\|\Psi_\varepsilon\| = 1$ and $\lim_{\varepsilon\to 0} \|\Psi_\varepsilon - \Psi_0\| = 0$ ('resonance eigenfunction'), and $E_\varepsilon \in \mathbf{C}$ with $\operatorname{Im} E_\varepsilon \leq 0$, ('resonance position') satisfying with some accuracy the exponential decay law for the survival amplitude:

$$\langle \Psi_\varepsilon, e^{-itH_\varepsilon}\Psi_\varepsilon\rangle \simeq e^{-itE_\varepsilon}. \tag{1.1}$$

If the bound state survives after turning on the perturbation, then the resonance pair is given by the corresponding bound state eigenfunction and eigenvalue of H_ε for which equality is realized in (1.1). If the eigenvalue disappears for $\varepsilon > 0$, then the situation is by far less clear. First of all, as is well known, the semi-boundedness of H_ε forbids the equality in (1.1) so we are left with the problem of finding $(\Psi_\varepsilon, E_\varepsilon)$ such that:

$$\sup_{t\geq 0} |\langle \Psi_\varepsilon, e^{-itH_\varepsilon}\Psi_\varepsilon\rangle - e^{-itE_\varepsilon}| \leq \delta(\varepsilon), \quad \lim_{\varepsilon\to 0}\delta(\varepsilon) = 0, \quad \lim_{\varepsilon\to 0}\|\Psi_\varepsilon - \Psi_0\| = 0. \tag{1.2}$$

Clearly, (1.2) does not define the pair $(\Psi_\varepsilon, E_\varepsilon)$ uniquely and this adds to the difficulty of the subject; for the moment, the best one can do is to search for pairs $(\Psi_\varepsilon, E_\varepsilon)$ leading to a $\delta(\varepsilon)$ as small as possible.

A natural candidate for Ψ_ε is just the unperturbed eigenvector Ψ_0. With the exception of [17] all the existing rigorous results are related to the (quasi-)exponential decay law for $\langle \Psi_0, e^{-itH_\varepsilon}\Psi_0\rangle$, at least as far as we know. The story started in the early days of quantum mechanics with the computation by Dirac of the decay rate in second order time-dependent perturbation theory, leading to the well known exponential decay law, $e^{-2\varepsilon^2\Gamma t}$, for the survival probability. Here Γ is given by the famous Fermi Golden Rule (FGR) constant:

$$\Gamma \sim |\langle \Psi_0, W\Psi_{\mathrm{cont},E_0}\rangle|^2, \tag{1.3}$$

where Ψ_{cont,E_0} is a generalized eigenfunction corresponding to E_0 in the continuous spectrum (assumed to have multiplicity one). The FGR formula has been so influential that the common wisdom in theoretical physics is that the decay law for the resonances produced by perturbation of non-degenerate bound states is exponential. However, since the decay law cannot be exactly exponential at the rigorous level (for semi-bounded Hamiltonians), the crucial problem is the estimation of the errors. This proved to be a hard problem, and only during the past decades consistent rigorous results have been obtained. It turns out that (see [5, 6, 17, 19, 24, 30, 36, 38] and the references given there) the decay law is indeed (quasi-)exponential, i.e. exponential up to error terms vanishing in the limit $\varepsilon \to 0$, if the resolvent of the unperturbed Hamiltonian is sufficiently smooth, when projected onto the subspace orthogonal to the eigenvalue under consideration. First of all, in the dilation analytic setting of the Balslev-Combes theory [4] there is a mathematically well defined candidate for the resonance position, E_ε, namely the perturbed eigenvalue of the dilated Hamiltonian. In this context Hunziker [17] proved that

$$|\langle \Psi_0, e^{-itH_\varepsilon}\Psi_0\rangle - e^{-itE_\varepsilon}| \lesssim \varepsilon^2. \tag{1.4}$$

Since dilation analyticity is a strong assumption, much effort has been devoted to the extension of the above result to the case when analyticity is relaxed to some smoothness conditions (see [5, 6, 19, 24, 28, 30, 36] and the references given there). More precisely, if the resolvent of the unperturbed Hamiltonian is sufficiently smooth, when projected onto the subspace orthogonal to the spectral subspace corresponding to the eigenvalue under consideration, it has been proved that one can *find* E_ε, $\operatorname{Im} E_\varepsilon \leq 0$, such that (1.4) holds true, see see Theorem 4.2ii in [19] and its slight refinement in the present Section 2. Moreover, it turns out that if Γ given by the FGR is nonzero, then $\operatorname{Im} E_\varepsilon = -\varepsilon^2\Gamma + \mathcal{O}(\varepsilon^3)$, which is consistent with the FGR formula. The discussion in the next paragraphs strongly suggests that the error term in (1.4) is optimal with respect to the power of ε.

The problem considered here is whether the error term can be made *smaller* by choosing a better ansatz for the initial state by replacing Ψ_0 with a properly chosen, ε-dependent, resonance eigenfunction. In the Balslev-Combes dilation analytic setting this question has been already addressed by Hunziker [17]. In that context (see [37]) the resonance position has a clear cut spectral meaning. Hunziker proved that *if* E_0 *is isolated* and the formal Rayleigh-Schrödinger (R-S) perturbation expansion for the perturbed eigenfunction is well defined up to order ε^N (as it is the case for the atomic Stark effect), then by using as the resonance eigenfunction the normalized truncated R-S series, one can improve the error term to be of order ε^{2N+2}. Here we also have $|\operatorname{Im} E_\varepsilon| \lesssim \varepsilon^{2N+2}$. For the embedded case (even in the dilation analytic case), the problem of improving the error term by choosing a better ansatz for the initial state remained open. Due to Hunziker's results, the natural conjecture is that under appropriate smoothness conditions, the existence of the formal R-S perturbation expansion up to order N should lead to an exponential decay law with a smaller error term. For the case of embedded eigenvalues the R-S series generally breaks down already at order $N = 1$. Hence there are two problems to be solved. The first one is to seek conditions under which the R-S perturbation expansion exists up to some order $N \geq 1$ and then to construct the 'corrected' resonance eigenfunction. The second (harder) one is to prove that under appropriate smoothness conditions, the new error term is indeed smaller.

The main result of the paper is a positive answer to both questions for $N = 1$ in the R-S expansion, see Proposition 2.11 and Theorem 2.15 below. More precisely, suppose Γ as given by the FGR vanishes, while the second derivative with respect to the energy of the generalized eigenfunction(s) of the unperturbed Hamiltonian exists in a neighborhood of E_0 and is θ-Hölder continuous with some $\theta > 0$. Then the formal R-S perturbation expansion exists to order $N = 1$, and for the corresponding initial value one can prove an exponential decay similar to (1.4) with both decay rate, $\operatorname{Im} E_\varepsilon$, and error term of order ε^4 or smaller.

The contents of the paper is as follows. In Section 2 we give the main results with an outline of proofs. Section 3 contains the technical details. In Section 4 we present a class of two channel Schrödinger operators for which our abstract theory applies. In two Appendices we collect, in a form appropriate for us, some known facts about Hölder properties of the Cauchy integral transform, and about resolvent smoothness and Γ-operator for one body Schrödinger operators, respectively.

2 The results and outline of proofs

Throughout the paper 's sufficiently small' is a shorthand for 'there exists $s_0 > 0$ such that for $0 < s < s_0$'. Also, for $A, B \geq 0$, we write $A \lesssim B$ instead of 'there exists a constant $0 < C < \infty$, independent of A and B, such that $A \leq CB$'.

Our results are model independent, in the sense that only the boundedness of the perturbation and some smoothness of the resolvent of the unperturbed Hamiltonian are demanded. For example, in Theorem 2.15 we require that the second derivative with respect to the spectral parameter of the generalized eigenfunctions of H exists and is Hölder continuous. Therefore we develop the theory at the abstract level and verify these assumptions for each concrete application.

Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and E_0 a non-degenerate eigenvalue of H, while P_0 is the corresponding orthogonal projection:

$$HP_0 = E_0 P_0, \ \ \dim P_0 = 1, \quad P_0 \Psi_0 = \Psi_0, \quad \|\Psi_0\| = 1, \quad Q_0 = 1 - P_0.$$

Without loss of generality we can take $E_0 = 0$ in what follows. We denote by $P(\Delta)$ the spectral measure of H. The first basic assumption is that except eigenvalue zero H only has absolutely continuous spectrum in some neighborhood of the origin:

Assumption 2.1. *There exists $a > 0$ such that $J_a \cap \sigma_{\mathrm{pp}}(H) = \{0\}$ and $J_a \cap \sigma_{\mathrm{sc}}(H) = \emptyset$, where $J_a = (-a, a)$.*

Then we add a perturbation W of strength $\varepsilon > 0$, $\varepsilon \to 0$, and consider the perturbed operator

$$H_\varepsilon = H + \varepsilon W. \tag{2.1}$$

In order to keep the technicalities at a reasonable level we impose:

Assumption 2.2. *W is self-adjoint and bounded.*

Adding some supplementary conditions, one can extend the results of this paper to the case when W is only relatively compact with respect to H. The case of singular perturbations is much harder, and detailed results are only known in specific cases, as for example the Stark effect; we shall not consider the case of singular perturbations here.

The most natural candidate for the resonance eigenfunction is Ψ_0. Thus the most studied object has been the amplitude of the survival probability of the unperturbed eigenfunction:

$$A^0(\varepsilon, t) := \langle \Psi_0, e^{-itH_\varepsilon} \Psi_0 \rangle. \tag{2.2}$$

Let us first discuss the trivial case when $E_0 = 0$ is isolated and lies in the resolvent set of $Q_0 H Q_0$. In this case the natural choice for E_ε in (1.4) is just the perturbed eigenvalue:

$$H_\varepsilon \Psi_\varepsilon = E_\varepsilon \Psi_\varepsilon, \ \ \|\Psi_\varepsilon\| = 1, \ \ \|\Psi_\varepsilon - \Psi_0\| \lesssim \varepsilon. \tag{2.3}$$

Using $(e^{\pm itH_\varepsilon} - e^{\pm itE_\varepsilon})\Psi_\varepsilon = 0$ we have:

$$A^0(\varepsilon, t) - e^{-itE_\varepsilon} = \langle (e^{-itH_\varepsilon} - e^{-itE_\varepsilon})(\Psi_0 - \Psi_\varepsilon), \Psi_0 - \Psi_\varepsilon \rangle, \tag{2.4}$$

which implies:

$$|A^0(\varepsilon, t) - e^{-itE_\varepsilon}| \leq 2\|\Psi_0 - \Psi_\varepsilon\|^2 \lesssim \varepsilon^2. \tag{2.5}$$

Due to the following result the estimate in (2.5) is optimal, if Ψ_0 is not replaced with a better choice:

Proposition 2.3. *Suppose that for sufficiently small ε there exists exactly one (possibly embedded) eigenvalue E_ε, while the singular continuous spectrum is empty. Assume that there exists $\Psi_1 \neq 0$ such that $\|\Psi_\varepsilon - \Psi_0 - \varepsilon\Psi_1\| = o(\varepsilon)$ and $\langle \Psi_0, \Psi_1\rangle = 0$. Then there exists $C > 0$ such that*

$$\sup_{t\geq 0} |A^0(\varepsilon, t) - e^{-itE_\varepsilon}| \geq C\varepsilon^2.$$

Note that the existence of a Ψ_1 in the above Proposition is guaranteed, if $E_0 = 0$ is isolated and Ψ_ε is obtained by applying to Ψ_0 the Sz.-Nagy unitary between the unperturbed projection P_0 and the perturbed one P_ε (see [23]). In the embedded case proving that the eigenvalue can survive is highly nontrivial [2]. In [11] a class of perturbations W is considered such that *if the eigenvalue survives* then the FGR constant must be zero and Ψ_1 can be constructed.

