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OPTIMAL MAGNETIC ATTITUDE CONTROL

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Abstract: Magnetic torquing is attractive as means of control for small satellites. The actuation principle is to use the interaction between the earth's magnetic field and a magnetic field generated by a coil set in the satellite. This control principle is inherently time-varying, and difficult to use because control torques can only be generated perpendicular to the local geomagnetic field vector. This has been a serious obstacle for using magnetorquer based control for three-axis attitude control. This paper deals with three-axis stabilization of a low earth orbit satellite. The problem of controlling the spacecraft attitude using only magnetic torquing is realized in the form of a infinite horizon periodic controller.

Keywords: Attitude control, satellite control, time-varying systems, quaternion feedback.

1. INTRODUCTION

Several control methods have been developed over the past years since the first satellite was launched. Generally speaking all those techniques may be classified as active or passive. Active techniques are needed for missions where high pointing accuracy is vital, and typical actuators are: gas jets, reaction or momentum wheels and electromagnetic coils. Magnetic torquing addressed in this paper is attractive for small, cheap satellites in low Earth orbits. Magnetic control systems are relatively lightweight, require low power and are inexpensive.

There is extensive literature covering satellite attitude control design. Though most of the algorithms presented assume application of reaction wheels and/or thrusters. Attitude control with use of magnetorquers has the significant limitation that the control torque is always perpendicular to the local geomagnetic field vector. The problem of three-axis magnetic control was addressed in Martel et al. (1988), where the linearized time-varying satellite motion model was approximated by a linear time-invariant counterpart. Three-axis stabilization with use of magnetic torquing of a satellite without appendages was treated by Wisniewski (1998), where sliding control was proposed to stabilize a tumbling satellite. General concept presented in the previous work was to calculate desired control torque and project it on the plane perpendicular to the geomagnetic field vector.

In this paper the attitude control problem is solved by an infinite horizon periodic control. The controller gain is a periodically changing dependent on the local geomagnetic field. The paper is organized as follows. Section 2 addresses linearization technique for a LEO satellite motion. It is shown that a satellite on a near polar orbit actuated by a set of perpendicular magnetorquers may be considered as a periodic system. The solution of the Riccati equation is essential for the control synthesis of the periodic systems. Section 3 is devoted to study of the periodic Riccati equation and related problems. The features of the periodic Riccati equation: properties of its solutions, the methods to calculate the solutions are very similar to the algebraic Riccati equation. The problem of infinite horizon controller is formulated in 4 and the simulation results are given in 5.

2. MAGNETIC ACTUATED SATELLITE AS A LINEAR PERIODIC SYSTEM

The satellite considered in this study is modeled as a rigid body in the Earth gravitational field influenced by the aerodynamic drag torque and the control torque generated by the magnetorquers

$$\boldsymbol{N}_{\mathrm{ctrl}}(t) = \boldsymbol{m}(t) \times \boldsymbol{B}(t), \tag{1}$$

where $\boldsymbol{N}_{\mathrm{ctrl}}$ is the control torque, and $\boldsymbol{B}(t)$ is the magnetic flux vector of the Earth. The magnetic moment, $\boldsymbol{m}$, is considered as the control signal in the following.

The attitude is parameterized by the unit quaternion providing a singularity free representation of the kinematics. The details about mathematical modeling of a LEO satellite can be found in Wisniewski (1996). In this paper only a satellite linear model will be investigated.

2.1 *Coordinate Systems*

The motion of a spacecraft is related to three coordinate systems: Principal Coordinate System (PCS), built on the spacecraft principal axes, a Local-Vertical-Local-Horizontal Coordinate System (LVLH) referring to the current position of the satellite in orbit, and an Earth Centered Inertial Coordinate System (SCI), an inertial frame with the origin in the Earth's center of mass. These coordinate systems are denoted in the text by the following indices: PCS by p, LVLH by o SCI by i, thus e.g. $\boldsymbol{\Omega}_{\mathrm{pi}}$ is the velocity of the PCS rotation in SCI.

2.2 *Linearized Equation of Motion*

The attitude control problem can be considered from a linear and purely nonlinear point of view. If performance of the control system is analyzed in the neighbourhood of the reference point (steady-state performance) a linear model of the spacecraft's motion is sufficient.

