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***Abstract*— Binary message-passing decoders for LDPC codes are analyzed using EXIT charts. For the analysis, the variable node decoder performs all computations in the L-value domain. For the special case of a hard decision channel, this leads to the well-know Gallager B algorithm, while the analysis can be extended to channels with larger output alphabets. By increasing the output alphabet from hard decisions to four symbols, a gain of more than 1.0 dB is achieved using optimized codes. For this code optimization, the mixing property of EXIT functions has to be modified to the case of binary message-passing decoders.**

I. Introduction

When Gallager introduced low-density parity-check (LDPC) codes [1], [2], he also presented binary message-passing decoding algorithms that exchange only binary messages between the variable and check node decoder. These algorithms are called Gallager A and Gallager B and we refer to [1], [2] for a description of them. The advantages of these algorithms are the reduced memory requirements and the low complexity implementation, especially of the check node decoder, making them promising candidates for high-speed applications. However, these advantages come with the cost of a significant loss in performance.

In this work, we apply extrinsic information transfer (EXIT) charts [3] in the analysis of binary message-passing algorithms. For binary messages, the mutual information describes the probability densities of the messages uniquely. In this case, the EXIT functions are exact. In contrast, this is not the case if the messages are approximated by Gaussian distributions. Furthermore, the EXIT functions for binary message-passing algorithms can be derived analytically, avoiding the need of Monte-Carlo simulations.

Binary message-passing algorithms were studied in [4] where the authors proved that optimum algorithms must satisfy certain symmetry and isotropy conditions. In contrast to majority based decision rules, we assume that the variable node decoder converts all incoming messages to L-values [5], performs decoding in the L-value domain and applies a hard decision on the result.[1] This general approach assures that the symmetry and isotropy conditions are satisfied and we are able to extend the algorithms for systems where the channel provides more information than hard decisions, while the variable and check node decoder still exchange binary messages only. This reduces the gap between optimum decoding and binary message-passing decoding, while still keeping the complexity low.

The rest of this paper is organized as follows. In Section II, we introduce basics and definitions which are used in Section III to derive the EXIT functions of the variable and check node decoders. In Section IV, we show how the EXIT functions can be used to optimize the code and the results are verified with simulations in Section V.

[1]Note that for these algorithms, there always exist majority decision rules that can be derived in a straightforward way.

II. Preliminaries

For binary message-passing decoders, the extrinsic channel [3] of the variable and check node decoder is represented as a binary symmetric channel (BSC) with crossover probability ϵ which we assume to be smaller than or equal to 0.5. Since there is a one-to-one mapping between mutual information and crossover probability for the BSC, we can equivalently describe those channels using their capacities

$$I = 1 - h_b(\epsilon), \tag{1}$$

where $h_b(\cdot)$ denotes the binary entropy function.

The *reliability* associated with a BSC is defined as

$$R = \log \frac{1-\epsilon}{\epsilon} \geq 0. \tag{2}$$

The L-value can be expressed in terms of the reliability as

$$L = y \cdot R, \tag{3}$$

where y denotes the output of a BSC which takes on values from $\{+1, -1\}$.

III. EXIT Functions of Component Decoders

A check node of degree d_c of a binary message-passing algorithm computes the output as the modulo-2 sum of the other $d_c - 1$ inputs. Let $\epsilon_{ac} = h_b^{-1}(1 - I_{ac})$ denote the a-priori crossover probability at the input of the check node.