Under the assumptions of Proposition 2.3, if there is a vector $\Psi^N(\varepsilon)$ with $\|\Psi^N(\varepsilon)\| = 1$ and $\|\Psi^N(\varepsilon) - \Psi_\varepsilon\| = \mathcal{O}(\varepsilon^N)$, then the quantity

$$A^N(\varepsilon, t) := \langle \Psi^N(\varepsilon), e^{-itH_\varepsilon}\Psi^N(\varepsilon)\rangle \tag{2.6}$$

will obey

$$\sup_{t\geq 0} |A^N(\varepsilon, t) - e^{-itE_\varepsilon}| \leq C\varepsilon^{2N}.$$

Of course, when taking an eigenvector Ψ_ε as the initial value, the error term vanishes.

In the nontrivial cases, i.e. when either E_0 is embedded in the continuous spectrum of H, or W is singular with respect to H as in the Stark effect, the generic phenomenon is that E_0 is moved out of the real axis and becomes a resonance, E_ε, with $\operatorname{Im} E_\varepsilon < 0$. Again, one expects that Ψ_0 is a good candidate for the resonance eigenfunction i.e.

$$|\langle \Psi_0, e^{-itH_\varepsilon}\Psi_0\rangle - e^{-itE_\varepsilon}| \leq \delta(\varepsilon), \tag{2.7}$$

uniformly in time, with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$. Proving (2.7) is much harder; in particular, in the general case there is no obvious candidate for E_ε. The situation is fully understood in the dilation analytic case, where an analogue of the Kato-Rellich perturbation theory has been developed, see [37]. In particular, the resonance position, E_ε, is unambiguously defined as an eigenvalue of the dilated Hamiltonian. In the dilation analytic setting Hunziker [17] proved that (2.7) holds true with $\delta(\varepsilon) \sim \varepsilon^2$, i.e. the error term has the same size as in the isolated eigenvalue case.

For the smooth case, i.e. when the resolvent of Q_0HQ_0 has smooth limit values on the real axis in a neighborhood of 0, the situation is again satisfactory, as (2.7) with $\delta(\varepsilon) \sim \varepsilon^2$ was proved under fairly weak smoothness conditions (see Assumption 2.5 below).

As an example, we give Theorem 2.6 below. It is a slight improvement of Theorem 4.1ii in [19]. For related results giving the same size of the error term, see [5, 6].

We use a factored form of the perturbation W, defined as follows.

Assumption 2.4. Assume there exist a Hilbert space $\mathcal{K}$ and two bounded operators $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $D \in \mathcal{B}(\mathcal{K})$, such that D is a self-adjoint involution and such that

$$W = A^*DA \tag{2.8}$$

We note that this type of assumption is very flexible, since it allows us in the Schrödinger operator case to consider a W which is a sum of a multiplicative potential and a finite rank operator. Using finite rank operators it is easy to construct examples with eigenvalues embedded in the continuous spectrum. See [20] for some examples in the threshold case.

Define

$$G(z) = AQ_0(H-z)^{-1}Q_0A^*, \tag{2.9}$$

and

$$\begin{aligned} F^0(z,\varepsilon) = {} & \varepsilon\langle\Psi_0, W\Psi_0\rangle - z \\ & - \varepsilon^2\langle\Psi_0, A^*D\{G(z) - \varepsilon G(z)[D+\varepsilon G(z)]^{-1}G(z)\}DA\Psi_0\rangle. \end{aligned} \tag{2.10}$$

Notice that $F^0(z,\varepsilon) = \overline{F^0(\overline{z},\varepsilon)}$. Then using the Stone formula, the Schur-Livsic-Feshbach-Grushin (SLFG) formula, and the Kato-Rellich regular perturbation theory, one obtains the starting formula for the stationary approach to the decay law problem (see [19] for details and references):

$$A^0(\varepsilon,t) = \lim_{\eta\searrow 0}\frac{1}{2\pi i}\int_{\mathbf{R}} dx\, e^{-ixt}\Big(\frac{1}{F^0(x+i\eta,\varepsilon)} - \frac{1}{F^0(x-i\eta,\varepsilon)}\Big). \tag{2.11}$$

In justifying the r.h.s. of (2.10), and evaluating the r.h.s. of (2.11), it is important to ensure that $G(z)$ is uniformly bounded and smooth in the norm topology in the rectangle

$$D_a = \{z = x+i\eta \in \mathbf{C} \,|\, x\in J_a = (-a,a),\ 0<\eta<1\}. \tag{2.12}$$

Let $\omega\colon [0,\infty) \mapsto [0,\infty)$ be a modulus of continuity, i.e. continuous and increasing, with $\omega(0)=0$. Let $\omega_\theta(x) = x^\theta$, $\theta\in(0,1)$, denote the Hölder modulus of continuity. We denote by $\mathcal{C}^{n,\omega}(D_a;B)$ the set of all functions $F(\cdot,\varepsilon)$, parametrically depending on ε, defined on D_a with values in some fixed Banach space B, which are n times continuously differentiable, and whose n^{th} derivative satisfies uniformly on D_a

$$\sup_{0<\varepsilon\le\varepsilon_0}\|F^{(n)}(x+i\eta,\varepsilon) - F^{(n)}(y+i\eta,\varepsilon)\|_B \lesssim \omega(|x-y|). \tag{2.13}$$

Assumption 2.5. We have for $G(z)$ given by (2.9) that

$$G(\cdot)\in\mathcal{C}^{1,\omega}(D_a;\mathcal{B}(\mathcal{K})) \quad\text{and}\quad \int_0^1\frac{\omega(x)}{x}dx<\infty. \tag{2.14}$$

Theorem 2.6. *Under Assumption 2.5 and $0<\varepsilon<\varepsilon_0$ taken sufficiently small we have $F^0(\cdot,\varepsilon)\in C^{1,\omega}(D_a;\mathbf{C})$.*

In particular, this function has a restriction to the real axis with the same smoothness properties $F^0(x,\varepsilon) := \lim_{\eta\searrow 0}F^0(x+i\eta,\varepsilon)\in C^{1,\omega}(J_a;\mathbf{C})$.

Let $R^0(x,\varepsilon)$ and $I^0(x,\varepsilon)$ be the real and imaginary part of $F^0(x,\varepsilon)$, respectively,

$$F^0(x,\varepsilon) =: R^0(x,\varepsilon) + iI^0(x,\varepsilon).$$

For a fixed ε sufficiently small the equation

$$R^0(x,\varepsilon) = 0 \tag{2.15}$$

has a unique solution $x^0(\varepsilon)$ in $J_{a/2}$, which obeys the estimate $|x^0(\varepsilon)|\lesssim\varepsilon$. Define

$$E^0_\varepsilon := x^0(\varepsilon) + iI(x^0(\varepsilon),\varepsilon). \tag{2.16}$$

Then for sufficiently small ε we have:

$$|A^0(\varepsilon,t) - e^{-itE^0_\varepsilon}| \lesssim \varepsilon^2. \tag{2.17}$$

Remark 2.7. We would like to stress the following three facts:

(i) The estimate (2.17) has exactly the same form as (2.5) for the case of eigenvalues, with the perturbed eigenvalue replaced by the 'resonance position', E^0_ε. In particular, Proposition 2.3 shows that in general, the error in (2.17) cannot be made smaller.

(ii) The computation of $I(x^0(\varepsilon),\varepsilon)$, using $|x^0(\varepsilon)| \lesssim \varepsilon$, leads to $I(x^0(\varepsilon),\varepsilon) = -\varepsilon^2\Gamma_{FGR} + \mathcal{O}(\varepsilon^3)$, with

$$\Gamma_{FGR} := \pi\langle\Psi_0, W\delta(Q_0HQ_0)W\Psi_0\rangle, \tag{2.18}$$

which coincides with the result given by the Dirac computation. Notice that no condition $I(x^0(\varepsilon),\varepsilon) < 0$ is needed, in particular one can have $\Gamma_{FGR} = 0$.

(iii) In the analytic case, the resonance position is spectrally defined as a pole of the analytically continued resolvent and coincides with the zero $z_r = x_r + iy_r$ of the analytic continuation of $F^0(z,\varepsilon)$. In the smooth case its definition also involves Ψ_0 since it is given via the limit values of $F^0(z,\varepsilon)$. Comparing the decay law given by Theorem 2.6 with the one given by Hunziker in the analytic case, one can show that $|E^0_\varepsilon - z_r| \lesssim \varepsilon^2|y_r|$, i.e. up to some order in ε, E^0_ε is indeed a spectral object of the family H_ε, see [5] and [22].

We now turn to the main question considered in this paper: Can the error term can be made *smaller* by choosing a better ansatz for the resonance eigenfunction? The heuristics put forward above indicates that in the case of an isolated eigenvalue that a better ansatz should be related to the R-S expansion of the perturbed eigenfunction. In the dilation analytic, Hunziker proved that if E_0 is *isolated* (but becomes a resonance due to the singularity of the perturbation) and the formal R-S expansion for the perturbed eigenfunction is well defined up to order ε^N, then by using as a resonance function the normalized truncated R-S series, one can prove that both $\operatorname{Im} E_\varepsilon$ and the error term are of order ε^{2N+2}. In particular, for the atomic Stark effect N can be taken arbitrarily large. It follows that in this case we have $|\operatorname{Im} E_\varepsilon| \lesssim \varepsilon^p$ for any integer p, which is consistent with the known fact that the imaginary part of the resonance position is exponentially small. Furthermore, the error term can be made smaller than any power of ε.

The problem with the embedded eigenvalues (even in the analytic case) is that generically, the R-S expansion for the perturbed eigenfunction is already ill defined for $N = 1$. In order to be able to make a better ansatz, one needs to find conditions for the existence of the R-S expansion up to some order $N \geq 1$. Our first result states that under appropriate smoothness conditions the vanishing of the FRG constant $\Gamma_{FGR} = \pi\langle\Psi_0, W\delta(Q_0HQ_0)W\Psi_0\rangle = 0$ insures that the R-S expansion for the perturbed eigenfunction is well defined for $N = 1$, see Proposition 2.11.

The main result of this paper is a generalization to embedded eigenvalues and smooth setting of Hunziker's results for $N = 1$, see Theorem 2.15.

Now we state the smoothness properties we need in order to introduce the Kuroda Γ operator [26]. Keep in mind that only the smoothness in a neighborhood of the origin matters.

Let $P(\Delta)$ be the spectral measure of H, and $Q(J_a) := P(J_a) - P_0$; in particular, $Q_0 = Q(J_a) + P(\mathbf{R} \setminus J_a)$. We can write the decomposition:

$$\mathcal{H} = P_0\mathcal{H} \oplus Q(J_a)\mathcal{H} \oplus P(\mathbf{R} \setminus J_a)\mathcal{H} =: P_0\mathcal{H} \oplus \mathcal{H}_< \oplus \mathcal{H}_>.$$

In order to keep the notation and technicalities at a reasonable level, we supplement Assumption 2.1 by:

Assumption 2.8. The multiplicity of the absolutely continuous spectrum of H in J_a is constant.

This assumption means that there exists a Hilbert space h and a unitary map

$$\widetilde{\Gamma}\colon Q(J_a)\mathcal{H} \to L^2(J_a, \mathsf{h}), \tag{2.19}$$

such that

$$(\widetilde{\Gamma} H \widetilde{\Gamma}^* \psi)(\lambda) = \lambda\psi(\lambda), \quad \psi(\lambda) \in \mathsf{h}, \quad \lambda \in J_a. \tag{2.20}$$

We will always see $\widetilde{\Gamma}$ as a partial isometry extended by zero outside the range of $Q(J_a)$. In all the cases we consider, $\widetilde{\Gamma}$ is constructed in the following way. We assume that there exists a dense subset $\mathcal{D} \subset \mathcal{H}$ and a family of operators

$$\Gamma(\lambda) : \mathcal{D} \mapsto \mathsf{h}, \quad \lambda \in J_a,$$

such that if $f \in \mathcal{D}$ then the map

$$J_a \ni \lambda \mapsto \Gamma(\lambda) f \in \mathsf{h}$$

is continuous. For the free Laplacian, $\Gamma(\lambda)$ is constructed with the help of the Fourier transform (see (B.3)). For Schrödinger operators with short range potentials, $\Gamma(\lambda)$ is closely related to the generalized eigenfunctions constructed through a Lippmann-Schwinger type argument starting from the free plane-waves (see (B.8)).