The satellite motion is considered in a neighbourhood of the following reference: the angular velocity of the satellite rotation (PCS) in LVLH is zero ($\boldsymbol{\Omega}_{\mathrm{po}} = \mathbf{0}$), and the attitude is such that PCS coincides with LVLH, i.e. the attitude quaternion describing the rotation of PCS in LVLH equals the identity, ${}^{\mathrm{p}}_{\mathrm{o}}\boldsymbol{q} = \boldsymbol{e}$.

The angular velocity of PCI in LVLH (Ω_{pi}) belongs to $\mathbb{E}^3$ therefore linearization of the angular velocity is commonplace and based on the first order extension of the Taylor series

$$\left(\boldsymbol{\Omega}_{\mathrm{pi}}\right)_{\mathrm{p}} = {}^{\mathrm{p}}_{\mathrm{o}}\mathbf{A}[0 \; -\omega_{\mathrm{o}} \; 0]^{\mathrm{T}} + \Delta\boldsymbol{\Omega}, \tag{2}$$

where $\Delta\boldsymbol{\Omega}$ is an infinitesimal perturbation of the angular velocity $\boldsymbol{\Omega}_{\mathrm{pi}}$ from the reference, ${}^{\mathrm{p}}_{\mathrm{o}}\mathbf{A}$ is the rotation matrix from LVLH to PCS, ω_{o} is the orbital rate. The subscript p means that the angular velocity vector is resolved in PCS.

Linearization of the attitude parameters is different. They geometrically form a differential manifold $\mathrm{SO}_3(\mathbb{R})$ for the rotation matrices and $\mathbb{S}^3$ for the unit quaternions. infinitesimal perturbations of a real orthogonal matrix can be written as $\mathbf{A} = \mathbf{E} + \epsilon$, where ϵ is an infinitesimal anti-symmetric 3 by 3 matrix

$$\epsilon = \begin{bmatrix} 0 & \theta_{\mathrm{z}} & -\theta_{\mathrm{z}} \\ -\theta_{\mathrm{z}} & 0 & \theta_{\mathrm{z}} \\ \theta_{\mathrm{y}} & -\theta_{\mathrm{x}} & 0 \end{bmatrix}. \tag{3}$$

The nonzero components of ϵ: θ_{x}, θ_{y} and θ_{z} can be used for parameterization of a linear approximation of the spacecraft motion. In fact, it can be shown that the sum of the quaternion $\frac{1}{2}\boldsymbol{\Theta}$ (having zero scalar part) and the unit quaternion represents infinitesimal rotation. For this purpose define

$$\left[\theta_{\mathrm{x}} \; \theta_{\mathrm{y}} \; \theta_{\mathrm{z}}\right]^{\mathrm{T}} = \boldsymbol{\Theta} \equiv 2\,\Delta\boldsymbol{q} \tag{4}$$

and consider a quaternion $\Delta\tilde{\boldsymbol{q}} = \boldsymbol{e} + \Delta\boldsymbol{q}$, see Jurdjevic (1997). This is a unit quaternion since $\Delta\boldsymbol{q}$ is infinitesimal, hence $\langle\Delta\boldsymbol{q}, \Delta\boldsymbol{q}\rangle = 0$. Now the mapping $\mathbf{A} : \mathbb{S}^2 \rightarrow \mathrm{SO}_3(\mathbb{R})$ is applied in order to show that that the unit quaternion $\Delta\tilde{\boldsymbol{q}}$ can be used for representation of a infinitesimal rotation.

$$\mathbf{A}(\Delta\tilde{\boldsymbol{q}}) = \mathbf{E} + 2\mathbf{S}(\Delta\boldsymbol{q}) = \epsilon, \tag{5}$$

where $\mathbf{E}$ is 3 by 3 identity matrix and $\mathbf{S}$ is a mapping of a vector to an anti-symmetric matrix, such that $S(\boldsymbol{\Theta}) = \epsilon$ in Eq. (3). The quaternion $\Delta\tilde{\boldsymbol{q}}$ has a nice physical interpretation

$$\Delta\tilde{\boldsymbol{q}} = \cos\frac{\Delta\phi}{2}\boldsymbol{e} + \sin\frac{\Delta\phi}{2}\boldsymbol{\epsilon}, \tag{6}$$

where $\boldsymbol{\epsilon}$ gives the axis of rotation, and $\Delta\phi$ is the angle of rotation. But for a small angle $\Delta\phi$ ($\sin\frac{\Delta\phi}{2} = \frac{\Delta\phi}{2}$ and $\cos\frac{\Delta\phi}{2} = 1$)

$$\Delta\tilde{\boldsymbol{q}} = \boldsymbol{e} + \frac{\Delta\phi}{2}\boldsymbol{\epsilon}. \tag{7}$$

Therefore, only the vector par of the unit quaternion will be used for the local attitude representation.

