An Efficient Formulation of the Elasto-plastic Constitutive Matrix on Yield Surface Corners

Clausen, Johan; Andersen, Lars; Damkilde, Lars

Published in:
Proceedings of the Twenty Second Nordic Seminar on Computational Mechanics

Publication date:
2009

Document Version
Publisher's PDF, also known as Version of record

Proceedings of the Twenty Second Nordic Seminar on Computational Mechanics
 | ISSN 1901-7278 DCE Technical Memorandum No. 11

An efficient formulation of the elasto-plastic constitutive matrix on yield surface corners

Johan Clausen*, **Lars Andersen and Lars Damkilde**
Department of Civil Engineering
Aalborg University, Aalborg, Denmark
e–mail: jc@civil.aau.dk

Summary A formulation for the elasto-plastic constitutive matrices on discontinuities on yield surfaces is presented, for use in finite element calculations. The formulation entails no rounding of the yield surface or the plastic potential, as it is done in most other formulations, and therefore exact analytical solutions can be approached. Computational examples are given with the Mohr-Coulomb, the Modified Mohr-Coulomb and the Hoek-Brown models.

Introduction

In practical geotechnical engineering most design calculations on soil structures are carried out with the Mohr-Coulomb material model, with the well-known hexagonal shaped yield criterion in principal stress space, see Fig. 1b. For clays the Tresca criterion, Fig. 1a, is used, and for rocks and concrete the Modified Mohr-Coulomb and the Hoek-Brown criteria are often used, see Fig. 1c and d.

As can be seen from the figure these criteria possess corners and apices, which explicitly have to be taken into account when formulating the constitutive matrices used for formulating the global stiffness matrix. This is especially true for 3D-calculations where all the different corner and apex discontinuities may come into play. One option of dealing with these discontinuities is to perform a local rounding of the yield criterion and/or the plastic potential, see e.g. [1, 2]. This option seems to work but the obtained numerical results do no longer converge towards the exact analytical results.