Then for every function $\phi \in C_0^\infty(J_a)$ and for every $f \in \mathcal{D}$ we have:

$$\langle f, \phi(H) f\rangle_{\mathcal{H}} = \phi(0)|\langle \Psi_0, f\rangle|^2 + \int_{J_a} \phi(\lambda)\langle \Gamma(\lambda) f, \Gamma(\lambda) f\rangle_{\mathsf{h}} d\lambda. \tag{2.21}$$

By a standard limiting argument this implies that for every $J \subseteq J_a$ and for every $f \in \mathcal{D}$ we have

$$||Q(J)f||^2 = \langle f, Q(J) f\rangle = \int_J \langle \Gamma(\lambda) f, \Gamma(\lambda) f\rangle_{\mathsf{h}} d\lambda. \tag{2.22}$$

The next step is to define

$$J_a \ni \lambda \mapsto (\widetilde{\Gamma} f)(\lambda) := \Gamma(\lambda) f \in \mathsf{h}, \quad f \in \mathcal{D}, \tag{2.23}$$

which due to (2.22) can be extended by continuity to a partial isometry between $\mathcal{H}$ and $L^2(J_a, \mathsf{h})$ such that

$$\int_{J_a} ||(\widetilde{\Gamma}\psi)(\lambda)||_{\mathsf{h}}^2 d\lambda = ||Q(J_a)\psi||_{\mathcal{H}}^2.$$

From now on we will make the following assumption.

Assumption 2.9. We have that $A^*\mathcal{K} \subseteq \mathcal{D}$ and furthermore:

$$\Gamma(\cdot)A^* \in \mathcal{C}^{2,\omega_\theta}(J_a; \mathcal{B}(\mathcal{K}, \mathsf{h})). \tag{2.24}$$

For $\operatorname{Im} z \neq 0$ we define

$$\begin{aligned} Q_0(H-z)^{-1}Q_0 &= Q(J_a)(H-z)^{-1}Q(J_a) + P(\mathbf{R}\setminus J_a)(H-z)^{-1}P(\mathbf{R}\setminus J_a) \\ &=: S_<(z) + S_>(z) =: S(z), \end{aligned} \tag{2.25}$$

where these operators act on $\mathcal{H}$. Notice that $S_>(z)$ is analytic in $|z| < a$. For every $g \in \mathcal{H}$ we have

$$S_<(z)g = \widetilde{\Gamma}^* \Big(\frac{1}{\cdot - z} (\widetilde{\Gamma} g)(\cdot) \Big), \quad ||S_<(z)g||^2 = \int_{J_a} \frac{1}{|\lambda - z|^2} \|(\widetilde{\Gamma} g)(\lambda)\|_{\mathsf{h}}^2 d\lambda. \tag{2.26}$$

The operator

$$S_< := Q(J_a) H^{-1} Q(J_a), \tag{2.27}$$

is well defined on the domain:

$$\mathcal{D}(S_<) := \big\{ g \in \mathcal{H} : \int_{J_a} \frac{\|(\widetilde{\Gamma} g)(\lambda)\|_{\mathsf{h}}^2}{\lambda^2} d\lambda < \infty \big\}. \tag{2.28}$$

The operator $S_<$ is self-adjoint and unbounded because 0 belongs to the continuous spectrum of H. For any $g \in \mathcal{D}(S_<)$ we have:

$$S_< g = \widetilde{\Gamma}^* \Big(\frac{1}{\cdot} (\widetilde{\Gamma} g)(\cdot) \Big). \tag{2.29}$$

We now define the operator:

$$S := S_< + S_>(0), \tag{2.30}$$

which is self-adjoint on $\mathcal{D}(S_<)$.

Due to (2.8) and Assumption 2.9 we know that $\Gamma(\cdot) W P_0 \in \mathcal{C}^{2,\omega_\theta}(J_a; \mathcal{B}(\mathcal{H}, \mathsf{h}))$ for some $\theta \in (0,1)$. Thus the Fermi Golden Rule constant (see(2.18)) reads as $\Gamma_{FGR} = \pi \|\Gamma(0) W \Psi_0\|_{\mathsf{h}}^2$, hence the assumption that $\Gamma_{FGR} = 0$ is equivalent with:

Assumption 2.10. We have $\Gamma(0) W P_0 = 0$ as an operator from $\mathcal{H}$ to h.

Excluding the trivial case when E_0 is isolated, the operator S is self-adjoint but unbounded, i.e. SWP_0 is not bounded in general, and this is the reason for the breakdown of the R-S expansion in the embedded case. Our first simple but important result says that SWP_0 remains bounded if Assumptions 2.9 and 2.10 hold true.

Proposition 2.11. *Suppose Assumptions 2.9 holds true. Then $WP_0\mathcal{H} \subseteq \mathcal{D}(S_<)$ if and only if Assumption 2.10 holds true. In particular if Assumptions 2.9 and 2.10 hold true, then SWP_0 is bounded and its adjoint is the extension by continuity of $P_0 WS$.*

Now consider:

$$T_1 := -SWP_0 - P_0 WS, \tag{2.31}$$

which in regular perturbation theory gives the first order correction in the expansion of the perturbed eigenprojection, and

$$T_\varepsilon := P_0 + \varepsilon T_1. \tag{2.32}$$

By a simple computation we obtain:

$$T_\varepsilon^2 - T_\varepsilon = \varepsilon^2 T_1^2, \tag{2.33}$$

thus T_ε is an 'almost orthogonal projection'. There exists a whole family of orthogonal projections which are in norm close to T_ε up to errors of order ε^2. The following one is distinguished by the fact that it is given by the following algebraic formula, assuming $\|T_\varepsilon^2 - T_\varepsilon\| < \frac{1}{4}$, see [32] and [29]:

$$P_\varepsilon = T_\varepsilon + (T_\varepsilon - 1/2)[(1 + 4(T_\varepsilon^2 - T_\varepsilon))^{-\frac{1}{2}} - 1]. \tag{2.34}$$

Notice that $P_\varepsilon - P_0 = \mathcal{O}(\varepsilon)$, $P_\varepsilon - T_\varepsilon = \mathcal{O}(\varepsilon^2)$. Let now U_ε be the Sz.-Nagy unitary intertwining P_ε and P_0:

$$U_\varepsilon = \frac{1}{(1-(P_\varepsilon - P_0)^2)^{\frac{1}{2}}}(P_\varepsilon P_0 + (1-P_\varepsilon)(1-P_0)), \quad P_\varepsilon = U_\varepsilon P_0 U_\varepsilon^*. \tag{2.35}$$

To establish the improved exponential decay law we take as our ansatz for the initial state:

$$\Psi_\varepsilon^1 := U_\varepsilon \Psi_0 = \Psi_0 - \varepsilon SW\Psi_0 + \mathcal{O}(\varepsilon^2). \tag{2.36}$$

In other words, we now have to estimate:

$$A^1(\varepsilon, t) := \langle \Psi^1(\varepsilon), e^{-itH_\varepsilon}\Psi^1(\varepsilon)\rangle. \tag{2.37}$$

A remark is in order here. One can equally well use the following (simpler at first sight) ansatz: $\widetilde{\Psi}_\varepsilon^1 = T_\varepsilon\Psi_0/\|T_\varepsilon\Psi_0\|$. The reasons for choosing (2.36) are that the proofs are somewhat simpler, and more importantly, the procedure extends unchanged to the degenerate case.

The aim in what follows is to find E_ε^1 such that:

$$|A^1(\varepsilon,t) - e^{-itE_\varepsilon^1}| \lesssim \varepsilon^4. \tag{2.38}$$

Using the Stone and SLFG formulae one obtains as in [19]:

$$A^1(\varepsilon,t) = \lim_{\eta\searrow 0}\frac{1}{2\pi i}\int_{\mathbf{R}} dx\, e^{-ixt}\Big(\frac{1}{F^1(x+i\eta,\varepsilon)} - \frac{1}{F^1(x-i\eta,\varepsilon)}\Big), \tag{2.39}$$

where

$$F^1(z,\varepsilon) := \langle \Psi_\varepsilon^1, H_\varepsilon\Psi_\varepsilon^1\rangle - z - \langle \Psi_\varepsilon^1, P_\varepsilon H_\varepsilon Q_\varepsilon (Q_\varepsilon H_\varepsilon Q_\varepsilon - z)^{-1} Q_\varepsilon H_\varepsilon P_\varepsilon \Psi_\varepsilon^1\rangle$$

and $Q_\varepsilon = 1 - P_\varepsilon$. In order to follow the line of the proof of Theorem 2.6 in [19], it is convenient to use (2.35), (2.36) and rewrite (2.40) as

$$F^1(z,\varepsilon) = \varepsilon\langle \Psi_0, \widetilde{H}_\varepsilon\Psi_0\rangle - z - \langle \Psi_0, P_0\widetilde{H}_\varepsilon Q_0(Q_0\widetilde{H}_\varepsilon Q_0 - z)^{-1}Q_0\widetilde{H}_\varepsilon P_0\Psi_0\rangle, \tag{2.40}$$

where

$$\widetilde{H}_\varepsilon = U_\varepsilon^* H_\varepsilon U_\varepsilon. \tag{2.41}$$

Using the notation

$$\widetilde{W}_\varepsilon := \frac{1}{\varepsilon}(\widetilde{H}_\varepsilon - H), \tag{2.42}$$

the formula (2.40) becomes

$$F^1(z,\varepsilon) = \varepsilon\langle \Psi_0, \widetilde{W}_\varepsilon\Psi_0\rangle - z - \varepsilon^2\langle \Psi_0, P_0\widetilde{W}_\varepsilon Q_0(Q_0(H+\widetilde{W}_\varepsilon)Q_0 - z)^{-1}Q_0\widetilde{W}_\varepsilon P_0\Psi_0\rangle. \tag{2.43}$$

The following lemma lists some of the important structural properties of $\widetilde{W}_\varepsilon$.

Lemma 2.12. *The following results hold for $\widetilde{W}_\varepsilon$.*

(i)

$$\begin{aligned}\widetilde{W}_\varepsilon =& (W - P_0WQ_0 - Q_0WP_0 + \varepsilon\big(-P_0WSW - WSWP_0 + SWP_0W + WP_0WS \\ &+ P_0WSWP_0 - \frac{1}{2}Q_0WP_0WS - \frac{1}{2}SWP_0WQ_0\big) + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{2.44}$$

In particular

$$D_\varepsilon := \frac{1}{\varepsilon}P_0\widetilde{W}_\varepsilon Q_0 = -P_0WSWQ_0 - P_0WP_0WS + \mathcal{O}(\varepsilon) \tag{2.45}$$

is uniformly bounded as $\varepsilon \to 0$.

(ii) *There exist some operators $V_{jk}(\varepsilon)$, $1 \le j,k \le 4$ which are uniformly bounded as $\varepsilon \to 0$, such that if $\mathcal{V}(\varepsilon) := \big[V_{jk}(\varepsilon)\big]$ denotes the obvious operator valued matrix, we have:*

$$\widetilde{W}_\varepsilon = \begin{bmatrix} SWP_0 & Q_0A^* & P_0WS & P_0 \end{bmatrix} \mathcal{V}(\varepsilon) \begin{bmatrix} P_0WS \\ AQ_0 \\ SWP_0 \\ P_0 \end{bmatrix}. \tag{2.46}$$

If we denote by $X\colon \mathcal{H} \to \mathcal{H} \oplus \mathcal{K} \oplus \mathcal{H} \oplus \mathcal{H}$ the map

$$X(f) := \begin{bmatrix} P_0WSf \\ AQ_0f \\ SWP_0f \\ P_0f \end{bmatrix},$$

then we can write:

$$\widetilde{W}_\varepsilon = X^* \mathcal{V}(\varepsilon) X. \tag{2.47}$$

Using (2.45), we can express $F^1(z,\varepsilon)$ as:

$$F^1(z,\varepsilon) = \varepsilon\langle \Psi_0, \widetilde{W}_\varepsilon \Psi_0\rangle - z - \varepsilon^4 \langle \Psi_0, P_0 D_\varepsilon Q_0 (Q_0(H + \widetilde{W}_\varepsilon)Q_0 - z)^{-1} Q_0 D_\varepsilon^* P_0 \Psi_0\rangle. \tag{2.48}$$

The next Lemma, whose proof is based on a standard perturbation theory argument, is a direct consequence of Lemma 2.12 and reduces the smoothness problem of the r.h.s. of (2.48) to the smoothness of four ε-independent expressions. In this Lemma and the next $\mathcal{B}$ is a shorthand for the four spaces $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{K})$, $\mathcal{B}(\mathcal{H},\mathcal{K})$, and $\mathcal{B}(\mathcal{K},\mathcal{H})$, as appropriate.