2.3 *Linearized Dynamics*

The equation of dynamics is divided into the cross coupling, the contribution of the gravity gradient torque, aerodynamic torque and the part due to control torque. Modeling of the satellite dynamics is treated in details in Wertz (1990), here only the aerodynamic drag model is addressed.

2.3.1. *Aerodynamic Torque* The aerodynamic drag is the main disturbance torque acting on LEO spacecraft. Its magnitude can be as large as 10^{-1} Nm for orbits with 100 km altitude. Assuming that the energy of the molecules is totally absorbed on impact with the spacecraft, the force $\mathrm{d}\boldsymbol{f}_{\mathrm{aero}}$ on a surface element $\mathrm{d}A$ is described by

$$\mathrm{d}\boldsymbol{f}_{\mathrm{aero}} = -\frac{1}{2}\mathrm{C_D}\rho \mathrm{v}^2(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v}\mathrm{dA}, \qquad (8)$$

where $\boldsymbol{n}$ is an outward normal to the surface, $\boldsymbol{v}$ is the unit vector in the direction of the translational velocity of the surface element relative to the incident stream of the molecules. The atmospheric density is denoted by ρ, and the drag coefficient by C_{D}. The total aerodynamic torque is the sum of the torques acting on individual parts of the satellite

$$\boldsymbol{N}_{\mathrm{aero}} = \sum_{i=1}^{k} \boldsymbol{r}_{\mathrm{i}} \times \boldsymbol{F}_{\mathrm{i}}, \qquad (9)$$

where $\boldsymbol{r}_{\mathrm{i}}$ is the vector from the spacecraft center of mass to the center of pressure of the i-th element.

Eq. (9) can be furthermore decomposed into a sum of 3 surfaces: A_1 perpendicular to the x-axis of PCS, the cross section surface perpendicular to the y-axis of PCS, A_2, and the cross section surface perpendicular to the z-axis of PCS, A_3.

$$\begin{aligned}\boldsymbol{N}_{\mathrm{aero}} = \frac{1}{2}C_{\mathrm{D}}\rho v^2 &\left(A_1\left([1\ 0\ 0]^{\mathrm{T}}\cdot\boldsymbol{i}_{\mathrm{o}}\right)\boldsymbol{i}_{\mathrm{o}}\times\boldsymbol{r}_1\right.\\ &+ A_2\left([0\ 1\ 0]^{\mathrm{T}}\cdot\boldsymbol{i}_{\mathrm{o}}\right)\boldsymbol{i}_{\mathrm{o}}\times\boldsymbol{r}_2\\ &\left.+ A_3\left([0\ 0\ 1]^{\mathrm{T}}\cdot\boldsymbol{i}_{\mathrm{o}}\right)\boldsymbol{i}_{\mathrm{o}}\times\boldsymbol{r}_3\right), \end{aligned} \qquad (10)$$

where $\boldsymbol{i}_{\mathrm{o}}$ is the unit vector in the direction of the translational velocity vector (the direction of the x-axis of LVLH). Thus the linear approximation of the aerodynamic torque is

$$\boldsymbol{N}_{\mathrm{aero}} = \frac{1}{2}C_{\mathrm{D}}\rho v^2 A_1\left[[0\ -r_{1\mathrm{z}}\ r_{1\mathrm{y}}\right]^{\mathrm{T}} \qquad (11)$$
$$+C_{\mathrm{D}}\rho v^2\begin{bmatrix}0 & -A_1 r_{1\mathrm{y}} & -A_1 r_{1\mathrm{z}}\\ 0 & A_1 r_{1\mathrm{x}} - A_3 r_{3\mathrm{z}} & A_2 r_{2\mathrm{z}}\\ 0 & A_3 r_{3\mathrm{y}} & A_1 r_{1\mathrm{x}} - A_2 r_{2\mathrm{y}}\end{bmatrix}\Delta\boldsymbol{q}.$$

Eq. (12) consists of two parts: constant and dependent on the spacecraft's attitude.

