Effective Stresses in Soil and Rock and Consolidation in Three Dimensions

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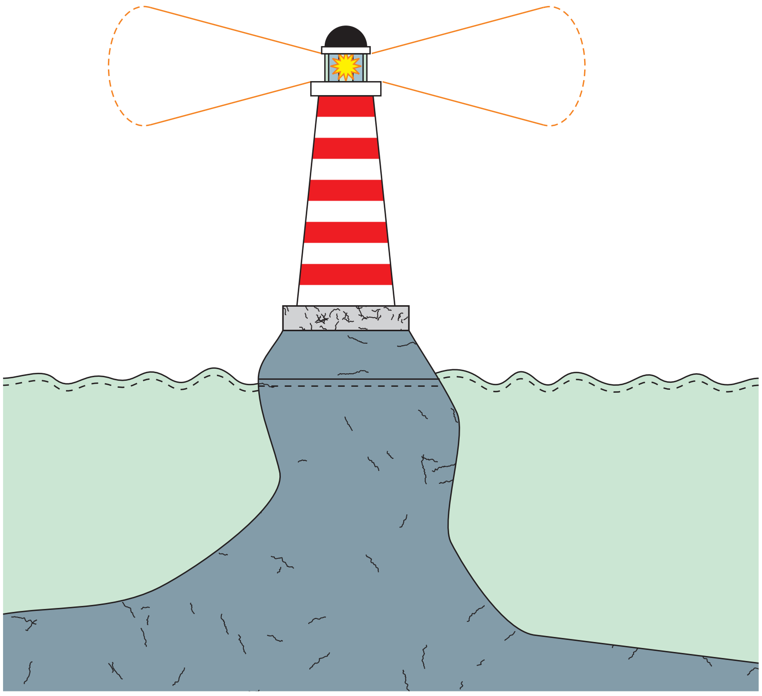
Lars Andersen



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Aalborg University
Department of Civil Engineering
Water and Soil

DCE Lecture Notes No. 14

Effective Stresses in Soil and Rock and Consolidation in Three Dimensions

by

Lars Andersen

March 2007

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Effective Stresses in Soil and Rock and Consolidation in Three Dimensions

LARS ANDERSEN

ABSTRACT. In the following, the continuum model for a fully saturated porous material is presented. The theory is mainly due to M.A. Biot [3, 4]. We shall only consider a two-phase system consisting of a solid skeleton and a single pore fluid, *e.g.* water. The theory for three-dimensional consolidation is developed. Anisotropic permeability of the material is allowed, but for simplicity the analysis is restricted to isotropic linear elastic material behaviour. However, the theory is easily extended to elastoplasticity. Finally, it will be shown that the effective stresses in a porous material may in general not be calculated as proposed by Terzaghi. Whereas highly accurate results are achieved for residual soils, *i.e.* sand, silt and clay, poor results are obtained for cemented materials such as concrete and rock. Here it is recommended to follow the stress approach proposed by Biot.

1. BASIC DEFINITIONS

A porous material, or matrix, with the total volume V is considered. The material is fully saturated, and the pores are assumed to be distributed randomly in space so that the material on a macroscopic level may be described as a continuum. The volume is divided into two parts,

$$V = V_s + V_f, \tag{1}$$

where V_s is the volume of the solid phase, *i.e.* the grain skeleton, and V_f is the volume of fluid. In geotechnical engineering, the subscript f is generally substituted by the subscript w, since the pore fluid is usually water. In saturated porous materials, *e.g.* soil, a part of the pore fluid is constrained. For example, a part of the water in clay is chemically bound to the clay mineral, and in rock or granular soil some of the water may be trapped in cracks that are not connected to the primary system of pores. This part of the fluid belongs to the solid phase, *i.e.* to V_s, since it cannot move relatively to the solid matrix. Hence, only the volume of interconnected voids is included in the definition of V_f, cf. Fig. 1. Unfortunately, in real soil or concrete etc. it may be difficult to determine which part of the pore fluid is free to move relatively to the solid skeleton.

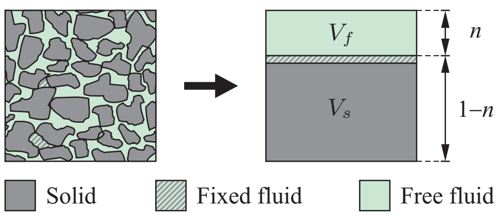

FIGURE 1. Definition of the porosity, n, in a saturated porous material. The volume of the free fluid in the interconnected pores is $V_f = nV$, and the volume of the solid (including fixed fluid) is $V_s = (1 - n)V$.

The *porosity* of the porous material, or matrix, is defined as

$$n = V_f/V, \tag{2}$$

i.e. as the volume fraction taken up by interconnected pores. Occasionally, in the international literature on porous materials, the porosity is denoted f, ϕ or β, but in the Danish geotechnical literature the symbol n is usually applied. Given that the soil is fully saturated, the mass density of the matrix material constituted by the solid and the fluid becomes

$$\rho = (1-n)\rho_s + n\rho_f, \tag{3}$$

where ρ_s is the mass density of the solid phase, whereas ρ_f is the mass density of the fluid phase. In standard geotechnical engineering ρ_s is most often the average density of the minerals constituting the grains in the soil. This is not the case in the present formulation, since any fluid that is not allowed to move freely between the grains is considered part of the solid phase as illustrated in Fig. 1. In other words, n is the volume fraction of interconnected pores. With this definition, n is occasionally referred to as the *effective porosity*.

In the present theory, it is assumed that the pores are distributed randomly, so that the matrix material may be considered homogeneous on a macroscopic level. Hence, in accordance with Eq. (2) for a cross section with the total area A, the area $A_f = nA$ will be constituted by the free pore fluid, whereas the solid phase (including fixed pore water) constitutes the area $A_s = (1-n)A$.

Next, the pore pressure, $p = p(\mathbf{x}, t)$, is defines as the pressure in the free pore fluid. Whereas the mean total stress, $\sigma = \sigma_{kk}$, is defined as positive in tension, the pore pressure is positive in compression. This definition is common practice in geotechnical engineering. The total stresses σ_{ij} are now divided into two parts,

$$\sigma_{ij} = \sigma'_{ij} - p\delta_{ij}. \tag{4}$$

Here σ'_{ij} are referred to as the *effective stresses*. While the pore pressure is present in both the fluid and the solid phase, the effective stresses are carried solely by the solid skeleton. Given that the solid phase only constitutes the fraction $1-n$ of the entire matrix, the total stress in the solid phase is actually $\sigma'_{ij}/(1-n) - p\delta_{ij}$, whereas the total stress in the pore fluid is $-p\delta_{ij}$, cf. Fig. 2. In Section 3 the definition of effective stresses is further discussed, in particular with respect to the formulation of constitutive laws.

The displacements of the grain skeleton and the fluid in the interconnected pores are denoted u_i and v_i, respectively. In addition to the full displacement of the pore fluid, a relative velocity is introduced in the form

$$\dot{w}_i = \frac{\partial w_i}{\partial t} = n\left(\frac{\partial v_i}{\partial t} - \frac{\partial u_i}{\partial t}\right), \qquad w_i = n\,(v_i - u_i)\,. \tag{5}$$

The quantity $\dot{w}_i = \dot{w}_i(\mathbf{x}, t)$ is referred to as the *seepage velocity*. Evidently, $\dot{w}_i$ is the average relative velocity of the fluid flow in the matrix including both the fluid and the solid phase. Thus, in the particular case $u_i = 0$ the definition $\dot{w}_i = q_i/A$ applies, where q_i is the fluid flow, or flux, through the cross section area A in coordinate direction i, cf. Fig 3.