Lemma 2.13. *Suppose that the following four families of operators*

$$P_0WSS(z)SWP_0,\quad AQ_0S(z)SWP_0,\quad P_0WSS(z)Q_0A^*,\quad AQ_0S(z)Q_0A^*, \tag{2.49}$$

belong to $\mathcal{C}^{1,\omega}(D_{a/2};\mathcal{B})$. Then:

$$\langle \Psi_0, P_0 D_\varepsilon Q_0 (Q_0(H + \varepsilon\widetilde{W}_\varepsilon)Q_0 - z)^{-1} Q_0 D_\varepsilon^* P_0 \Psi_0\rangle \in \mathcal{C}^{1,\omega}(D_{a/2};\mathbf{C}). \tag{2.50}$$

The next lemma shows that the hypothesis of Lemma 2.13 are indeed satisfied in the case when $\omega = \omega_\theta$:

Lemma 2.14. *Suppose Assumptions 2.9 and 2.10 hold true. Then the four families of Lemma 2.13 belong to $\mathcal{C}^{1,\omega_\theta}(D_{a/2};\mathcal{B})$.*

A consequence of Lemmas 2.13 and 2.14 is that the map $F^1(\cdot,\varepsilon)$ lies in $\mathcal{C}^{1,\omega_\theta}(D_{a/2})$. Now we can formulate the main result of the paper, showing that the replacement of Ψ_0 with Ψ_ε^1 leads to an improvement of the error term in the exponential decay law, namely the error term is of order at most ε^4.

Theorem 2.15. *Suppose Assumptions 2.9 and 2.10 hold true. Then for sufficiently small ε we have*

$$F^1(\cdot,\varepsilon) \in C^{1,\omega_\theta}(D_{a/2};\mathbf{C}).$$

In particular, it has well defined limit values

$$F^1(x,\varepsilon) := \lim_{\eta\searrow 0} F^1(x+i\eta,\varepsilon) \in C^{1,\omega_\theta}(J_{a/2};\mathbf{C}).$$

Let $R^1(x,\varepsilon)$ and $I^1(x,\varepsilon)$ be the real and imaginary part of $F^1(x,\varepsilon)$, respectively:

$$F^1(x,\varepsilon) =: R^1(x,\varepsilon) + iI^1(x,\varepsilon). \tag{2.51}$$

For a fixed ε the equation

$$R^1(x,\varepsilon) = 0 \tag{2.52}$$

has a unique solution $x^1(\varepsilon)$ in $J_{a/2}$, with $|x^1(\varepsilon)| \lesssim \varepsilon$. Define

$$E^1_\varepsilon := x^1(\varepsilon) + iI(x^1(\varepsilon),\varepsilon). \tag{2.53}$$

Then for sufficiently small ε we have

$$|A^1(\varepsilon,t) - e^{-itE^1_\varepsilon}| \lesssim \varepsilon^4. \tag{2.54}$$

The proofs of both theorems heavily rely on a careful estimate of the integrals in the r.h.s. of (2.39) and (2.11), respectively. The following technical lemma provides an abstract setting which can be used directly in both theorems (note that $\int_0^1 \frac{\omega_\theta}{x}dx < \infty$ for all $\theta > 0$).

Lemma 2.16. *Consider the function*

$$F(z,\varepsilon) = a(\varepsilon) - z - \gamma(\varepsilon)f(z,\varepsilon), \tag{2.55}$$

for which the following five conditions hold true:

(i) *$f(z,\varepsilon)$ is an analytic function on $\{z \in \mathbf{C} : \operatorname{Im} z > 0\}$, and $\overline{f(\overline{z},\varepsilon)} = f(z,\varepsilon)$;*

(ii) *Let $a(\varepsilon)$ be real valued, $\gamma(\varepsilon) > 0$ and $\lim_{\varepsilon\to 0}(|a(\varepsilon)| + \gamma(\varepsilon)) = 0$;*

(iii) *$\operatorname{Im} f(z,\varepsilon) \geq 0$ for $\operatorname{Im} z > 0$;*

(iv) *$\lim_{\eta\searrow 0} \frac{1}{\pi}\int_{\mathbf{R}} \frac{\operatorname{Im} F(x+i\eta,\varepsilon)}{|F(x+i\eta,\varepsilon)|^2}dx = 1$;*

(v) *$f(\cdot,\varepsilon) \in \mathcal{C}^{1,\omega}(D_b;\mathbf{C})$ for some $b > 0$ and $\int_0^1 \frac{\omega(x)}{x}dx < \infty$.*

Let $R(x,\varepsilon)$, $I(x,\varepsilon)$ be the real and imaginary part of $\lim_{\eta\searrow 0} F(x+i\eta,\varepsilon)$ for $x \in J_a$. Then the following statements hold true:

(a) *For a sufficiently small ε, the equation*

$$R(x,\varepsilon) = 0 \tag{2.56}$$

has a unique solution $x(\varepsilon) \in J_b$, obeying:

$$|x(\varepsilon)| \lesssim |a(\varepsilon)| + \gamma(\varepsilon). \tag{2.57}$$

(b) *If*

$$E_\varepsilon := x(\varepsilon) + iI(x(\varepsilon),\varepsilon), \tag{2.58}$$

then for sufficiently small ε we have:

$$\limsup_{\eta\searrow 0}\Big|\frac{1}{2\pi i}\int_{\mathbf{R}} e^{-ixt}\Big(\frac{1}{F(x+i\eta,\varepsilon)} - \frac{1}{F(x-i\eta,\varepsilon)}\Big)dx - e^{-itE_\varepsilon}\Big| \lesssim \gamma(\varepsilon). \tag{2.59}$$

In both Theorem 2.6 and Theorem 2.15 two functions $F^0(z,\varepsilon)$ and $F^1(z,\varepsilon)$ appear, and the above lemma has to be applied to each of them.

It is important to remember that up to an application of the SLFG formula, we have that $1/F^0(z,\varepsilon) = \langle \Psi_0, (H+\varepsilon W - z)^{-1}\Psi_0\rangle$ and $1/F^1(z,\varepsilon) = \langle \Psi_0, (H+\varepsilon \widetilde{W}_\varepsilon - z)^{-1}\Psi_0\rangle$. In this case, the limit $\lim_{\eta\searrow 0}\frac{1}{2\pi i}\int_{\mathbf{R}} e^{-ixt}(\frac{1}{F(x+i\eta,\varepsilon)} - \frac{1}{F(x-i\eta,\varepsilon)})dx$ exists due to the Stone formula (see (2.11) and (2.39)). Moreover, the same Stone formula shows that the condition (iv) in the lemma is nothing but the fact that when $t=0$, the evolution group equals the identity, hence $A^0(\varepsilon,0) = A^1(\varepsilon,0) = 1$.

We will see that in the case of Theorem 2.6 we have $\gamma(\varepsilon) = \varepsilon^2$, while in the case of Theorem 2.15 we have $\gamma(\varepsilon) = \varepsilon^4$. An outline of the proof of Lemma 2.16 for the case $\omega(x) = x^\theta$ has been given in [19]. For completeness, we shall provide the proof in the next section.

There are many open questions in this area and we close this section by mentioning two of them.

(i) The condition $\Gamma_{FGR} = 0$ is crucial in proving Theorem 2.15. A natural question is whether in case $\Gamma_{FGR} \neq 0$ one can still find a better resonance function than Ψ_0 (e.g. by truncating the corresponding Gamov vector [39]) leading to an error term of order ε^q, $q > 2$. We believe that this is not possible.

(ii) Theorem 2.15 gives $\operatorname{Im} E^1_\varepsilon = -\varepsilon^4\Gamma^1_{FGR} + \mathcal{O}(\varepsilon^5)$ and moreover, Γ^1_{FGR} can be computed explicitly. At the formal level, one can show that via the result in [16], $\Gamma_{FGR} = \Gamma^1_{FGR} = 0$ insures that the R-S expansion is well defined up to $N=2$ so that the procedure can be iterated, leading to an even smaller error term. Unfortunately, the proof of the needed smoothness properties becomes prohibitively complex.

3 Proofs

3.1 Proof of Proposition 2.3

Let P_ε be the orthogonal projection corresponding to Ψ_ε. Since $\langle\Psi_0,\Psi_1\rangle = 0$, we have $P_0\Psi_1 = 0$ and $Q_0\Psi_1 = \Psi_1$.

From (2.4) we get:

$$A^0(\varepsilon,t) - e^{-itE_\varepsilon} = \varepsilon^2\langle\Psi_1,(e^{-itH_\varepsilon} - e^{-itE_\varepsilon})\Psi_1\rangle + o(\varepsilon^2).$$

Let $Q_\varepsilon = 1 - P_\varepsilon$. We have that $\|P_\varepsilon - P_0\| = \mathcal{O}(\varepsilon)$, $\|P_\varepsilon Q_0\| = \mathcal{O}(\varepsilon)$ and $\|P_\varepsilon\Psi_1\| = \mathcal{O}(\varepsilon)$. Then

$$\langle\Psi_1,(e^{-itH_\varepsilon} - e^{-itE_\varepsilon})\Psi_1\rangle = \langle Q_\varepsilon\Psi_1, e^{-itH_\varepsilon}Q_\varepsilon\Psi_1\rangle - e^{-itE_\varepsilon}\|\Psi_1\|^2 + \mathcal{O}(\varepsilon).$$

Now, as $Q_\varepsilon\mathcal{H}$ maps into the subspace of absolute continuity for H_ε the first term in the r.h.s. vanishes in the limit $t\to\infty$. Summing up:

$$\liminf_{t\to\infty}|\langle\Psi_1,(e^{-itH_\varepsilon} - e^{-itE_\varepsilon})\Psi_1\rangle| = \|\Psi_1\|^2 + \mathcal{O}(\varepsilon)$$

and the proof is finished as $\|\Psi_1\| \neq 0$.

3.2 Proof of Proposition 2.11

If $g \in \mathcal{H}$ with $\|g\| = 1$, then $h(\lambda) := (\widetilde{\Gamma} W P_0 g)(\lambda) \in \mathsf{h}$. We see that $h(\lambda) = \Gamma(\lambda) W P_0 g$ because of (2.23) and Assumption 2.9. In addition

$$h(0) = \Gamma(0)W\Psi_0\langle\Psi_0, g\rangle, \quad \|h(\lambda) - h(0)\|_{\mathsf{h}} \le C|\lambda|, \quad \lambda \in J_a, \tag{3.1}$$

where $C < \infty$ is a constant which is independent of g. From (2.28) and (2.29) we see that WP_0g belongs to the domain of $S_<$ if and only if

$$\int_{J_a} \frac{\|h(\lambda)\|_{\mathfrak{h}}^2}{\lambda^2} d\lambda < \infty.$$

From (3.1) it follows that if Assumption 2.10 also holds true, then $h(0) = 0$ and $S_< WP_0$ is bounded. If Assumption 2.10 does not hold true, then $W\Psi_0$ does not belong to the domain of $S_<$.