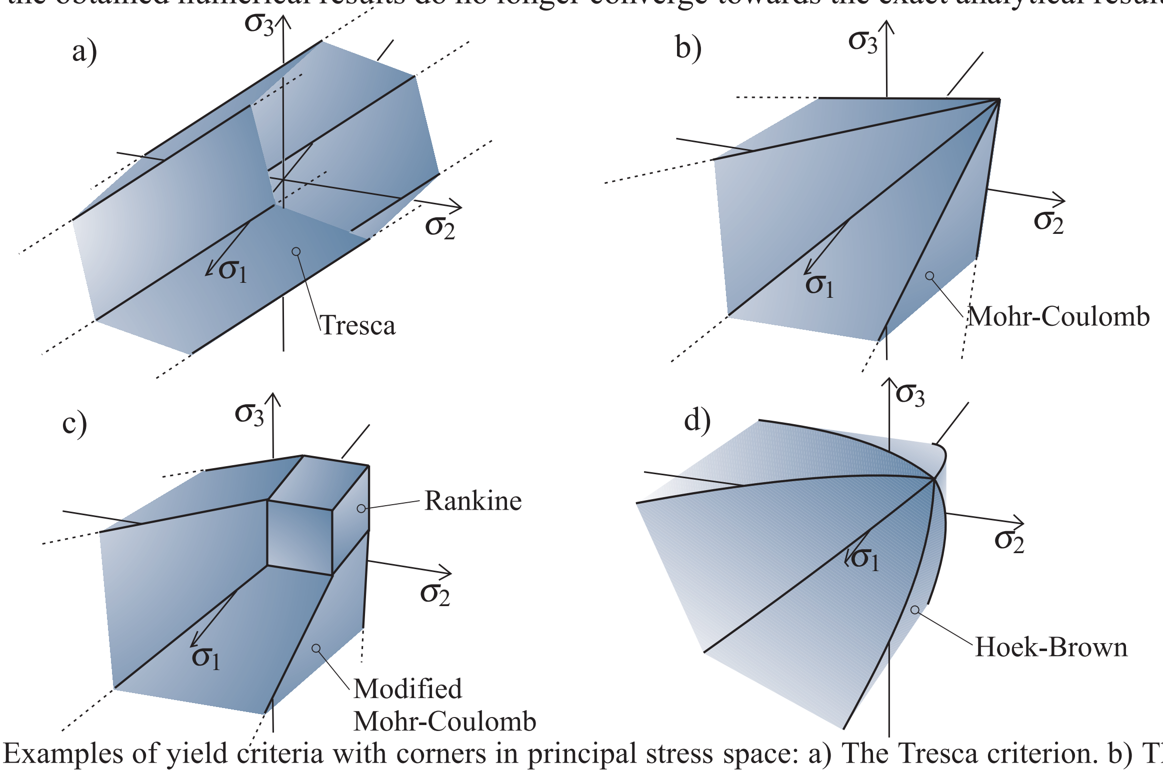

Figure 1: Examples of yield criteria with corners in principal stress space: a) The Tresca criterion. b) The Mohr-Coulomb Criterion. c) The Modified Mohr-Coulomb criterion. d) The Hoek-Brown criterion.

In this paper a formulation is presented that does not include a rounding of the corners or apices. It is also shown that the numerical solution for a footing on a Mohr-Coulomb soil converges towards the exact analytical solution.

Constitutive matrix on a surface

When a stress point is located on a yield surface the elasto-plastic constitutive matrix is found as

$$\mathbf{D}^{ep} = \mathbf{D} - \frac{\mathbf{D}\mathbf{b}^{\mathrm{T}}\mathbf{a}\mathbf{D}}{\mathbf{a}^{\mathrm{T}}\mathbf{D}\mathbf{b}} \tag{1}$$

where $\mathbf{a} = \partial f / \partial \boldsymbol{\sigma}$, $\mathbf{b} = \partial g / \partial \boldsymbol{\sigma}$ and $\mathbf{D}$ is the elastic constitutive matrix. f and g is the yield function and the plastic potential, respectively. Note that $\mathbf{D}^{ep}$ is singular with respect to $\mathbf{b}$, i.e. $\mathbf{D}^{ep}\mathbf{b} = \mathbf{0}$.

Constitutive matrix on a corner and an apex

When the stress point is located on a corner the constitutive matrix must be singular with respect to both $\mathbf{b}_1$ and $\mathbf{b}_2$. In Fig. 3 a direction vector of a yield surface corner, $\bar{\boldsymbol{\ell}}$ is shown. This can be regarded as a direction vector of any of the lines defining the yield criteria in Fig. 1. In Fig. 3 the direction vector of the plastic potential corner, $\bar{\boldsymbol{\ell}}^g$, is also shown. From these direction vectors it is shown in [3] that the doubly singular constitutive matrix on a line in principal stress space can be expressed as:

$$\bar{\mathbf{D}}^{ep}_{\ell} = \frac{\bar{\boldsymbol{\ell}}(\bar{\boldsymbol{\ell}}^g)^{\mathrm{T}}}{\bar{\boldsymbol{\ell}}^{\mathrm{T}}\bar{\mathbf{D}}^{-1}\bar{\boldsymbol{\ell}}^g} = \frac{(\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2)(\bar{\mathbf{b}}_1 \times \bar{\mathbf{b}}_2)^{\mathrm{T}}}{(\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2)^{\mathrm{T}}\,\bar{\mathbf{D}}^{-1}(\bar{\mathbf{b}}_1 \times \bar{\mathbf{b}}_2)} \tag{2}$$

The overbar indicates the the vectors and matrices are expressed with respect to the principal coordinates without the shear component terms, i.e. the vectors have three components and the matrices three by three. The $\times$ symbol indicates the cross product. The shear part is added,

$$\hat{\mathbf{D}}^{ep}_{\ell} = \begin{bmatrix} \bar{\mathbf{D}}^{ep}_{\ell} & \\ & \bar{\mathbf{G}} \end{bmatrix} \tag{3}$$

and the matrix is transformed from the principal stress coordinate system into the *xyz*-coordinate system. In the above equation the hat, ˆ, signifies that the matrix includes all six by six components and is expressed in the principal coordinate system. The matrix $\bar{\mathbf{G}}$ is the shear part of the elastic constitutive matrix.

