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Total domination in partitioned graphs by Allan Frendrup, Preben Dahl Vestergaard and Anders Yeo R-2009-06 April 2009

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Abstract. We present results on total domination in a partitioned graph G = (V, E). Let $\gamma_t(G)$ denote the total dominating number of G. For a partition $V_1, V_2, \ldots, V_k, k \ge 2$, of V, let $\gamma_t(G; V_i)$ be the cardinality of a smallest subset of V such that every vertex of V_i has a neighbour in it and define the following

 $f_t(G; V_1, V_2, \dots, V_k) = \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \dots + \gamma_t(G; V_k)$ $f_t(G; k) = \max\{f_t(G; V_1, V_2, \dots, V_k) \mid V_1, V_2, \dots, V_k \text{ is a partition of } V\}$ $g_t(G; k) = \max\{\Sigma_{i=1}^k \gamma_t(G; V_i) \mid V_1, V_2, \dots, V_k \text{ is a partition of } V\}$

We summarize known bounds on $\gamma_t(G)$ and for graphs with all degrees at least δ we derive the following bounds for $f_t(G; k)$ and $g_t(G; k)$.

- (i) For $\delta \geq 2$ and $k \geq 3$ we prove $f_t(G; k) \leq 11|V|/7$ and this inequality is best possible.
- (ii) for $\delta \geq 3$ we prove that $f_t(G; 2) \leq (5/4 1/372)|V|$. That inequality may not be best possible, but we conjecture that $f_t(G; 2) \leq 7|V|/6$ is.
- (iii) for $\delta \geq 3$ we prove $f_t(G; k) \leq 3|V|/2$ and this inequality is best possible.
- (iv) for $\delta \geq 3$ the inequality $g_t(G; k) \leq 3|V|/4$ holds and is best possible.

Key words. Total domination, Partitions and Hypergraphs.

1. Notation

By G = (V, E) we denote a graph G with vertex set V = V(G) and edge set E = E(G). The order of G is |V(G)| = n. For $x \in V(G)$ we denote by $N_G(x)$ the set of neighbours to x and $N_G[x] = \{x\} \cup N_G(x)$. Indices may be omitted if clear from context. The degree of x is $d_G(x) = |N_G(x)|$, the number of neighbours to x. We let $\delta(G) = \delta$ denote the minimum degree in G and $\Delta(G) = \Delta$ the maximum degree. A hypergraph H = (V, E) has vertex set V = V(H) and its set of hyperedges, or edges for short, is E = E(H). Each hyperedge e is a subset of V, $e \subseteq V(H)$. A vertex v is incident with an edge e if $v \in e$, the degree of

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v is the number of hyperedges in H containing v. We let $\delta(H) = \delta$ denote the minimum degree in H and $\Delta(H) = \Delta$ the maximum degree. H is r-regular if each vertex has degree r, i.e. $d_H(x) = r$, or equivalently, x is contained in precisely r edges. H is k-uniform if each hyperedge contains exactly k vertices. Two edges e_1 and e_2 are said to be overlapping if $|V(e_1) \cap V(e_2)| \geq 2$. Let $Y \subseteq V(H)$ then E(Y) denotes all hyperedges, e, contained in Y (i.e. $V(e) \subseteq Y$).

For a hypergraph H a hitting set or a transversal \mathcal{T} is a set of vertices $\mathcal{T} \subseteq V(H)$ such that $e \cap \mathcal{T} \neq \emptyset$ for each hyperedge e in E(H), i.e. each edge e contains at least one vertex from \mathcal{T} . $\mathcal{T}(H)$ denotes the minimum cardinality of a transversal for the hypergraph H. For sets $S, T \subseteq V$, in a graph G the set S totally dominates T if every vertex in T is adjacent to some vertex of S. The minimum number of vertices needed to totally dominate V is the total domination number $\gamma_t(G)$. For a subset S of V we let $\gamma_t(G; S)$ denote the smallest number of vertices in G which totally dominates S. A partition $V = (V_1, V_2, \ldots, V_k)$ of V(G) into k disjoint sets, $k \geq 2$, has $V = \bigcup_{i=1}^k V_i, V_i \cap V_j = \emptyset$, $1 \leq i < j \leq k$. For a partition (V_1, V_2, \ldots, V_k) of V, we define the following.

$$f_t(G; V_1, V_2, \dots, V_k) = \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \dots + \gamma_t(G; V_k)$$

$$g_t(G; V_1, V_2, \dots, V_k) = \gamma_t(G; V_1) + \gamma_t(G; V_2) + \dots + \gamma_t(G; V_k)$$

We furthermore define $f_t(G; k)$ and $g_t(G; k)$ as follows.

 $f_t(G;k) = \max\{f_t(G;V_1, V_2, \dots, V_k) \mid V_1, V_2, \dots, V_k \text{ is a partition of } V\}$ $g_t(G;k) = \max\{g_t(G;V_1, V_2, \dots, V_k) \mid V_1, V_2, \dots, V_k \text{ is a partition of } V\}$

For further notation we refer to Chartrand and Lesniak [1].

2. Introduction

The theory of domination is outlined in two books by Haynes, Hedetniemi and Slater [5, 6]. A combination of domination and partitions is treated by Hartnell and Vestergaard [7], Seager [14], Tuza and Vestergaard [17], Henning and Vestergaard [11]. There has been an upsurge in the study of total domination. New results on total domination are given by Henning, Kang, Shan, Thomassé and Yeo in [10, 12, 15, 18]. In [9] Henning surveys recent results on total domination. Here we shall study total domination in partitioned graphs.

3. Bounds on γ_t

We summarize in Theorem 1 results found by Henning, Thomassé and Yeo. If C_{10} : $v_1, v_2, \ldots, v_{10}, v_1$ is the circuit with 10 vertices then let G_{10} denote the graph obtained from C_{10} by addition of the edge v_1v_6 and let H_{10} denote the graph obtained from C_{10} by addition of the edge v_1v_6 and let H_{10} denote the graph obtained from C_{10} by addition of the edges v_1v_6 and v_2v_7 .

Theorem 1. Let G be a connected graph with n vertices and minimum degree $\delta(G) = \delta$. Then

- $\delta \ge 2 \text{ implies } \gamma_t(G) \le 4n/7 \text{ for } G \notin \{C_3, C_5, C_6, C_{10}, G_{10}, H_{10}\} \text{ ([8, Corollary 6], [9, Theorem 27]).}$
- $\delta \geq 3 \text{ implies } \gamma_t(G) \leq n/2.$ ([15]).

