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# Modal and Mixed Specifications: Key Decision Problems and their Complexities

Adam Antonik[1], Michael Huth[2†], Kim G. Larsen[3‡], Ulrik Nyman[3], Andrzej Wąsowski[4‡]

[1] *CNRS, Ecole Normale Supérieure de Cachan, France*

[2] *Department of Computing, Imperial College London, United Kingdom*

[3] *Department of Computer Science, Aalborg University, Denmark*

[4] *IT University of Copenhagen, Denmark*

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Modal and mixed transition systems are specification formalisms that allow mixing of over- and under-approximation. We discuss three fundamental decision problems for such specifications: whether a set of specifications has a common implementation, whether a sole specification has an implementation, and whether all implementations of one specification are implementations of another one. For each of these decision problems we investigate the worst-case computational complexity for the modal and mixed case. We show that the first decision problem is EXPTIME-complete for modal as well as for mixed specifications. We prove that the second decision problem is EXPTIME-complete for mixed specifications (while it is known to be trivial for modal ones). The third decision problem is furthermore demonstrated to be EXPTIME-complete for mixed specifications.

## 1. Introduction

Labeled transition systems are frequently used to define semantics of modeling languages, and then to reason about models in these languages. It often happens that a single transition system cannot serve multiple purposes. For example, an over-approximating transition system can be used to soundly establish safety properties, but not liveness properties. Similarly an under-approximating transition system, can be used to prove liveness, but not safety, properties. A simple remedy for that problem is to use two transition systems in a verification process that requires capturing both viewpoints: one describing an over-approximation, and one describing an under-approximation of the same behavior.

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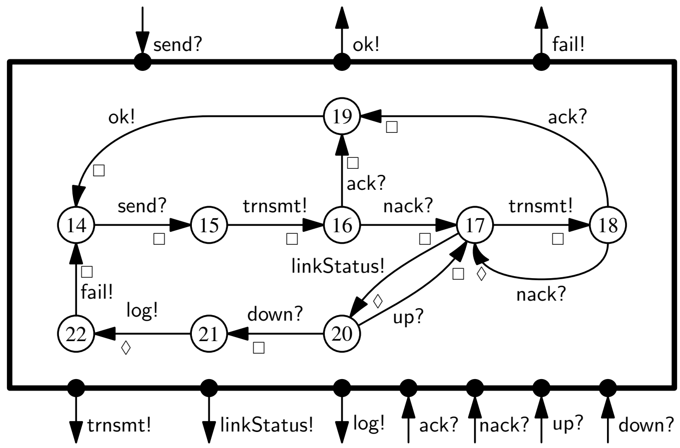



Fig. 1. An interface of a simple communication module, originally presented in (Larsen et al., 2007a)

This solution introduces a lack of precision, caused by decoupling the states of one abstraction from those of the other, which makes it impossible to verify nested properties typical for recursive logics. For example one cannot prove that a state in which a certain liveness property holds, is unreachable. To deal with the problem, model checkers such as Yasm (Gurfinkel et al., 2006) handle over- and under-approximation in one structure based on a single transition systems.

This idea can be traced back to the late eighties, when Larsen and Thomsen proposed modal transition systems (Larsen and Thomsen, 1988), also known as modal specifications (Larsen, 1989) and mixed specifications (Dams, 1996). Modal specifications combine over and under- approximation in a single transition system, using two transition relations, but a single set of states. Inconsistencies may arise in such specifications, if some behaviour is both required and disallowed. We chose to call the general, possibly inconsistent form of specifications *mixed specifications*, and the subset that syntactically enforces consistency *modal specifications* (in modal specifications the required transition relation is included in the allowed transition relation). This naming is a convention that we adopt here for clarity, remarking that this is not always followed in the existing literature.

Mixed specifications have since been applied as suitable abstractions in, among others, program analysis (Huth et al., 2001; Schmidt, 2001), model checking (Godefroid et al., 2001; Børjesson et al., 1993; Gurfinkel et al., 2006), verification (Larsen et al., 1995; Bruns, 1997), solving process algebraic equation systems (Larsen and Xinxin, 1990), compositional reasoning with interface theories (Larsen et al., 2007a), modeling of variability in software product lines (Larsen et al., 2007a; Fischbein et al., 2006) and other model management areas such as model merging (Uchitel and Chechik, 2004; Brunet et al., 2006).

As an example, let us briefly consider a model originating in interface theories, that can be used to explain the motivation of our work. Figure 1 shows an interface of a communication component. This interface models communication components that retry transmission at least once after a failure, and that optionally can check the link status upon a failure (to react appropriately). Specifically, the interface specifies five output actions (ok, fail, trnsmt, linkStatus, log) and five input actions (send, ack, nack, up, down), all enumerated on the rectangular frame. The interior of the frame contains an automaton specifying the desired and allowed behavior. Transitions labeled by $\Box$ are *required* by the interface, transitions labeled by $\Diamond$ are *allowed* by the interface. Thus assuming that the state labelled 14 is the initial state, the component must await a send request, and then it is obliged to transmit a message, awaiting an acknolwedgement. If the acknowledgement arrives (state 19), the component successfully closes the communication notifying the requester ok!, if an error message arrives (state 17) the component needs to retransmit, or alternatively it *may* check the status of the underlying link. After the second attempt to transmit (state 18) the component either disallows failure, or *may* retry again (the nack transition).

Such interface models have been described in (Larsen et al., 2007a) and in (Raclet, 2008). Here let us just mention that the underlying semantic model is that of modal specifications. Thus decision procedures for interfaces often relate to decision procedures for modal specifications. For example, if a component needs to implement several interfaces, the question arises whether the interfaces are consistent. A similar question is whether a certain interface is a proper generalization of another one, i.e. does every component implementing the former also implement the latter. In the present paper we discuss the computational complexity of these questions, formulating them for both "mixed" and "modal" specifications implicitly:

- C Is a single specification consistent, i.e. can it be implemented?
- CI Is a collection of specifications consistent, i.e. does there exist a common implementation of them?
- TR Does one specification thoroughly refine the other, i.e. is every implementation of the former an implementation of the latter?

Our results are obtained as follows. First, we argue that all three decision problems are in EXPTIME for modal as well as for mixed specifications. Then we prove three reductions from which we obtain lower bounds:

1 We show that the EXPTIME-complete problem of acceptance of an input in a linearly bounded alternating Turing machine reduces to CI for *modal* specifications. From this we learn that CI is EXPTIME-hard for modal and so for mixed specifications as well.

2 We show that CI for modal specifications reduces to C for mixed specifications, and so the EXPTIME-hardness of CI renders EXPTIME-hardness of C for mixed specifications.

3 Finally, we show that C for mixed specifications reduces to TR for mixed specifications, obtaining thus EXPTIME-hardness of TR for mixed specifications from the EXPTIME-hardness of C for mixed specifications.

This reduction chain begins with modal specifications but has to resort to mixed, non-modal specifications for C. Therefore, we are only able to infer that CI is EXPTIME-complete for modal specifications, being unable to reveal any new lower bounds on TR for modal specifications.

The paper proceeds as follows. In Section 2 we give the needed background for appreciating the technical development of the paper. In Section 3 we discuss related work. In Sections 4, 5, and 6 we state the three reductions that render EXPTIME-completeness of CI (for modal and mixed specifications), of C (for mixed specifications), and of TR (for mixed specifications). In Section 7 we put these results into context, and we summarize the paper in Section 8.

## 2. Background

Let us now formally define the basic models of interest in our study (Larsen, 1989; Dams, 1996; Clarke et al., 1994):

**Definition 1.** Let $\Sigma$ be a finite alphabet of actions.

1 A *mixed specification* $M$ is a triple $(S, R^\square, R^\diamond)$, where $S$ is a finite set of states and $R^\square, R^\diamond \subseteq S \times \Sigma \times S$ are the must- and may- transitions relations (respectively).

2 A *modal* specification is a mixed specification satisfying $R^\square \subseteq R^\diamond$; all its must-transitions are also may-transitions.

3 A *pointed* mixed specification $(M, s)$ is a mixed specification $M$ with a designated initial state $s \in S$.

4 The size $|\,M\,|$ of a mixed specification $M$ is defined as $|\,S\,| + |\,R^\square \cup R^\diamond\,|$.

**Remark 1.** Throughout this paper references to "mixed" specifications also apply to "modal" ones, as in the last two items of Definition 1 – unless stated otherwise.

Refinement (Larsen, 1989; Dams, 1996; Clarke et al., 1994), called "modal refinement" in (Larsen et al., 2007b), is a co-inductive relationship between two mixed specifications that verifies that one such specification is more abstract than the other. This generalizes the co-inductive notion of bisimulation (Park, 1981) to mixed specifications:

**Definition 2.** A pointed, mixed specification $(N, t_0) = ((S_N, R_N^\square, R_N^\diamond), t_0)$ *refines* another pointed, mixed specification $(M, s_0) = ((S_M, R_M^\square, R_M^\diamond), s_0)$ over the same alphabet $\Sigma$, written $(M, s_0) \prec (N, t_0)$, iff there is a relation $Q \subseteq S_M \times S_N$ containing $(s_0, t_0)$ such that whenever $(s, t) \in Q$ then

1 for all $(s, a, s') \in R_M^\square$ there exists some $(t, a, t') \in R_N^\square$ with $(s', t') \in Q$

2 for all $(t, a, t') \in R_N^\diamond$ there exists some $(s, a, s') \in R_M^\diamond$ with $(s', t') \in Q$

Deciding whether one finite-state, pointed, mixed specification refines another one is in PTIME, and can be implemented by a standard fixpoint algorithm as used for checking simulation or bisimilarity.

**Example 1.** Pointed, mixed specification $(M, s_0)$ and pointed, modal specification $(N, t_0)$ in Figure 2 have the same set of implementations $I(M, s_0) = I(N, t_0)$ (defined shortly) and we have $(M, s_0) \prec (N, t_0)$, given by

$$Q = \{(s_0, t_0), (s_1, t_1), (s_2, t_2), (s_3, t_2), (s_4, t_3)\}$$

But we do not have $(N, t_0) \prec (M, s_0)$. To see this, assume that there is a relation $Q$ with $(t_0, s_0) \in Q$ satisfying the properties in Definition 2. Then from $(s_0, \pi, s_2) \in R_M^\diamond$ we infer

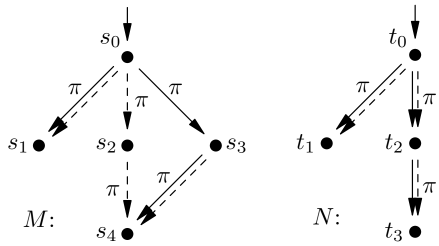



Fig. 2. Pointed, mixed ($(M, s_0)$) and pointed, modal ($(N, t_0)$) specifications over alphabet $\Sigma = \{\pi\}$ with $I(M, s_0) = I(N, t_0)$ but not $(N, t_0) \prec (M, s_0)$. Throughout figures showing specifications, solid arrows denote must-transitions, whereas dashed arrows depict may-transitions.

that there must be some $x$ with $(t_0, \pi, x) \in R^{\diamond}_N$ and $(x, s_2) \in Q$. In particular, $x$ can only be $t_1$ or $t_2$. If $x$ is $t_1$, then since $(s_2, \pi, s_4) \in R^{\diamond}_M$ and $(t_1, s_2) \in Q$ there has to be some $R^{\diamond}_N$ transition out of $t_1$, which is not the case. If $x$ is $t_2$, then $(t_2, \pi, t_3) \in R^{\Box}_N$ and $(t_2, s_2) \in Q$ imply that there is some $R^{\Box}_M$ transition out of $s_2$, which is not the case. In conclusion, there cannot be such a $Q$ and so $(N, t_0) \not\prec (M, s_0)$.