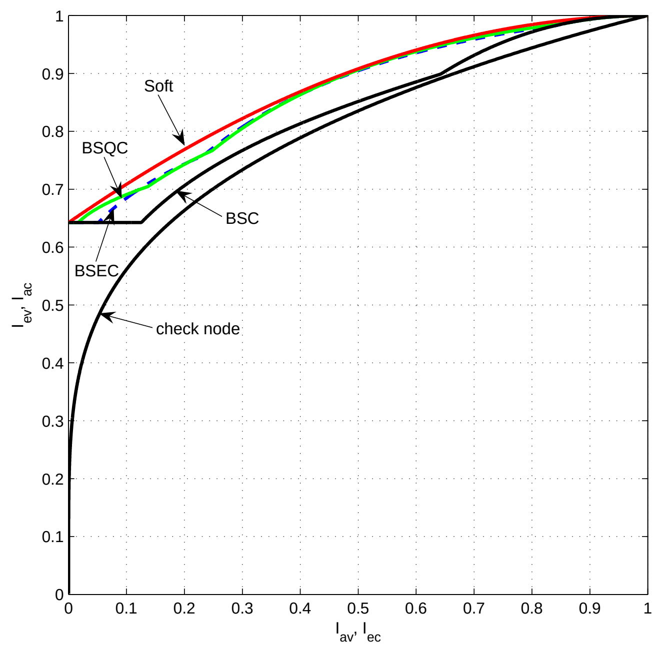

Fig. 1. EXIT Function of Check Nodes with $d_c = 6$ and Variable Nodes with $d_v = 4$ and $\sigma = 0.67$ for BSC, BSEC, BSQC and Soft Output.

Then the crossover probability at the output reads [2, Lemma 4.1]

$$\epsilon_{ec} = f_c(\epsilon_{ac}; d_c) = \frac{1 + (1 - 2\epsilon_{ac})^{d_c - 1}}{2}, \tag{4}$$

where f_c is the EXIT function of a check node parametrized by d_c. Using (4) and (1) leads to $I_{ec} = 1 - h_b(\epsilon_{ec})$. The inverse of the EXIT function in (4) takes the form

$$\epsilon_{ac} = f_c^{-1}(\epsilon_{ec}; d_c) = \frac{1 - (1 - 2\epsilon_{ec})^{\frac{1}{d_c - 1}}}{2}. \tag{5}$$

An example of an EXIT function of a check node decoder with $d_c = 6$ is shown in Figure 1.

For a variable node of degree d_v, the computation of the outgoing messages consists of the summation of all other $d_v - 1$ incoming messages and the channel message in the L-value domain [6]. In order to be able to perform this summation, all messages are converted to L-values using (2) and (3), assuming that the variable node decoder knows the parameters of the communication and the extrinsic channel. We show in Section V how the decoder can be implemented without this knowledge. In the following, we assume an additive white Gaussian noise (AWGN) communication channel with noise variance σ^2 where the received values y_{ch} are converted to L-values as

$$L_{ch} = \frac{2}{\sigma^2} y_{ch}, \tag{6}$$

before being quantized. The unquantized L-values are Gaussian random variables with variance $\sigma_{ch}^2 = 4/\sigma^2$ and mean $\mu_{ch} = \sigma_{ch}^2/2$ [7]. In the following sections, we derive the EXIT functions for various quantization schemes.

A. Hard Decision Channel

Consider the case where the receiver performs hard decisions. Then the communication channel can be modeled as a BSC with crossover probability ϵ_{ch}. Let R_{ch} and R_{av} denote the reliabilities of the communication and extrinsic channel respectively. In order to compute an outgoing message, the variable node decoder converts all incoming messages to L-values and computes the sum of the channel L-value and all other $(d_v - 1)$ incoming L-values as

$$L_{ev} = L_{ch} + \sum_{i=1}^{d_v - 1} L_{av,i}. \tag{7}$$

The outgoing message transmitted to the check node decoder is the hard decision of L_{ev}. The probability that this message is in error is

$$\begin{aligned} \epsilon_{ev} &= f_v(\epsilon_{av}; d_v, \epsilon_{ch}) \qquad (8) \\ &= 1 - \epsilon_{ch} B\left(\left\lfloor \frac{R_{av}(d_v - 1) - R_{ch}}{2R_{av}} \right\rfloor; d_v - 1, \epsilon_{av}\right) \\ &- (1 - \epsilon_{ch}) B\left(\left\lfloor \frac{R_{av}(d_v - 1) + R_{ch}}{2R_{av}} \right\rfloor; d_v - 1, \epsilon_{av}\right), \end{aligned}$$

where

$$B(k; n, p) = \sum_{i=0}^{k} \binom{n}{i} p^i (1 - p)^{n-i} \tag{9}$$

denotes the binomial cumulative distribution. The first product term in (8) represents the probability that the channel message is in error but the messages from the check nodes are able to correct it, and the second product term represents the probability that the channel message is correct and is not changed by the check node messages.