 $\delta \geq 4$ implies $\gamma_t(G) \leq 3n/7$ ([15]) and there exists some $\epsilon > 0$ such that $\gamma_t(G) \leq (3/7 - \epsilon)n$ for $G \neq G_{14}$, where G_{14} is an incidence bipartite graph of order 14 derived from the Fano plane ([19]).

It is a conjecture that $\delta \geq 5$ implies $\gamma_t(G) \leq 4n/11$.

Theorem 2 and Theorem 3 below, give conditions for equality in Theorem 1.

Theorem 2. ([9, Theorem 29]) Let G be a connected graph of order n > 14 with $\delta \ge 2$. Then $\gamma_t(G) = 4n/7$ if and only if G can be obtained from a connected graph F of order at least three by adding |V(F)| disjoint copies of C_6 , one corresponding to each $v \in V(F)$, such that either v is joined by a new edge to a vertex in its corresponding C_6 or by two new edges to two vertices at distance two apart in its corresponding C_6 .

The family $\mathcal{G} \cup \mathcal{H}$ is constructed in [3] as follows. Take two copies $a_1b_1a_2b_2\ldots a_kb_k$ and $c_1d_1c_2d_2\ldots c_kd_k$, of the path $P_{2k}, k \geq 2$, and add edges a_id_i , b_ic_i for $i = 1, 2, \ldots, k$. ¿From this the graph of order 4k belonging to the infinite family \mathcal{G} is obtained by adding a_1c_1 and b_kd_k , while the graph of order 4k in \mathcal{H} is obtained by adding a_1b_k and c_1d_k , The generalized Petersen graph GP_{16} is obtained from two circuits $u_1u_2u_3\ldots u_7u_8$ and $v_1v_2v_3\ldots v_7v_8$ by addition of edges $u_1v_1, u_2v_4, u_3v_7, u_4v_2, u_5v_5, u_6v_8, u_7v_3, u_8v_6$.

Theorem 3. ([12, Theorem 5]) Let G be a connected graph with $\delta(G) \geq 3$. Then $\gamma_t(G) = n/2$ if and only if $G \in \mathcal{G} \cup \mathcal{H}$ or $G = GP_{16}$.

4. f_t for k-partitioned graphs with $\delta \geq 2$

We have that f_t increases with the number of partition classes, i.e., $f_t(G;k) \leq f_t(G;k+1)$. Therefore we get a weaker inequality if we partition V into more than two classes. That is demonstrated in Theorem 4 below.

Theorem 4. Let G be a connected graph of order n with $\delta(G) \ge 2$ and $G \notin \{C_3, C_5, C_6, C_{10}\}$. If $k \ge 2$ then $f_t(G; k) \le 11n/7$.

- If k = 2 then $f_t(G; k) \leq 3n/2$. Equality holds if and only if G is a circuit of length zero modulo four, $G = C_{4t}, t \geq 1$.
- If k = 3 then $f_t(G; k) \le 11n/7$. For n > 14 equality holds if and only if G can be obtained from a circuit or a path of order at least three by joining each of its vertices by one edge to disjoint copies of C_6 .
- If $k \ge 4$ then $f_t(G;k) \le 11n/7$ and for n > 14 equality holds if and only if $\Delta(G) \le k$ and G can be obtained from a connected graph F having order at least three and $g_t(F;k) = |V(F)|$ by adding disjoint copies of C_6 , one corresponding to each $v \in V(F)$, such that either v is joined by a new edge to one vertex in its corresponding C_6 or by two new edges to two vertices at distance two apart in its corresponding C_6 .

Proof. By Theorem 1 we have $\gamma_t(G) \leq 4n/7$ and assigning to each vertex its own class dominator we have $g_t(G;k) \leq n$. Therefore $f_t(G;k) = \gamma_t(G) + g_t(G;k) \leq 11n/7$. The result for k = 2 is proven by Frendrup, Henning and Vestergaard in [4, Theorem 2]. For $k \geq 3$ the equality $f_t(G;k) = 11n/7$ implies $\gamma_t(G) = 4n/7$ and $g_t(G;k) = n$ and therefore G has the structure described in Theorem 2. Since $g_t(G;k) = n$ each subgraph H of G must satisfy $g_t(H;k) = |V(H)|$ and further $\Delta(G) \leq k$. Let H_1 be the graph obtained from a circuit $C_6: v_1v_2...v_6$ by adding a new vertex x and the edge xv_1 and let $H_2:=H_1+xv_3$. Observe for k=3 that $g_t(H_1;k) = |V(H_1)|$ (obtainable from partitioning $x, v_1, v_2..., v_6$ into classes indexed 1122133 or 1221133) while $g_t(H_2;k) < |V(H_2)|$. For $k \ge 4$ we can easily show that $g_t(H_i;k) = |V(H_i)|, i = 1, 2$. This proves for $k \ge 3$ that $f_t(G;k) = 11n/7$ implies G has the structure described in this theorem. Conversely, assume first that k = 3 and that G is obtainable as a disjoint union of H_1 's with edges added between the vertices named x, so they span F, where F is a path or circuit. We must exhibit a partition of V(G) proving that $f_t(G;k) = 11n/7$, i.e. that $g_t(G;k) = |V(G)|$. It is easy to find a partition V'_1, V'_2, V'_3 of V(F) such that $g_t(F;k) = |V(F)|$. If k = 3 we can extend this partition to all the H_1 's such that the following holds, which proves that $g_t(G;V'_1, V'_2, V'_3) = n$.

- $-N(x) = N_F(x) \cup \{v_1\}$ contains at most one vertex from each V'_1, V'_2, V'_3 (just put v_1 in the partition set which doesn't contain any of the two vertices in $N_F(x)$).
- $-N(v_1) = \{x, v_2, v_6\}$ contains one vertex from each V'_1, V'_2, V'_3 (just put v_2 and v_6 in the partition sets such that this holds).
- $-N(v_3), N(v_5) \subset \{v_2, v_4, v_6\}$, which contains one vertex from each V'_1, V'_2, V'_3 (just put v_4 in the same set as x).
- $-N(v_2), N(v_4), N(v_6) \subset \{v_1, v_3, v_5\}$, which contains one vertex from each V'_1, V'_2, V'_3 (just put v_3 and v_5 in the partition sets such that this holds).