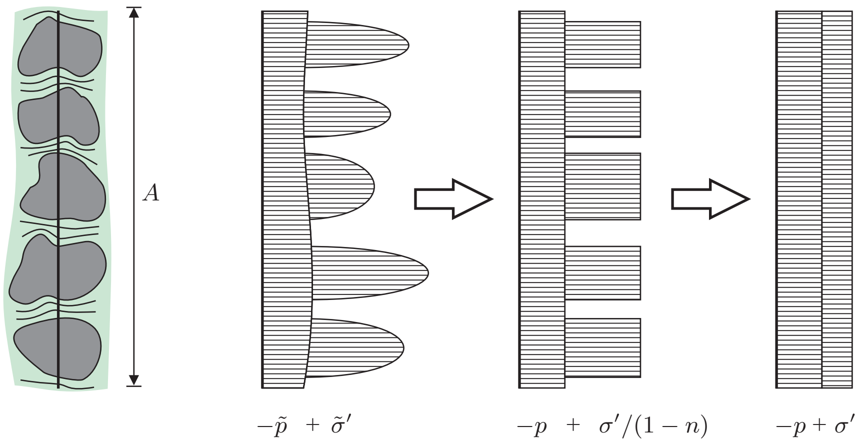

FIGURE 2. Stresses acting on a cross section in a poroelastic medium.

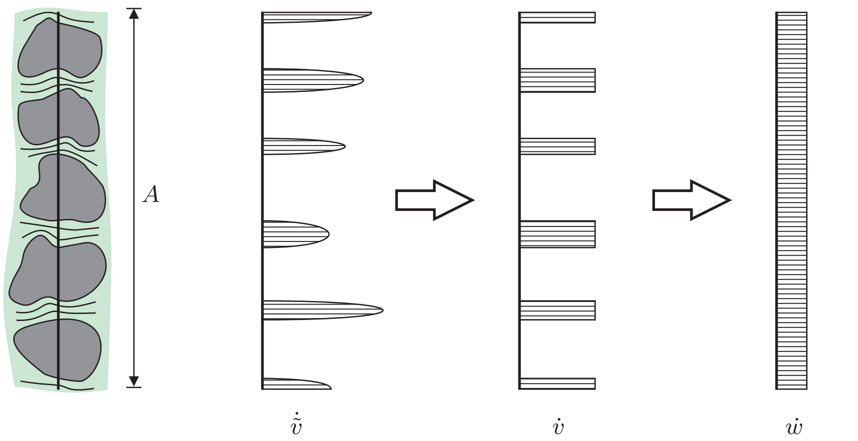

FIGURE 3. Flow velocity over a cross section in a poroelastic medium.

2. Constitutive Laws in Poroelasticity

Firstly, an elastic material is considered. A linear relationship between the strain and stress rates is assumed. Hence, the constitutive law may be expressed in terms of the *generalized Hooke's law*,

$$\dot{\sigma}_{ij} = C_{ijkl}\dot{\epsilon}_{kl}. \tag{6}$$

Here $\dot{\epsilon}_{kl} = \dot{\epsilon}_{kl}(\mathbf{x}, t)$ is the *strain rate tensor*,

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right), \qquad \dot{u}_i = \frac{\partial u_i}{\partial t}. \tag{7}$$

where $u_i = u_i(\mathbf{x}, t)$ denotes the displacement field. In general, the *tangent stiffness tensor* $C_{ijkl} = C_{ijkl}(\mathbf{x}, t)$ may account for elastoplastic material behaviour as well as rheological and thermoelastic response [**?**]. However, in any case only the elastic strains will contribute to the development of stresses, and for the present purpose an elastic model will suffice. In particular, for an isotropic and linear elastic material, the constitutive law simplifies to

$$\dot{\sigma}_{ij} = \lambda \dot{\Delta}_s \delta_{ij} + 2\mu \dot{\epsilon}_{ij}, \tag{8}$$

where $\dot{\Delta}_s = \dot{\Delta}_s(\mathbf{x}, t)$ is the *dilation rate*,

$$\dot{\Delta}_s = \frac{\partial \dot{u}_k}{\partial x_k}, \tag{9}$$

δ_{ij} is the *Kronecker delta*,

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{10}$$

and $\lambda = \lambda(\mathbf{x})$ and $\mu = \mu(\mathbf{x})$ are the so-called *Lamé constants*, which are related to Young's modulus $E = E(\mathbf{x})$ and Poisson's ratio $\nu = \nu(\mathbf{x})$ as

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{11}$$

The inverse relationships are given as

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{12}$$

The Lamé constant μ is also identified as the *shear modulus*, which is often denoted G.

Secondly, in a porous material the effective stresses are provided by the constitutive law for the matrix material, *i.e.* the saturated grain skeleton. In the case of linear isotropic material behaviour the stress–strain relationship is similar to Eq. (8). However, it must be taken into consideration that a change in the pore pressure will lead to a change in the volume of the solid constituent, *i.e.* the grains or the porous solid, which does not involve a change in the effective stresses. Hence,

$$\dot{\sigma}'_{ij} = \lambda \dot{\epsilon}'_{kk} \delta_{ij} + 2\mu \dot{\epsilon}'_{ij}, \qquad \text{where} \qquad \dot{\epsilon}'_{ij} = \dot{\epsilon}_{ij} - \frac{1}{3} \dot{\Delta}^p_s \delta_{ij}, \quad \dot{\Delta}^p_s = -\frac{\dot{p}}{K_s}. \tag{13}$$

Here $\dot{\epsilon}_{ij}$ are the components of the strain rate tensor given by Eq. (7), and λ and μ are the Lamé constants of the matrix material. Note that these may be substantially different from the Lamé constants λ_s and μ_s of the solid material as further discussed below. Finally, $\dot{\Delta}^p_s$ is the dilation rate in the solid phase due to the pore pressure (positive in expansion), whereas K_s is the bulk modulus of the solid constituent. In a linear elastic isotropic material the following relationship applies between K_s and the Lamé constants λ_s and μ_s of the solid constituent, *e.g.* the grains or the rock mineral:

$$K_s = \lambda_s + \frac{2}{3} \mu_s. \tag{14}$$

In granular materials such as residual soils, λ_s and μ_s are typically much greater than λ and μ, *e.g.* the Lamé constants of sand are much smaller than those of quarts crystal. Hence, the reduction in the volume of the solid phase resulting from an increase of the pore pressure is generally negligible. In this situation $\dot{\Delta}^p_s$ may be disregarded and the approximation $\dot{\epsilon}'_{ij} \approx \dot{\epsilon}_{ij}$ applies. On the other hand, in concrete, intact rock and similar cemented materials the bulk modulus of the solid constituent, K_s, and that of the matrix material, K, are typically of the same order of magnitude. Therefore, in these materials $\dot{\Delta}^p_s$ becomes significant, and the full definition provided by Eq. (13) applies. Finally, it is noted that in the general nonlinear case, an additional contribution to the total strain rate $\dot{\epsilon}_{ij}$, but not to $\dot{\epsilon}'_{ij}$, stems from creep and thermoelastic behaviour. However, this irreversible term is disregarded in the present theory.