An example of this EXIT function is shown in Figure 1. It can be observed that the decoder changes its behavior depending on I_{av}. This corresponds to the Gallager B algorithm [1], [2] where the majority decision rule is changed depending on the crossover probability. Compared with channels using a larger output alphabet, this EXIT function serves as a lower bound. Using the L-value representation we are able to generalize this algorithm to channels with larger output alphabets.

B. Larger Output Alphabets

We consider the case where the channel messages stem from a binary input additive noise channel with a K-ary quantizer. The quantizer provides the sign of the received values and the magnitude where the quantization scheme is described by the vector $\boldsymbol{\zeta} = [\zeta_0, \ldots, \zeta_K]$ where $0 \le \zeta_0 < \zeta_1 < \cdots < \zeta_K$. Let k be the sub-channel indicator defined as

$$k = \underset{k'}{\operatorname{argmin}} |L_{ch}| < \zeta_{k'}, \tag{10}$$

and let $L_{ch,K}$ be the quantized channel message

$$L_{ch,K} = \operatorname{sign}(L_{ch}) \cdot k, \tag{11}$$

where $\operatorname{sign}(\cdot)$ is the signum function.

Following [8], this channel quantization scheme can be decomposed as $(K+1)$ BSCs. Sub-channel k is used with probability p_k and has cross-over probability $\epsilon_{ch,k}$. We define sub-channel zero as a BSC with crossover probability 0.5 [8]. The parameters for sub-channel zero are

$$p_0 = \int_{-\zeta_0}^{\zeta_0} g(l)dl, \quad \text{and} \quad \epsilon_{ch,0} = \frac{1}{2}, \tag{12}$$

where $g(l) = p_{L|X}(l|X=+1)$ is the conditional transition probability of the channel. For $k>0$ we have

$$\begin{aligned} p_k &= \int_{\zeta_{k-1}}^{\zeta_k} g(l)dl + \int_{-\zeta_k}^{-\zeta_{k-1}} g(l)dl \qquad (13) \\ &= \int_{-\zeta_k}^{\zeta_k} g(l)dl - p_{k-1} \end{aligned}$$

and

$$\epsilon_{ch,k} = \frac{1}{p_k}\int_{-\zeta_k}^{-\zeta_{k-1}} g(l)dl. \tag{14}$$

The EXIT function of the overall channel is the average EXIT function of the sub-channels

$$I_{ev} = \sum_{k=0}^{K} p_k I_{ev,k}. \tag{15}$$

Example 1 (BSEC): The output of the binary symmetric erasure channel (BSEC) takes on values from $\{+1, 0, -1\}$. This quantization can be represented using

$$\boldsymbol{\zeta} = \left[\zeta_0, \quad \infty\right]. \tag{16}$$

In the case of an erasure from the channel, the variable node decoder has to rely completely on the messages from the check node decoder in order to compute its outgoing message. Therefore, the EXIT function can be below the channel capacity for small values of I_{av}. However, we will assume in the following that the variable node decoder has always knowledge of the hard decision of the channel message and therefore the EXIT function is always larger than or equal to the capacity of the channel. The EXIT function of the variable node decoder for a BSEC with $\zeta_0 = 1.69$ is shown in Figure 1, where ζ_0 was chosen such that the area below the EXIT function is maximized. For code design, this parameter has to be optimized jointly with the degree distributions of the code.

Example 2 (BSQC): The output of a binary symmetric quaternary output channel (BSQC) takes on values from $\{-2, -1, +1, +2\}$ which can be represented by a quantization using

$$\boldsymbol{\zeta} = \left[0, \quad \zeta_1, \quad \infty\right]. \tag{17}$$

The EXIT function of the variable node decoder for this channel with $\zeta_1 = 1.90$ (which maximizes the area below the EXIT function) is shown in Figure 1.