2.4 *Linearized Equation of Spacecraft's Motion*

The matrix form of the linearized spacecraft's motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\Delta\boldsymbol{\Omega}\\ \Delta\boldsymbol{q}\end{bmatrix} = \mathbf{A}\begin{bmatrix}\Delta\boldsymbol{\Omega}\\ \Delta\boldsymbol{q}\end{bmatrix} + \mathbf{B}(t)\mathbf{m}, \qquad (12)$$

where

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix},$$

$$\mathbf{A}_{11} = \begin{bmatrix}0 & 0 & -\sigma_{\mathrm{x}}\omega_{\mathrm{o}}\\ 0 & 0 & 0\\ -\omega_{\mathrm{o}}\sigma_{\mathrm{z}} & 0 & 0\end{bmatrix}, \ \mathbf{A}_{21} = \frac{1}{2}\mathbf{E},$$

$$\mathbf{A}_{22} = \begin{bmatrix}0 & 0 & \omega_{\mathrm{o}}\\ 0 & 0 & 0\\ -\omega_{\mathrm{o}} & 0 & 0\end{bmatrix}, \ \mathbf{A}_{12} =$$

$$\begin{bmatrix}-6\omega_{\mathrm{o}}^2\sigma_{\mathrm{x}} & -k_{1\mathrm{y}}/I_{\mathrm{x}} & -k_{1\mathrm{z}}/I_{\mathrm{x}}\\ 0 & 6\omega_{\mathrm{o}}^2\sigma_{\mathrm{y}} + (k_{1\mathrm{x}} - k_{3\mathrm{z}})/I_{\mathrm{y}} & k_{2\mathrm{z}}/I_{\mathrm{y}}\\ 0 & k_{3\mathrm{y}}/I_{\mathrm{z}} & (k_{1\mathrm{x}} - k_{2\mathrm{y}})/I_{\mathrm{z}}\end{bmatrix},$$

$$\mathbf{B}(t) = \begin{bmatrix}\mathbf{I}^{-1}\begin{bmatrix}0 & -B_{\mathrm{oz}}(t) & B_{\mathrm{oy}}(t)\\ B_{\mathrm{oz}}(t) & 0 & -B_{\mathrm{ox}}(t)\\ -B_{\mathrm{oy}}(t) & B_{\mathrm{ox}}(t) & 0\end{bmatrix}\\ \mathbf{0}\end{bmatrix},$$

$k_{\mathrm{ij}} = C_{\mathrm{D}}\rho v^2 A_{\mathrm{i}} r_{\mathrm{ij}}$ for $\mathrm{i} = 1, 2, 3$ and $\mathrm{j} = \mathrm{x}, \mathrm{y}, \mathrm{z}$. ω_{o} is the orbital rate, $\mathbf{I} = \mathrm{diag}([\mathrm{I_x}\ \mathrm{I_y}\ \mathrm{I_z}]^{\mathrm{T}})$ is the inertia tensor resolved in PCS. $\sigma_{\mathrm{x}} = \frac{I_{\mathrm{y}}-I_{\mathrm{z}}}{I_{\mathrm{x}}}$, $\sigma_{\mathrm{y}} = \frac{I_{\mathrm{z}}-I_{\mathrm{x}}}{I_{\mathrm{y}}}$, $\sigma_{\mathrm{z}} = \frac{I_{\mathrm{x}}-I_{\mathrm{y}}}{I_{\mathrm{z}}}$. The time-varying matrix $\mathbf{B}$ consists of the components of the local geomagnetic field vector resolved in LVLH.

The following observation is used for the design of the infinite horizon attitude controller. The geomagnetic field on a near polar orbit is approximately periodic with a period $T = 2\pi/\omega_0$. Due to periodic nature of the geomagnetic field, seen from LVLH, the linearized model of the satellite can be considered as periodic. The solution of the infinite horizon problem is based on the solution of a periodic Riccati equation addressed in the next section.

3. PERIODIC RICCATI EQUATION

The Riccati equation was a theme of an extensive research work from early seventies. The interested reader is referred to Bittanti (1991) for a collection of studies on the subject. Here, only the main results are stated, which are directly related to the design of a periodic attitude controller.

Consider the Riccati equation

$$-\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^{\mathrm{T}}(t)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t) + \mathbf{C}^{\mathrm{T}}(t)\mathbf{C}(t), \quad (13)$$

where the matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$ are T-periodic, i.e. $\mathbf{A}(t+T) = \mathbf{A}(t)$, $\mathbf{B}(t+T) = \mathbf{B}(t)$, and $\mathbf{C}(t+T) = \mathbf{C}(t)$.