Thirdly, for the pore pressure, a constitutive equation equivalent to Eq. (13) is achieved by consideration of the volumetric strain of the matrix, *i.e.* the dilatation. Making use of Eqs. (7) and (5), the total dilatation rate for the matrix, $\dot{\Delta}_m$, may be expressed as

$$\dot{\Delta}_m = (1-n)\dot{\Delta}_s + n\dot{\Delta}_f = \frac{\partial \dot{u}_k}{\partial x_k} + \frac{\partial \dot{w}_k}{\partial x_k}, \quad \text{where} \quad \dot{\Delta}_s = \frac{\partial \dot{u}_k}{\partial x_k}, \quad \dot{\Delta}_f = \frac{\partial \dot{v}_k}{\partial x_k}. \quad (15)$$

Defining K_f as the bulk modulus of the fluid, the following constitutive laws are obtained with regard to the volumetric strains of the solid and the fluid phase, respectively:

$$\dot{\Delta}_s = \frac{1}{1-n}\frac{\dot{\sigma}'}{K_s} - \frac{\dot{p}}{K_s} = \dot{\Delta}'_s + \dot{\Delta}^p_s, \qquad \dot{\sigma}' = \frac{\dot{\sigma}'_{kk}}{3}, \qquad \dot{\Delta}_f = -\frac{\dot{p}}{K_f}. \quad (16)$$

Here $\dot{\sigma}'$ is the mean effective stress rate, and the quantity $\dot{\sigma}'/(1-n) - \dot{p}$ is identified as the actual mean stress rate in the solid phase, given that the effective stresses are carried by the solid skeleton alone as discussed above.

Insertion of Eq. (16) into Eq. (15) provides a relation which defines the pore pressure rate in terms of the matrix material velocity $\dot{u}_i$ and the seepage velocity $\dot{w}_i$,

$$\dot{\Delta}_m = \frac{\partial \dot{u}_k}{\partial x_k} + \frac{\partial \dot{w}_k}{\partial x_k} = \frac{\dot{\sigma}'}{K_s} - (1-n)\frac{\dot{p}}{K_s} - n\frac{\dot{p}}{K_f} \approx -n\frac{\dot{p}}{K_f}. \quad (17)$$

Note that only the volumetric strain rate, *i.e.* the dilation rate, occurs in Eq. (17), that is the constitutive law for the pore fluid is independent of any shear deformations in the solid constituent or the fluid. On the righthand side of Eq. (17) it has been assumed that the volumetric strain rate in the solid phase is much smaller than the dilatation rate in the pore fluid. This approximation is valid for most granular materials such as soil, given that $nK_s \gg (1-n)K_f$. This is the case for sand, where n is typically around 0.2 to 0.3 and $K_s \approx 20K_f$ ($K_f \approx 2$ GPa for water). However, for rock-like materials n may be close to zero, and in this case the contributions to the total dilatation from the solid phase and the fluid phase may be of the same order of magnitude.

3. Alternative Definition of Effective Stresses—Biot Theory

The total stresses, σ_{ij}, may be divided into the pore pressure, p and the so-called effective stresses, σ'_{ij}. According to Eq. (4)

$$\sigma'_{ij} = \sigma_{ij} + p\delta_{ij}. \quad (18)$$

This definition of effective stresses was proposed by K. Terzaghi, and the components σ'_{ij} are occasionally referred to as the *Terzaghi effective stresses*. It is noted that p is positive in compression, whereas the stresses are defined as positive in tension.

With the definition given by Eq. (18), the constitutive law for the effective stresses is given in terms of "effective strains" according to Eq. (13). This implies that the volumetric strain of the solid skeleton due to a change in the pore pressure will not provide any change in the effective stresses. Alternatively, for an isotropic linear elastic material one may define the rate of change in the effective stresses as

$$\dot{\sigma}''_{ij} = \lambda\dot{\epsilon}_{kk}\delta_{ij} + 2\mu\dot{\epsilon}_{ij}, \quad (19)$$

i.e. in terms of the *total* strain rate. A comparison of Eqs. (13) and (19) implies that in the general case $\dot{\sigma}''_{ij} \neq \dot{\sigma}'_{ij}$, and therefore $\sigma''_{ij} \neq \sigma_{ij} + p\delta_{ij}$. Hence, Terzaghi's definition of effective stresses is invalid when the constitutive equation is defined in terms of effective stresses and total strains. In order to prove this, the mean effective stress rate $\dot{\sigma}'$ is computed from Eq. (13). Making use of the fact that $\delta_{kk} = 3$, the following result is obtained:

$$\dot{\sigma}' = \frac{1}{3}\dot{\sigma}'_{kk} = \frac{1}{3}\lambda\dot{\epsilon}'_{ii}\delta_{kk} + \frac{2}{3}\mu\dot{\epsilon}'_{kk} = K\dot{\epsilon}'_{kk} = K\left(\dot{\epsilon}_{kk} + \frac{\dot{p}}{K_s}\right). \tag{20}$$

Here K and K_s are the bulk moduli of the matrix material and the solid constituent, respectively,

$$K = \lambda + \frac{2}{3}\mu, \qquad K_s = \lambda_s + \frac{2}{3}\mu_s. \tag{21}$$

Next, similarly to Eq. (20), the mean effective stress rate $\dot{\sigma}''$ is found as

$$\dot{\sigma}'' = \frac{1}{3}\dot{\sigma}''_{kk} = \frac{1}{3}\lambda\dot{\epsilon}_{ii}\delta_{kk} + \frac{2}{3}\mu\dot{\epsilon}_{kk} = K\dot{\epsilon}_{kk}. \tag{22}$$

Comparison of Eqs. (20) and (23) provides the result that $\dot{\sigma}'' = \dot{\sigma}' - (K/K_s)\dot{p}$. Since a change in p provides the same change in all normal stress components and no change in the shear stress components,

$$\dot{\sigma}''_{ij} = \dot{\sigma}'_{ij} - \delta_{ij}\frac{K}{K_s}\dot{p}. \tag{23}$$

Next, Eq. (23) is integrated over time in the interval $\tau \in [t_0, t]$, where τ is the integration variable. This leads to the equation

$$\sigma''_{ij}(\mathbf{x}, t) - \sigma''_{ij}(\mathbf{x}, t_0) = \sigma'_{ij}(\mathbf{x}, t) - \sigma'_{ij}(\mathbf{x}, t_0) + \delta_{ij}\frac{K}{K_s}\left(p(\mathbf{x}, t) - p(\mathbf{x}, t_0)\right). \tag{24}$$

Since t_0 may be chosen arbitrarily, the stresses at time t must fulfill the equation

$$\sigma''_{ij} = \sigma'_{ij} - \delta_{ij}\frac{K}{K_s}p. \tag{25}$$

Thus, as an alternative to the Terzaghi effective stresses given by Eq. (23), one may apply the definition

$$\sigma''_{ij} = \sigma_{ij} + \beta p\delta_{ij}, \qquad \beta = 1 - \frac{K}{K_s}. \tag{26}$$

This formulation was originally proposed by Biot. A comparison of Eqs. (13) and (19) suggests that the *Biot effective stresses* lead to much simpler constitutive models than the Terzaghi effective stresses which must be defined in terms of "effective" strains.

It is worthwhile to note that for both granular soil and clay saturated with water, the bulk modulus of the minerals constituting the solid part of the material is much greater than the bulk modulus of the matrix, *i.e.* $K_s \gg K$. In particular, for normally or under consolidated clayey soil with large water contents $\beta \approx 1$, see Table 1. Hence, the Biot and the Terzaghi effective stresses are equivalent.