C. Soft Decision Channel

In the limit of no quantization of the output of an AWGN channel, the crossover probability at the output of the variable node decoder can be derived as

$$\begin{aligned} \epsilon_{ev} = 1 - \sum_{z=0}^{d_v-1} \quad & b(z; d_v-1, \epsilon_{av}) \cdot \\ & Q\left(\frac{R_{av}(d_v-1-2z)-\mu_{ch}}{\sigma_{ch}}\right), \end{aligned}$$

where $Q(\cdot)$ is defined as

$$Q(\phi) = \frac{1}{\sqrt{2\pi}}\int_{\phi}^{\infty} e^{-\frac{\psi^2}{2}} d\psi, \tag{18}$$

and

$$b(k;n,p) = \binom{n}{k} p^k (1-p)^{n-k}, \tag{19}$$

denotes the binomial probability mass function.

The EXIT function for this type of channel is shown in Figure 1. Since every quantized channel can be derived from the soft output channel, this EXIT function serves as an upper bound.

IV. Code Optimization

In this section we describe how to maximize the code rate for a given channel by optimizing the variable node degree distribution λ [7] of the code (we consider only check regular codes). In the case of binary message-passing decoders, the EXIT function of the mixture of codes can not be computed as the average of the EXIT functions of the component codes as presented in [3]. To show this, we prove the following theorem which was mentioned in [9], [10], [11].

Theorem 1: Let $g(l)$ denote the conditional probability density function at the output of a decoder averaged with respect to the component nodes. If $g(l)$ satisfies $g(-l) = e^l g(l)$ then the average EXIT functions of the component codes equals the EXIT function of the average distribution.

Proof: Using $g(l) = p_{L|X}(l|X=+1)$, the mutual information between the binary input X and the output L of a channel, can be written as

$$\begin{aligned} I(X;L) &= h(L) - h(L|X) \\ &= \int_0^{\infty} -\left[g(l)+g(-l)\right]\log\frac{g(l)+g(-l)}{2} \\ &\quad + g(l)\log g(l) + g(-l)\log g(-l)dl. \end{aligned}$$

Using the symmetry condition $g(-l) = e^{-l}g(l)$ this simplifies to

$$I(X;L) = \int_0^{\infty} g(l)\left\{e^{-l}\log\frac{2}{1+e^l} + \log\frac{2}{1+e^{-l}}\right\}dl,$$

which is a linear operation on $g(l)$. Therefore, one can invert the order of computation of the mutual information and of the average over the component codes. ■

In the case of binary message passing decoders, the density of the computed L-values of the i^{th} component code consists of two nonzero values at $+R_{ev,i}$ and $-R_{ev,i}$. The mixture of

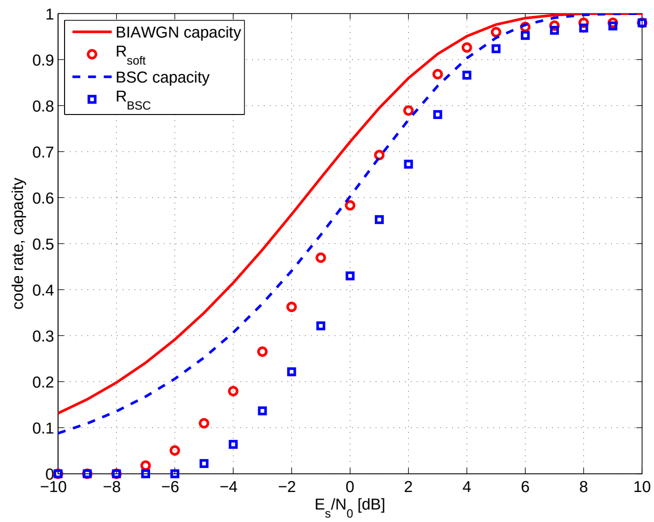

Fig. 2. Thresholds of optimized Codes for Soft Channel Information and Hard Decision Channel (BSC).

these densities has more than two nonzero values, but will be quantized to $\{+1, -1\}$ before being transmitted to the check nodes. This nonlinear operation prohibits the exchange of averaging and the computation of the mutual information.