Now let us assume that both Assumptions 2.9 and 2.10 hold. Since $S_> WP_0$ is also bounded, it follows that SWP_0 is bounded. On the other hand, for $f \in \mathcal{D}(S_<)$, we have by duality

$$\|P_0 W S_< f\| = \sup_{\|g\|=1} |\langle S_< WP_0 g, f\rangle| \le (\sup_{\|g\|=1} \|S_< WP_0 g\|)\|f\| \le \|S_< WP_0\|\|f\|,$$

and the proof is finished.

3.3 Proof of Lemma 2.12

The proof requires tedious computations using (2.41), (2.42), (2.34), and (2.35). We have to prove that $\widetilde{W}_\varepsilon$ is uniformly bounded as $\varepsilon \to 0$, even though H is not supposed to be bounded. We will show that $\widetilde{W}_\varepsilon$ has a norm expansion in ε and compute this expansion up to errors of order $\mathcal{O}(\varepsilon^3)$.

To proceed we need to expand the term

$$\Delta_\varepsilon := P_\varepsilon - P_0. \tag{3.2}$$

From now on we shall take ε sufficiently small such that Proposition 2.11, (2.31), (2.32), and (2.33) imply

$$\|T_\varepsilon^2 - T_\varepsilon\| = \varepsilon^2 \|T_1\|^2 < \tfrac{1}{4}, \tag{3.3}$$

and then one can expand in powers of $T_\varepsilon^2 - T_\varepsilon$ in the r.h.s. of (2.34) and obtain

$$\Delta_\varepsilon = \varepsilon T_1 + \varepsilon^2 \big(1 - 2P_0\big) T_1^2 + \varepsilon^3 \big(P_0 E_0(\varepsilon) + T_1 E_1(\varepsilon)\big) T_1 \tag{3.4}$$

with $E_j(\varepsilon)$ uniformly bounded as $\varepsilon \to 0$.

The following identities follow from the fact that P_ε and P_0 are projections.

$$[\Delta_\varepsilon^2, P_\varepsilon] = [\Delta_\varepsilon^2, P_0] = 0, \tag{3.5}$$

$$P_\varepsilon P_0 + (1 - P_\varepsilon)(1 - P_0) = 1 - (2P_\varepsilon - 1)\Delta_\varepsilon = 1 + \Delta_\varepsilon(2P_0 - 1). \tag{3.6}$$

For ε sufficiently small such that in addition to (3.3) we have $\|\Delta_\varepsilon\| < 1$. Then

$$(1 - \Delta_\varepsilon^2)^{-\frac{1}{2}} =: 1 + \Delta_\varepsilon^2 N_\varepsilon =: 1 + \tfrac{1}{2}\Delta_\varepsilon^2 + \Delta_\varepsilon^2 \widetilde{N}_\varepsilon \Delta_\varepsilon^2, \tag{3.7}$$

with N_ε, $\widetilde{N}_\varepsilon$ uniformly bounded as $\varepsilon \to 0$, and commuting with both P_0 and P_ε. Inserting (3.2) in (2.35) and using (3.7), (3.5), (3.6) one obtains

$$U_\varepsilon = 1 + \tfrac{1}{2}\Delta_\varepsilon^2 + \Delta_\varepsilon(2P_0 - 1) + \Delta_\varepsilon^2 \widetilde{N}_\varepsilon \Delta_\varepsilon^2 - \Delta_\varepsilon^2 N_\varepsilon (2P_\varepsilon - 1)\Delta_\varepsilon. \tag{3.8}$$

Then we write

$$B_\varepsilon := U_\varepsilon - 1 =: \Delta_\varepsilon(2P_0 - 1) + \tfrac{1}{2}\Delta_\varepsilon^2 + \Delta_\varepsilon^2 M_\varepsilon \Delta_\varepsilon \tag{3.9}$$

with M_ε uniformly bounded as $\varepsilon \to 0$.

By direct computation we obtain (see (2.41) and (2.42)):

$$\widetilde{W}_\varepsilon = B_\varepsilon^* H + H B_\varepsilon + B_\varepsilon^* H B_\varepsilon + \varepsilon\big(W + B_\varepsilon^* W + W B_\varepsilon + B_\varepsilon^* W B_\varepsilon\big). \tag{3.10}$$

To proceed further, we list the following identities which follow from $E_0 = 0$ and (2.31).

$$HP_0 = P_0 H = 0, \quad HT_1 = -Q_0 W P_0, \quad T_1 Q_0 = P_0 T_1 = -P_0 W S. \tag{3.11}$$

From (3.4) and (3.11) we get

$$H\Delta_\varepsilon = HP_\varepsilon = -\varepsilon Q_0 W P_0 + \varepsilon^2 Q_0 W P_0 W S - \varepsilon^3 Q_0 W P_0 E_1(\varepsilon) T_1, \tag{3.12}$$

$$\Delta_\varepsilon Q_0 = -\varepsilon P_0 W S + \varepsilon^2\big(1 - 2P_0\big) T_1^2 Q_0 - \varepsilon^3\big(P_0 E_0(\varepsilon) + T_1 E_1(\varepsilon)\big) P_0 W S. \tag{3.13}$$

From this point onwards the proof of Lemma 2.12 is a somewhat long but straightforward computation using (3.11). Consider for example the term HB_ε. From (3.9) and (3.12) we get

$$\begin{aligned} HB_\varepsilon = \big(&-\varepsilon Q_0 W P_0 + \varepsilon^2 Q_0 W P_0 W S - \varepsilon^3 Q_0 W P_0 E_1(\varepsilon) T_1\big) \\ &\cdot \big(2P_0 - 1 + \tfrac{1}{2}\Delta_\varepsilon + \Delta_\varepsilon M_\varepsilon \Delta_\varepsilon\big), \end{aligned} \tag{3.14}$$

and using (3.4) one can see by inspection that it has the structure in (2.46) and also compute explicitly its expansion up to terms of order $\mathcal{O}(\varepsilon^2)$.

3.4 Proof of Lemma 2.13

Take $\operatorname{Im} z > 0$. We want to write $\big(Q_0 \widetilde{H}_\varepsilon Q_0 - zQ_0\big)^{-1}$ in the Hilbert space $Q_0\mathcal{H}$ in a different way. Using (2.47), by a standard re-summation argument we obtain

$$\begin{aligned} \big(Q_0 \widetilde{H}_\varepsilon Q_0 - zQ_0\big)^{-1} &= \big(Q_0 H Q_0 - zQ_0\big)^{-1} - \varepsilon\big(Q_0 H Q_0 - zQ_0\big)^{-1} Q_0 X^* \\ &\cdot \big\{1 + \varepsilon \mathcal{V}_\varepsilon X Q_0 (Q_0 H Q_0 - zQ_0)^{-1} Q_0 X^*\big\}^{-1} \mathcal{V}_\varepsilon X Q_0 \big(Q_0 H Q_0 - zQ_0\big)^{-1}. \end{aligned} \tag{3.15}$$

Note that both X^* and X contain localizing factors. If $z \in D_a$ and ε is small enough, then due to our hypothesis on the four families of operators we have the uniform bound

$$\varepsilon \| \mathcal{V}_\varepsilon X Q_0 \big(Q_0 H Q_0 - zQ_0\big)^{-1} Q_0 X^* \| \le \tfrac{1}{2},$$

hence the representation (3.15) makes sense in D_a up to the real line. Now by standard resolvent identities we can transfer the smoothness of $XQ_0(Q_0HQ_0 - zQ_0)^{-1}Q_0X^*$ to the inverse

$$\big\{1 + \varepsilon \mathcal{V}_\varepsilon X Q_0 (Q_0 H Q_0 - zQ_0)^{-1} Q_0 X^*\big\}^{-1}.$$

Also, from (3.12), (3.10), (3.9), and (3.4) we conclude that $P_0 D_\varepsilon Q_0$ contains localizing factors, uniformly in ε. Using again (3.15), we obtain:

$$\langle \Psi_0, P_0 D_\varepsilon Q_0 \big(Q_0(H + \widetilde{W}_\varepsilon) Q_0 - zQ_0\big)^{-1} Q_0 D_\varepsilon^* P_0 \Psi_0 \rangle \in C^{1,\omega_\theta}(D_{a/2}; \mathbf{C}).$$

3.5 Proof of Lemma 2.14

It is sufficient to consider the operators in (2.49) with S and $S(z)$ replaced by $S_<$ and $S_<(z)$, respectively. We only consider $P_0WS_<S_<(z)Q_0A^*$, the others can be treated similarly. From (2.26), (2.29), (2.23) and the fact that A^* maps $\mathcal{K}$ into $\mathcal{D}$ we have:

$$\langle f, P_0WS_<S_<(z)Q_0A^*g\rangle = \int_{J_a} \frac{1}{\lambda(\lambda - z)} \langle \Gamma(\lambda)WP_0f, \Gamma(\lambda)A^*g\rangle_{\mathsf{h}} d\lambda.$$

Let us consider (we also use Assumption 2.10):

$$\begin{aligned}\Phi(\lambda) &= \frac{1}{\lambda}\big\langle [\Gamma(\lambda) - \Gamma(0)]WP_0f, \Gamma(\lambda)A^*g\big\rangle_{\mathsf{h}} \\ &= \int_0^1 \langle [\Gamma'(u\lambda)WP_0]f, \Gamma(\lambda)A^*g\rangle_{\mathsf{h}}\, du.\end{aligned}$$

Thus there exists a constant $C > 0$ such that

$$\max\{|\Phi(\lambda_1) - \Phi(\lambda_2)|, |\Phi'(\lambda_1) - \Phi'(\lambda_2)|\} \leq C\|f\|\|g\|\, \omega_\theta(|\lambda_1 - \lambda_2|), \tag{3.16}$$

for all $\lambda_1, \lambda_2 \in J_a$. Let $\chi \in C^\infty(J_a)$, $0 \leq \chi(\lambda) \leq 1$, $\chi(\lambda) = 1$ for $|\lambda| < \frac{3}{4}a$, $\chi(\lambda) = 0$ for $|\lambda| > \frac{7}{8}a$, and write

$$\begin{aligned}\langle f, P_0WS_<S_<(z)Q_0A^*g\rangle &= \int_{J_a} \frac{\Phi(\lambda)}{\lambda - z} d\lambda \\ &= \int_{J_a} (1 - \chi(\lambda))\frac{\Phi(\lambda)}{\lambda - z} d\lambda + \int_{J_a} \frac{\Phi(\lambda)\chi(\lambda)}{\lambda - z} d\lambda.\end{aligned} \tag{3.17}$$

The first term in the r.h.s. of (3.17) is analytic in $|z| < \frac{3}{4}a$, while for the second one we use that the Cauchy integral transform a compactly supported function from $C^{n,\omega_\theta}(J_a; \mathbf{C})$ is continuously mapped to $C^{n,\omega_\theta}(D_{a/2}; \mathbf{C})$ (see Appendix A). Due to (3.16) we can lift the weak estimate to a norm estimate, and the proof is over.