Assume next that $k \ge 4$. Then a vertex $x \in F$ may belong to a unit H_1 or H_2 . Again there is a partition V'_1, V'_2, \ldots, V'_k of V(F) such that $g_t(F; k) = |V(F)|$ and similarly to above we can extend this partition to all of G, such that the neighbourhood of every vertex in G contains at most one vertex from any partition set. The details are left to the reader. This proves that $g_t(G; k) = n$.

5. g_t for two-partitioned graphs with $\delta \geq 3$

Chvátal and McDiarmid [2] and Tuza [16] independently established the following result about transversals in hypergraphs (see also Thomassé and Yeo [15] for a short proof of this result).

Theorem 5. ([2,16,15]) If H is a hypergraph with all edges of size at least three, then $\mathcal{T}(H) \leq (|V(H)| + |E(H)|)/4$.

Theorem 6. Let G be a graph of order n with $\delta \geq 3$. Then $g_t(G; 2) \leq 3n/4$.

Proof. ¿From the two-partitioned graph G, we define for $i = 1, 2, H_i$ to be the hypergraph on n vertices and m_i edges where $V(H_i) = V(G)$ and the hyperedges of H_i are the sets of neighbourhoods of class i vertices. In other words, $e \in E(H_i)$ precisely if, for some vertex v in V_i , $e = N_G(v)$. Each edge in H_i has at least three vertices because $\delta(G) \geq 3$. In Gwe see that a set \mathcal{T}_i of vertices totally dominates V_i if and only if \mathcal{T}_i is a transversal of H_i . Applying Theorem 5 to H_1 and H_2 separately we obtain transversals \mathcal{T}_i of H_i , i = 1, 2, satisfying

$$|\mathcal{T}_1| \le \frac{m_1 + n}{4} \qquad \qquad |\mathcal{T}_2| \le \frac{m_2 + n}{4}$$

Since $m_1+m_2 = n$ we obtain $|\mathcal{T}_1|+|\mathcal{T}_2| \leq \frac{m_1+n}{4} + \frac{m_2+n}{4} = \frac{3n}{4}$. This proves Theorem 6. \Box An example of graphs with equality $g_t(G; 2) = 3n/4$ is given in the next section.

6. An infinite family of graphs extremal for Theorem 6

We have the following theorem.

Theorem 7. For each integer $r \ge 1$ there exists a connected bipartite graph G_r of order n = 16r with $\delta(G_r) = 3$ such that $g_t(G_r; 2) = 3|V(G_r)|/4$ and $f_t(G_r; 2) \ge 9|V(G_r)|/8$.

Proof. We define the graph G_r as follows. Define the vertex set of G_r to be $V(G_r) = W_r \cup A_r \cup B_r$, where

$$W_r = \{w_0, w_1, w_2, \dots, w_{8r-1}\}$$

$$A_r = \{a_0, a_1, a_2, \dots, a_{4r-1}\}$$

$$B_r = \{b_0, b_1, b_2, \dots, b_{4r-1}\}$$

We define the edge set of G_r such that the following holds, for all $i \in \{0, 1, 2, ..., r-1\}$ (where $b_{-1} = b_{4r-1}$ by definition):

 $N(w_{8i}) = \{a_{4i}, a_{4i+1}, b_{4i}\}$ $N(w_{8i+2}) = \{a_{4i}, a_{4i+2}, b_{4i}\}$ $N(w_{8i+4}) = \{a_{4i+2}, b_{4i+1}, b_{4i+2}\}$ $N(w_{8i+6}) = \{a_{4i+3}, b_{4i+1}, b_{4i+3}\}$ $N(w_{8i+6}) = \{a_{4i+3}, b_{4i+1}, b_{4i+3}\}$ $N(w_{8i+6}) = \{a_{4i+3}, b_{4i+1}, b_{4i+3}\}$ $N(w_{8i+6}) = \{a_{4i+3}, b_{4i+2}, b_{4i+3}\}$ $N(w_{8i+6}) = \{a_{4i+3}, b_{4i+2}, b_{4i+3}\}$

We now assume $r \ge 1$ is fixed, and therefore omit the subscripts of the above sets and graph. Define V_1 and V_2 as follows.

 $V_1 = A \cup \bigcup_{i=0}^{r-1} \{ w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+5} \}$ $V_2 = B \cup \bigcup_{i=0}^{r-1} \{ w_{8i}, w_{8i+4}, w_{8i+6}, w_{8i+7} \}$

We will now show that if S_i is a set such that every vertex in V_i has a neighbour in S_i , then $|S_i| \ge 3|V(G)|/8$, for i = 1, 2. This would imply that $f_t(G; 2) \ge 9|V(G)|/8$ and $g_t(G) \ge 6|V(G)|/8$ when k = 2 (as clearly the above would also imply that $\gamma_t(G) \ge 3|V(G)|/8$). From Theorem 6 follows that $g_t(G) = 3|V(G)|/4$.

Let S_1 be a set that totally dominates V_1 (i.e. every vertex in V_1 has a neighbour in S_1). As w_{8i+5} has a neighbour in S_1 we note that $|S_1 \cap \{a_{4i+3}, b_{4i+1}, b_{4i+2}\}| \ge 1$, for all $i = 0, 1, 2, \ldots, r-1$. As w_{8i+1}, w_{8i+2} and w_{8i+3} all have a neighbour in S_1 we note that $|S_1 \cap \{a_{4i}, a_{4i+1}, a_{4i+2}, b_{4i}, b_{4i-1}\}| \ge 2$, for all $i = 0, 1, 2, \ldots, r-1$ (recall that $b_{-1} = b_{4r-1}$). As the above sets are all disjoint we note that $|S_1 \cap (A \cup B)| \ge 3|A \cup B|/8$.

As a_{4i+3} has a neighbour in S_1 we note that $|S_1 \cap \{w_{8i+5}, w_{8i+6}, w_{8i+7}\}| \ge 1$, for all $i = 0, 1, 2, \ldots, r-1$. As a_{4i} , a_{4i+1} and a_{4i+2} all have a neighbour in S_1 we note that $|S_1 \cap \{w_{8i}, w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+4}\}| \ge 2$, for all $i = 0, 1, 2, \ldots, r-1$. As the above sets are all disjoint we note that $|S_1 \cap W| \ge 3|W|/8$. This implies the desired result for S_1 .