*Labeled transition systems* over an alphabet $\Sigma$ are pairs $(S, R)$ where $S$ is a non-empty set of states and $R \subseteq S \times \Sigma \times S$ is a transition relation. We identify labelled transition systems $(S, R)$ with modal specifications $(S, R, R)$. The set of implementations $I(M, s)$ of a pointed, mixed specification $(M, s)$ are all pointed labelled transition systems $(T, t)$ refining $(M, s)$. Note that $I(M, s)$ may be empty in general, but is guaranteed to be non-empty if $M$ is a modal specification.

**Definition 3.** Let $(N, t)$ and $(M, s)$ be pointed, mixed specifications. As in (Larsen et al., 2007b) we define *thorough refinement* $(M, s) \prec_{th} (N, t)$ to be the predicate $I(N, t) \subseteq I(M, s)$.

Refinement approximates this notion: $(M, s) \prec (N, t)$ implies $(M, s) \prec_{th} (N, t)$ since refinement is transitive. The converse is known to be false (Hüttel, 1988; Xinxin, 1992; Schmidt and Fecher, 2007) – contrary to what has been claimed in (Huth, 2005b); Figure 2 provides a counterexample.

We shall now formally define the decision problems informally stated above. Each decision problem has two instances – one for modal and one for mixed specifications.

- *Common implementation* (CI): given $k > 1$ specifications $(M_i, s_i)$, is the intersection $\bigcap_{i=1}^{k} I(M_i, s_i)$ non-empty?
- *Consistency* (C): Is $I(M, s)$ non-empty for a specification $(M, s)$?
- *Thorough refinement* (TR): Does a specification $(N, t)$ thoroughly refine a specification $(M, s)$, i.e., do we have $I(N, t) \subseteq I(M, s)$?

As far as these decision problems are concerned, the restriction to finite implementations, which follows from restricting our definitions to finite specifications, causes no loss of generality, as already explained in (Antonik et al., 2008b): A mixed specification $(M, s)$ is consistent in the infinite sense iff its characteristic modal $\mu$-calculus formula $\Psi_{(M,s)}$

(Huth, 2005a) is satisfiable. In general a transition system satisfying a modal $\mu$-calculus formula may be infinite. Due to the small model theorem for $\mu$-calculus (Kozen, 1988), $\Psi_{(M,s)}$ is satisfiable iff it is satisfiable over finite-state implementations. Hence reasoning about consistency does not require reasoning about infinite structures. We can reason in a similar manner about common implementation and thorough refinement, which justifies the restriction to finite-state implementations. The restriction to finite-state specifications is needed in order to do complexity analysis.

Now we establish an EXPTIME upper bound for our key decision problems for modal and mixed specifications.

**Lemma 4.** The decision problems CI, C, and TR are in EXPTIME for modal as well as for mixed specifications, in the sum of their sizes.

*Proof sketch.* Mixed and modal specifications $(M, s)$ have characteristic formulæ $\Psi_{(M,s)}$ (Huth, 2005a) in the modal $\mu$-calculus such that pointed labeled transition systems $(L, l)$ are implementations of $(M, s)$ iff $(L, l)$ satisfies $\Psi_{(M,s)}$. The common implementation and consistency problem, CI and C, reduce to satisfiability checks of $\bigwedge_i \Psi_{(M_i,s_i)}$ and $\Psi_{(M,s)}$, respectively. The thorough refinement problem of whether $(M, s)\prec_{th}(N, t)$ reduces to a validity check of $\neg\Psi_{(N,t)} \vee \Psi_{(M,s)}$.

Validity checking of such vectorized modal $\mu$-calculus formulæ is in EXPTIME. One way in which this membership in EXPTIME can be seen is by translating the problem into alternating tree automata. It is well known that a formula $\Psi_{(M,s)}$ can be efficiently translated (Wilke, 2001) into an alternating tree automata $\mathsf{A}_{(M,s)}$ (with parity acceptance condition) that accept exactly those pointed labeled transition systems that satisfy $\Psi_{(M,s)}$. Since non-emptiness, intersection, and complementation of languages is in EXPTIME for alternating tree automata, we get our EXPTIME upper bounds if these automata have size polynomial in $|\,M\,|$.

Since the size of $\Psi_{(M,s)}$ may be exponential in $|\,M\,|$ we require a direct translation from $(M, s)$ into a version of $\mathsf{A}_{(M,s)}$. The formulæ $\Psi_{(M,s)}$ can be written as a system of recursive equations (Larsen, 1989) $X_s = body_s$ for each state $s$ of $M$. We can therefore construct all $\mathsf{A}_{(M,s)}$ in a compositional manner: whenever $X_s$ refers in its $body_s$ to some $X_t$, then $\mathsf{A}_{(M,s)}$ has a transition to the initial state of $\mathsf{A}_{(M,t)}$ at that point. This $\mathsf{A}_{(M,s)}$ generates the same language as the one constructed from $\Psi_{(M,s)}$, by appeal to the existence of memoryless winning strategies in parity games (Zielonka, 1998). The system of equations is polynomial in $|\,M\,|$, and so the compositional version of $\mathsf{A}_{(M,s)}$ is polynomial in the size of that system of equations.

For full details we refer the reader to (Wilke, 2001) and (Larsen, 1989).

**Remark 2.** Throughout this paper we work with Karp reductions, i.e. many-one reductions computable by deterministic Turing machines in polynomial time. This choice is justified since we reduce problems that are EXPTIME-complete or PSPACE-hard.

## 3. Related work

In the following we briefly state research directly relevant to this paper.

The workshop paper (Antonik et al., 2008c) contains a sketch of the reduction of $\mathrm{ATM}_{\mathrm{LB}}$, i.e. the acceptance of input for a linearly bounded alternating Turing machine, to CI for modal specifications. This reduction was discovered, independently, by Antonik and Nyman in their PhD dissertation work (Antonik, 2008; Nyman, 2008). This reduction constitutes an improvement over the reduction to CI for modal specifications from the PSPACE-complete problem of Generalized Geography, that reduction appeared in (Antonik et al., 2008b) already.

The conference paper (Antonik et al., 2008b) also contains the reductions of C for mixed to CI for modal specifications, and of TR for mixed to C for mixed specifications – but the stronger reduction to alternating Turing machines makes these reductions stronger by transitivity.

The conference paper (Antonik et al., 2008b) also shows that TR for modal specifications is PSPACE-hard. This result is completely orthogonal to the techniques and results reported in this paper.

We refer the interested reader to an invited concurrency column (Antonik et al., 2008a), that provides more motivation and potential applications of the decision problems studied in this paper.

The prime numbers construction in the example of Section 4 has been originally proposed by Antonik, and published in (Antonik, 2008). Only after the fact, we have learned that the same technique has been used by Berwanger and colleagues in two other works, published around the same time (Berwanger et al., 2008; Berwanger and Doyen, 2008). In these papers, the technique of multiplication of small prime numbers is used to (i) show that imperfect information games require exponential strategies and to (ii) reduce imperfect information parity games to imperfect information safety games.

## 4. Common Implementation

We would like to develop an intuition as to why the CI problem is hard. Before attempting a formal proof of this fact we demonstrate a set of specifications, whose size is exponentially smaller than its smallest common implementation. The succinctness of specifications as a representation in itself does not prove hardness of the problem, but, we think, it makes it quite evident that the problem is hard.

**Example 2.** The construction used below originates in (Antonik, 2008). Let $I$ be a finite set of natural indices and, and for $i \in I$ let $M_i$ be modal specifications consisting of:

— states $s_i^j$, $j = 1 \dots i$, such that $(s_i^j, \pi, s_i^{j+1}) \in R^\Box, R^\Diamond$ for $1 \le j \le i-1$ and $(s_i^i, \pi, s_i^1) \in R^\Diamond$.

— an extra deadlock state $d$, such that $(s_i^j, a_l, d) \in R^\Diamond$ if $l \in I - \{i\}$, while $(s_i^i, a_i, d) \in R^\Diamond, R^\Box$.

Figure 3 shows an example of specifications $M_2$, $M_3$ and $M_5$ for $I = \{2, 3, 5\}$. Observe that each $M_i$ is a counter that counts $i-1$ transitions labeled by $\pi$, allowing the implementation to stop after $i-1$ $\pi$-steps (or any multiple of that). In any state $M_i$ is allowed to make an $a_j$ transition to a deadlocking state, but only in its topmost state (see figure) it is allowed and required to be able to make an $a_i$ transition to this state.

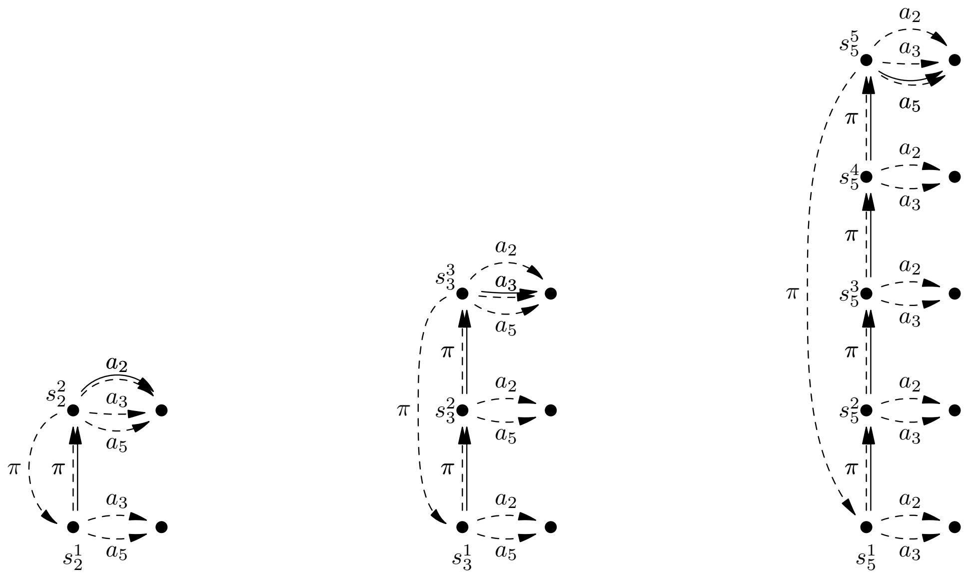



Fig. 3. Pointed specifications $(M_2, s_2^1)$, $(M_3, s_3^1)$ and $(M_5, s_5^1)$ whose common implementation has at least $2 \cdot 3 \cdot 5 = 30$ states.