Finally, in saturated porous rock and concrete, K and K_s are of the same order of magnitude, and values of β as low as 0.5 may be expected. This fact is often neglected in geotechnical engineering practice. Here Eqs. (18) and (19) are usually applied with the erroneous assumption that $\sigma''_{ij} = \sigma'_{ij}$. Unfortunately, the Terzaghi effective normal stresses provided by Eq. (18) are smaller than, or equal to, the Biot effective normal stresses given by Eq. (26). Therefore, elastoplastic constitutive laws based on Terzaghi effective stresses

TABLE 1. Compressibility of soil, rock and concrete [1].

	Bulk Modulus[a] (MPa)		
Material	K	K_s	$\beta = (1 - K/K_s)$
Quartzitic Sandstone	17,000	37,000	0.54
Quincy Granite (30 m deep)	13,000	52,000	0.75
Vermont Marble	5,600	71,000	0.92
Concrete (approximately)	5,000	40,000	0.88
Dense Sand	56	36,000	0.9985
Loose Sand	11	36,000	0.9997
London Clay (over consolidated)	13	50,000	0.99975
Gosport Clay (normally consolidated)	1.7	50,000	0.99997

[a] Bulk moduli at $p = 98$ kPa (atmospheric pressure); water: $K_f = 2040$ MPa.

and total strains will generally provide an overestimation of the shear strength of saturated granular materials such as soil, and in particular rock, below the phreatic surface.

4. Three-dimensional Consolidation in Soil and Rock

A poroelastic continuum is considered. Disregarding any inertia forces, the incremental form of the equilibrium equation for the matrix material may be put as

$$\frac{\partial \dot{\sigma}_{ij}}{\partial x_j} + \rho \dot{g}_i = 0, \qquad \dot{g}_i = \frac{\partial g_i}{\partial t}, \tag{27}$$

where $g_i = g_i(\mathbf{x}, t)$ are the specific body forces. In soil mechanics g_i is usually the gravity field. Further, $\rho = \rho(\mathbf{x})$ is the mass density of the matrix material, cf. Eq. (3).

In a similar manner, the equilibrium for the fluid phase is provided by the equation

$$-\dot{w}_i = k_{ij} \left(\frac{\partial p}{\partial x_j} + \rho_f g_j \right), \tag{28}$$

where $\dot{w}_i$ is the seepage velocity, cf. Eq. (5), and $k_{ij} = k_{ij}(\mathbf{x})$ is a second order tensor with the SI units (m^3·s·kg^{-1}) representing the permeability of the material. In the general case k_{ij} is fully populated and asymmetric. However, in orthotropic materials only the diagonal terms have none-zero values. Typically, in stratified soil $k_{11} = k_{22} \neq k_{33}$, that is the vertical permeability is different from the horizontal permeability. In the particular simple case of isotropic porous materials $k_{ij} = \delta_{ij} k$. Thus, the permeability is defined by a single parameter, k.

Equation (28) is identified as the generalized Darcy's law for fluid flow in a porous medium. It is observed that a gradient in the pore pressure, p, which is not counterbalanced by gravitation forces, will lead to seepage with the velocity $\dot{w}_i$. The speed of the fluid flow increases with the permeability of the matrix. Further, in anisotropic materials the flow may not necessarily be in the opposite direction of the gradient. This is evidently the case for isotropic and orthotropic materials, since k_{ij} only contains diagonal terms.

In contrast to the *hydraulic conductivity*, $\chi = g \rho_f k$, with SI units (m·s^{-1}), the parameter k is independent of the fluid density and the gravitational acceleration, g. However, the components of k_{ij} still depend on the dynamic viscosity of the pore fluid, μ_f, with the SI

TABLE 2. Permeability of various soil and rock materials [2].

<table>
<tr><td>Relative Permeability</td><td colspan="4">Pervious</td><td colspan="4">Semi-Pervious</td><td colspan="5">Impervious</td></tr>
<tr><td>Unconsolidated Sand & Gravel</td><td colspan="2">Well Sorted Gravel</td><td colspan="3">Well Sorted Sand or Sand & Gravel</td><td colspan="4">Very Fine Sand, Silt Loess, Loam</td><td colspan="4"></td></tr>
<tr><td>Unconsolidated Clay & Organic</td><td colspan="4"></td><td colspan="2">Peat</td><td colspan="3">Layered Clay</td><td colspan="4">Fat / Unweathered Clay</td></tr>
<tr><td>Consolidated Rocks</td><td colspan="4">Highly Fractured Rock</td><td colspan="3">Oil Reservoir Rocks</td><td colspan="2">Fresh Sandstone</td><td colspan="2">Fresh Limestone, Dolomite</td><td colspan="2">Fresh Granite</td></tr>
<tr><td>κ (cm^2)</td><td>10^{-3}</td><td>10^{-4}</td><td>10^{-5}</td><td>10^{-6}</td><td>10^{-7}</td><td>10^{-8}</td><td>10^{-9}</td><td>10^{-10}</td><td>10^{-11}</td><td>10^{-12}</td><td>10^{-13}</td><td>10^{-14}</td><td>10^{-15}</td></tr>
<tr><td>κ (milli-darcies)</td><td>10^{8}</td><td>10^{7}</td><td>10^{6}</td><td>10^{5}</td><td>10^{4}</td><td>10^{3}</td><td>100</td><td>10</td><td>1</td><td>0.1</td><td>0.01</td><td>10^{-3}</td><td>10^{-4}</td></tr>
</table>

units (kg$\cdot$m$^{-1}\cdot$s^{-1}). An alternative measure of the permeability that only depends on the geometry of the soil skeleton may be defined as

$$\kappa = \mu_f k. \tag{29}$$

This parameter is coined the *permeability coefficient* and has the SI units (m^2), but it is usually measured in *darcy* (d), or more commonly *milli-darcy* (md) (1 darcy $\sim 10^{-12}$ m^2). Typical values for soil and rock are listed in Table 2.

Finally, mass balance of the flow is ensured by Eq. (17). Making use of Eqs. (7) and (28), the balance equation (17) may in turn be written

$$\frac{\partial}{\partial x_i}\left\{k_{ij}\left(\frac{\partial p}{\partial x_j}+\rho_f g_j\right)\right\}+\frac{\dot{\sigma}'}{K_s}-(1-n)\frac{\dot{p}}{K_s}-n\frac{\dot{p}}{K_f}-\dot{\Delta}_s=0, \tag{30}$$

where $\dot{\sigma}'$ is the effective mean stress rate, cf. Eq. (20), and $\dot{\Delta}_s$ is the dilation rate in the solid phase, cf. (15).

Next, for the purpose of analysing consolidation in a fully saturated porous elastic material it is useful to divide the total pore pressure $p(\mathbf{x}, t)$ into the steady state pore pressure $p_s = p_s(\mathbf{x})$ and the excess pore pressure $p_e = p_e(\mathbf{x}, t)$, that is

$$p = p_s + p_e. \tag{31}$$

In the steady state, the volume of pore fluid inside a give control volume will be constant, *i.e.* independent of time. Therefore, according to Eq. (28),

$$-\left.\frac{\partial \dot{w}_i}{\partial x_i}\right|_{\textit{steady state}} = \frac{\partial}{\partial x_i}\left\{k_{ij}\left(\frac{\partial p_s}{\partial x_j}+\rho_f g_j\right)\right\}=0. \tag{32}$$

Inserting this result into Eq. (30), the governing equation for the development of the excess pore pressure, p_e, over time is achieved in the form

$$\frac{\partial}{\partial x_i}\left\{k_{ij}\frac{\partial p_e}{\partial x_j}\right\}+\frac{\dot{\sigma}'}{K_s}-(1-n)\frac{\dot{p}_e}{K_s}-n\frac{\dot{p}_e}{K_f}-\dot{\Delta}_s=0, \tag{33}$$

where use has been made of the fact that $\dot{p}_s = 0$, *i.e.* $\dot{p} = \dot{p}_e$.