The fact that if S_2 totally dominates V_2 , then $|S_2| \ge 3|V(G)|/8$ is proved analogously to above. We now just need to show that G is connected. Let $P_i = \{w_{8i}, w_{8i+1}, \ldots, w_{8i+7}\}$ and let $Q_i = \{a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}, b_{4i}, b_{4i+1}, b_{4i+2}, b_{4i+3}\}$ for all $i = 0, 1, 2, \ldots, r-1$. Note that $G[P_i \cup Q_i]$ is connected. As the edges $w_{8i+3}b_{4i-1}$, for all $i = 0, 1, 2, \ldots, r-1$ connects P_i with Q_{i-1} ($Q_{-1} = Q_{r-1}$) we are done.

7. $f_t(G)$ for two-partitioned graphs with $\delta \geq 3$

Let G be a graph of order n with $\delta(G) \geq 3$.

From Theorems 1 and 6 it follows immediately that $f_t(G; 2) = \gamma_t(G) + g_t(G; k) \leq n/2 + 3n/4 = 5n/4$ when $\delta(G) \geq 3$. We shall in Theorem 8 below prove a slightly stronger result and later pose an even stronger conjecture.

The following result is known (see for example [13]).

Lemma 1. ([13]) If G is a 3-regular graph, then there exists a matching M in G, such that $|M| \geq \frac{7}{16}|V(G)|$.

Lemma 2. Let *H* be a 2-regular 3-uniform hypergraph with no two edges overlapping. Then $\mathcal{T}(H) \leq \frac{|V(H)+|E(H)|}{4} - \frac{|V(H)|}{24}$.

Proof. Let H be a 2-regular 3-uniform hypergraph with no overlapping edges. Define the graph G_H as follows $V(G_H) = E(H)$ and $E(G_H) = \{e_1e_2 : |V(e_1) \cap V(e_2)| = 1\}$. As there are no overlapping edges and H is 2-regular and 3-uniform, we note that G_H is a 3-regular graph. By Lemma 1, there exists a matching M in G_H , such that $|M| \ge \frac{7}{16}|V(G_H)|$.

If $e_1e_2 \in M$, then by the definition of G_H we note that $V(e_1) \cap V(e_2) = \{x_{e_1e_2}\}$ for some $x_{e_1e_2} \in V(H)$. Let $X = \{x_f \mid f \in M\}$ and note that 2|M| edges in H contain a vertex from X (as M was a matching). Let X' be a set of vertices of order |E(H)| - 2|M|containing a vertex from every edge in H, which does not contain a vertex from X. Note that $X \cup X'$ is a transversal of H of order |M| + (|E(H)| - 2|M|). By the above bound on |M| we get the following, as $3|E(H)| = \sum_{x \in V(H)} d(x) = 2|V(H)|$.

$$\begin{aligned} \mathcal{T}(H) &\leq |E(H)| - |M| \leq |E(H)| - \frac{7}{16}|E(H)| \\ &= \frac{|E(H)|}{4} + \frac{5|E(H)|}{16} = \frac{|E(H)|}{4} + \frac{5}{16} \times \frac{2|V(H)|}{3} \\ &= \frac{|V(H)| + |E(H)|}{4} - \frac{|V(H)|}{24} \end{aligned}$$

Lemma 3. Let H be a 3-uniform hypergraph, where multiple edges are allowed. For each edge and vertex in H we assign a non-empty subset of $\{0, 1, 2\}$. Let this subset be denoted by L(q) for all $q \in V(H) \cup E(H)$. Let H_i be the 3-uniform hypergraph containing vertex-set $V_i = \{v : i \in L(v) \text{ and } v \in V(H)\}$ and edge-set $E_i = \{e : i \in L(v) \text{ and } e \in E(H)\}$, for i = 0, 1, 2. Let $Y \subseteq V(H)$ be arbitrary and assume that the following holds.

 $\begin{array}{l} (a): \ \Delta(H_1), \ \Delta(H_2) \leq 2\\ (b): \ \Delta(H - E(Y)) \leq 4.\\ (c): \ There \ are \ no \ overlapping \ edges \ in \ H_i, \ i \in \{1, 2\}.\\ (d): \ If \ e \in E(H) - E(Y), \ then \ 0 \in L(e) \ and \ |L(e)| \geq 2. \end{array}$

This implies that the following holds.

$$\sum_{i=0}^{2} \mathcal{T}(H_i) \le \left(\sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4}\right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]}{372}$$

Remark. We assume here in Lemma 3 that the assignment of a set L(q) to each q is done such that H_0, H_1, H_2 really are hypergraphs, i.e., such that each hyperedge in E_i consists of vertices from V_i , i = 0, 1, 2. This requirement will be satisfied in the proof of Theorem 8 where the lemma is applied.

Proof. Assume that the lemma is false, and that H is a counterexample with minimum $|E_0| + |E_1| + |E_2|$. Clearly $|E_0| + |E_1| + |E_2| > 0$, as otherwise $\sum_{i=0}^{2} \mathcal{T}(H_i) = 0$. For simplicity we will use the following notation:

$$T^* = \sum_{i=0}^{2} \mathcal{T}(H_i)$$

$$S^* = \sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4}$$

$$V^* = V(H_0) \cap V(H_1) \cap V(H_2)$$

We recall that H was assumed to be a "minimal" counterexample to $T^* \leq S^* - (|V^* \setminus N_H[Y]|)/372$. We will now prove a few claims, which end in a contradiction, thereby proving the lemma. For H the left hand side of the inequality, ℓ , and the right hand side of the inequality, r, in Lemma 3 satisfies $\ell > r$. We shall construct smaller H' which also satisfies (a)-(d) and which therefore has $\ell' \leq r'$ by the minimality of H. H' is to be constructed such that there exist $\alpha \leq \beta$ for which $\ell - \alpha \leq \ell'$ and $r' \leq r - \beta$. Those inequalities combine to give the desired contradiction $\ell \leq r$.

Claim A: If we add a vertex to Y, then N[Y] does not increase by more than 9 vertices.

Proof of Claim A: This follows from the fact that H is 3-uniform and $\Delta(H - E(Y)) \leq 4$, by (b) in the statement of the lemma.

Claim B: There is no $e = \{v_1, v_2, x\} \in E_i$, such that $d_{H_i}(v_1) = d_{H_i}(v_2) = 1$ and $d_{H_i}(x) = 2$, for i = 0, 1, 2.