It is not hard to see that if we take a collection of $M_i$ models for $i = p_1, \ldots, p_n$, ranging over the first $n$ primes, then any implementation has at least $\prod_{i=1}^{n} p_i > \prod_{i=1}^{n} 2 = 2^n$ states. Thus the size of any common implementation of the family of models for the first $n$ primes is exponential in $n$. It remains to argue that the total size of the specifications themselves remains polynomial in $n$.

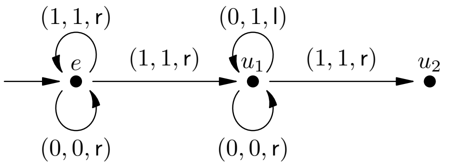
By a theorem of Chebyshev (Chebyshev, 1852), there exists a constant $\theta > 0$ such that the number of primes less than a given $k$ is at least $\theta k / \log k$. Since for sufficiently large $k$ we have that $\log k < k^{1/2}$, we have that the number of primes is greater than $\theta k^{1/2}$. In order to ensure at least $n$ primes in the range $[0, x]$ it suffices to take $x$ larger than $(\frac{n}{\theta})^2$. The total size of $M_i$ specifications corresponding to these numbers is $O(n(\frac{n}{\theta})^2) = O(n^3)$. Thus the set of specifications has size polynomial in $n$, while its common implementations are at least exponential in $n$. We remark that this construction can be easily adapted to only use a binary alphabet.

In the remaining part of this section, we present a formal reduction, demonstrating EXPTIME-hardness of CI. Let us begin with a definition of the decision problem used in the lower bound proof for common implementation.

An *Alternating Turing Machine* (Chandra et al., 1981), or an ATM, is a tuple $T = (Q, \Gamma, \delta, q_0, \mathsf{mode})$, where $Q$ is a non-empty finite set of control states, $\Gamma$ is an alphabet of tape symbols, $\mathsf{null} \notin \Gamma$ is a special symbol denoting empty cell contents,

$$\delta \colon Q \times (\Gamma \cup \{\mathsf{null}\}) \to \mathcal{P}(Q \times \Gamma \times \{\mathsf{l}, \mathsf{r}\})$$

is a transition relation, $q_0 \in Q$ is the initial control state, and $\mathsf{mode} \colon Q \to \{\mathsf{Univ}, \mathsf{Exst}\}$ is a labeling of control states as respectively universal or existential. Universal and existential




$$
\begin{aligned}
\delta(e,\ 0) &= \{(e,\ 0,\mathsf{r})\}\\
\delta(e,\ 1) &= \{(e,\ 1,\mathsf{r}),(u_1,1,\mathsf{r})\}\\
\delta(u_1,0) &= \{(u_1,1,\mathsf{l}),(u_1,0,\mathsf{r})\}\\
\delta(u_1,1) &= \{(u_2,1,\mathsf{r})\}\\
\delta(u_2,0) &= \delta(u_2,1)=\{\}
\end{aligned}
$$

Fig. 4. The transition relation of an ATM as a labelled graph and as a function.

states with no successors are called accepting and rejecting states (respectively). Each ATM $T$ has an infinite tape of cells with a leftmost cell. Each cell can store one symbol from $\Gamma$. A head points to a single cell at a time, which can then be read or written to. The head can then move to the left or right: $(q', a', \mathsf{r}) \in \delta(q, a)$, e.g., says "if the head cell (say $c$) reads $a$ at control state $q$, then a successor state can be $q'$, in which case cell $c$ now contains $a'$ and the head is moved to the cell on the right of $c$." The state of the tape is an infinite word over $\Gamma \cup \{\mathsf{null}\}$.

We provide a simple example that we use throughout this paper for sake of illustration.

**Example 3.** Figure 4 presents an example of an ATM $T$ over a binary alphabet $\Gamma = \{0, 1\}$ where arrows $q \xrightarrow{(a,a',d)} q'$ denote $(q', a', d) \in \delta(q, a)$. The initial control state $e$ is an existential one, and both $u_i$ control states are universal.

**Definition 5.**

1 Configurations of an ATM $T$ are triples $\langle q, i, \tau\rangle$ where $q \in Q$ is the current control state, the head is on the $i$th cell from the left, and $\tau \in (\Gamma \cup \mathsf{null})^\omega$ is the current tape state.

2 For input $w \in \Gamma^*$, the initial configuration is $\langle q_0, 1, w\mathsf{null}^\omega\rangle$.

3 The recursive and parallel execution of all applicable† transitions $\delta$ from initial configuration $\langle q_0, 1, w\mathsf{null}^\omega\rangle$ yields a computation tree $\mathsf{T}_{\langle T,w\rangle}$. We say that ATM $T$ accepts input $w$ iff the tree $\mathsf{T}_{\langle T,w\rangle}$ accepts, where the latter is a recursive definition:

— Subtree $\mathsf{T}_{\langle T,w\rangle}$ with root $\langle q, i, \tau\rangle$ and $\mathsf{mode}(q) = \mathsf{Exst}$ accepts iff there is a successor $\langle q', i', \tau'\rangle$ of $\langle q, i, \tau\rangle$ in $\mathsf{T}_{\langle T,w\rangle}$ such that the sub-tree with root $\langle q', i', \tau'\rangle$ accepts

— Subtree $\mathsf{T}_{\langle T,w\rangle}$ with root $\langle q, i, \tau\rangle$ and $\mathsf{mode}(q) = \mathsf{Univ}$ accepts iff for all successors $\langle q', i', \tau'\rangle$ of $\langle q, i, \tau\rangle$ in $\mathsf{T}_{\langle T,w\rangle}$ the sub-tree with root $\langle q', i', \tau'\rangle$ accepts (in particular, this is the case if there are no such successors)

**Example 4.** The ATM of Figure 4 accepts the regular language $(0 + 1)^*10^*1(0 + 1)^*$. Observe that $u_2$ is the only accepting state. Intuitively the part of $T$ rooted in $e$ accepts the prefix $(0+1)^*1$ — the semantics of existential states is locally that of states in nondeterministic Turing machines. The part of $T$ rooted in $u_1$ consumes a series of 0 symbols until 1 is reached, which leads to acceptance. The suffix of the input word after the last

† Transitions $(\_, \_, \_, \_, \mathsf{l})$ are not applicable in configurations $\langle \_, 1, \_\rangle$ as the head cannot move over the left boundary of the tape, where we use _ as a wildcard.

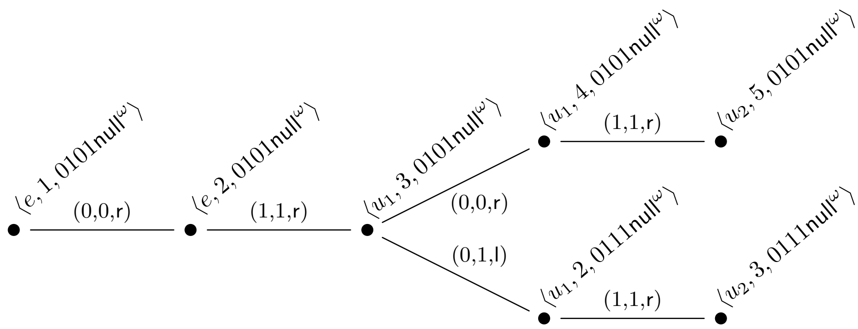



Fig. 5. An accepting computation tree $\mathsf{T}_{\langle T, 0101\mathsf{null}^\omega\rangle}$ for the ATM $T$ of Example 4.

1 is ignored. Note that the computation forks in $u_1$ whenever a 0 is seen. However, the top branch would reach the earlier 1 eventually and accept. Figure 5 shows one possible accepting tree for this ATM and the word $0101\mathsf{null}^\omega$.

An ATM $T$ is *linearly bounded* iff for all words $w \in \Gamma^*$ accepted by $T$, the accepting part of the computation tree $\mathsf{T}_{\langle T,w\rangle}$ only contains configurations $\langle q, i, v\mathsf{null}^\omega\rangle$, where the length of $v \in \Gamma^*$ is no greater than the length of $w$. That is to say, by choosing exactly one accepting successor for each existential configuration in $\mathsf{T}_{\langle T,w\rangle}$, and by removing all the remaining successors and configurations unreachable from the root, one can create a smaller tree that only contains configurations with $\langle q, i, v\mathsf{null}^\omega\rangle$ where $|v| \leq |w|$. We refer to such pruned computation trees simply as "computations".

Our notion of "linear boundedness" follows (Landweber, 1963) and (Laroussinie and Sproston, 2007) in limiting the tape size to the size of the input. This limitation does not change the hardness of the acceptance problem (see below). In addition we assume that linearly bounded ATMs have no infinite computations since any linearly bounded ATM can be transformed into another linearly bounded ATM, which accepts the same language, but also counts the number of computation steps used, rejecting any computation whose number of steps exceeds the number of possible configurations. This is possible because ASPACE = EXPTIME (Sipser, 1996, Thm. 10.18).

**Fact 1.** Consider the formal language

$$\mathrm{ATM}_{\mathrm{LB}} = \{\langle T, w\rangle \mid w \in \Gamma^* \text{ is accepted by linearly bounded ATM } T\}$$

The problem of deciding whether for an arbitrary linearly bounded ATM $T$ and an input $w$ the pair $\langle T, w\rangle$ is in $\mathrm{ATM}_{\mathrm{LB}}$ is EXPTIME-complete (Chandra et al., 1981).

We are now in a position to prove our first EXPTIME-hardness result, for the decision problem of common implementations of *modal* specifications.

**Theorem 6.** Let $\{(M_l, s_l)\}_{l\in\{1\ldots k\}}$ be a finite family of modal specifications over the same action alphabet $\Sigma$. Deciding whether there exists an implementation $(I, i)$ such that $(M_l, s_l) \prec (I, i)$ for all $l = 1 \ldots k$ is EXPTIME-hard.

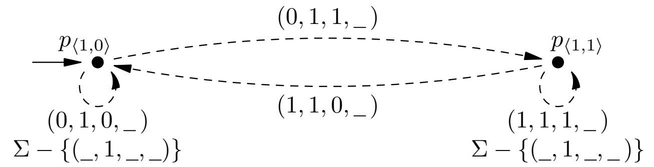



Fig. 6. Specification $M_1$ of the first tape cell in our running example, assuming $w_1 = 0$. In figures we visualize multiple transitions with the same source and target as single arrows labeled with sets of actions. Several labels placed by the same arrow denote a union of sets. Wildcards (the '_' symbol) are used to generate sets of actions that match the pattern in the usual sense.