Equation (33) may be solved simultaneously with the equation of equilibrium (27). For complicated geometries, the coupled equations (27) and (33) may be discretized over the volume and solved by means of, for example, the finite difference or the finite element method. Initial values must be provided for the displacement field and the seepage over

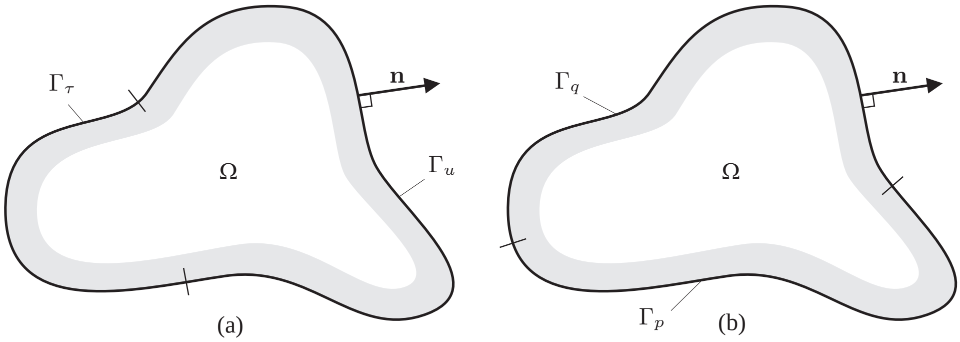

FIGURE 4. Poroelastic domain, Ω, with division of the boundary, Γ, for the definition of boundary conditions concerning (a) force equilibrium, and (b) seepage.

the entire volume. Further, Dirichlet or Neumann conditions must be supplied for both the displacement and the seepage velocity along the boundary Γ, cf. Fig. 4. Thus, for the equilibrium equation (27) the boundary conditions must be given in the form:

$$\left.\begin{aligned} u_i &= \bar{u}_i \quad \text{for} \quad \mathbf{x} \in \Gamma_u \\ \tau_i &= \bar{\tau}_i \quad \text{for} \quad \mathbf{x} \in \Gamma_\tau \end{aligned}\right\} \quad \text{where} \quad \Gamma_u \cup \Gamma_\tau = \Gamma \quad \wedge \quad \Gamma_u \cap \Gamma_\tau = \emptyset, \tag{34}$$

where $\tau_i = \sigma'_{ij} n_j$ defines the surface traction and $(\bar{u}_i, \bar{\tau}_i)$ denote the prescribed values. Here $n_j = n_j(\mathbf{x})$ are the components of the unit outward normal to the boundary Γ. Similarly, for the seepage velocity

$$\left.\begin{aligned} p_i &= \bar{p}_i \quad \text{for} \quad \mathbf{x} \in \Gamma_p \\ q_i &= \bar{q}_i \quad \text{for} \quad \mathbf{x} \in \Gamma_q \end{aligned}\right\} \quad \text{where} \quad \Gamma_p \cup \Gamma_q = \Gamma \quad \wedge \quad \Gamma_p \cap \Gamma_q = \emptyset, \tag{35}$$

Here $q_i = \partial \dot{w}_i / \partial x_j \, n_j$ is the flux of pore water through the boundary. Along impermeable surfaces, the flux is $\dot{q}_i = 0$, *i.e.* there is no flow of pore fluid through the boundary. On the other hand the pore pressure will be known along free surfaces. Finally it is noted that in general there is no relation between the boundary subsets (Γ_u, Γ_τ) and (Γ_p, Γ_q). However, since the traction and the pore pressure are likely to be described along the same part of the boundary, *e.g.* at a free surface, most often $\Gamma_\tau = \Gamma_p$ and $\Gamma_u = \Gamma_q$.

5. Drained and Undrained Conditions

As indicated in Table 2, the permeability of highly fractured rock and well sorted sand and gravel is very high. Therefore, the pore water drains away almost immediately when the matrix material is subjected to stress. In other words, there is no excess pore pressure, *i.e.* $p_e = \dot{p}_e = 0$, and the pore pressure $p = p_s$ is governed by Eq. (32). Clearly, the steady state pore pressure is decoupled from the displacements of the skeleton. Thus, p may be calculated first, and subsequently the displacement field u is computed by solution of the equation

$$\frac{\partial \sigma'_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i = 0, \qquad \sigma'_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} + \frac{K}{K_s} p \delta_{ij} \tag{36}$$

with known p. For the analysis of nonlinear behaviour of the soil skeleton, Eq. (36) may instead be solved in incremental form. Here it may be utilized that $\dot{p} = 0$, that is the entire stress increment is carried by the soil skeleton as effective stresses.

Finally, clay subject to rapid loading is almost fully undrained. Thus, $k_{ij} \approx 0$ (see Table 2) and according to Eq. (28) the seepage velocity is $\dot{w}_i = 0$. Hence, by Eqs. (7) and (17) it follows that

$$\frac{\dot{\sigma}'}{K_s} - \dot{\epsilon}_{kk} = (1-n)\frac{\dot{p}}{K_s} + n\frac{\dot{p}}{K_f} \qquad \Rightarrow \qquad \dot{p} \approx -\frac{K_f}{n}\dot{\epsilon}_{kk} \tag{37}$$

In the approximation it has been assumed that the bulk modulus of the solid phase is very high compared with the other quantities. This is realistic for clay minerals.

Now, by Eqs. (4) and (13) the total stress in the matrix material may be written as

$$\dot{\sigma}_{ij} = \lambda\dot{\epsilon}_{kk}\delta_{ij} + 2\mu\dot{\epsilon}_{ij} + \left(\frac{K}{K_s} - 1\right)\dot{p}\delta_{ij}. \tag{38}$$

Inserting the pore pressure defined by Eq. (38), the governing equation for the undrained poroelastic material is expressed in terms of total stresses and strains:

$$\frac{\partial\dot{\sigma}_{ij}}{\partial x_j} + \rho\dot{g}_i = 0, \qquad \dot{\sigma}_{ij} \approx \lambda\dot{\epsilon}_{kk}\delta_{ij} + 2\mu\dot{\epsilon}_{ij} + \frac{K_f}{n}\dot{\epsilon}_{kk}\delta_{ij}. \tag{39}$$

Here, it has been assumed that $K_s \gg K$. As discussed above, this is a fair approximation for clay with a high porosity; but for intact rock the first term in the parenthesis in Eq. (38) has to be included. However, the approximation in Eq. (37) may still be valid.

6. Conclusions

The theory for a saturated poroelastic material has been presented. Firstly, the definition of effective stresses has been discussed, and it has been shown that the stress approach originally proposed by K. Terzaghi is accurate for clay, sand and similar residual soils. However, constitutive models for saturated rock and concrete are better formulated in terms of the effective stress measure proposed by M.A. Biot. Secondly, the theory for the consolidation process in three dimensions has been explained. Based on Darcy's law for the quasi-static flow in a porous medium, the governing equations for the development of excess pore pressure have been derived. Finally, the equilibrium equations for the particular cases of perfectly drained and undrained behaviour have been presented.