Proof of Claim B: Assume that there is such an edge $e = \{v_1, v_2, x\} \in E_i$. Let $e' = \{w_1, w_2, x\}$ be the other edge in H_i containing x. Now delete v_1, v_2, x, e and e' from H_i and add $\{v_1, v_2, x, w_1, w_2\}$ to Y. Note that (a)-(d) still hold and that T^* decreases by 1 as we simply add x to any transversal in the new H_i in order to get a transversal in the old H_i . By Claim A the set N[Y] does not increase by more than 45 vertices. As V^* does not decrease by more than 3 vertices and S^* decreases by 5/4, we are done by the "minimality" of H (as $\alpha = 1 \leq 5/4 - 48/372 = \beta$ in the argument above Claim A).

Claim C: There is no $e = \{x, v_1, v_2\} \in E_i$, such that $d_{H_i}(v_1) = d_{H_i}(v_2) = 2$ and $d_{H_i}(x) = 1$, for i = 1, 2.

Proof of Claim C: Assume that there is such an edge $e = \{x, v_1, v_2\} \in E_i$. Let $e_1 = \{w_1, w_2, v_1\}$ be the other edge in H_i containing v_1 and let $e_2 = \{u_1, u_2, v_2\}$ be the other edge in H_i containing v_2 . As there are no overlapping edges in H_i (by (c) in the statement of the lemma) we note that $e_1 \neq e_2$ and $|\{w_1, w_2, u_1, u_2\}| \geq 3$. Let S be any subset of $\{w_1, w_2, u_1, u_2\}$ such that |S| = 3. We now separately consider the cases when addition of S as a new hyperedge to H_i causes overlapping edges in H_i , and when it doesn't.

Assume that adding S to E_i does not cause overlapping edges in $H_i - e_1 - e_2$. Now delete x, v_1 , v_2 , e, e_1 and e_2 from H_i and add the edge S to H_i (and H). Furthermore add $\{x, v_1, v_2, w_1, w_2, u_1, u_2\}$ to Y. Note that (a)-(d) still hold. If T' is a transversal in the new H_i then due to the edge S we either have $\{u_1, u_2\} \cap T' \neq \emptyset$, in which case $T' \cup \{v_1\}$ is a transversal in the old H_i or $\{w_1, w_2\} \cap T' \neq \emptyset$, in which case $T' \cup \{v_2\}$ is a transversal in the old H_i . Therefore T^* decreases by at most one. By Claim A we have that N[Y] does not increase by more than 63 vertices. As V^* does not decrease by more than 3 and S^* decreases by 5/4, we are done by the "minimality" of H (as $1 \leq 5/4 - 66/372$).

So now assume that the above addition of S would cause overlapping edges in $H_i - e_1 - e_2$. This can only happen if there is an edge $e' \in E_i$ such that $|S \cap V(e')| \ge 2$. Note that by (a) the degree in H_i is two for all vertices in $S \cap V(e')$ (they only lie in S and e'). Now delete the vertices $\{x, v_1, v_2\} \cup (S \cap V(e'))$ from H_i and delete the edges e, e_1, e_2 and e' from H_i (do not add the edge S to H_i). Furthermore add $\{x, v_1, v_2, w_1, w_2, u_1, u_2\} \cup (V(e') - S)$ to Y. Note that (a)-(d) still hold. By a similar argument to above we note that T^* decreases

by at most two. By Claim A we see that N[Y] does not increase by more than 72 vertices. As V^* does not decrease by more than 6 and S^* decreases by at least 9/4, we are done by the "minimality" of H (as $2 \le 9/4 - 78/372$).

Claim D: There is no $e = \{x, v_1, v_2\} \in E_0$, such that $d_{H_0}(v_1) = d_{H_0}(v_2) = 2$ and $d_{H_0}(x) = 1$ and $|N_{H_0}[V(e)]| \ge 6$.

Proof of Claim D: Assume that there is such an edge $e = \{x, v_1, v_2\} \in E_0$. Let $e_1 = \{w_1, w_2, v_1\}$ be the other edge in H_0 containing v_1 and let $e_2 = \{u_1, u_2, v_2\}$ be the other edge in H_0 containing v_2 . If $e_1 = e_2$, then $|N_{H_0}[V(e)]| \leq 4$, a contradiction. So assume that $e_1 \neq e_2$. As $|N_{H_0}[V(e)]| \geq 6$ we note that $|\{w_1, w_2, u_1, u_2\}| \geq 3$. We are now done analogously to Claim C.

Claim E: $\Delta(H_1), \Delta(H_2) \leq 1$.

Proof of Claim E: Assume that $\Delta(H_1) \geq 2$. By (a) we have $\Delta(H_1) = 2$. By Claim B and Claim C we note that there is a 2-regular component, R, in H_1 . There are no overlapping edges in R by (c). By Lemma 2 there is a transversal T_R in R of order at most (|V(R)| + |E(R)|)/4 - |V(R)|/24. So delete all edges and vertices in R and add all vertices in R to Y. By Claim A we have that N[Y] increases by at most 9|V(R)| vertices. We now have a contradiction to the "minimality" of H, as $|V(R)|/24 \geq 9|V(R)|/372$. Analogously we can show that $\Delta(H_2) \leq 1$.

Claim F: Assume $e_1, e_2 \in E(H_0)$ overlap and $e_i = (x_1, x_2, u_i)$ for i = 1, 2, where $u_1 \neq u_2$. If $d_{H_0}(x_1) = d_{H_0}(x_2) = 2$, then there is an edge $e' \in E(H_0)$ such that $\{u_1, u_2\} \subseteq V(e')$.

Proof of Claim F: Let e_1 and e_2 be defined as in the Claim, and assume that there is no edge $e' \in E(H_0)$ such that $\{u_1, u_2\} \subseteq V(e')$. Delete e_1, e_2, x_1, x_2 and u_1 from H_0 . For every edge, e'', in H_0 that contains u_1 , delete e'' and add the edge $(e'' - \{u_1\}) \cup \{u_2\}$ instead. Furthermore add $\{x_1, x_2, u_1, u_2\}$ and V(e'') from all transformed edges, to Y. As there is at most 4 edges containing u_1 in $H_0 - E(Y)$ we note that Y increases by at most 10 (the neighbours of u_1 in $H_0 - E(Y)$ and $\{u_1, u_2\}$). Therefore $V^* - N[Y]$ decreases by at most 3 + 90, by Claim A. We also note that S^* decreases by 5/4.

We now show that T^* decreases by at most one. If $u_2 \in T'$ then $T' \cup \{u_1\}$ is a transversal in the old H_0 . If $u_2 \notin T'$ then $T' \cup \{x_1\}$ is a transversal in the old H_0 . As (a)-(d) still holds after the above operations, we have a contradiction to the "minimality" of H, as $1 \leq 5/4 - 93/372$.