We prove Theorem 6 by demonstrating a PTIME reduction from $\mathrm{ATM}_{\mathrm{LB}}$. Given an ATM $T$ and an input word $w$ of length $n$ we synthesize a collection of (pointed) modal specifications

$$\mathcal{M}_w^T = \{M_i \mid 1 \leq i \leq n\} \cup \{M_{\mathrm{head}}, M_{\mathrm{ctrl}}, M_{\mathrm{exist}}\} \tag{1}$$

whose sum of sizes is polynomial in $n$ and in the size of $T$, such that $T$ accepts $w$ iff there exists a (pointed) implementation $I$ refining all members of $\mathcal{M}_w^T$.

Specifications $M_i$, $M_{\mathrm{head}}$, $M_{\mathrm{ctrl}}$, and $M_{\mathrm{exist}}$ model tape cell $i$, the current head position, the finite control of $T$, and acceptance (respectively). Common implementations of these specifications model action synchronization to agree on what symbol is being read from the tape, what is the head position, what is the symbol written to the tape, in what direction the head moves, what are the transitions taken by the finite control, and whether a computation is accepting. The achieved effect is that any common implementations of these specifications correspond to an accepting computation of $T$ on input $w$. More precisely, any common implementation will correspond to different unfoldings of the structure of the finite control into a computation tree based on the content of the tape cells and the tape head position.

We now describe the specifications in $\mathcal{M}_w^T$ both formally and through our running example in Figure 4. All specifications in $\mathcal{M}_w^T$ have the same alphabet. Actions are of the form $(a_1, i, a_2, d)$ and denote that the machine's head is over the $i$th cell of the tape, which contains the $a_1$ symbol, and that it shall be moved one cell in the direction $d$ after writing $a_2$ in the current cell. In addition, two special actions, $\exists$ and $\pi$, are used to encode logical constraints like disjunction and conjunction. The alphabet for our running example is

$$\{\pi, \exists\} \cup (\{0, 1\} \times \{1..n\} \times \{0, 1\} \times \{\mathsf{l}, \mathsf{r}\})$$

We remark, that a stricter and more complex reduction to CI of modal specifications over a *binary* alphabet is possible by encoding actions in binary form.

**Encoding Tape Cells.** For each tape cell $i$, specification $M_i$ represents the possible contents of cell $i$. It has $|\Gamma|$ states $\{p_{\langle i,a\rangle}\}_{a\in\Gamma}$ and initial state $p_{\langle i,w_i\rangle}$, representing the initial contents of the $i$th cell. There are no must-transitions:

$$R^{\square} = \emptyset$$

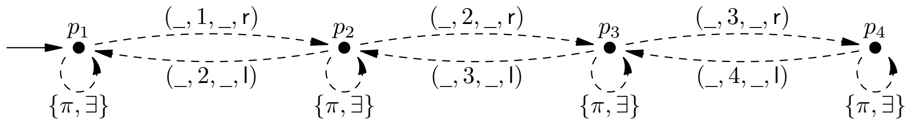



Fig. 7. Example of the head specification $M_{\text{head}}$ assuming $|w| = 4$.

The may-transition relation connects any two states:

$$\text{for all symbols } a_1,\, a_2 \text{ in } \Gamma \text{ we have } (p_{\langle i,a_1\rangle}, (a_1, i, a_2, \_), p_{\langle i,a_2\rangle}) \in R^\diamond$$

Changes in cells other than $i$ are also consistent with $M_i$:

$$\text{for all } a \in \Gamma \text{ if } i \neq j \text{ with } 1 \leq j \leq n, \text{ then } (p_{\langle i,a\rangle}, (\_, j, \_, \_), p_{\langle i,a\rangle}) \in R^\diamond$$

Finally the $\pi$ and $\exists$ actions may be used freely as they do not affect the contents of the cell:

$$(p_{\langle i,a\rangle}, \pi, p_{\langle i,a\rangle}) \in R^\diamond \text{ and } (p_{\langle i,a\rangle}, \exists, p_{\langle i,a\rangle}) \in R^\diamond \text{ for any } a \in \Gamma$$

There are no more may-transitions in $M_i$.

Figure 6 presents a specification $M_1$ for the leftmost cell of an ATM over a binary alphabet.

**Encoding The Head.** Specification $M_{\text{head}}$, which tracks the current head position, has $n$ states labeled $p_1$ to $p_n$ — one for each possible position. Initially, the head occupies the leftmost cell, so $p_1$ is the initial state of $M_{\text{head}}$. There are no must-transitions:

$$R^\square = \emptyset$$

May-transitions are consistent with any position changes based on the direction encoded in observed actions. More precisely,

$$\text{for every position } 1 \leq i < n \text{ we have } (p_i, (\_, i, \_, \mathsf{r}), p_{i+1}) \in R^\diamond$$

$$\text{for every position } 1 < i \leq n \text{ we have } (p_i, (\_, i, \_, \mathsf{l}), p_{i-1}) \in R^\diamond$$

The $\pi$ and $\exists$ transitions may again be taken freely, but in this case without moving the machine's head:

$$(p_i, \pi, p_i) \in R^\diamond \text{ and } (p_i, \exists, p_i) \in R^\diamond \text{ for each position } 1 \leq i \leq n$$

There are no more may-transitions in $M_{\text{head}}$. Note that the head of $T$ is only allowed to move between the first and $n$th cell in any computation. Figure 7 shows specification $M_{\text{head}}$ for our running example.

**Encoding The Finite Control.** Specifications $M_{\text{ctrl}}$ and $M_{\text{exist}}$ model the finite control of the ATM $T$. Specification $M_{\text{exist}}$ is independent of the ATM $T$. It is defined in Figure 8. It ensures that a $\pi$-transition is taken after every $\exists$-transition. Specification $M_{\text{ctrl}}$ mimics the finite control of $T$ almost directly. Each control state $q_s \in Q$ is identified with a state in $M_{\text{ctrl}}$ of the same name. Additional internal states of $M_{\text{ctrl}}$ encode existential

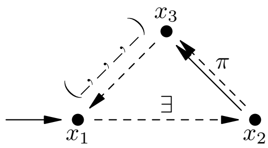



Fig. 8. Specification $M_{\mathrm{exist}}$, which enforces a $\pi$-transition after each $\exists$-transition.

and universal branching:

for each $q_s$ a state $q_{s\exists}$ with two $\exists$-transitions $(q_s, \exists, q_{s\exists}) \in R^\diamond \cap R^\Box$ is added

Dependent on $\mathsf{mode}(q_s)$, additional states and transitions are created:

— If $\mathsf{mode}(q_s) = \mathsf{Exst}$: for each $1 \leq i \leq n$, $a_{\mathrm{old}} \in \Gamma$, and for each transition $(q_t, a_{\mathrm{new}}, d) \in \delta(q_s, a_{\mathrm{old}})$ add a may $\pi$-transition from $q_{s\exists}$ to a new intermediate state uniquely named $\langle q_s a_{\mathrm{old}} i a_{\mathrm{new}} d q_t \rangle$, and add a must-transition labeled $(a_{\mathrm{old}}, i, a_{\mathrm{new}}, d)$ from that intermediate state to $q_t$. Formally:

$$(q_{s\exists}, \pi, \langle q_s a_{\mathrm{old}} i a_{\mathrm{new}} d q_t \rangle) \in R^\diamond$$
$$(\langle q_s a_{\mathrm{old}} i a_{\mathrm{new}} d q_t \rangle, (a_{\mathrm{old}}, i, a_{\mathrm{new}}, d), q_t) \in R^\diamond \cap R^\Box$$

Figure 9 shows this encoding for the state $e$ of our running example.

— If $\mathsf{mode}(q_s) = \mathsf{Univ}$: for each $1 \leq i \leq n$, $a_{\mathrm{old}} \in \Gamma$, and for each transition $(q_t, a_{\mathrm{new}}, d) \in \delta(q_s, a_{\mathrm{old}})$ add a may $\pi$-transition from $q_{s\exists}$ to an intermediate state named $\langle q_s a_{\mathrm{old}} i \rangle$, and add a must-transition labeled $(a_{\mathrm{old}}, i, a_{\mathrm{new}}, d)$ from the intermediate state $\langle q_s a_{\mathrm{old}} i \rangle$ to $q_t$. Formally:

$$(q_{s\exists}, \pi, \langle q_s a_{\mathrm{old}} i \rangle) \in R^\diamond$$
$$(\langle q_s a_{\mathrm{old}} i \rangle, (a_{\mathrm{old}}, i, a_{\mathrm{new}}, d), q_t) \in R^\diamond \cap R^\Box$$

The initial state of $M_{\mathrm{ctrl}}$ is its state named $q_0$, where $q_0$ is the initial state of $T$. Figure 10 demonstrates the encoding of the state $u_1$ of the ATM in Figure 4. The complete specification $M_{\mathrm{ctrl}}$ for our running example is shown in Figure 11.

Notice how the two specifications $M_{\mathrm{ctrl}}$ and $M_{\mathrm{exist}}$ cooperate to enforce the nature of alternation. For example, for an existential state, $M_{\mathrm{ctrl}}$ forces every implementation to have an $\exists$-transition, which may be followed by a $\pi$-transition. Simultaneously $M_{\mathrm{exist}}$ allows an $\exists$-transition but subsequently requires a $\pi$-transition. Effectively at least one of the $\pi$ branches from $M_{\mathrm{ctrl}}$ must be implemented (which is an encoding of a disjunction).

This concludes the description of all specifications from set $\mathcal{M}_w^T$ in (1). All these specifications are modal by construction. Since the sum of their sizes is bounded by a polynomial in $n$ and in the size of $T$, the remainder of the proof for Theorem 6 follows from the following lemma:

**Lemma 7.** For each linearly bounded ATM $T$ and an input $w$, $T$ accepts $w$ iff the set of modal specifications $\mathcal{M}_w^T$ has a common implementation.