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EXERCISE: THE LIGHTHOUSE ON THE ROCK

An old lighthouse stands on a small granite rock in the Atlantic Sea near Stavanger, Norway. The rock raises 25 metres above the seabed and as indicated on Fig. 5 it is permeable and fully saturated with seawater—also above the phreatic level. Four triaxial compression tests have been carried out on a dry intact cylindrical specimen of the rock. The chamber pressure σ_3 in the four tests is equal to 100, 200, 300 and 400 kPa, respectively. Note that *compression is defined as positive*. In each test, the chamber pressure is kept constant while the piston pressure, σ_1, is increased. This results in a stress deviation $q = \sigma_1 - \sigma_3$ which will eventually lead to failure in the material. The stress–strain curves obtained in the four compression tests are plotted in Fig. 6 along with the volumetric strain history. Since the specimen is dry, all stresses in the triaxial test are effective, *e.g.* $q = q'$.

Question 1. Based on the triaxial compression tests, determine E and ν for the matrix material, *i.e.* the rock. Further, verify that the bulk modulus $K \approx 15$ GPa. Do you get the same result from the hydrostatic step and the triaxial compression step?

Question 2. Failure in the rock is assumed to be governed by the Mohr-Coulomb criterion. In terms of the effective stress deviation, $q' = \sigma_1' - \sigma_3'$, and the effective confining pressure, σ_3', the failure criterion takes the form (compressive stresses are defined as positive):

$$q'\,(1 - \sin\phi) - 2\sigma_3' \sin\phi - 2c\cos\phi = 0.$$

Determine the value of the cohesion, c, and the angle of friction, ϕ, from Fig. 6.

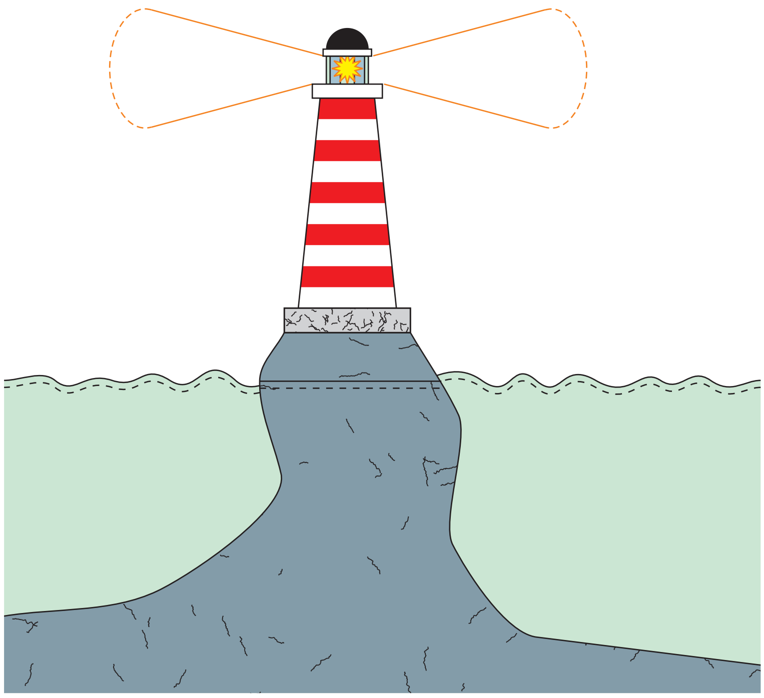

FIGURE 5. A lighthouse on a nearly cylindrical rock.

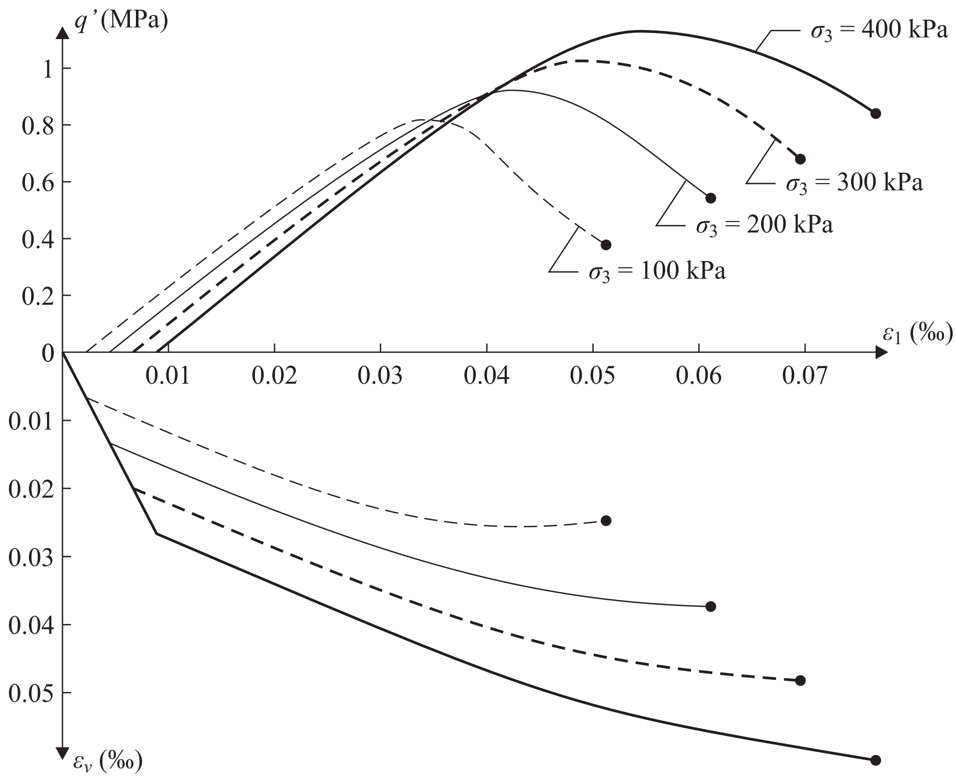

FIGURE 6. Stress–strain curves and volumetric strain recorded in triaxial compression tests for different values of the chamber pressure σ_3.

The rock is simplified as a circular cylinder with the diameter 10 m. It is fully saturated and extends 5 metres above the phreatic level and 20 metres below. It is assumed that the vertical normal stresses are distributed uniformly over a horizontal cross section and that no effective stresses are developed in the horizontal directions, *i.e.* $\sigma_2' = \sigma_3' = 0$. Further, the lighthouse has the total weight G, and the solid granite has the bulk modulus $K_s = 50$ GPa and the density $\rho_s = 2700$ kg/m^3. Finally, the porosity is $n = 0.05$.

Question 3. Sketch the pore pressure and the total vertical normal stress in the rock as functions of the depth below the sea level. The actual values cannot be determined at this stage, since the weight of the lighthouse, G, is yet unknown.

Question 4. Plot the vertical *Terzaghi* effective normal stress, σ_1', as function of the depth beneath the sea level. What is the maximum possible weight of the lighthouse before the rock collapses based on σ_1' and σ_3'?

Question 5. Plot the vertical *Biot* effective normal stress, σ_1'', as function of the depth beneath the sea level. Determine the maximum possible weight of the lighthouse G on the basis of σ_1'' and σ_3''. Compare and discuss the results of Questions 4 and 5.