Definition G: Let $x \in V^* - N[Y]$ be arbitrary. The vertex x exists since otherwise we would be done by Theorem 5.

Claim H: $d_{H_1}(u) = d_{H_2}(u) = 1$ for all $u \in N_{H_0}[x]$, where x is defined in Definition G.

Proof of Claim H: Assume that $u \in N_{H_0}[x]$ has $d_{H_2}(u) = 0$ or $u \notin V(H_2)$, which are the only possibilities for u, if $d_{H_2}(u) \neq 1$ (by Claim E). If $u \in V(H_2)$ and $d_{H_2}(u) = 0$, then delete u from $V(H_2)$. We are now done as T^* is unchanged, S^* decreases by 1/4and $V^* - N[Y]$ does not decrease by more than one. So we may assume that $u \notin V(H_2)$. Since $x \in V^*$ we note that $x \in V(H_1)$ and $x \in V(H_2)$, which by the above argument implies that $d_{H_1}(x) = d_{H_2}(x) = 1$ and $u \neq x$. Let $e_1 = \{x, u, q\}$ be the edge in H_1 (and H_0) containing u and x. Let e_2 be the edge in H_2 (and H_0) that contains x. Note that $d_{H_0}(x) = 2$ and $d_{H_0}(u) = 1$. If $d_{H_0}(q) = 1$ then we are done by Claim B. So $d_{H_0}(q) \geq 2$. However as any edge containing q must also lie in H_1 or H_2 , as $q \notin Y$, we note that

 $d_{H_0}(q) = 2$. Let e_q be the edge in H_2 that contains q. Note that $e_q \neq e_2$, by Claim F. As e_q and e_2 do not intersect we note that $|N_{H_0}[V(e)]| = 7 \ge 6$, so we are done by Claim D.

Claim I: Let $e_1 \in E_1$ and $e_2 \in E_2$ be the edges containing x (defined in Definition G). They exist by Claim H. Then $V(e_1) \cap V(e_2) = \{x\}$.

Proof of Claim I: Assume for the sake of contradiction that $|V(e_1) \cap V(e_2)| \geq 2$. If $|V(e_1) \cap V(e_2)| = 3$, then we delete e_1 from H_0 and add $V(e_1)$ to Y. This contradicts the "minimality" of H, as T^* remains unchanged, S^* decreases by 1/4 and N[Y] increases from Claim A by at most 27. Therefore assume that $|V(e_1) \cap V(e_2)| = 2$. Let $e_1 = \{x, v, w\}$ and let $e_2 = \{x, v, y\}$ where $w \neq y$. As $d_{H_0}(x) = d_{H_0}(v) = 2$, there is an edge, e', in H_0 such that $\{w, y\} \subseteq V(e')$, by Claim F. However $e' \notin E(H_1)$ and $e' \notin E(H_2)$ by Claim E. This is however a contradiction to (d), as $w, y \notin Y$.

Claim J: We now obtain a contradiction.

Proof of Claim J: Let $e_1 \in E_1$ and $e_2 \in E_2$ be the edges containing x (defined in Definition G). They exist by Claim H and $V(e_1) \cap V(e_2) = \{x\}$, by Claim I. Let $e_1 = \{x, v_1, v_2\}$ and let $e_2 = \{x, w_1, w_2\}$. Let e'_1 be the edge in H_1 containing w_1 and let e''_1 be the edge in H_1 containing w_2 (they exist by Claim H). Let e'_2 be the edge in H_2 containing v_1 and let e''_2 be the edge in H_2 containing v_2 (they exist by Claim H).

If $e'_1 = e''_1$, then $V(e'_1) \cap V(e_2) = \{w_1, w_2\}$ and $e'_1 = \{w_1, w_2, r\}$ for some $r \in V(H_0)$. By Claim F, there is an edge in H_0 that contains x and r. But this is a contradiction, as neither e_1 or e_2 contain r, by Claim H. Therefore $e'_1 \neq e''_1$. Analogously we can show that $e'_2 \neq e''_2$.

We now delete e_1, e'_1, e''_1 from H, H_0 and H_1 . Delete e_2, e'_2, e''_2 from H, H_0 and H_2 . Delete $V(e_1) \cup V(e'_1) \cup V(e''_1)$ from $V(H_1)$ and delete $V(e_2) \cup V(e'_2) \cup V(e''_2)$ from $V(H_2)$. Delete $V(e_1) \cup V(e_2)$ from H and H_0 . Let S_1 be any subset of size three in $V(e'_1) \cup V(e''_1) - \{w_1, w_2\}$ and let S_2 be any subset of size three in $V(e'_2) \cup V(e''_2) - \{v_1, v_2\}$. Add the edges S_1 and S_2 to H and H_0 . Finally add all vertices in $V(e'_1) \cup V(e''_1) \cup V(e''_2) \cup V(e''_2) - \{w_1, w_2, v_1, v_2, x\}$ to Y.

We first show that T^* decreases by at most 8. It is clear that the transversal size drops by three in both H_1 and H_2 . So assume that T' is a transversal of the new H_0 . As in the proof of Claim C we note that one of the three edges e_1, e'_2, e''_2 are already covered by a vertex in T' (due to S_2) and the other two edges can be covered by one additional vertex. Similarly by adding one more vertex to T' we can make sure that e_2, e'_1, e''_1 are all covered. Therefore the transversal size drops by at most two in H_0 .

Note that S^* drops by 33/4 as we delete 9 vertices in each of H_1 and H_2 and we delete 5 vertices in H_0 . We also delete three edges in each of H_1 and H_2 and six edges in H_0 . But we also add two edges in H_0 .

N[Y] increases by at most 72 vertices by Claim A, as $|V(e'_1) \cup V(e''_1) \cup V(e'_2) \cup V(e''_2) - \{w_1, w_2, v_1, v_2, x\}| \le 8$. As V^* decreases by at most 13, we note that $V^* - N[Y]$ decreases by at most 85. We note that (a)-(d) still holds after the above operations. We therefore have a contradiction to the "minimality" of H, as $8 \le 33/4 - 85/372$.

Theorem 8. If G is a graph with $\delta(G) \ge 3$ then $f_t(G; 2) \le (\frac{5}{4} - \frac{1}{372})|V(G)|$.