The solution of the Riccati equation can be reduced to an iterative solution of the Lyapunov equation. This method is known as Bellman's quasi linearization. In the method the solution to the Riccati equation is found using a Newton-type algorithm.

Consider an operator

$$\mathrm{Ric}: \ \mathbf{P}(t) \longmapsto \dot{\mathbf{P}}(t) + \mathbf{A}^{\mathrm{T}}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t) + \mathbf{C}^{\mathrm{T}}(t)\mathbf{C}(t). \quad (14)$$

A symmetric solution to the Riccati equation satisfies the operator equation

$$\mathrm{Ric}(\mathbf{P}(t)) = 0. \quad (15)$$

Suppose that $\mathbf{P}_i(t)$ is a symmetric matrix function approximating the solution of Eq. (15) with a certain accuracy. A Newton algorithm can then be used for computing a new and more accurate approximation

$$\mathbf{P}_{\mathrm{i}+1}(t) = \mathbf{P}_{\mathrm{i}}(t) + \Delta\mathbf{P}_{\mathrm{i}}(t). \quad (16)$$

More precisely, $\mathbf{P}_{\mathrm{i}+1}(t)$ is computed from $\mathbf{P}_{\mathrm{i}}(t)$ by solving the differential equation

$$-\dot{\mathbf{P}}_{i+1}(t) = \mathbf{A}_{\mathrm{i}}^{\mathrm{T}}(t)\mathbf{P}_{\mathrm{i}+1}(t) + \mathbf{P}_{\mathrm{i}+1}(t)\mathbf{A}_{\mathrm{i}}(t) + \mathbf{Q}(t) + \mathbf{K}_{\mathrm{i}}^{\mathrm{T}}(t)\mathbf{K}_{\mathrm{i}}(t), \quad (17)$$

where

$$\mathbf{A}_{\mathrm{i}}(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}_{\mathrm{i}}(t), \qquad \mathbf{K}_{\mathrm{i}}(t) = \mathbf{B}^{\mathrm{T}}(t)\mathbf{P}_{\mathrm{i}}(t). \quad (18)$$

It turns out that Eq. (17) is the Lyapunov equation, and the Riccati equation can be considered as a limit of a series of the Lyapunov equations. Furthermore, the stabilizability of the pair $(\mathbf{A}(t), \mathbf{B}(t))$ is sufficient for the existence of a periodic, positive semidefinite solution to the Riccati equation. Before a theorem showing this results will be formulated the notion of a maximal and a strong solution has to be introduced.

Definition 3.1. A solution $\mathbf{P}_{\mathrm{M}}(t)$ is maximal if for any symmetric and periodic solution $\mathbf{P}(t)$ of the Riccati equation the following inequality is true $\mathbf{P}_{\mathrm{M}}(t) \geq \mathbf{P}(t)$. A solution $\mathbf{P}(t)$ is strong if the closed loop system has its characteristic multipliers belonging to the closed unit disk.

Theorem 3.1. Suppose that the pair $(\mathbf{A}(t), \mathbf{B}(t))$ is stabilizable and consider the sequence of the periodic Lyapunov equations defined in Eqs. (17) and (18). Let $\mathbf{K}_0(t)$ be a T-periodic matrix function such that $\mathbf{A}_0(t)$ is stable. Then

(1) For each $i \geq 0$, there exists a unique symmetric periodic and semidefinite solution $\mathbf{P}_{\mathrm{i}+1}(t)$ to (17) and $\mathbf{A}_{\mathrm{i}+1}(t)$ is stable.
(2) The sequence $\{\mathbf{P}_{\mathrm{i}}(t)\}$ is a monotonically nonincreasing sequence of symmetric periodic positive semidefinite matrices, i.e., $\mathbf{0} \leq \mathbf{P}_{\mathrm{i}+1}(t) \leq \mathbf{P}_{\mathrm{i}}(t)$.
(3) The sequence $\{\mathbf{P}_{\mathrm{i}}(t)\}$ is such that $\lim_{i\to\infty} \mathbf{P}_{\mathrm{i}}(t) = \mathbf{P}_{\mathrm{M}}(t)$, where $\mathbf{P}_{\mathrm{M}}(t)$ is a maximal and strong solution to the Riccati equation.