The proof of this lemma can be found in Appendix A. We mention here some points of interest. From an accepting computation tree $\mathsf{T}_{\langle T,w \rangle}$ one can construct a specification

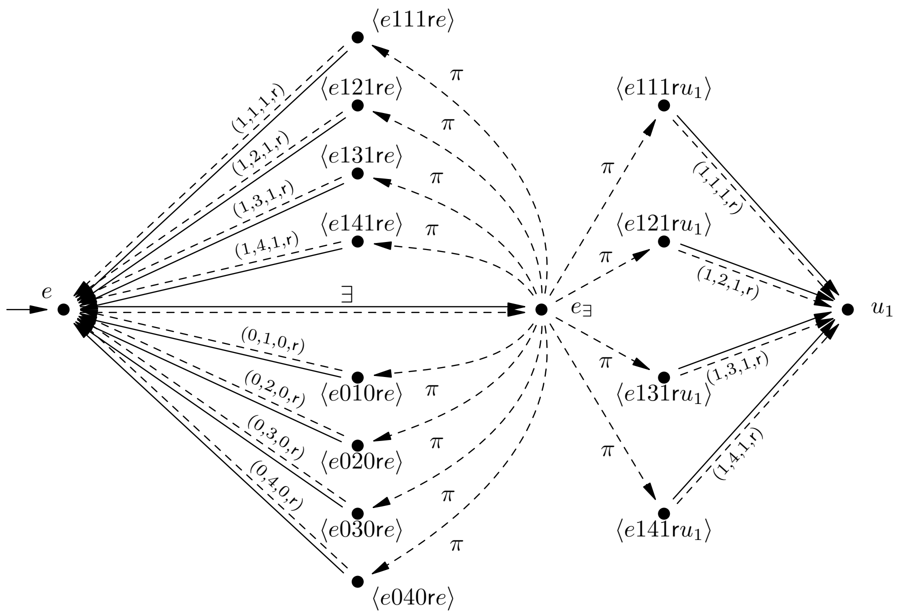



Fig. 9. Encoding for the existential state of the running example, assuming $|w| = 4$.

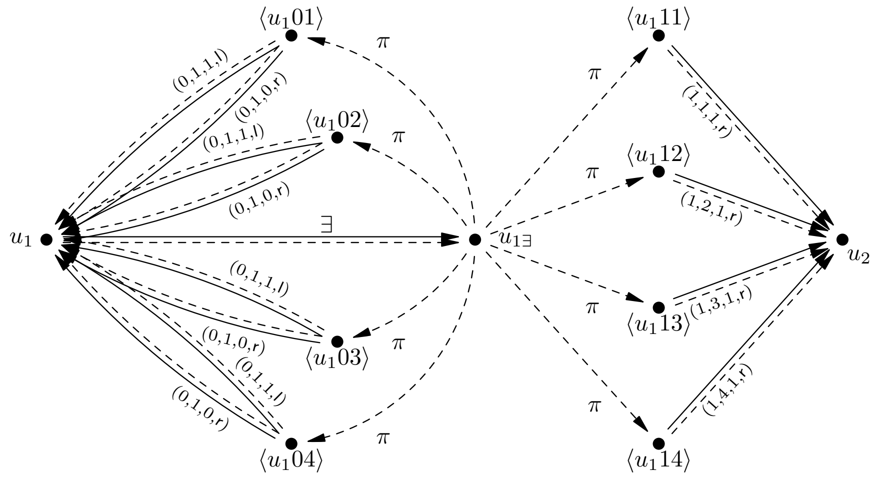



Fig. 10. Encoding for the universal state $u_1$ of the running example, assuming $|w| = 4$.

$N$ by structural induction on $\mathsf{T}_{\langle T,w\rangle}$. This $N$ effectively adds to $\mathsf{T}_{\langle T,w\rangle}$ some new states and labeled transitions so that the computation encoded in $\mathsf{T}_{\langle T,w\rangle}$ then interlocks with the action synchronization of specifications in $\mathcal{M}_w^T$. Since $N$ is of the form $(S, R, R)$ it suffices to show that $N$ is a common refinement of all members in $\mathcal{M}_w^T$. This is a lengthy but routine argument.

For the converse, a common implementation of $\mathcal{M}_w^T$ is cycle-free by our assumption that $T$ never repeats a configuration. So that pointed common implementation is a DAG

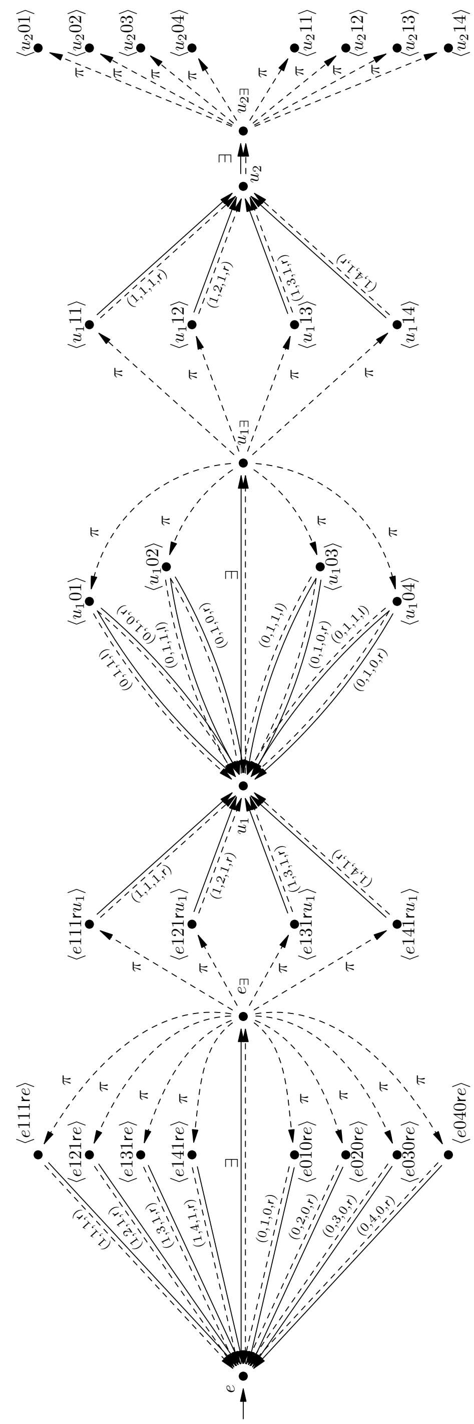


Fig. 11. The entire specification $M_{\mathrm{ctrl}}$ for the example of Figure 4 assuming $|w| = 4$.

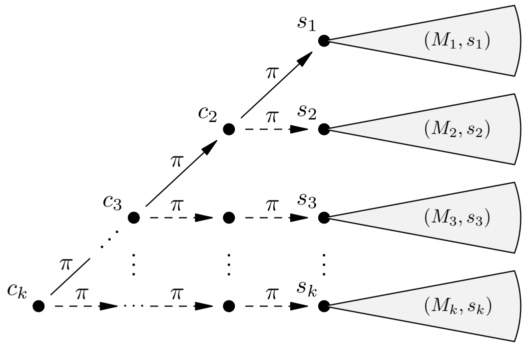



Fig. 12. Conjunction of $k$ mixed specifications into one mixed specification

and we use structural induction on that DAG to synthesize an accepting computation tree of $T$ for input $w$. This makes use of the fact that the head of $T$ never reaches a cell that was not initialized by input $w$.

We can now deduce EXPTIME-completeness for the decision problem CI for modal as well as for mixed specifications.

**Corollary 8.** The decision problem CI is EXPTIME-complete for modal as well as for mixed specifications, in the sum of their sizes.

*Proof.* Theorem 6 states EXPTIME-hardness of CI for *modal* specifications. Since modal are also mixed specifications, this renders EXPTIME-hardness of CI for mixed specifications, too. From Lemma 4 we know that both instances of CI are in EXPTIME. □

## 5. Consistency for Mixed Specifications

The decision problem C is of course trivial for modal specifications as all such specifications have implementations by construction. Given a pointed, modal specification $((S, R^\square, R^\diamond), s_0)$, one such implementation is $(S, R^\square, R^\square, s_0)$. In contrast, let us now show that consistency of a single mixed specification is EXPTIME-hard in its size. We achieve this by appealing to Theorem 6, and reducing CI for several modal specifications to the decision problem C for a single mixed specification.

**Theorem 9.** Consistency of a mixed specification is EXPTIME-hard in its size.

*Proof.* By Theorem 6, it suffices to show how $k > 1$ mixed specifications $(M_i, s_i)$ can be conjoined into one mixed specification $(M, c_k)$ with $|\,M\,|$ being polynomial in $\sum_i |\,M_i\,|$ such that $(M, c_k)$ has an implementation iff all $(M_i, s_i)$ have a common implementation.

Figure 12 illustrates the construction, which originates in (Larsen et al., 2007b), by showing a conjunction of states $s_1$, $s_2$, $s_3$ up to $s_k$. In order to conjoin two states $s_1$ and $s_2$, two new $\pi$-transitions are added from a fresh state $c_2$ to each of $s_1$ and $s_2$. One of the $\pi$-transitions is a $R^\diamond \setminus R^\square$ $\pi$-transition and the other is a $R^\square$ $\pi$-transition. Only two states can be conjoined directly in this way, but the process can be iterated as many times as

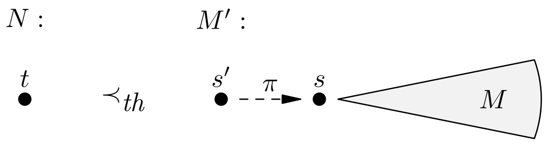



Fig. 13. Reduction of C for mixed specification $(M, s)$ to TR for mixed specifications $(N, t)$ and $(M', s')$: mixed specification $(M, s)$ is consistent iff not $(N, t) \prec_{th} (M', s')$.

needed, as seen in the figure, by adding a corresponding number of $\pi$-transitions to the newly conjoined systems. Observe that the resulting specification is properly mixed (not modal) since it contains $\pi$-transitions that are in $R^\diamond \setminus R^\Box$. Its size is linear in $\sum_i |M_i|$ and quadratic in $k$, which itself is $O(\sum_i |M_i|)$.

If the specifications that are being conjoined have a common implementation, then the new specification will also have an implementation which is the same implementation prefixed with a sequence of $k-1$ $\pi$-transitions. Conversely if the new mixed specification has an implementation, then this implementation will contain at least a sequence of $k-1$ $\pi$-transitions, followed by an implementation that must individually satisfy all the systems that have been conjoined. $\Box$

## 6. Thorough Refinement for Mixed Specifications

We show EXPTIME-hardness of the decision problem TR for mixed specifications, by appeal to Theorem 9 and by a reduction of consistency checks to thorough refinement checks.

**Theorem 10.** Thorough refinement of mixed specifications is EXPTIME-hard in the size of these specifications.

*Proof.* By Theorem 9, deciding C for a mixed specification is EXPTIME-hard. Therefore it suffices to reduce C for mixed specifications to TR for mixed specifications. Let $(M, s)$ be a pointed, mixed specification over $\Sigma$. Consider a pointed, modal specification $(N, t)$ over $\Sigma \cup \{\pi\}$ with $N = (\{t\}, \{\}, \{\})$, which only has a single state and no transitions. From $(M, s)$ construct the mixed specification $(M', s')$ over $\Sigma \cup \{\pi\}$ by prefixing $s$ with a new state $s'$ and a single transition $(s', \pi, s) \in R^\diamond_{M'} \setminus R^\Box_{M'}$. This construction is depicted in Figure 13.