SOLUTION

Question 1. Young's modulus, E, is determined as the ratio between the increment in axial stresses and strains at constant chamber pressure, $\sigma_2 = \sigma_3 =$ constant. The stress–strain curves for the triaxial compression tests have almost identical slopes, independently of the chamber pressure. From the (ϵ_1, q')-curve for $\sigma_3 = 400$ kPa (see Fig. 7) we get that

$$dq' = E d\epsilon_1 \qquad \Rightarrow \qquad E = \frac{dq'}{d\epsilon_1} \approx \frac{0.84 \cdot 10^6}{0.037 \cdot 10^{-3} - 0.008 \cdot 10^{-3}} \approx 29 \text{ GPa.}$$

Secondly, an increment in the volume strain, i.e. the dilation, during the elastic part of the triaxial compression test is given as

$$d\epsilon_2 = d\epsilon_3 = -\nu d\epsilon_1 \qquad \Rightarrow \qquad d\epsilon_v = d\epsilon_1 + d\epsilon_2 + d\epsilon_3 = (1 - 2\nu)\, d\epsilon_1.$$

Hence, Poisson's ratio may be found from the slope of the volumetric strain curve,

$$\nu = \frac{1}{2}\left(1 - \frac{d\epsilon_v}{d\epsilon_1}\right) \approx \frac{1}{2}\left(1 - \frac{0.045 \cdot 10^{-3} - 0.026 \cdot 10^{-3}}{0.037 \cdot 10^{-3} - 0.008 \cdot 10^{-3}}\right) \approx 0.17.$$

A Poisson ratio of about 0.15 to 0.2 is expected for cemented rock, *i.e.* the value $\nu = 0.17$ appears to be realistic.

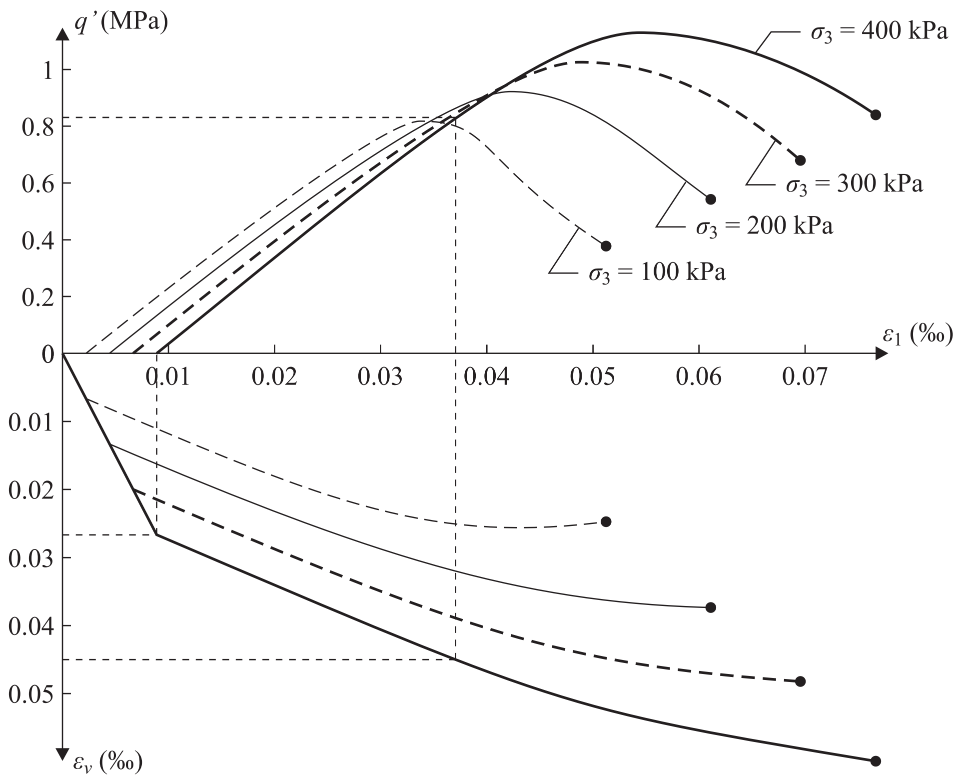

FIGURE 7. Points on the stress–strain curve and the volumetric strain curve utilised for the determination of Young's modulus and Poisson's ratio, respectively.

Finally, based on Young's moduls and Poisson's ratio, the bulk modulus of the matrix material is determined as

$$K = \frac{E}{3\,(1-2\nu)} \approx 15 \text{ GPa}$$

Alternatively, the bulk modulus is determined from the hydrostatic part of the triaxial test. Again making use of the results for $\sigma_3 = 400$ kPa, we get

$$d\sigma_3 = d\sigma = K d\epsilon_v \quad \Rightarrow \quad K = \frac{d\sigma_3}{d\epsilon_v} \approx \frac{400 \cdot 10^3}{0.026 \cdot 10^{-3}} \approx 15 \text{ GPa},$$

where σ is the mean stress. Clearly we obtain the same results in the hydrostatic step and the triaxial compression step.

Question 2. By inspection of the results of the triaxial compression tests, we observe that $q'_u(\sigma_3 = 100 \text{ kPa}) \approx 0.81$ MPa $= 810$ kPa and $q'_u(\sigma_3 = 300 \text{ kPa}) \approx 1.02$ MPa $= 1020$ kPa (see Fig. 8). Then, we get the linear relationship

$$q'_u(\sigma_3) = \frac{1020-810}{300-100}\sigma_3 + 810 \text{ kPa} - \frac{1020-810}{300-100} \cdot 100 \text{ kPa} = 1.05\sigma_3 + 705 \text{ kPa}.$$

Next, the failure criterion may be written as

$$q'\,(1-\sin\phi) - 2\sigma'_3 \sin\phi - 2c\cos\phi = 0 \quad \Rightarrow \quad q'(\sigma_3) = \frac{2\sin\phi}{1-\sin\phi}\sigma_3 + \frac{2\cos\phi}{1-\sin\phi}c.$$

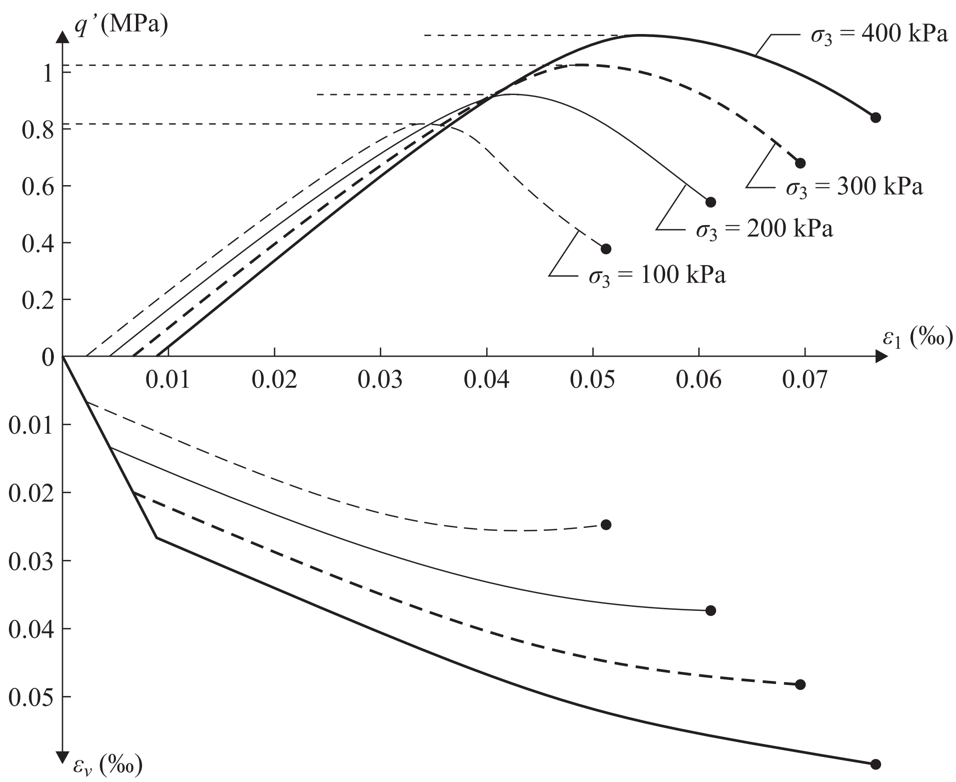

FIGURE 8. Points on the stress–strain curve used for evaluation of the friction angle and the cohesion of rock.