Stabilizability of the pair $(\mathbf{A}(t), \mathbf{B}(t))$ does not solve the problem of existence of a stable solution since the characteristic multiplier can lay on the unit circle, the solution $\tilde{\mathbf{P}}_{\mathrm{M}}(t)$ is strong. If additionally no unit modulus characteristic multipliers of $\mathbf{A}(t)$ is $(\mathbf{A}(t), \mathbf{C}(t))$ unobservable then there exists a stabilizing symmetric periodic solution. As a matter of fact these two conditions are necessary and sufficient.

Theorem 3.2. There exists a stabilizing symmetric periodic solution $\mathbf{P}_{+}(t)$ to the Riccati Equation (13) if, and only if $((\mathbf{A}(t), \mathbf{B}(t)))$ is stabilizable and no unit-modulus characteristic multipliers of $\mathbf{A}(t)$ are $(\mathbf{A}(t), \mathbf{C}(t))$ unobservable.

The last objective of this section is to show that "the steady state" solution to the Riccati equation is periodic. This statement is a conclusion from Theorem 3.1.

Theorem 3.3. Let $\mathbf{P}(t)$ be solution to the Riccati Equation (13) for positive semidefinite final condition defined at infinity and $\mathbf{P}_M(t)$ be the periodic solution to the Riccati equation defined in Theorem 3.1, then

$$\lim_{t\to 0} \mathbf{P}(t) = \mathbf{P}_M(t).$$

4. INFINITE HORIZON CONTROL

After the results on periodic Riccati equation have been established the problem of periodic infinite horizon control can be formulated. Let a dynamic system be described by

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}(t)\boldsymbol{x}(t) + \mathbf{B}(t)\boldsymbol{u}(t)$$
$$\boldsymbol{z}(t) = \mathbf{C}(t)\boldsymbol{x}(t) + \mathbf{D}(t)\boldsymbol{u}(t), \quad (19)$$

where $\mathbf{A}(T+t) = \mathbf{A}(t)$, $\mathbf{B}(T+t) = \mathbf{B}(t)$, $\mathbf{C}(T+t) = \mathbf{C}(t)$, and $\mathbf{D}(T+t) = \mathbf{D}(t)$. Assume that the full state information is available. Furthermore suppose that the periodic system (25) fulfills the following assumptions:

(1) $\mathbf{D}(t)$ has full column rank with $[\mathbf{D}(t)\ \mathbf{D}_{\perp}(t)]$ being unitary.
(2) The pair $(\mathbf{A}(t), \mathbf{B}(t))$ is stabilizable
(3) No unit-modulus characteristic multipliers of
$$\tilde{\mathbf{A}}(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{D}^{\mathrm{T}}(t)\mathbf{C}(t) \qquad (20)$$
are $(\tilde{\mathbf{A}}(t), \mathbf{D}_{\perp}^{T}(t)\mathbf{C}(t))$ unobservable.

The assumption (1) is technical, it states that $\mathbf{D}^{\mathrm{T}}(t)\mathbf{D}(t) = \mathbf{E}$ and $\mathbf{D}(t)\mathbf{D}^{\mathrm{T}}(t) + \mathbf{D}_{\perp}(t)\mathbf{D}_{\perp}^{\mathrm{T}}(t) = \mathbf{E}$, where $\mathbf{E}$ is the identity matrix of a compatible dimension.

The assumption (2) together with (3) states that the periodic Riccati equation

$$\begin{aligned}-\dot{\mathbf{P}}(t) = &\mathbf{P}(t)\tilde{\mathbf{A}}(t) + \tilde{\mathbf{A}}^{\mathrm{T}}(t)\mathbf{P}(t)\\ &- \mathbf{P}(t)\mathbf{B}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t)\\ &+ \mathbf{C}^{\mathrm{T}}(t)\mathbf{D}_{\perp}(t)\mathbf{D}_{\perp}^{\mathrm{T}}(t)\mathbf{C}(t)\end{aligned} \qquad (21)$$

has a stabilizing symmetric periodic solution, see Theorem 3.2. Let $\mathbf{P}(t)$ be a stabilizing periodic solution of the Riccati equation (21). If we define periodic feedback gain

$$\mathbf{F}(t) = -(\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t) + \mathbf{D}^{\mathrm{T}}(t)\mathbf{C}(t)) \qquad (22)$$

then the closed loop system $\mathbf{A}_{\mathrm{F}}(t) = \mathbf{A}(t) + \mathbf{B}(t)\mathbf{F}(t)$ is stable and it minimizes the L_2 norm of the signal to be regulated, $\boldsymbol{z}(t)$

$$\min_{\mathbf{u}\in L_2[0,\infty)} \| \boldsymbol{z}(t) \|_2^2 = \boldsymbol{x}^{\mathrm{T}}(t_0)\mathbf{P}(t_0)\boldsymbol{x}(t_0). \qquad (23)$$