We show that $(M, s)$ is consistent iff not $(N, t) \prec_{th} (M', s')$. (It is easily seen, but immaterial to this proof, that the converse $(M', s') \prec_{th} (N, t)$ always holds.)

1° If $(M, s)$ is consistent, then it has an implementation $(L, l)$, from which we get an implementation $(L', l')$ of $(M', s')$ by creating a new state $l'$ with a transition $(l', \pi, l)$. But then $(M', s')$ has an implementation that is not allowed by $(N, t)$ and so $I(M', s') \not\subseteq I(N, t)$.

2° Conversely, if $I(M', s') \not\subseteq I(N, t)$ then there exists an implementation $(L, l')$ of $(M', s')$, which is not an implementation of $(N, t)$ – and so $(L, l')$ has a transition $(l', \pi, l)$. Moreover $(L, l)$ refines $(M, s)$ since $(L, l')$ refines $(M', s')$ and $s$ is the unique successor of $s'$ in $M'$. Thus $(M, s)$ is consistent.

□

**Remark 3.** Observe that argument 1° above would also work for refinement instead of thorough refinement. However we would not be able to get the second implication for refinement in item 2° above – due to the fact that thorough refinement does not generally imply refinement.

Also note that we have just shown EXPTIME-completeness not only for deciding whether a *mixed* specification thoroughly refines another *mixed* specification, but also for deciding whether a *mixed* specification thoroughly refines a *modal* specification.

## 7. Discussion

We first summarize the complexity results obtained in this paper.

**Corollary 11.** The worst-case computational complexities shown in Table 1 are correct.

Corollary 11 leaves one complexity gap in Table 1, that for TR for *modal* specifications. We studied this fairly extensively without being able to settle the exact complexity of this decision problem. Recently we have learned that this problem has been determined to be EXPTIME-complete as well (Beneš et al., 2009). It would be of interest to see whether the proof of that result can shed any light on the complexity of the validity problem for formulae given in the vectorized form of (Larsen, 1989), since the latter is one way in which one can reexpress TR for modal and mixed specifications alike.

Interestingly, we can reduce thorough refinement to a universal version of generalized model checking (Bruns and Godefroid, 2000). In loc. cit. Bruns and Godefroid consider judgments $\mathsf{GMC}(M, s, \varphi)$ which are true iff there exists an implementation of $(M, s)$ satisfying $\varphi$. They remark that this generalizes both model checking (when $(M, s)$ is an implementation) and satisfiability checking (when $(M, s)$ is such that all labeled transition systems refine it). This existential judgment has a universal dual (see e.g. (Antonik and Huth, 2009)), $\mathsf{VAL}(M, s, \varphi)$ which is true iff all implementations of $(M, s)$ satisfy $\varphi$, thus generalizing model checking and validity checking. The former judgment is useful for finding counter-examples, the latter one for verification; e.g. both uses can be seen in the CEGAR technique for program verification of (Godefroid and Huth, 2005). Since $(M, s) \prec_{th} (N, t)$ directly reduces to $\mathsf{VAL}(N, t, \Psi_{(M,s)})$, it would be of interest to under-

Table 1. *Tabular summary of the results provided in this paper.*

| | Modal specifications | Mixed specifications |
|---|---|---|
| Common impl. | **EXPTIME-complete** | **EXPTIME-complete** |
| Consistency | **trivial** | **EXPTIME-complete** |
| Thorough ref. | **EXPTIME** | **EXPTIME-complete** |

stand the exact complexity of $\mathsf{VAL}(N, t, \varphi)$ for modal specifications $(N, t)$ when $\varphi$ ranges over characteristic formulæ $\Psi_{(M,s)}$ in vectorized form.

## 8. Conclusion

We have revisited modal and mixed specifications. Such specifications consist of state spaces with two transition relations that can serve as over-approximations and under-approximations (respectively) of transition relations in labeled transition systems. We then discussed three fundamental decision problems for modal and mixed specifications:

— common implementation: do finitely many specifications have a common implementation?

— consistency: does a specification have an implementation?

— thorough refinement: are all implementations of one specification also implementations of another?

For these three decision problems we investigated their worst-case computational complexity for both cases of modal and mixed specifications. In the case of mixed specifications, we showed that all three decision problems are EXPTIME-complete in the sizes of these systems. In the case of modal specifications, we proved the decision problem of common implementation to be also EXPTIME-complete in the size of these systems. (The decision problem of consistency for modal specifications is known to be trivial.) However, for the decision problem of modal specifications for throrough refinement we could not give any new results as our reductions for TR work only for mixed specifications.

In securing these results, we have crucially appealed to a new reduction of input acceptance for linearly bounded alternating Turing machines to the existence of a common implementation for modal specifications.

## Acknowledgments

We thank Nir Piterman for having pointed out to us that the constructions for prime numbers in (Berwanger et al., 2008; Berwanger and Doyen, 2008) are similar to the constructions for prime numbers given in Example 2. We also thank Jiří Srba and Jan Křetínský for sharing with us their recent discovery that TR for modal specifications is EXPTIME-complete.

## References

Antonik, A. (2008). *Decision problems for partial specifications: empirical and worst-case complexity.* PhD thesis, Imperial College, London.

Antonik, A. and Huth, M. (2009). On the complexity of semantic self-minimization. In *Proc. AVOCS 2007*, volume 250 of *ENTCS*, pages 3–19.

Antonik, A., Huth, M., Larsen, K. G., Nyman, U., and Wasowski, A. (2008a). 20 years of modal and mixed specifications. *Bulletin of EATCS*, 95. Available at `http://processalgebra.blogspot.com/2008/05/concurrency-column-for-beatcs-june-2008.html`.

Antonik, A., Huth, M., Larsen, K. G., Nyman, U., and Wasowski, A. (2008b). Complexity of decision problems for mixed and modal specifications. In *FoSSaCS'08*, volume 4962 of *Lecture Notes in Computer Science*. Springer.

Antonik, A., Huth, M., Larsen, K. G., Nyman, U., and Wasowski, A. (2008c). Exptime-complete decision problems for modal and mixed specifications. In *15th International Workshop on Expressiveness in Concurrency*. ENTCS.

Beneš, N., Křetínský, J., Larsen, K. G., and Srba, J. (2009). Checking thorough refinement on modal transition systems is EXPTIME-complete. In *Proceedings of the 6th International Colloquium on Theoretical Aspects of Computing*, volume 5684 of *LNCS*, pages 112–126. Springer-Verlag.

Berwanger, D., Chatterjee, K., Doyen, L., Henzinger, T. A., and Raje, S. (2008). Strategy construction for parity games with imperfect information. In *Proceedings of the 19th International Conference on Concurrency Theory (CONCUR'08), Toronto, Canada, August 2008*, volume 5201 of *LNCS*, pages 325–339. Springer.

Berwanger, D. and Doyen, L. (2008). On the power of imperfect information. In *Proceedings of the 28th Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS'08), Bangalore, India, December 2008.* Springer.

Børjesson, A., Larsen, K. G., and Skou, A. (1993). Generality in design and compositional verification using tav. In *FORTE '92 Proceedings*, pages 449–464, Amsterdam, The Netherlands, The Netherlands. North-Holland Publishing Co.

Brunet, G., Chechik, M., and Uchitel, S. (2006). Properties of behavioural model merging. In Misra, J., Nipkow, T., and Sekerinski, E., editors, *FM*, volume 4085 of *Lecture Notes in Computer Science*, pages 98–114. Springer.

Bruns, G. (1997). An industrial application of modal process logic. *Sci. Comput. Program.*, 29(1-2):3–22.

Bruns, G. and Godefroid, P. (2000). Generalized model checking: Reasoning about partial state spaces. In Palamidessi, C., editor, *CONCUR*, volume 1877 of *Lecture Notes in Computer Science*, pages 168–182. Springer.

Chandra, A. K., Kozen, D., and Stockmeyer, L. J. (1981). Alternation. *J. ACM*, 28(1):114–133.

Chebyshev, P. (1852). La totalité des nombres premiers inférieurs a une limite donnée. *Journal de Mathematiques Pures et Appliques*, (17):341–365.

Clarke, E. M., Grumberg, O., and Long, D. E. (1994). Model checking and abstraction. *ACM Trans. Program. Lang. Syst.*, 16(5):1512–1542.

Dams, D. (1996). *Abstract Interpretation and Partition Refinement for Model Checking.* PhD thesis, Eindhoven University of Technology.

Fischbein, D., Uchitel, S., and Braberman, V. (2006). A foundation for behavioural conformance in software product line architectures. In *ROSATEA '06 Proceedings*, pages 39–48, New York, NY, USA. ACM Press.

Godefroid, P. and Huth, M. (2005). Model checking vs. generalized model checking: Semantic minimizations for temporal logics. In *LICS*, pages 158–167. IEEE Computer.

Godefroid, P., Huth, M., and Jagadeesan, R. (2001). Abstraction-based model checking using modal transition systems. *Lecture Notes in Computer Science*, 2154:426+.

Gurfinkel, A., Wei, O., and Chechik, M. (2006). Yasm: A software model-checker for verification and refutation. In Ball, T. and Jones, R. B., editors, *CAV*, volume 4144 of *Lecture Notes in Computer Science*, pages 170–174. Springer.

Huth, M. (2005a). Labelled transition systems as a Stone space. *Logical Methods in Computer Science*, 1(1):1–28.

Huth, M. (2005b). Refinement is complete for implementations. *Formal Asp. Comput.*, 17(2):113–137.

Huth, M., Jagadeesan, R., and Schmidt, D. (2001). Modal transition systems: A foundation for three-valued program analysis. *Lecture Notes in Computer Science*, 2028.

Hüttel, H. (1988). Operational and denotational properties of modal process logic. Master's thesis, Computer Science Department. Aalborg University.

Kozen, D. (1988). A finite model theorem for the propositional $\mu$-calculus. *Studia Logica*, 47(3):233–241.

Landweber, P. S. (1963). Three theorems on phrase structure grammars of type 1. *Information and Control*, 6(2):131 – 136.

Laroussinie, F. and Sproston, J. (2007). State explosion in almost-sure probabilistic reachability. *Inf. Process. Lett.*, 102(6):236–241.

Larsen, K. G. (1989). Modal specifications. In Sifakis, J., editor, *Automatic Verification Methods for Finite State Systems*, volume 407 of *Lecture Notes in Computer Science*, pages 232–246. Springer.

Larsen, K. G., Nyman, U., and Wasowski, A. (2007a). Modal I/O automata for interface and product line theories. In Nicola, R. D., editor, *ESOP*, volume 4421 of *Lecture Notes in Computer Science*, pages 64–79. Springer.