3.6 Proof of Lemma 2.16

Write for $z = x + i\eta \in D_b$:

$$F(x + i\eta, \varepsilon) = \operatorname{Re} F(x + i\eta, \varepsilon) + i \operatorname{Im} F(x + i\eta, \varepsilon) =: R(x, \varepsilon, \eta) + iI(x, \varepsilon, \eta). \tag{3.18}$$

From (2.55) and the properties fo $f(z, \varepsilon)$ it follows that for $|x| < b/2$ and ε, η sufficiently small the equation $R(x, \varepsilon, \eta) = 0$ has a unique solution $x(\varepsilon, \eta)$ in $|x| < b/2$, $|x(\varepsilon, \eta)| \lesssim |a(\varepsilon)| + \gamma(\varepsilon)$. In addition, $\lim_{\eta \searrow 0} x(\varepsilon, \eta) = x(\varepsilon)$. From condition (iii) it follows that $I(x, \varepsilon, \eta) < 0$ and in addition from (ii) and (v) that

$$\lim_{\eta \searrow 0} I(x, \varepsilon, \eta) = I(x, \varepsilon). \tag{3.19}$$

Notice also that on D_b

$$0 < -I(x, \varepsilon, \eta) \lesssim \eta + \gamma(\varepsilon). \tag{3.20}$$

Now fix ε sufficiently small. Due to (3.20) one can find $C > 0$ such that for η sufficiently small

$$J_{\varepsilon,\eta} = \big[x(\varepsilon, \eta) - C\frac{\Gamma(\varepsilon, \eta)}{\gamma(\varepsilon)},\ x(\varepsilon, \eta) + C\frac{\Gamma(\varepsilon, \eta)}{\gamma(\varepsilon)}\big] \subset (-b/2, b/2) \tag{3.21}$$

where

$$\Gamma(\varepsilon,\eta) := -I(x(\varepsilon,\eta),\varepsilon,\eta). \tag{3.22}$$

Define

$$L(x,\varepsilon,\eta) := -(x - x(\varepsilon,\eta)) - i\Gamma(\varepsilon,\eta). \tag{3.23}$$

The technical core of the proof is to show that uniformly in $t \in \mathbf{R}$ and $\eta \searrow 0$ we have

$$\Big|\int_{J_{\varepsilon,\eta}} e^{-ixt}\Big(\frac{1}{F(x+i\eta,\varepsilon)} - \frac{1}{L(x,\varepsilon,\eta)}\Big)dx\Big| \lesssim \gamma(\varepsilon). \tag{3.24}$$

Let us prove this. By construction we have $L(x(\varepsilon,\eta),\varepsilon,\eta) = F(x(\varepsilon,\eta)+i\eta,\varepsilon)$, thus:

$$\begin{aligned} L(x,\varepsilon,\eta) - F(x+i\eta,\varepsilon) &= \int_{x(\varepsilon,\eta)}^{x} \frac{d}{du}(L(u,\varepsilon,\eta) - F(u+i\eta,\varepsilon))du \\ &= (x - x(\varepsilon,\eta))\frac{d}{du}(L(u,\varepsilon,\eta) - F(u+i\eta,\varepsilon))|_{u=x(\varepsilon,\eta)} \\ &\quad + \int_{x(\varepsilon,\eta)}^{x} \Big\{\frac{d}{du}(L(u,\varepsilon,\eta) - F(u+i\eta,\varepsilon)) \\ &\qquad - \frac{d}{du}(L(u,\varepsilon,\eta) - F(u+i\eta,\varepsilon))|_{u=x(\varepsilon,\eta)}\Big\}du. \end{aligned} \tag{3.25}$$

From (2.55) and (3.23) we get

$$\Big|\frac{d}{du}(L(u,\varepsilon,\eta) - F(u+i\eta,\varepsilon))|_{u=x(\varepsilon,\eta)}\Big| \lesssim \gamma(\varepsilon) \tag{3.26}$$

and from(2.55), (3.23) and (v) we then get

$$\begin{aligned} &\Big|\int_{x(\varepsilon,\eta)}^{x} \Big\{\frac{d}{du}(L(u,\varepsilon,\eta) - F(u+i\eta,\varepsilon)) \\ &\qquad - \frac{d}{du}(L(u,\varepsilon,\eta) - F(u+i\eta,\varepsilon))|_{u=x(\varepsilon,\eta)}\Big\}du\Big| \\ &\qquad \lesssim \gamma(\varepsilon)\int_{x(\varepsilon,\eta)}^{x} \omega(|u - x(\varepsilon,\eta)|)du \\ &\qquad \lesssim \gamma(\varepsilon)|x - x(\varepsilon,\eta)|\omega(|x - x(\varepsilon,\eta)|). \end{aligned} \tag{3.27}$$

Now we write

$$\begin{aligned} \frac{1}{F(x+i\eta,\varepsilon)} - \frac{1}{L(x,\varepsilon,\eta)} &= \frac{L(x,\varepsilon,\eta) - F(x+i\eta,\varepsilon)}{L(x,\varepsilon,\eta)^2} \\ &\quad + \frac{\big(L(x,\varepsilon,\eta) - F(x+i\eta,\varepsilon)\big)^2}{L(x,\varepsilon,\eta)^2F(x+i\eta,\varepsilon)}, \end{aligned} \tag{3.28}$$

and estimate the terms in the r.h.s. of (3.28). From the first equality in (3.25), (3.23), (3.26) and the fact that

$$|x - x(\varepsilon,\eta)| \lesssim |R(x,\varepsilon,\eta)| \le |F(x+i\eta,\varepsilon)|, \tag{3.29}$$

we have

$$\Big|\frac{(L(x,\varepsilon,\eta) - F(x+i\eta,\varepsilon))^2}{L(x,\varepsilon,\eta)^2F(x+i\eta,\varepsilon)}\Big| \lesssim \gamma(\varepsilon)^2\frac{|x - x(\varepsilon,\eta)|}{|x - x(\varepsilon,\eta)|^2 + \Gamma(\varepsilon,\eta)^2}$$

and then from (3.21) (remember that $\lim_{\varepsilon\searrow 0}\gamma(\varepsilon)=0$):

$$\Big|\int_{J_{\varepsilon,\eta}}\frac{(L(x,\varepsilon,\eta)-F(x+i\eta,\varepsilon))^2}{L(x,\varepsilon,\eta)^2F(x+i\eta,\varepsilon)}dx\Big|$$
$$\lesssim \gamma(\varepsilon)^2\int_0^{C\frac{\Gamma(\varepsilon,\eta)}{\gamma(\varepsilon)}}\frac{y}{y^2+\Gamma(\varepsilon,\eta)^2}dy\lesssim \gamma(\varepsilon)^2\ln(1+\gamma(\varepsilon)^{-2}). \tag{3.30}$$

We now come to the first term in the r.h.s. of (3.28). We claim that

$$\Big|\int_{J_{\varepsilon,\eta}}e^{-ixt}\frac{x(\varepsilon,\eta)-x}{L(x,\varepsilon,\eta)^2}dx\Big|\lesssim 1. \tag{3.31}$$

Indeed, write

$$\frac{x(\varepsilon,\eta)-x}{L(x,\varepsilon,\eta)^2}=\frac{1}{L(x,\varepsilon,\eta)}+\frac{i\Gamma(\varepsilon,\eta)}{L(x,\varepsilon,\eta)^2}. \tag{3.32}$$

Using $\frac{1}{\pi}\int_{\mathbf{R}}\frac{\Gamma(\varepsilon,\eta)}{x^2+\Gamma(\varepsilon,\eta)^2}dx=1$ one has

$$\Big|\int_{J_{\varepsilon,\eta}}e^{-ixt}\frac{x(\varepsilon,\eta)-x}{L(x,\varepsilon,\eta)^2}dx\Big|\le\Big|\int_{J_{\varepsilon,\eta}}e^{-ixt}\frac{1}{L(x,\varepsilon,\eta)}dx\Big|+\pi. \tag{3.33}$$

Further (see (3.21))

$$\Big|\int_{J_{\varepsilon,\eta}}e^{-ixt}\frac{1}{L(x,\varepsilon,\eta)}dx\Big|=\Big|\int_{-C/\gamma(\varepsilon)}^{C/\gamma(\varepsilon)}\frac{e^{-it\Gamma(\varepsilon,\eta)u}}{u+i}du\Big|\lesssim 1, \tag{3.34}$$

where the last inequality is obtained by closing the contour in the upper complex plane with a semicircle. Putting together (3.32), (3.33), (3.34) one obtains (3.31).

From (3.31) and (3.26):

$$\Big|\int_{J_{\varepsilon,\eta}}e^{-ixt}\frac{(x-x(\varepsilon,\eta))[\frac{d}{dx}(L(x,\varepsilon,\eta)-F(x+i\eta,\varepsilon))]_{x=x(\varepsilon,\eta)}}{L(x,\varepsilon,\eta)^2}dx\Big|\lesssim\gamma(\varepsilon) \tag{3.35}$$

while from (3.27):

$$\Big|\int_{J_{\varepsilon,\eta}}e^{-ixt}\frac{1}{L(x,\varepsilon,\eta)^2}\int_{x(\varepsilon,\eta)}^{x}\Big\{\frac{d}{du}\big(L(u,\varepsilon,\eta)-F(u+i\eta,\varepsilon)\big)$$
$$-\frac{d}{du}\big(L(u,\varepsilon,\eta)-F(u+i\eta,\varepsilon)\big)\big|_{u=x(\varepsilon,\eta)}\Big\}dudx\Big|$$
$$\le\gamma(\varepsilon)\int_{J_{\varepsilon,\eta}}\frac{|x-x(\varepsilon,\eta)|\omega(|x-x(\varepsilon,\eta)|)}{|L(x,\varepsilon,\eta)^2|}dx\lesssim\gamma(\varepsilon). \tag{3.36}$$

Finally, gathering together (3.28),(3.30), (3.25), (3.35) and (3.36) one obtains (3.24).

By taking the complex conjugate in (3.24) and replacing t with $-t$ one also obtains

$$\Big|\int_{J_{\varepsilon,\eta}}e^{-ixt}\Big(\frac{1}{\overline{F(x+i\eta,\varepsilon)}}-\frac{1}{\overline{L(x,\varepsilon,\eta)}}\Big)dx\Big|\lesssim\gamma(\varepsilon). \tag{3.37}$$

We now claim that if $t\ge 0$ we have

$$\Big|\frac{1}{\pi}\int_{J_{\varepsilon,\eta}}e^{-ixt}\frac{\Gamma(\varepsilon,\eta)}{(x-x(\varepsilon,\eta))^2+\Gamma(\varepsilon,\eta)^2}dx-e^{-it(x(\varepsilon,\eta)-i\Gamma(\varepsilon,\eta))}\Big|\lesssim\gamma(\varepsilon). \tag{3.38}$$

Indeed, by direct computation we get

$$\Big|\Big(\int_{\mathbf{R}} - \int_{J_{\varepsilon,\eta}}\Big) e^{-ixt} \frac{\Gamma(\varepsilon,\eta)}{(x-x(\varepsilon,\eta))^2+\Gamma(\varepsilon,\eta)^2} dx\Big| \leq 2\int_{C\frac{\Gamma(\varepsilon,\eta)}{\gamma(\varepsilon)}}^{\infty} \frac{\Gamma(\varepsilon,\eta)}{x^2+\Gamma(\varepsilon,\eta)^2} dx \lesssim \gamma(\varepsilon). \quad (3.39)$$

On the other hand, by exact integration, we get

$$\frac{1}{\pi}\int_{\mathbf{R}} e^{-ixt} \frac{\Gamma(\varepsilon,\eta)}{(x-x(\varepsilon,\eta))^2+\Gamma(\varepsilon,\eta)^2} dx = e^{-it(x(\varepsilon,\eta)-i\Gamma(\varepsilon,\eta))},$$

which together with (3.39) gives (3.38).