To prove this statement denote $\mathbf{C}_{\mathrm{F}}(t) = \mathbf{C}(t) + \mathbf{D}(t)\mathbf{F}(t)$ then the Riccati equation (21) can be rearranged into a Lyapunov equation

$$-\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{A}_{\mathrm{F}}(t) + \mathbf{A}_{\mathrm{F}}^{\mathrm{T}}(t)\mathbf{P}(t) + \mathbf{C}_{\mathrm{F}}^{\mathrm{T}}(t)\mathbf{C}_{\mathrm{F}}(t). \qquad (24)$$

After the change of variable $\boldsymbol{v}(t) = \boldsymbol{u}(t) - \mathbf{F}(t)\boldsymbol{x}(t)$ the system (25) can be rewritten as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \mathbf{A}_{\mathrm{F}}(t)\boldsymbol{x}(t) + \mathbf{B}(t)\boldsymbol{u}(t)\\ \boldsymbol{z}(t) &= \mathbf{C}_{\mathrm{F}}(t)\boldsymbol{x}(t) + \mathbf{D}(t)\boldsymbol{v}(t),\end{aligned} \qquad (25)$$

now we define the following quadratic function $l(t) = \boldsymbol{x}^{\mathrm{T}}(t)\mathbf{P}(t)\boldsymbol{x}(t)$. The time derivative of $l(t)$ is given by

$$\begin{aligned}\frac{d}{dt}l &= \dot{\boldsymbol{x}}^{\mathrm{T}}\mathbf{P}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}\mathbf{P}\dot{\boldsymbol{x}} + \boldsymbol{x}^{\mathrm{T}}\dot{\mathbf{P}}\boldsymbol{x}\\ &= \boldsymbol{x}^{\mathrm{T}}\mathbf{A}_{\mathrm{F}}^{\mathrm{T}}\mathbf{P}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{\mathrm{F}}\boldsymbol{x} + 2\boldsymbol{x}^{\mathrm{T}}\mathbf{P}\mathbf{B}\boldsymbol{v} + \boldsymbol{x}^{\mathrm{T}}\dot{\mathbf{P}}\boldsymbol{x}\\ &= -\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z} + \boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}.\end{aligned} \qquad (26)$$

Integrating both sides of Eq. (26) from 0 to ∞ and using the definition of the L_2 norm we get

$$\min \| \boldsymbol{z}(t) \|_2^2 = \boldsymbol{x}^{\mathrm{T}}(t_0)\mathbf{P}(t_0)\boldsymbol{x}(t_0) + \| v(t) \|_2^2 \,. \qquad (27)$$

Clearly the optimal control is given for $\boldsymbol{v}(t) = \mathbf{0}$ and $\boldsymbol{u}(t) = \mathbf{F}(t)\boldsymbol{x}(t)$. In the next section the control law (22) is implemented for the attitude control stabilization of a magnetic actuated satellite.

5. IMPLEMENTATION

Due to periodic nature of the geomagnetic field, seen from the LVLH, the linearized model of the satellite can be considered as periodic. It is though necessary to find an ideally periodic counterpart of the real magnetic field of the Earth. This is done by averaging the geomagnetic field over $N = 15$ number of orbits covering 24 *hours*. Furthermore, the geomagnetic field is parameterized by the mean anomaly M, since the geomagnetic field and the mean anomaly have the common period T

$$(\boldsymbol{B}_{ave}(M))_{\mathrm{o}} = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{B}(M))_{\mathrm{o}}, \qquad (28)$$

where the subscript o indicates that the the geomagnetic field vector is resolved in LVLH. The resultant linear periodic system is

$$\frac{d}{dt}\begin{bmatrix}\Delta\boldsymbol{\Omega}\\ \Delta\boldsymbol{q}\end{bmatrix} = \mathbf{A}\begin{bmatrix}\Delta\boldsymbol{\Omega}\\ \Delta\boldsymbol{q}\end{bmatrix} + \hat{\mathbf{B}}(M)\boldsymbol{m}, \qquad (29)$$

where $\hat{\boldsymbol{B}}(M)$ is given in Eq. (12) after substituting the symbol $\boldsymbol{B}(t)$ for $\hat{\boldsymbol{B}}(M)$, and the components of the vector $(\boldsymbol{B}(t))_{\mathrm{o}}$ for the components of $(\boldsymbol{B}_{ave}(M))_{\mathrm{o}}$.