There are two different forms of constitutive matrix on an apex. If the stress point is located on an apex on the hydrostatic line the constitutive matrix must be singular with respect to all stress directions, i.e.

$$\hat{\mathbf{D}}^{ep}_{a,1} = \mathbf{D}^{ep}_{a,1} = \mathbf{0} \tag{4}$$

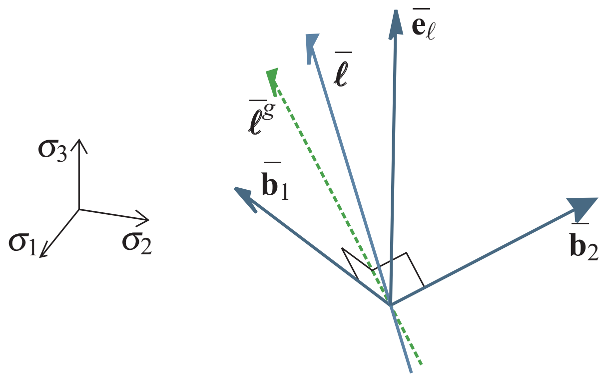

Figure 2: A direction vector, $\bar{\boldsymbol{\ell}}$, of an intersection line in principal stress space. The corresponding potential curve direction vector is denoted $\bar{\boldsymbol{\ell}}^g$. An elastic strain direction vector is denoted $\bar{\mathbf{e}}_\ell$. The vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are perpendicular to the direction vector of the plastic potential intersection line, $\bar{\boldsymbol{\ell}}^g$.

This is the case on the Mohr-Coulomb apex, the Hoek-Brown apex and one of the Modified Mohr-Coulomb apices, see Fig. 1. If, on the other hand, the stress point is located on an apex not on the hydrostatic line it is singular only in the normal directions, i.e. its composition in the principal coordinates is

$$\hat{\mathbf{D}}^{ep}_{a,2} = \begin{bmatrix} \mathbf{0} & \\ & \bar{\mathbf{G}} \end{bmatrix} \tag{5}$$

This is the case for stress points located on the Modified Mohr-Coulomb apices outside the hydrostatic line, see Fig 1c.

Improved formulation

The formulations for the constitutive matrices given above works well for two-dimensional models where the (instant) friction angle is not too high, see e.g. [3, 4]. But for high friction angles and/or three dimensional problems the above formulations can cause the global stiffness matrix to become ill-conditioned. This is due to many stress points located on either corners or apices which add many singularities to the global stiffness matrix. This problem can be mended by adding a small stiffness in appropriate directions.

Improved formulation on the apex

When the elasticity of the material is linear the implicit stress integration can written in the "return mapping" formulation,

$$\bar{\boldsymbol{\sigma}}_{\mathrm{C}} = \bar{\boldsymbol{\sigma}}_{\mathrm{B}} - \Delta\bar{\boldsymbol{\sigma}}^{p}, \qquad \text{with} \quad \boldsymbol{\sigma}_{\mathrm{B}} = \boldsymbol{\sigma}_{\mathrm{A}} + \mathbf{D}\Delta\boldsymbol{\varepsilon} \quad \text{and} \quad \Delta\bar{\boldsymbol{\sigma}}^{p} = \bar{\mathbf{D}}\Delta\bar{\boldsymbol{\varepsilon}}^{p} \tag{6}$$

Here $\bar{\boldsymbol{\sigma}}_{\mathrm{C}}$ is the updated stress point on the yield surface, $\bar{\boldsymbol{\sigma}}_{\mathrm{B}}$ is the elastic predictor stress and $\Delta\bar{\boldsymbol{\sigma}}^{p}$ is the plastic corrector stress, all three expressed in the principal stress space as indicated by the overbar. The total strain increment is denoted $\Delta\boldsymbol{\varepsilon}$ and the plastic strain increment in principal coordinates is $\Delta\bar{\boldsymbol{\varepsilon}}^{p}$.