Proof. Let G be any graph with $\delta(G) \geq 3$ and let (W_1, W_2) be a partition of V(G). Define the hypergraph H_G , such that $V(H_G) = V(G)$ and $E(H_G)$ is obtained by selecting for each $v \in V(G)$ one set of three vertices from $N_G(v)$ to form a hyperedge. $E(H_G) =$ $\{e_v : v \in V(G)\}, e_v = \{x_v, y_v, z_v\} \subseteq N_G(v)$. Furthermore for every hyperedge, $e \in E(H_G)$ let L(e) be the set $\{0, i\}$ if $v \in W_i$. For reasons which will be clear later we let $L(v) = \{0, 1, 2\}$ for every $v \in V(H_G)$. Let H_i be the 3-uniform hypergraph containing vertex-set $V_i = \{v : i \in L(v) \text{ and } v \in V(H)\}$ and edge-set $E_i = \{e : i \in L(e) \text{ and } e \in E(H)\}$, for i = 0, 1, 2. Note that a transversal of H_0 corresponds to a total dominating set in G and a transversal of H_i ($i \in \{1, 2\}$) corresponds to a total dominating set in G of the set W_i . Therefore we would be done if we could show that $\mathcal{T}(H_0) + \mathcal{T}(H_1) + \mathcal{T}(H_2) \leq (\frac{5}{4} - \frac{1}{372})|V(G)|$. Let Y be an empty set. We note that $|E_1| + |E_2| = |E_0| = |V_0| = |V_1| = |V_2| = |V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]| = |V(G)|$ and therefore the inequality above is equivalent to

(*)
$$\sum_{i=0}^{2} \mathcal{T}(H_i) \le \left(\sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4}\right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}$$

For simplicity we will use the following notation: $T^* = \sum_{i=0}^{2} \mathcal{T}(H_i)$ $S^* = \sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4}$

 $V^* = V(H_0) \cap V(H_1) \cap V(H_2)$

We will now do a few transformations on H, H_0, H_1, H_2 .

Transformation 1: While there is some vertex $x \in V(H)$ with $d_{H_0}(x) \geq 5$ (or equivalently $d_H(x) \geq 5$), delete x and all edges incident with x from H (and therefore also from H_0 , H_1 and H_2).

Claim A: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim A: We note that T^* drops by at most three, as we may place x in the transversal of the new H_i 's in order to get transversals in the old H_i 's. We note that S^* decreases by at least 13/4, as we delete x from H_0 , H_1 , H_2 and 5 edges from H_0 plus a total of 5 edges from H_1 and H_2 . As V^* decreases by one and $N_H[Y] = \emptyset$ remains unchanged, we are done.

Transformation 2: While there is a vertex $x \in V(H)$ with $d_{H_1}(x) \geq 3$, delete x and all edges incident to x from H_0 and H_1 . Also delete these edges from H (but do not delete x or any edges incident to x in H_2). If $d_{H_2}(x) = 0$ then delete x from H_2 (i.e. delete 2 from L(x)). If $d_{H_2}(x) > 0$ then note that $d_{H_2}(x) = 1$ (as we have performed transformation 1 as long as we could) and put $N_{H_2}[x]$ in Y.

Claim B: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim B: We note that T^* drops by at most two, as we may place x in the transversal of the new H_0 and H_1 in order to get transversals in the old H_0 and H_1 . We note that S^* decreases by at least 9/4, as we delete 3 edges and 1 vertex from H_0 and H_1 and we either delete a vertex in H_2 or 4 edges from H_0 . As V^* decreases by one and $N_H[Y]$ increases by at most 21 (as $\Delta(H) \leq 4$, after Transformation 1), we are done.

Transformation 3: While there is a vertex $x \in V(H)$ with $d_{H_2}(x) \geq 3$, then do the following. Delete x and all edges incident to x from H_0 and H_2 . Also delete these edges from H (but do not delete x or any edges incident to x in H_1). Furthermore delete any

vertices in H_2 , which get degree zero by the above transformation. If $d_{H_1}(x) = 0$ then delete x from H_1 . If $d_{H_1}(x) > 0$, then we put $N_{H_1}[x]$ in Y.

Claim C: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim C: We note that T^* drops by at most two, as we may place x in the transversal of the new H_0 and H_2 in order to get transversals in the old H_0 and H_2 . Lets count any edge, e, in H_1 , which does not lie in H_0 as contributing $1 + |V(e) \cap V(H_0)|/3$ to the sum S^* . We note that there are no such edges when we start the transformation 3's.

We note that S^* now decreases by at least 25/12, because of the following. For every edge containing x in H_2 , which does not lie in H_0 there is a vertex of degree one in the edge, due to the above transformations. Therefore we either delete an edge in H_0 or a vertex in H_2 for each of the edges containing x in H_2 . As we also delete the edges in H_2 and the vertex x in H_0 and H_2 we note that S^* drops by at least 8/4. So if $d_{H_1}(x) = 0$ then S^* decreases by at least 9/4 as claimed. If $d_{H_1}(x) > 0$ and the edge, e, containing x in H_1 also lies in H_0 , then we are done as we delete an extra edge in H_0 and the edge left in H_1 is counted as at most 1 + 2/3. If $d_{H_1}(x) > 0$ and the edge, e, containing x in H_1 does not lie in H_0 , then we decrease the value of e by 1/3 as $1 + |V(e) \cap V(H_0)|/3$ decreases. This shows that S^* decreases by at least 25/12.

As V^* decreases by one and N[Y] increases by at most 21 (as $\Delta(H) \leq 4$, after Transformation 1), we are done.

Transformation 4: If $e_1, e_2 \in E(H_i)$ and $|V(e_1) \cap V(e_2)| \ge 2$ for some $i \in \{1, 2\}$, then we do the following.

If $|V(e_1) \cap V(e_2)| = 3$, then if $e_1, e_2 \in E_0$ we delete e_2 from both H_0 and H_i . If $e_j \notin E_0$ $(j \in \{1, 2\})$ then we delete e_j from H_i (in this case $V(e_j) \subseteq Y$). So now assume that $|V(e_1) \cap V(e_2)| = 2$ and $e_1 = (u_1, x, y)$ and $e_2 = (u_2, x, y)$, where $u_1 \neq u_2$,

If $d_{H_i}(u_1) = d_{H_i}(u_2) = 2$, then by the above transformations we note that $e_1, e_2 \in E_0$. We now add a new vertex q to H, H_0 and H_i . We delete e_1 and e_2 from H, H_i and H_0 and add the edges $\{q, x, y\}$ to H, H_i and H_0 .