From (3.24), (3.37), and (3.39) one obtains for all $t \geq 0$

$$\Big|\frac{1}{2\pi i}\int_{J_{\varepsilon\eta}} e^{-ixt}\Big(\frac{1}{F(x+i\eta,\varepsilon)} - \frac{1}{\overline{F(x+i\eta,\varepsilon)}}\Big) dx - e^{-it(x(\varepsilon,\eta)-i\Gamma(\varepsilon,\eta))}\Big| \lesssim \gamma(\varepsilon). \quad (3.40)$$

We now finish the proof of Lemma 2.16 by using a trick going back to Hunziker [17]. Write

$$\begin{aligned} I(\varepsilon,\eta,t) &= \frac{1}{\pi}\int_{\mathbf{R}} e^{-itx} \frac{\operatorname{Im} F(x+i\eta,\varepsilon)}{|F(x+i\eta,\varepsilon)|^2} dx \\ &= \frac{1}{\pi}\Big(\int_{J_{\varepsilon,\eta}} + \int_{\mathbf{R}\setminus J_{\varepsilon,\eta}}\Big) e^{-itx} \frac{\operatorname{Im} F(x+i\eta,\varepsilon)}{|F(x+i\eta,\varepsilon)|^2} dx \\ &=: I_1(\varepsilon,\eta,t) + I_2(\varepsilon,\eta,t). \end{aligned}$$

By assumption (iv) we have that

$$\lim_{\eta\searrow 0} I(\varepsilon,\eta,0) = 1, \quad (3.41)$$

while from (3.40), uniformly in $\eta \searrow 0$,

$$|I_1(\varepsilon,\eta,0) - 1| \lesssim \gamma(\varepsilon). \quad (3.42)$$

It follows that, uniformly in $\eta \searrow 0$

$$\begin{aligned} I_2(\varepsilon,\eta,0) = |I_2(\varepsilon,\eta,0)| &\leq |I(\varepsilon,\eta,0) - 1| + |I_1(\varepsilon,\eta,0) - 1| \\ &\lesssim |I(\varepsilon,\eta,0) - 1| + \gamma(\varepsilon), \end{aligned} \quad (3.43)$$

and then from (3.41) and the fact that $|I_2(\varepsilon,\eta,t)| \leq |I_2(\varepsilon,\eta,0)|$:

$$\limsup_{\eta\searrow 0} |I_2(\varepsilon,\eta,t)| \lesssim \gamma(\varepsilon). \quad (3.44)$$

Finally, using (3.44) and the fact that

$$\lim_{\eta\searrow 0}\big(x(\varepsilon,\eta) - i\Gamma(\varepsilon,\eta)\big) = E_\varepsilon,$$

we obtain for an arbitrary $t \geq 0$

$$\begin{aligned} \limsup_{\eta\searrow 0} |I(\varepsilon,\eta,t) - e^{-itE_\varepsilon}| &\leq \limsup_{\eta\searrow 0} |I_1(\varepsilon,\eta,t) - e^{-it(x(\varepsilon,\eta)-i\Gamma(\varepsilon,\eta))}| \\ &\quad + \limsup_{\eta\searrow 0} |I_2(\varepsilon,\eta,t)| \\ &\lesssim \gamma(\varepsilon), \end{aligned}$$

and the proof is finished.

4 Application to two-channel Schrödinger operators

We apply the abstract theory developed in the previous sections to a certain class of two-channel Schrödinger operators in arbitrary dimensions, as they appear for example in the theory of Feshbach resonances in atomic physics; see e.g. [35, 25] and references given there. The model has been also considered in [19] and [9] in connection with FGR at thresholds. We follow the setting and notations in Section 5 of [19].

Our two-channel Schrödinger operator has a non-degenerate bound state in the 'closed' channel, whose Hilbert space is modelled with $\mathbf{C}$. The total Hilbert space is $\mathcal{H} = L^2(\mathbf{R}^d) \oplus \mathbf{C}$. As the unperturbed Hamiltonian we take

$$H = \begin{bmatrix} -\Delta + V - E_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_0 > 0, \tag{4.1}$$

where V satisfies for some $\gamma > 0$ to be specified later

$$\langle \cdot \rangle^\gamma V \in L^\infty(\mathbf{R}^d). \tag{4.2}$$

Here $\langle x \rangle = (1 + x^2)^{1/2}$ as usual. The perturbation is

$$W = \begin{bmatrix} W_{11} & |W_{12}\rangle\langle 1| \\ |1\rangle\langle W_{12}| & b \end{bmatrix}, \tag{4.3}$$

which is a shorthand for

$$W \begin{bmatrix} f(x) \\ \xi \end{bmatrix} = \begin{bmatrix} W_{11}(x)f(x) + W_{12}(x)\xi \\ \int_{\mathbf{R}^d} \overline{W_{12}(x)} f(x) dx + b\xi \end{bmatrix}. \tag{4.4}$$

Here we assume

$$\langle \cdot \rangle^\gamma W_{11} \in L^\infty(\mathbf{R}^d), \quad \langle \cdot \rangle^{\gamma/2} W_{12} \in L^\infty(\mathbf{R}^d), \tag{4.5}$$

and furthermore that W_{11} is real-valued and $b \in \mathbf{R}$. We introduce the weight function

$$\rho_\gamma = \langle \cdot \rangle^{-\gamma/2} \tag{4.6}$$

the weight operator

$$B = \begin{bmatrix} \rho_{-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.7}$$

and define the bounded self-adjoint operator, C, and its polar decomposition with $D = D^* = D^{-1}$

$$C = BWB = |C|^{1/2} D |C|^{1/2}. \tag{4.8}$$

In the factorization (see (2.8)) of W we take $\mathcal{K} = \mathcal{H}$ and

$$A = |C|^{1/2} B^{-1}, \tag{4.9}$$

so that

$$W = B^{-1} |C|^{1/2} D |C|^{1/2} B^{-1}. \tag{4.10}$$

Notice that in our case

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_0 = 1 - P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.11}$$

i.e. Q_0 is the orthogonal projection onto $L^2(\mathbf{R}^d)$. The key point of the above factorization is that

$$[B, Q_0] = 0, \tag{4.12}$$

so that (see (2.9)):

$$G(z) = AQ_0(H-z)^{-1}Q_0A^* = |C|^{1/2}\begin{bmatrix}\rho_\gamma(-\Delta+V-E_0-z)^{-1}\rho_\gamma & 0\\ 0 & 0\end{bmatrix}|C|^{1/2}. \tag{4.13}$$

Concerning Assumption 2.9, notice that in our case

$$Q(J_a) = \begin{bmatrix}E(J_a) & 0\\ 0 & 0\end{bmatrix}, \tag{4.14}$$

where $E(\Delta)$ is the spectral measure of $-\Delta+V$ in $L^2(\mathbf{R}^d)$. Thus the verification of Assumption 2.9 boils down to the verification of the corresponding assumption for $-\Delta+V-E_0$ in $L^2(\mathbf{R}^d)$ with A^* replaced by $\langle\cdot\rangle^{-\gamma/2}$ and $\Gamma(\lambda)$ replaced by the corresponding operator for $-\Delta+V$ in $L^2(\mathbf{R}^d)$.

We summarize the above discussion in the following lemma.

Lemma 4.1. (i) *Assumption 2.5 is implied by*

$$\rho_\gamma(-\Delta+V-E_0-z)^{-1}\rho_\gamma \in C^{1,\omega}(D_a;\mathcal{B}(L^2(\mathbf{R}^d))), \quad \int_0 \frac{\omega(x)}{x}dx < \infty. \tag{4.15}$$

(ii) *Assumption 2.9 is implied by*

$$\Gamma(\cdot)\langle\cdot\rangle^{-\gamma/2} \in C^{2,\omega_\theta}(J_a;\mathcal{B}(L^2(\mathbf{S}^{d-1})))$$

for some $\theta \in (0,1)$. Here $\mathbf{S}^{d-1}$ is the unit sphere in $\mathbf{R}^d$ with the induced measure and $\Gamma(\cdot)$ is the trace operator of $-\Delta+V$ corresponding to J_a.

Both these facts are well known in spectral theory of d-dimensional Schödinger operators with rapidly decaying potential and for the convenience of the reader we shall recall some of these results in Appendix B. In particular, from these results one has that the conclusion of Theorem 2.6 holds true for $\gamma > 3$ and if the FGR constant vanishes the conclusion of Theorem 2.15 holds true for $\gamma > 5$.

A Hölder continuity for Cauchy integral transform

The Cauchy integral transform preserves θ-Hölder continuity for $\theta \in (0,1)$. Even though the result is known, we prove it here in a form which is appropriate for our needs. The argument is a generalization of the one in [13, Ch. I, §5].

Let $\Phi(\cdot)$ be a complex valued function satisfying

$$\operatorname{supp}\Phi \subset (-1,1), \quad \Phi \in C^{n,\omega_\theta}(\mathbf{R};\mathbf{C}),\ \theta\in(0,1). \tag{A.1}$$

There exists a constant $C_{n,\theta}$ such that:

$$|\Phi^{(n)}(x) - \Phi^{(n)}(y)| \le C_{n,\theta}\,|x-y|^\theta, \quad \forall x,y\in(-1,1).$$

Define

$$|||\Phi|||_{n,\theta} := \max\{\|\Phi\|_\infty, \|\Phi^{(1)}\|_\infty, \dots, \|\Phi^{(n)}\|_\infty, C_{n,\theta}\}.$$

Define for $z = x+i\eta \in \mathbf{C}\setminus[-1,1]$the Cauchy transform

$$\Psi(z) = \int_{\mathbf{R}} \frac{\Phi(x)}{x-z}dx. \tag{A.2}$$

Proposition A.1. *The map* Ψ *is holomorphic on* $\mathbf{C}\setminus[-1,1]$. *For every* $k=0,1,...,n$ *the limits* $\Psi^{(k)}(x)=\lim_{\eta\searrow 0}\Psi^{(k)}(x+i\eta)$ *exist. Moreover, there exists a constant* C *such that uniformly in* $x,y\in(-2,2)$, $\eta>0$ *and* $0\le k\le n$ *we have:*

$$|\Psi^{(k)}(x+i\eta)-\Psi^{(k)}(y+i\eta)|\le C\,|||\Phi|||_{n,\theta}\,|x-y|^{\theta}. \tag{A.3}$$

Proof. A finite number of constants appearing during the proof will be denoted by C. If $z\notin[-1,1]$ we have $\Psi^{(k)}(z)=\int_{\mathbf{R}}\frac{\Phi(x)}{(x-z)^{k+1}}dx$; integrating by parts, we can write $\Psi^{(k)}(z)=\int_{\mathbf{R}}\frac{\Phi^k(x)}{x-z}dx$, thus it is sufficient to prove the proposition for $n=0$. The argument for the existence of limit values is the standard principal value argument and it will not be repeated here. The argument for Hölder continuity is more elaborated. Consider

$$\widetilde{\Psi}(x+i\eta)=\int_{-11}^{11}\frac{\Phi(\tau)-\Phi(x)}{\tau-x-i\eta}d\tau=\Psi(x+i\eta)-\Phi(x)\ln\frac{11-x-i\eta}{-11-x-i\eta}. \tag{A.4}$$

Since the second term in the r.h.s. satisfies (A.3), it is sufficient to consider $\widetilde{\Psi}(x+i\eta)$. In what follows, $x,y\in(-2,2)$ and $\eta\in(0,1)$.

Denote by $L:=[-11,11]$. For a given pair $x_1<x_2$ in $(-2,2)$, we define

$$l:=(x_1-2|x_1-x_2|,x_2+2|x_1-x_2|)=:(a,b)\subset[-10,10]\subset L.$$

For $z_j=x_j+i\eta$, $j=1,2$ we have to estimate $\widetilde{\Psi}(z_2)-\widetilde{\Psi}(z_1)$. We write

$$\begin{aligned}\widetilde{\Psi}(z_2)-\widetilde{\Psi}(z_1)=&\int_l\frac{\Phi(\tau)-\Phi(x_2)}{\tau-z_2}d\tau-\int_l\frac{\Phi(\tau)-\Phi(x_1)}{\tau-z_1}d\tau\\&+\int_{L\setminus l}\Big(\frac{\Phi(\tau)-\Phi(x_2)}{\tau-z_2}-\frac{\Phi(\tau)-\Phi(x_1)}{\tau-z_1}\Big)d\tau.\end{aligned} \tag{A.5}$$

The first two integrals are easily estimated

$$\begin{aligned}\Big|\int_l\frac{\Phi(\tau)-\Phi(x_2)}{\tau-z_2}d\tau\Big|&\le C\int_l\frac{|\tau-x_2|^{\theta}}{|\tau-x_2|}d\tau\\&\le C\int_0^{3|x_1-x_2|}u^{\theta-1}d\tau=C\frac{|x_1-x_2|^{\theta}}{\theta},\end{aligned} \tag{A.6}$$

and similarly for the second one. In the third integral of (A.5) one uses the following identity

$$\begin{aligned}&\frac{\Phi(\tau)-\Phi(x_2)}{\tau-z_2}-\frac{\Phi(\tau)-\Phi(x_1)}{\tau-z_1}\\&\qquad=\frac{\Phi(x_1)-\Phi(x_2)}{\tau-z_1}+\frac{(\Phi(\tau)-\Phi(x_2))(z_2-z_1)}{(\tau-z_1)(\tau-z_2)}.\end{aligned} \tag{A.7}$$