A key point of the elasto-plastic constitutive matrix is that it must be singular in the direction of the plastic strain increment. A simple method to add a little stiffness in the formulation of $\mathbf{D}^{ep}$ on the apex is given as

$$\bar{\mathbf{D}}^{ep}_{a,\mathrm{mod}} = \frac{1}{\alpha}\left(\bar{\mathbf{D}} - \frac{\bar{\mathbf{D}}^{\mathrm{T}}\,\Delta\bar{\boldsymbol{\varepsilon}}^{p}(\Delta\bar{\boldsymbol{\varepsilon}}^{p})^{\mathrm{T}}\bar{\mathbf{D}}}{(\Delta\bar{\boldsymbol{\varepsilon}}^{p})^{\mathrm{T}}\bar{\mathbf{D}}\Delta\bar{\boldsymbol{\varepsilon}}^{p}}\right) \tag{7}$$

This matrix is singular in the plastic strain direction and depending on the value of α posesses a small stiffness in other directions. In the presentation a study on the optimal value of α will be given.

Improved formulation on a corner

When the updated stress point is located on a corner the basic formulation for the constitutive matrix is given by Eq. (2). Again a simple formulation that adds a little stiffness is

$$\bar{\mathbf{D}}^{epc}_{\ell} = \frac{\bar{\boldsymbol{\ell}}(\bar{\boldsymbol{\ell}}^{g})^{\mathrm{T}}}{\bar{\boldsymbol{\ell}}^{\mathrm{T}}(\bar{\mathbf{D}}^{c})^{-1}\bar{\boldsymbol{\ell}}^{g}} + \frac{1}{\beta}\frac{\bar{\mathbf{c}}\,\bar{\mathbf{c}}^{\mathrm{T}}}{\bar{\mathbf{c}}^{\mathrm{T}}(\bar{\mathbf{D}}^{c})^{-1}\bar{\mathbf{c}}} \tag{8}$$

The direction vector $\bar{\mathbf{c}}$ is the direction perpendicular to the plastic strain direction, $\Delta\bar{\varepsilon}^{p}$, and the line defining the corner, $\bar{\boldsymbol{\ell}}$, see Fig. 2. In the presentation different results indicating the optimal value of the scalar β will be given. β controls the amount of stiffness that will be added. The Ref. [5].

Computational example

To assess the validity of the formulation a calculation is carried out with a rough circular footing resting on a cohesionless Mohr-Coulomb soil with a friction angle of $\varphi = 30°$, and a selfweight of $\gamma = 20$ kN/m^3. For symmetry reasons only a quarter of the footing is modelled, see Fig. 3a.

a)

b)