Computing the resulting EXIT function of the variable node decoder can still be written in a linear manner by averaging over the crossover probabilities instead of the individual EXIT functions [12] as

$$\epsilon_{ev} = \sum_{i=1}^{d_{v,max}} \lambda_i \epsilon_{ev,i}, \tag{20}$$

and also formulating the constraint in terms of crossover probabilities

$$f_v(\epsilon) > f_c^{-1}(\epsilon), \quad \text{for all } \epsilon \in (0, 0.5). \tag{21}$$

Since the rate of the code is a linear function in λ and also the constraints are linear, we can apply linear programming to solve this optimization problem.

Using this procedure, we optimized codes and compared them with the capacity of the BIAWGN and BSC. For the optimization we set the maximum variable node degree to $d_{v,max} = 100$ and performed the optimization for check node degrees in the range between 2 and 100. The thresholds of these codes are shown in Figure 2.

It can be observed, that the gap to capacity decreases with increasing rate. This makes binary message-passing decoders attractive for applications which require a high code rate. For a rate of 0.9, the best code using soft channel information is as close as approximately 0.5 dB to capacity. Note that the soft channel and the hard decision channel serve as upper and lower bounds respectively for all quantization schemes.

V. Simulation Results

In Section III, we assumed that the variable node decoder has knowledge of the parameters of the extrinsic channel, i.e. it knows the crossover probability of the messages going from check nodes to variable nodes. Since this assumption can not be satisfied in practice, we present a method to implement the decoder without this knowledge.

By using the EXIT functions, we can predict the trajectory of the decoder. We assume a channel parameter where convergence in a maximum number of iterations is guaranteed (i.e. a bit above the threshold) and use the decoding trajectory to compute a sequence of crossover probabilities. This sequence is used instead of the genie-aided knowledge by the variable node decoder.

To verify our derivations, we optimized codes of rate 0.5 for the BSC, BSEC, BSQC and the soft information channel. The thresholds of these codes using the associated quantization schemes are shown in Table I and the bit error rate simulation results are shown in Figure 3 using codes of length $N = 10^4$ constructed with the PEG algorithm [13], [14].

Channel	E_b/N_0[dB]
BSC	3.62
BSEC	2.95
BSQC	2.62
Soft	2.28

TABLE I
Thresholds for Codes of Rate 0.5.

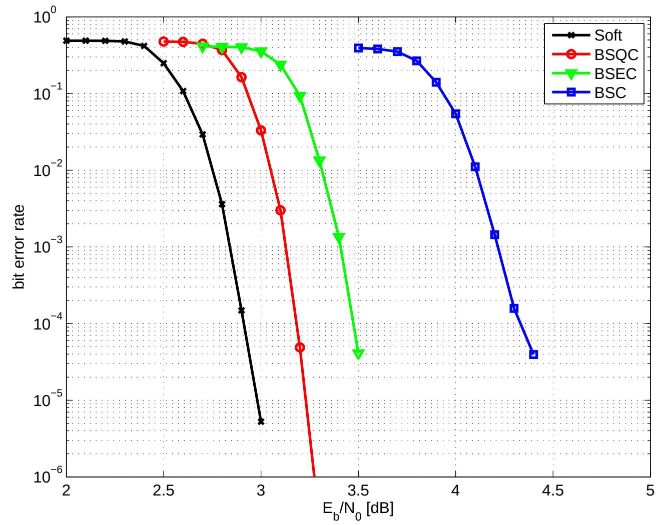

Fig. 3. Bit Error Rate Simulations for Codes of Rate 0.5.

The system with hard channel decisions (BSC) corresponds to the algorithm Gallager B. It can be seen, that by adding one more bit for the channel messages and quantize them according to a BSQC, the performance of this algorithm can be improved by more than 1.0 dB with only a small increase in complexity. A finer quantization of the channel messages will not result in a significant gain, since the gap to the unquantized system is only approximately 0.25 dB.

VI. Conclusions

We analyzed binary message-passing decoders using EXIT charts. For channels which deliver hard decisions, this analysis lead to an algorithm that is equivalent to Gallagers decoding algorithm B. The analysis of this algorithm was extended to channels with larger output alphabets including channels that deliver soft information. A small increase of the output alphabet size of the channel results in a significant gain in performance. Finally, we showed that the mixing property of EXIT functions does not apply directly to binary message-passing algorithms, and presented a modified mixing method in order to optimize codes.

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