Larsen, K. G., Nyman, U., and Wasowski, A. (2007b). On modal refinement and consistency. In Caires, L. and Vasconcelos, V. T., editors, *CONCUR*, volume 4703 of *Lecture Notes in Computer Science*, pages 105–119. Springer.

Larsen, K. G., Steffen, B., and Weise, C. (1995). A constraint oriented proof methodology based on modal transition systems. In *Tools and Algorithms for Construction and Analysis of Systems*, pages 17–40.

Larsen, K. G. and Thomsen, B. (1988). A modal process logic. In *Third Annual IEEE Symposium on Logic in Computer Science (LICS)*. IEEE Computer Society.

Larsen, K. G. and Xinxin, L. (1990). Equation solving using modal transition systems. In *Fifth Annual IEEE Symposium on Logics in Computer Science (LICS), 4–7 June 1990, Philadelphia, PA, USA*, pages 108–117.

Nyman, U. (2008). *Modal Transition Systems as the Basis for Interface Theories and Product Lines*. PhD thesis, Department of Computer Science, Aalborg University.

Park, D. (1981). Concurrency and automata on infinite sequences. In *Proceedings of the 5th GI-Conference on Theoretical Computer Science*, pages 167–183, London, UK. Springer-Verlag.

Raclet, J.-B. (2008). Residual for component specifications. *Electr. Notes Theor. Comput. Sci.*, 215:93–110.

Schmidt, D. (2001). From trace sets to modal-transition systems by stepwise abstract interpretation.

Schmidt, H. and Fecher, H. (2007). Comparing disjunctive modal transition systems with a one-selecting variant. *Submitted for publication.*

Sipser, M. (1996). *Introduction to the Theory of Computation*. International Thomson Publishing.

Uchitel, S. and Chechik, M. (2004). Merging partial behavioural models. In Taylor, R. N. and Dwyer, M. B., editors, *SIGSOFT FSE*, pages 43–52. ACM.

Wilke, Th. (2001). Alternating tree automata, parity games, and modal $\mu$-calculus. *Bull. Soc. Math. Belg.*, 8(2).

Xinxin, L. (1992). *Specification and Decomposition in Concurrency*. PhD thesis, Department of Mathematics and Comnputer Science, Aalborg University.

Zielonka, W. (1998). Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theor. Comput. Sci.*, 200(1-2):135–183.

## Appendix A. Proof of Lemma 7

We need to argue that if the linearly bounded ATM $T$ has an accepting computation on input $w$, then the set $\mathcal{M}_w^T$ of constructed modal specifications will have a common implementation; and, conversely, that if this set $\mathcal{M}_w^T$ of modal specifications has a common implementation, then this common implementation witnesses an accepting computation for the linearly bounded ATM $T$ on input $w$. We will prove each of these directions separately.

### A.1. *Acceptance implies existence of common implementation*

Let the ATM $T$ accept input $w$. We mean to show that $\mathcal{M}_w^T$ has a common implementation. Since we have assumed that $T$ does not repeat configurations on any computation path, we know that there exists a computation tree $\mathsf{T}_{\langle T,w\rangle}$ demonstrating that $T$ accepts $w$ in an exponentially bounded number of steps.

We shall use $\mathsf{T}_{\langle T,w\rangle}$ to construct a modal specification

$$N = (N_{\text{states}}, R_N, R_N)$$

over $\Sigma$, where $N_{\text{states}}$ is a set of states, $R_N$ is a transition relation and $\Sigma$ is the alphabet of specifications in $\mathcal{M}_w^T$. The argument that $N$ is indeed an implementation of all specifications in $\mathcal{M}_w^T$ will follow shortly after the construction.

Since $N$ has identical must- and may transition relations we shall just refer to transitions for $N$ without mentioning their type. States of $N$ are labeled by configurations of the computation tree $\mathsf{T}_{\langle T,w\rangle}$. More precisely, we distinguish three kinds of states:

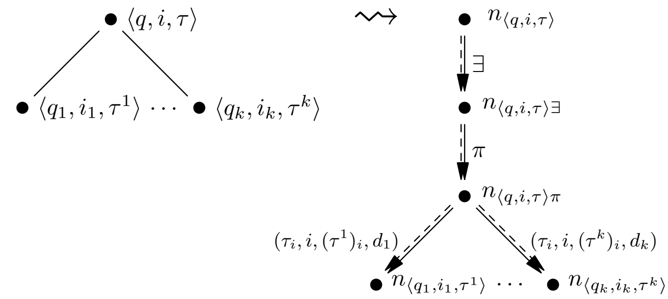
— type 1 states, indexed just with a configuraion of $\mathsf{T}_{\langle T,w\rangle}$, for example state $n_{\langle q_0,1,w\rangle}$
— type 2 states, indexed with a configuration and an extra subscript $\exists$, as in $n_{\langle q,i,\tau\rangle\exists}$
— type 3 states, indexed with a configuration and an extra subscript $\pi$, as in $n_{\langle q,i,\tau\rangle\pi}$

We construct $N$ in a recursive manner starting from the root of the accepting computation tree. We start by creating the initial state of $N$ labelled $n_{\langle q_0,1,w\rangle}$, where $\langle q_0, 1, w\rangle$ is the configuration of the root node in $\mathsf{T}_{\langle T,w\rangle}$. We shall be adding new successor states and transitions in a top-down fashion while progressing. Our recursive procedure accepts two parameters $(\langle q,i,\tau\rangle, n_{\langle q,i,\tau\rangle})$: a node from $\mathsf{T}_{\langle T,w\rangle}$ and a state from $N_{\text{states}}$. For any pair of parameters $(\langle q,i,\tau\rangle, n_{\langle q,i,\tau\rangle})$ proceed as follows:

— If $\mathsf{mode}(q) = \mathsf{Univ}$ create two new states $n_{\langle q,i,\tau\rangle\exists}$ and $n_{\langle q,i,\tau\rangle\pi}$ and a $\exists$-transition from $n_{\langle q,i,\tau\rangle}$ to $n_{\langle q,i,\tau\rangle\exists}$, and a $\pi$-transition from $n_{\langle q,i,\tau\rangle\exists}$ to $n_{\langle q,i,\tau\rangle\pi}$. Second, for each of the successors $\langle q',i',\tau'\rangle$ of $\langle q,i,\tau\rangle$ create a new state $n_{\langle q',i',\tau'\rangle}$ and a transition from $n_{\langle q,i,\tau\rangle\pi}$ to $n_{\langle q',i',\tau'\rangle}$ labelled by $(\tau_i, i, \tau'_i, d)$ where $d = \mathsf{r}$ if $i' = i+1$ and $d = \mathsf{l}$ otherwise.[‡] Then continue recursively for every successor $\langle q',i',\tau'\rangle$ of $\langle q,i,\tau\rangle$, and its corresponding state $n_{\langle q',i',\tau'\rangle}$. See also Figure 14(a).
— If $\mathsf{mode}(q) = \mathsf{Exst}$ create two new states $n_{\langle q,i,\tau\rangle\exists}$ and $n_{\langle q,i,\tau\rangle\pi}$ and an $\exists$-transition from $n_{\langle q,i,\tau\rangle}$ to $n_{\langle q,i,\tau\rangle\exists}$ and a $\pi$-transition from $n_{\langle q,i,\tau\rangle\exists}$ to $n_{\langle q,i,\tau\rangle\pi}$. Second, because

‡ We write $\tau_i$, meaning the $i$th symbol of the tape state $\tau$.

(a) Construction for a universal state $q$. For $l = 1..k$ we define $d_l = \mathsf{r}$ if $i_l = i + 1$ and $d_l = \mathsf{l}$ otherwise.




(b) Construction for an existential state $q$. Here a selected single $l \in 1..k$ is the index of the accepting successor and $d_l = \mathsf{r}$ if $i_l = i + 1$ and $d_l = \mathsf{l}$ otherwise.

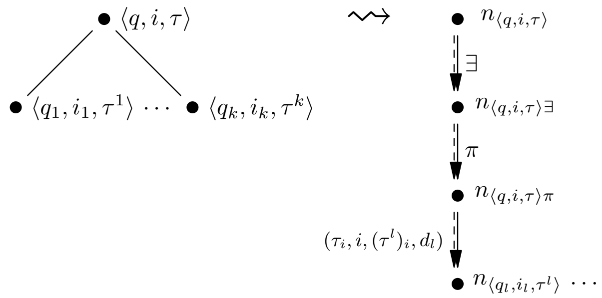



Fig. 14. Construction of a common implementation $N$ from fragments of the accepting computation tree $\mathsf{T}_{\langle T,w\rangle}$.

$\mathsf{T}_{\langle T,w\rangle}$ is accepting, we know that there exists at least one successor configuration $\langle q', i', \tau'\rangle$ for which the subtree with this configuration as root accepts. Select this one configuration and create a new state $n_{\langle q',i',\tau'\rangle}$ and a transition from $n_{\langle q,i,\tau\rangle\pi}$ to $n_{\langle q',i',\tau'\rangle}$ labelled by $(\tau_i, i, \tau'_i, d)$ where $d = \mathsf{r}$ if $i' = i + 1$ and $d = \mathsf{l}$ otherwise. Then continue recursively with $\langle q', i', \tau'\rangle$ and $n_{\langle q',i',\tau'\rangle}$. See also Figure 14(b).

Observe that the above recursive computation terminates in universal states with no successors, due to an iteration over an empty set. This is so since $\mathsf{T}_{\langle T,w\rangle}$ is an accepting computation tree and so we are guaranteed that the existential branch can always continue, and, because $T$ only allows execution of a bounded number of steps, every branch of the above recursive procedure will eventually terminate.

We shall now argue that specification $(N, n_{\langle q_0,1,w\rangle})$ refines each of the modal specifications in $\mathcal{M}^T_w$.