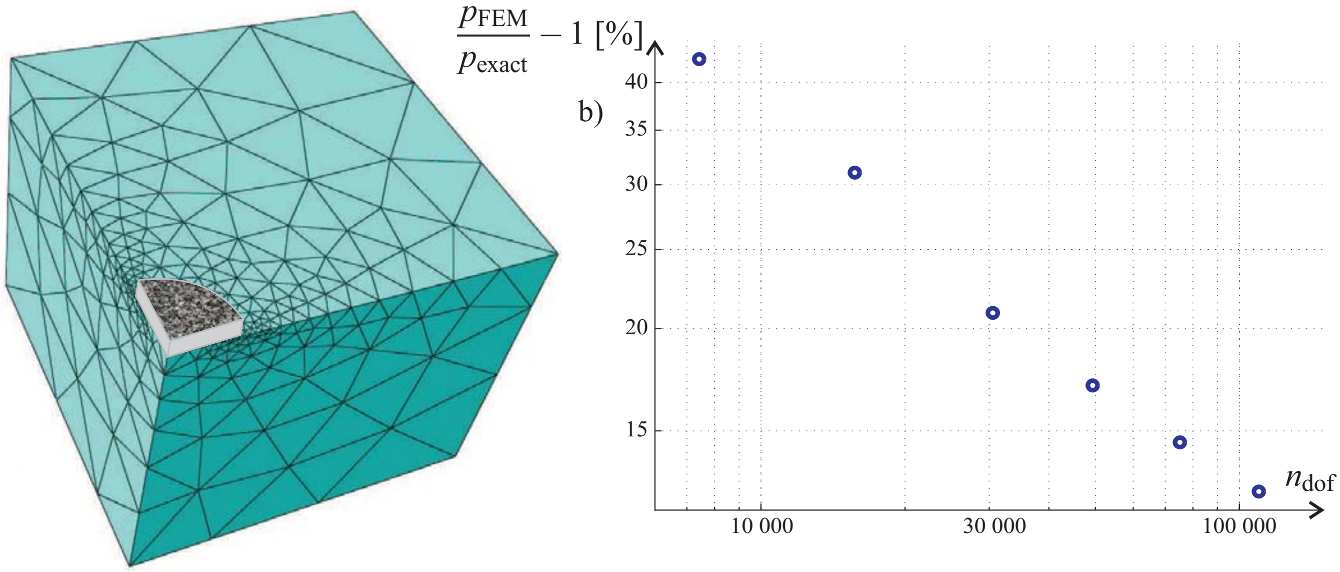

Figure 3: a) A quarter of a circular footing and an example of the element mesh with 7425 degrees of freedom. b) Results from the bearing capacity calculations compared to the exact value.

The elements are standard ten-node tetrahedrons. A vertical forced displacement is applied to the footing in steps and the bearing capacity is calculated from the sum of the maximum reaction forces at the footing nodes divided by the footing area. The exact bearing capacity is found in Ref. [6]. The result of the calculations can be seen in Fig. 3 for different mesh densities. It is seen that the calculated values converge toward the exact value. In the presentation results for the other material models shown in Fig. 1 will be given.

Conclusion

A formulation for elasto-plastic constitutive matrices on corner and apex singularities is given. The initial formulation is improved in order to make full 3D-calculations stable. It is shown that finite element calculations based on the formulation converge towards the exact solution.

References

[1] A. J. Abbo and S. W. Sloan. A smooth hyperbolic approximation to the mohr-coulomb yield criterion. *Computers & Structures*, 54(3):427–441, 1995.

[2] H. A. Taiebat and J. P. Carter. Flow rule effects in the tresca model. *Computers and Geotechnics*, 35:500–503, 2008.

[3] Johan Clausen, Lars Damkilde, and Lars Andersen. An efficient return algorithm for non-associated plasticity with linear yield criteria in principal stress space. *Computers & Structures*, 85:1795–1897, 2007.

[4] Johan Clausen, Lars Damkilde, and Lars Andersen. Efficient return algorithms for associated plasticity with multiple yield planes. *International Journal for Numerical Methods in Engineering*, 66(6):1036–1059, 2006.

[5] Johan Clausen. *Efficient Non-Linear Finite Element Implementation of Elasto-Plasticity for Geotechnical Problems*. PhD thesis, Esbjerg Institute of Technology, Aalborg University, 2007. http://vbn.aau.dk/fbspretrieve/14058639/JCthesis.pdf.

[6] C. M. Martin. User guide for ABC - analysis of bearing capacity, version 1.0. OUEL Report No. 2261/03, University of Oxford, 2004.