The difference between the time varying matrix $\mathbf{B}(t)$ and the ideal periodic counterpart $\hat{\mathbf{B}}(M(t))$ used for the controller design is considered an additional external disturbance torque acting on the satellite.

The controller gain is calculated from the steady state solution of the Riccati equation, which is periodic. The solution to the Riccati equation is calculated off-line and stored in the computer memory.

The periodic solution of the Riccati equation, $\mathbf{P}_{+}(t)$ is found from the periodic extension of the steady state solution $\mathbf{P}_{\infty}(t)$ of the Riccati equation (21).

$$\hat{\mathbf{P}}(t) = \begin{cases}\mathbf{P}_{\infty}(t) & \text{if } 0 \le t < T\\ \mathbf{0} & \text{otherwise}\end{cases} \qquad (30)$$

$$\mathbf{P}_{+}(t) = \sum_{k=0}^{\infty}\hat{\mathbf{P}}(t - kT) \qquad (31)$$

The solution $\mathbf{P}_{\infty}(t)$ is calculated using backward integration of the Riccati equation for an arbitrary final condition. This solution converges to the periodic solution, Theorem 3.3. The matrix function $\mathbf{P}_{\infty}(t)$ corresponding to one orbital passage is

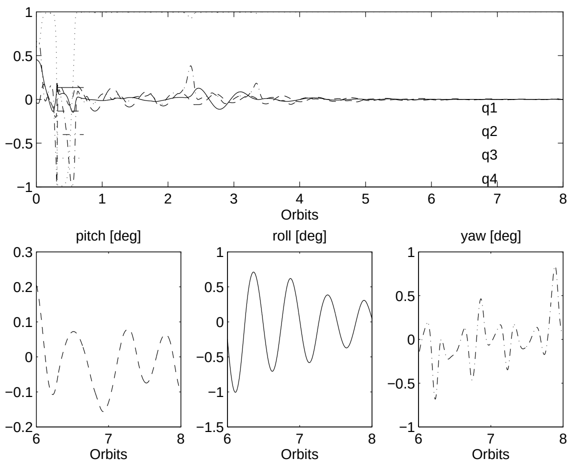

Fig. 1. Performance of the infinite horizon controller for the Ørsted satellite. The steady state attitude error is below 1 *deg*.

stored in the computer memory, and then used for the subsequent orbits.

Again, the mean anomaly M can be used for parameterization of $\mathbf{P}_+(M)$, since both $\mathbf{P}_+(t)$ and $M(t)$ are T-periodic. In fact the controller gain matrix is also T-periodic and can be parameterized by M

$$\mathbf{K}_+(M) = -\mathbf{B}(M)\mathbf{P}_+(M). \qquad (32)$$

An option is to parameterize $\mathbf{K}_+(M)$ in terms of the Fourier coefficients, benefiting in a reduction of the data stored. A satisfactory approximation of the gain matrix $\mathbf{K}_+$ has been obtained with 16th order Fourier series. For example 172800 floating point memory is needed for a sampling time of 10 sec and the orbital period of 6000 sec.

Simulation results of the infinite horizon attitude control are performed for the Ørsted satellite. The satellites moments of inertia are $\mathbf{I} = \text{diag}\begin{bmatrix}182 & 181 & 1\end{bmatrix}$. The Ørsted orbit is low earth near polar with apogee 850 km, perigee 450 km, and the inclination 96.4 deg. A simulation test of the infinite horizon controller for the Ørsted satellite is depicted in Fig. 1. It is seen that the steady state attitude error is kept below 1 deg.

6. CONCLUSION

This work is believed to contribute to the development of linear periodic feedback based only on magnetic torquing for low earth orbit satellites. A computational expense for the infinite horizon controller lies in the off-line numeric solution to the Riccati equation, but relatively large computer memory is required for keeping the gain data for one orbit. It is concluded that the infinite horizon magnetic controller is applicable for missions with pointing requirements of couple of degrees.

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