If $d_{H_i}(u_j) = 1$, for some $j \in \{1, 2\}$, then do the following. Delete e_1, e_2 and the vertices $\{u_j, x, y\}$ from H_i . Add the vertices $\{u_1, u_2, x, y\}$ to Y.

Claim D: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim D: In the case when $|V(e_1) \cap V(e_2)| = 3$ we note that T^* remains unchanged, S^* decreases by 1/4 and $V^* - N[Y]$ remains unchanged. We are now done with this case.

In the case when $d_{H_i}(u_1) = d_{H_i}(u_2) = 2$, we note that T^* , S^* and V^* remain unchanged and N[Y] can only grow by adding q to it, but $q \notin V^*$. We also note that the above transformation decreases the number of edges in H_i , so it cannot continue indefinitely. We are now done with this case.

In the case when $d_{H_i}(u_j) = 1$, we note that T^* decreases by at most one, S^* decreases by 5/4, V^* decreases by at most three and N[Y] increases by at most 24 (In $H - e_1 - e - 2$ we note that u_1 and u_2 have degree at most 3 while x and y have degree at most 2). As $1/4 \ge 27/372$ we are done with this case. Claim E: $\Delta(H_1), \Delta(H_2) \leq 2$ and $\Delta(H - E(Y)) \leq 4$ and there are no overlapping edges in $H_i, i \in \{1, 2\}$.

Proof of Claim E: The fact that $\Delta(H_1), \Delta(H_2) \leq 2$ follow from Transformations 2 and 3. As $\Delta(H) \leq 4$ after Transformation 1 and no other transformation increases $\Delta(H)$, we note that $\Delta(H - E(Y)) \leq \Delta(H) \leq 4$. There are no overlapping edges in $H_i, i \in \{1, 2\}$ due to Transformation 4.

Claim F: If $e \in E(H) - E(Y)$, then $0 \in L(e)$ and $|L(e)| \ge 2$.

Proof of Claim F: This was true before Transformation 1 as it was true for all edges. Transformation 1 clearly does not change this property. In Transformation 2, we only keep an edge, e, in H_i , where $i \in \{1, 2\}$ but delete it in H_0 if we put V(e) in Y. So the above still holds after Transformation 2. Analogously it also holds after Transformation 3. It is not difficult to check that it also holds after Transformation 4 (note that the above property holds for the edge we might add to H in Transformation 4).

We now see that (*) holds due to Lemma 3. That implies the theorem.

8. Possible strengthening of Theorem 8

No graph extremal for Theorem 8 is known and probably an inequality $f_t(G; 2) \leq \alpha |V(G)|$ can be obtained for some α smaller than $\frac{5}{4} - \frac{1}{372}$. Certainly α must be at least 9/8, that is demonstrated by the graphs of section 6.

There is a graph of order 12 having $f_t(H_{12}; 2) = 7n/6$, namely H_{12} from the family \mathcal{H} defined after Theorem 2, with the two P_6 's as its partition classes. Unless we, e.g., demand that the order of the graphs be large, H_{12} shows that we cannot get a better inequality than the following conjecture.

Conjecture 1. Let G be a graph of order n with $\delta \geq 3$ then $f_t(G;k) \leq 7n/6$.

9. Three partition classes

Theorem 9. Let G be a graph of order n with $\delta \geq 3$ then $f_t(G;3) \leq 3n/2$.

For arbitrarily large $n, n \equiv 0 \pmod{6}$, there exist graphs G_n with $g_t(G_n; 3) = n$, $\gamma_t(G_n) = n/3, f_t(G; 3) = 4n/3.$

Proof. By Theorem 1 we have that $\gamma_t(G) \leq n/2$, and $g_t(G;3) \leq n$ holds trivially, so by addition we get $f_t(G;3) \leq 3n/2$ as desired.

Assume a graph G has $g_t(G; 3) = n$. Then $\Delta(G) \leq 3$ and as $\delta(G) \geq 3$, G is cubic. Since each vertex has three neighbours, one in each partition class, we see for each i = 1, 2, 3, that vertices in class V_i span a matching in G.

Listing the 3 neighbours to each V_i -vertex we count each vertex of G once, so $3|V_i| = n$ giving $|V_1| = |V_2| = |V_3| = n/3$.

Each V_1 -vertex is adjacent to precisely one V_2 -vertex and that has no other V_1 -neighbour, so there is a perfect matching of V_1V_2 -edges and analogously G contains perfect matchings of V_1V_3 - and V_2V_3 -edges.

One partition class V_i totally dominates G so $\gamma_t(G) \leq n/3$. In fact, $\gamma_t(G) = n/3$ because each vertex in G can totally dominate at most its three neighbours.

Following the steps above, it is now easy for $n \equiv 0 \pmod{3}$ to construct a graph G_n with $g_t(G_n; 3) = n$. This graph has $f_t(G_n; 3) = \gamma_t(G_n) + g_t(G_n; 3) = 4n/3$.

We do not know if there, for $\delta \geq 3$, are graphs G with $4n/3 < f_t(G;3) \leq 3n/2$, but we pose the following conjecture.

Conjecture 2. There exists some positive ϵ such that the following holds. If G is a graph with $\delta(G) \geq 3$, then $f_t(G;3) \leq (3/2 - \epsilon)|V(G)|$.

Theorem 10. Let G be a graph of order n with $\delta \geq 3$ and let $k \geq 4$. $f_t(G;k) \leq 3n/2$ and there exists an infinite family of graphs with $f_t(G;k) = 3n/2$.

Proof. The inequality is proven as in Theorem 9. For a graph with $f_t(H; k) = 3n/2$ take $H \in \mathcal{H}$ (\mathcal{H} is defined after Theorem 2). Let $v_1, v_2, \ldots, v_{n/2}$ and $u_1, u_2, \ldots, u_{n/2}$ be two disjoint paths in H such that $\{v_1u_2, v_2u_1, v_1v_{n/2}, u_1u_{n/2}\} \subseteq E(H)$. Let V_1, V_2, V_3, V_4 be a partition of H such that $l(v_1), l(v_2), \ldots, l(v_{n/2}), \ldots = 1, 2, 3, 4, 1, 2, 3, 4, \ldots$ and $l(u_1), l(u_2), \ldots, l(u_{n/2}), \ldots = 4, 3, 2, 1, 4, 3, 2, 1, \ldots$ where l(x) = i if $x \in V_i$, then $f_t(H; V_1, V_2, V_3, V_4) = 3n/2$.

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