Thus, comparing the coefficient in the linear term, the angle of friction is obtained as

$$\frac{2\sin\phi}{1-\sin\phi} = 1.05 \quad \Rightarrow \quad \sin\phi = \frac{1.05}{2+1.05} \quad \Rightarrow \quad \phi \approx 20^\circ,$$

and by comparison of the constant terms we get the cohesion,

$$\frac{2\cos\phi}{1-\sin\phi}c = 705\ \text{kPa} \quad \Rightarrow \quad c = \frac{1-\sin\phi}{2\cos\phi} \cdot 705\ \text{kPa} \approx 250\ \text{kPa}.$$

Question 3. The mass density of the solid phase is $\rho_s = 2700$ kg/m^3 and the mass density of the pore fluid (sea water) is estimated as $\rho_f = 1020$ kg/m^3. Hence, with the porosity $n = 0.05$ and application of Eq. (3), the total density of the matrix material becomes

$$\rho = (1-n)\rho_s + n\rho_f = (1-0.05)\cdot 2700\ \text{kg/m}^3 + 0.05 \cdot 1020\ \text{kg/m}^3 = 2616\ \text{kg/m}^3.$$

The specific weight of rock and water is determined by multiplication with the gravitational acceleration, *i.e.*

$$\gamma_f = \rho_f g = 1020\ \text{kg/m}^3 \cdot 9.82\ \text{m/s}^2 = 10.02\ \text{kN/m}^3,$$
$$\gamma = \rho g = 2616\ \text{kg/m}^3 \cdot 9.82\ \text{m/s}^2 = 25.67\ \text{kN/m}^3.$$

The rock has the height $h_1 = 5$ m above the sea level and $h_2 = 20$ m below the surface. At the bottom of the cliff, the total vertical stress is computed as

$$\sigma_{1,bottom} = (h_1 + h_2)\gamma + G/A = 642\ \text{kPa} + G/A, \qquad A = \pi r^2,$$

where $r = 5$ m is the radius of the cylindrical rock and G is the weight of the lighthouse. Similarly, at the top of the rock, the total stress is equal to G/A.

The pore pressure is zero at the mean water table, which is the equivalent of the ground water table. Above the mean water table, the pore pressure is negative due to capillarity, and below the mean water table, the pore pressure is assumed to be equal to the hydrostatic pressure. Thus, any dynamic contributions from ocean waves or current are disregarded, and the pore pressure at the top and bottom of the cliff become

$$p_{top} = -h_1\gamma_f = -50\ \text{kPa} \quad \text{and} \quad p_{bottom} = h_2\gamma_f = 200\ \text{kPa},$$

respectively. These values are utilised to plot the variation of the pore pressure and total vertical stresses with depth in Fig. 9.

Question 4. A visualisation of the Terzaghi effective stresses is provided in Fig. 9. Clearly, the greatest value of σ_1' is expected at the bottom of the cliff. According to the derivations in the answer to Question 2 and with $\sigma_3' = 0$, the rock collapses if the vertical effective stress reaches the value $\sigma_1' = q_u' = 705$ kPa at the bottom of the sea. Then, with Terzaghi's definition of effective stresses, the maximum weight of the lighthouse is found as the solution to the equation

$$q_u' = \sigma_{1,bottom} - p_{bottom} = 642\ \text{kPa} - 200\ \text{kPa} + G/A = 705\ \text{kPa}.$$

We then get that $G = 20.7$ MN. It has been assumed that no other forces than the self-weight due to gravitation act on the lighthouse and the rock. This is obviously very unrealistic, and in reality a great contribution to the stresses will stem from wave and wind forces on the structure and the cliff.

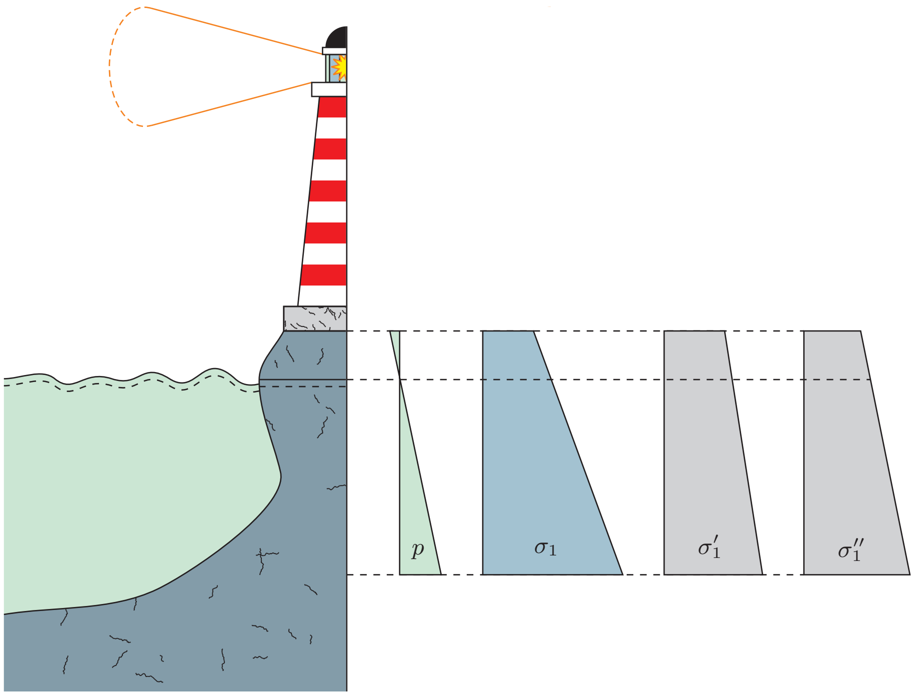

FIGURE 9. Vertical stresses and pore pressure as functions of the depth below the top of the cliff. Note that the Terzaghi effective stress $\sigma_1' = \sigma_1 - p$ is greater than the Biot effective stress $\sigma_1'' = \sigma_1 - \beta p$ at the top of the cliff and smaller at the bottom of the cliff.

Question 5. Firstly, we calculate $\beta = 1 - K/K_s = 1 - 15/50 = 0.7$. Next, with Biot's effective stresses are sketched in Fig. 9, and it is noted that increase of σ_1'' with depth is slightly greater than that of Terzaghi's effective stress, σ_1'. Hence, at the bottom of the cliff $\sigma_1'' > \sigma_1'$ and the weight of the lighthouse is determined by

$$q_u'' = \sigma_{1,bottom} - \beta p_{bottom} = 642 \text{ kPa} - 0.7 \cdot 200 \text{ kPa} + G/A = 705 \text{ kPa},$$

which provides $G = 15.9$ MN. Again it has been assumed that no other forces than gravity act on the lighthouse and the rock.

Comparing the results of the computations with the two definitions of effective stresses, it becomes clear that Terzaghi's definition leads to a significantly higher load-carrying capacity than Biot's definition. Assuming that the lighthouse is made of concrete, the construction material has a specific weight of approximately 25 kN/m^3. Then, Terzaghi's theory provide the erroneous result that approximately 830 m^3 of concrete can be used, whereas only about 640 m^3 can be used according to the exact theory by Biot.

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