For the integral involving the first term in the r.h.s. of (A.7) we observe that $|a-z_1|=|b-z_1|$ and

$$\int_{L\setminus l}\frac{1}{\tau-z_1}d\tau=\ln\Big|\frac{11-z_1}{11+z_1}\Big|+i\arg\Big(\frac{11-z_1}{z_1-b}\frac{a-z_1}{11+z_1}\Big),$$

which is uniformly bounded for $x_1\in(-2,2)$ and $\eta\in(0,1)$, hence

$$\Big|\int_{L\setminus l}\frac{\Phi(x_1)-\Phi(x_2)}{\tau-z_1}d\tau\Big|\le C\,|x_1-x_2|^{\theta}. \tag{A.8}$$

We are left with estimating the integral involving the second term in the r.h.s. of (A.7):

$$\Big|\int_{L\setminus l} \frac{(\Phi(\tau)-\Phi(x_2))(z_2-z_1)}{(\tau-z_1)(\tau-z_2)}d\tau\Big| \le C|x_1-x_2|\int_{L\setminus l}\frac{1}{|\tau-x_1||\tau-x_2|^{1-\theta}}d\tau$$
$$= C|x_1-x_2|\int_{L\setminus l}\Big|\frac{\tau-x_1}{\tau-x_2}\Big|^{1-\theta}\frac{1}{|\tau-x_1|^{2-\theta}}d\tau. \qquad \text{(A.9)}$$

Because $\frac{\tau-x_1}{\tau-x_2}$ is piecewise monotone as a function of τ, we have

$$\sup_{x_1\in(-2,2),\tau\in L\setminus l}\Big|\frac{\tau-x_1}{\tau-x_2}\Big| < \infty,$$

where the maximum is attained in the set $\{-11, 11, a, b\}$. Using this in (A.9) one obtains

$$|x_1-x_2|\int_{L\setminus l}\frac{|\tau-x_1|}{|\tau-x_2|}^{1-\theta}\frac{1}{|\tau-x_1|^{2-\theta}}d\tau$$
$$\le C|x_1-x_2|\int_{2|x_1-x_2|}^{11}u^{\theta-2}d\tau \le \frac{C}{1-\theta}|x_1-x_2|^{\theta}. \qquad \text{(A.10)}$$

Putting together (A.5), (A.6) - (A.10), the proof is finished. □

B Resolvent smoothness and Γ operator for one body d-dimensional Schrödinger operators

A convenient formalism for stationary scattering theory is given by the Γ operators or trace operators. These were introduced in [26] in an abstract setting. A presentation of applications to Schrödinger operators can be found in [27]. Extensive results on stationary scattering theory can be found in [40, 41], both in an abstract framework, and applied to a number of differential operators. Trace operators are used in many cases in these monographs.

We now describe the trace operators for Schrödinger Hamiltonians $-\Delta+V$ in $L^2(\mathbf{R}^d)$ for the convenience of the reader.

Let $H_0 = -\Delta$ on $\mathcal{H} = L^2(\mathbf{R}^d)$, with the domain $\mathcal{D}(H_0) = H^2(\mathbf{R}^d)$, the usual Sobolev space.

We need the weighted L^2-spaces. We have

$$L^{2,s}(\mathbf{R}^d) = \{f \mid \langle\cdot\rangle^s f \in L^2(\mathbf{R}^d)\}, \quad s \in \mathbf{R}. \qquad \text{(B.1)}$$

Here $\langle\cdot\rangle$ denotes multiplication by $\langle x\rangle = (1+x^2)^{1/2}$, $x \in \mathbf{R}^d$. We use the following convention for the Fourier transform.

$$\mathcal{F}\colon L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d), \quad (\mathcal{F}f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int e^{-ix\xi}f(x)dx. \qquad \text{(B.2)}$$

We also use the Fourier transform between other spaces. For example we have $\mathcal{F}(L^{2,s}(\mathbf{R}^d)) = H^s(\mathbf{R}^d)$, $s \in \mathbf{R}$, the Sobolev spaces.

We let $J = (0,\infty)$.

Definition B.1. The free Γ operator is defined for $f \in L^{2,s}(\mathbf{R}^d)$, $s > \frac{1}{2}$, as

$$(\Gamma_0(\lambda)f)(\omega) = 2^{-1/2}\lambda^{(d-2)/4}(\mathcal{F}f)(\lambda^{1/2}\omega), \quad \lambda \in J,\ \omega \in \mathbf{S}^{d-1}. \qquad \text{(B.3)}$$

We record some of the properties of Γ_0. We use the notation $\mathsf{h} = L^2(\mathbf{S}^{d-1})$. We also use the notation $\mathcal{B}(L^{2,s}(\mathbf{R}^d), \mathsf{h})$ for the bounded operators. Furthermore, we use the Hölder space

$$C^{n,\theta}(J, \mathcal{B}(L^{2,s}(\mathbf{R}^d), \mathsf{h})), \quad n \in \mathbf{N},\ 0 < \theta < 1, \tag{B.4}$$

and also the local Hölder space $C^{n,\theta}_{\mathrm{loc}}(J, \mathcal{B}(L^{2,s}(\mathbf{R}^d), \mathsf{h}))$, which means that the functions are Hölder continuous on any relatively compact open subinterval of J.

Proposition B.2. *For $\lambda \in J$ and $s > \frac{1}{2}$ we have $\Gamma_0(\lambda) \in \mathcal{B}(L^{2,s}(\mathbf{R}^d), \mathsf{h})$. If $s = \frac{1}{2} + n + \theta$, $n \in \mathbf{N}$, $0 < \theta < 1$, then $\Gamma_0 \in C^{n,\theta}_{\mathrm{loc}}(J, \mathcal{B}(L^{2,s}(\mathbf{R}^d), \mathsf{h}))$. If $s = \frac{1}{2} + n + 1$, then $\Gamma_0 \in C^{n,\theta}_{\mathrm{loc}}(J, \mathcal{B}(L^{2,s}(\mathbf{R}^d), \mathsf{h}))$ for all $0 < \theta < 1$.*

The result follows from the fact that $\mathcal{F}(L^{2,s}(\mathbf{R}^d)) = H^s(\mathbf{R}^d)$ and the trace theorem in Sobolev spaces, see for example [1, 41].

Using Γ_0 one then defines the spectral representation of H_0 as follows.

Definition B.3. The spectral representation of H_0 is defined for $f \in L^{2,s}(\mathbf{R}^d)$, $s > \frac{1}{2}$, by

$$(\mathcal{F}_0 f)(\lambda)(\omega) = (\Gamma_0(\lambda) f)(\omega), \quad \lambda \in J, \quad \omega \in \mathbf{S}^{d-1}. \tag{B.5}$$

Proposition B.4. *$\mathcal{F}_0$ extends to a unitary map from $\mathcal{H}$ to $L^2(J, \mathsf{h})$. Furthermore, we have $\mathcal{F}_0 H_0 = M_\lambda \mathcal{F}_0$, where M_λ is the operator of multiplication by λ in $L^2(J, \mathsf{h})$.*

This result is then the starting point for obtaining Γ operators for $H = H_0 + V$. We only outline some of the basic results. The definition relies on the boundary values of the resolvent $R(z) = (H - z)^{-1}$.

We will not deal with local singularities of the perturbation V, so we use the following assumption. The compact operators between two Hilbert spaces are denoted by $\mathcal{C}(\mathcal{H}, \mathcal{K})$.

Assumption B.5. Assume that V is a bounded selfadjoint operator on $\mathcal{H}$, such that for some $\beta > \frac{1}{2}$ it satisfies $V \in \mathcal{C}(L^{2,-\beta}(\mathbf{R}^d), L^{2,\beta}(\mathbf{R}^d))$.

Assumption B.6. Assume that the limiting absorption principle holds for $R(z)$ on J with boundary values

$$R(\lambda \pm i0) \in \mathcal{B}(L^{2,s}(\mathbf{R}^d), L^{2,-s}(\mathbf{R}^d)), \quad \text{for some } s > \tfrac{1}{2}.$$

Assume that the boundary values are locally Hölder continuous with exponent θ, $0 < \theta < s - \frac{1}{2}$ and $\theta < 1$.

Note that this formulation excludes positive eigenvalues for H. This assumption can be verified in different manners, for example by using Mourre theory, see [14, 3]. Differentiability of the boundary values and Hölder continuity of the highest derivative of the boundary values can also be verified, provided sufficiently strong assumptions are imposed on V.

Let us state a special case of the results in [14, 3].

Proposition B.7. *Let V be a real-valued function. Assume that there exist $\beta > \frac{1}{2}$ and $C > 0$ such that*

$$|V(x)| \le C \langle x \rangle^{-2\beta}, \quad x \in \mathbf{R}^d. \tag{B.6}$$

Let $H = H_0 + V$. If $\beta > \frac{1}{2} + n + \theta$, $n \in \mathbf{N}$, $0 < \theta < 1$, and $s > \beta$, then the boundary values exist and satisfy

$$R(\cdot \pm i0) \in C^{n,\theta}_{\mathrm{loc}}(J, \mathcal{B}(L^{2,s}(\mathbf{R}^d), L^{2,-s}(\mathbf{R}^d))). \tag{B.7}$$

With these preparations we state the following definition.

Definition B.8. Assume that Assumption B.5 is verified for some $\beta > \frac{1}{2}$ and that Assumption B.6 is satisfied for $s = \beta$. For $f \in L^{2,\beta}(\mathbf{R}^d)$ we define

$$\Gamma_\pm(\lambda)f = \Gamma_0(\lambda)(I - VR(\lambda \pm i0))f. \tag{B.8}$$

These operators then have the same Hölder continuity properties as Γ_0, with $s = \beta$.

Definition B.9. The spectral representations are defined for $f \in L^{2,\beta}(\mathbf{R}^d)$ by

$$(\mathcal{F}_\pm f)(\lambda)(\omega) = (\Gamma_\pm(\lambda)f)(\omega), \quad \lambda \in J,\ \omega \in \mathbf{S}^{d-1}. \tag{B.9}$$

We denote the projection onto the absolutely continuous subspace for H by $P_{\mathrm{ac}}(H)$. Then we can state the following result.

Proposition B.10. *The operators $\mathcal{F}_\pm$ extend to unitary operators from $P_{\mathrm{ac}}(H)\mathcal{H}$ to $L^2(J, \mathsf{h})$. Furthermore, we have $\mathcal{F}_\pm H = M_\lambda \mathcal{F}_\pm$, where M_λ is the operator of multiplication by λ in $L^2(J, \mathsf{h})$.*

Let $E_0(\lambda)$ and $E(\lambda)$ denote the spectral families of H_0 and H respectively. Then for $f, g \in L^{2,\beta}(\mathbf{R}^d)$ we have for $\lambda \in J$

$$\langle f, E_0'(\lambda)g\rangle = \langle \Gamma_0(\lambda)f, \Gamma_0(\lambda)g\rangle, \tag{B.10}$$
$$\langle f, E'(\lambda)g\rangle = \langle \Gamma_\pm(\lambda)f, \Gamma_\pm(\lambda)g\rangle. \tag{B.11}$$

This framework is used for the derivation of the stationary scattering for the pair of operators H_0 and H.

Remark B.11. The constructions above based on the Fourier transform can be applied to a number of constant coefficient pesudodifferential operators $H_0 = f(-i\nabla)$, provided sufficient information is available on the energy surfaces $\{\xi \in \mathbf{R}^d \,|\, f(\xi) = \lambda\}$. For example, one can construct free Γ operators and spectral representations for the free Dirac operator and for the relativistic Schrödinger operators $\sqrt{-\Delta + m^2}$, $m \geq 0$.

Using a different transform one can also construct the free Γ operator and the spectral representation for the free Stark operator $H_0 = -\Delta + x \cdot \mathcal{E}$, see [42] for the details and [42, 18] for resonances in the analytic continuation framework based on this spectral representation.

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