1 $(M_{\text{exist}}, x_1) \prec (N, n_{\langle q_0,1,w\rangle})$: Recall that the specification $M_{\text{exist}}$ has exactly three states named $x_1$, $x_2$ and $x_3$ (see Figure 8). Consider the following binary relation on states

of $M_{\text{exist}}$ and states of $N$:

$$Q_1 = \{(x_1, n_{\langle q_s,i,\tau\rangle}) \mid n_{\langle q_s,i,\tau\rangle} \in N_{\text{states}}\} \;\cup \\ \{(x_2, n_{\langle q_s,i,\tau\rangle\exists}) \mid n_{\langle q_s,i,\tau\rangle\exists} \in N_{\text{states}}\} \;\cup \\ \{(x_3, n_{\langle q_s,i,\tau\rangle\pi})) \mid n_{\langle q_s,i,\tau\rangle\pi} \in N_{\text{states}}\} \;.$$

We shall argue that $Q_1$ witnesses a refinement of $(M_{\text{exist}}, x_1)$ by $(N, n_{\langle q_0,1,w\rangle})$. First, observe that the pair of initial states $(x_1, n_{\langle q_0,1,w\rangle})$ of $M_{\text{exist}}$ and $N$ are related in $Q_1$. Second, check that $Q_1$ fulfils the conditions of Definition 2:

(1) We want to show for all pairs $(x,n) \in Q_1$ that for all states $x'$ of $M_{\text{exist}}$ if $(x,a,x') \in R^{\Box}_{M_{\text{exist}}}$ then there exists a state $n' \in N_{\text{states}}$ with $(n,a,n') \in R^{\Box}_N$ and $(x',n') \in Q_1$. A must-transition occurs in $R^{\Box}_{M_{\text{exist}}}$ only if $x = x_2$. In this case there is exactly one must $\pi$-transition going to $x_3$. We see from $Q_1$ that $x_2$ is paired only with states of form $n = n_{\langle q_s,i,\tau\rangle\exists}$. By construction of $N$, the latter state always has a must $\pi$-transition to some state $n' = n_{\langle q_s,i,\tau\rangle\pi}$ which gives us that $(x',n') \in Q_1$ by the construction of $Q_1$.

(2) We want to show for all pairs $(x,n) \in Q_1$ that for all states $n' \in N_{\text{states}}$ if $(n,a,n') \in R^{\Diamond}_N$ then there exists a state $x'$ of $M_{\text{exist}}$ such that $(x,a,x') \in R^{\Diamond}_{M_{\text{exist}}}$ with $(x',n') \in Q_1$. This argument is split into three sub-cases.

— If $n$ is of type 1, $n = n_{\langle q_s,i,\tau\rangle}$, then by $Q_1$'s construction $x = x_1$. By construction of $N$ any may-transition leaving $n$ will be labelled by $\exists$ and target a type 2 state $n' = n_{\langle q_s,i,\tau\rangle\exists}$. This can be matched by $(x_1, \exists, x_2) \in R^{\Diamond}_{M_{\text{exist}}}$ and, for $x' = x_2$ we get $(x',n') \in Q_1$ by construction of $Q_1$.

— If $n$ is of type 2, $n = n_{\langle q_s,i,\tau\rangle\exists}$, then by $Q_1$'s construction $x = x_2$. By construction of $N$ there is exactly one may $\pi$-transition leaving $n$. It targets a state $n'$ of type 3, so $n' = n_{\langle q_s,i,\tau\rangle\pi}$. This can be matched by $(x_2, \pi, x_3) \in R^{\Diamond}_{M_{\text{exist}}}$, so take $x' = x_3$ obtaining $(x',n') \in Q_1$ by construction of $Q_1$.

— If $n$ is of type 3, $n = n_{\langle q_s,i,\tau\rangle\pi}$, then by $Q_1$'s construction $x = x_3$. By construction of $N$ all possible may-transitions leaving $n$ target type 1 states of the form $n' = n_{\langle q_s,i,\tau\rangle}$. All these transitions have labels in $(\_,\_,\_,\_)$. These can all be matched by $(M_{\text{exist}}, x_3)$, as that specification contains all transitions of type $(\_,\_,\_,\_)$ going from $x_3$ to $x_1$. Since $x_1$ is paired with all states of type 1 in $Q_1$ this again gives us that $(x',n') \in Q_1$, for $x' = x_1$.

2 For each tape cell $1 \le i \le n$ show that $(M_i, p_{\langle i,w_i\rangle}) \prec (N, n_{\langle q_0,1,w\rangle})$. For any selection of $i$ above consider the following relation $Q_2^i$ over the states of $M_i$ and the states of $N$:

$$Q_2^i = \{(p_{\langle i,\tau_i\rangle}, n) \mid n = n_{\langle q_s,j,\tau\rangle} \text{ or } n = n_{\langle q_s,j,\tau\rangle\pi} \text{ or } n = n_{\langle q_s,j,\tau\rangle\exists},\ \text{for } 1 \le j \le n\}$$

First see that the initial states of the two specifications are related in $Q_2^i$. This is clearly the case since the initial state of each $M_i$ is $p_{\langle i,w_i\rangle}$, so by definition of $Q_2^i$ it is related to $n_{\langle q_0,1,w\rangle}$. It remains to be shown that given $(p,n) \in Q_2^i$ the refinement conditions are preserved.

(1) This condition is vacuously true since $M_i$'s have no must transitions.

(2) We want to show for all pairs $(p, n) \in Q_2^i$ that for all states $n' \in N_{\text{states}}$ if $(n, a, n') \in R_N^\diamond$ then there exists a state $p'$ of $M_i$ such that $(p, a, p') \in R_{M_i}^\diamond$ with $(p', n') \in Q_2^i$. With only one exception, whenever $N$ takes a may-transition $M_i$ will be able to match it. The exception is if the label contains as its old tape symbol, a symbol different from the one that $M_i$ has in its current state and where $i$ is the current position of the head in $n$, so $i = j$. Since the transitions of $N$ are created from a legal computation tree for the ATM $T$ we can conclude that $N$ will never change the content of the tape without writing to it and $N$ will thus never try to read something from a tape cell that is not in that given tape cell. It will also always update the new content of the tape cell correctly and we are thus assured that $(p', n') \in Q_2^i$.

3 Show that $(M_{\text{head}}, p_1) \prec (N, n_{\langle q_0,1,w\rangle})$: The relation $Q_3$ witnessing this refinement is defined as follows:

$$Q_3 = \{\, (p_i, n) \mid n = n_{\langle q_s,i,\tau\rangle} \text{ or } n = n_{\langle q_s,i,\tau\rangle\pi} \text{ or } n = n_{\langle q_s,i,\tau\rangle\exists} \,\} \ .$$

We first have to ensure that the initial states of the two specifications are in $Q_3$. This is the case since the initial state of $N$ has $i = 1$, which is $Q_3$-related to $p_1$, the initial state of $M_{\text{head}}$. We need to show, that for any given $(p, n) \in Q_3$ the two refinement conditions of Definition 2 are preserved.

(1) This condition is vacuously satisfied since $M_{\text{head}}$ has no must transitions.

(2) We need to show that whenever $(n, a, n') \in R_N^\diamond$ then there exists $p'$, a state of $M_{\text{head}}$, such that $(p, a, p') \in R_{M_{\text{head}}}^\diamond$ with $(p', n') \in Q_3$. We just discuss the case when $n$ is of type 3 here, so $n = n_{\langle q_s,i,\tau\rangle\pi}$. For the remaining two types the transitions leaving $n$ do not move the head and the preservation of refinement can be concluded directly.
By construction of $N$ whenever $n_{\langle q_s,i,\tau\rangle\pi}$ takes a may-transition then this transition is labeled $(\_, i, \_, d)$ targeting a type 1 state $n_{\langle q',i',\tau'\rangle}$, where $i' = i+1$ if $d = \mathsf{r}$ and $i' = i-1$ otherwise. Now by construction of $M_{\text{head}}$ the state $p_i$ can match such a transition moving to $p_{i'}$ accordingly. The only case where $M_{\text{head}}$ would not be able to match is when $N$ would try to move the head off either end of the tape, but this will never happen since $N$ is constructed from a legal accepting computation tree. Thus we conclude that the refinement condition is preserved.

4 In order to show that $(M_{\text{ctrl}}, q_0) \prec (N, n_{\langle q_0,1,w\rangle})$ consider the following binary relation $Q_4$ on states of $M_{\text{ctrl}}$ and $N$:

$$\begin{aligned} Q_4 = \{(q_s, n) \mid n = n_{\langle q_s,i,\tau\rangle}\} \cup &\\ \{(q_{s\exists}, n) \mid n = n_{\langle q_s,i,\tau\rangle\exists}\} \cup &\\ \{(\langle q_s\tau_i i\rangle, n_{\langle q_s,i,\tau\rangle\pi}) \mid \mathsf{mode}(q_s) = \mathsf{Univ}\} \ \cup &\\ \{(\langle q_s\tau_i i a_2 d q_t\rangle, n_{\langle q_s,i,\tau\rangle\pi}) \mid \mathsf{mode}(q_s) = \mathsf{Exst} \text{ and } &\\ (n_{\langle q_s,i,\tau\rangle\pi}, (\tau_i, i, a_2, d), n_{\langle q_t,i',\tau'\rangle}) \in R_N^\diamond\} \ . & \end{aligned}$$

First, observe that the initial states of the two specifications are in $Q_4$, as $q_0$ is the initial state of $M_{\mathrm{ctrl}}$ and $n_{\langle q_0,1,w\rangle}$ is the initial state of $N$ (see the first summand in the definition of $Q_4$). Second, we need to show that, given a pair $(q, n) \in Q_4$, the two refinement conditions of Definition 2 are preserved.

(1) We need to show that whenever $(q, a, q') \in R^{\Box}_{M_{\mathrm{ctrl}}}$ then there exists a state $n' \in N_{\mathrm{states}}$ such that $(n, a, n') \in R^{\Box}_N$ with $(q', n') \in Q_4$. The argument is split in four cases.

— If $q = q_s$ for some $q_s \in Q$ (a state of the ATM $T$) then there is exactly one must $\exists$-transition leaving it, which targets $q_{s\exists}$. This transition can be matched by an $\exists$-transition leaving $n_{\langle q_s,i,\tau\rangle}$ and targeting $n_{\langle q_s,i,\tau\rangle\exists}$. These new target states remain in relation $Q_4$, as per the above definition.

— If $q = q_{s\exists}$ for some $q_s \in Q$ (a state of the ATM $T$) then the condition is satisfied vacuously. There is simply no must-transition leaving $q$.

— If $q$ has the form $\langle q_s\tau_i i\rangle$, where $q_s$ is a universal state of the ATM $T$, then $n$ has the form $n_{\langle q_s,i,\tau\rangle\pi}$. But since $n_{\langle q_s,i,\tau\rangle\pi}$ was constructed by our recursive procedure from a universal configuration of an accepting computation tree we know that, for all must-transitions leaving $\langle q_s\tau_i i\rangle$ to some state $q_t$, there will be a matching must-transition in $N$ leaving $n_{\langle q_s,i,\tau\rangle\pi}$ and targeting $n_{\langle q_t,i',\tau'\rangle}$, which is in relation with $q_t$ as per the first summand in the definition of $Q_4$.

— If $q$ has the form $\langle q_s\tau_i i a_2 d q_t\rangle$, where $q_s$ is an existential state of the ATM $T$, then $n$ has the form $n_{\langle q_s,i,\tau\rangle\pi}$. The state $\langle q_s\tau_i i a_2 d q_t\rangle$ has exactly one must-transition labeled $(\tau_i, i, a_2, d)$ and targeting state $q_t$. Since $q_s$ is an existential state, we know that $n_{\langle q_s,i,\tau\rangle\pi}$ was constructed from an existential configuration and consequently there is a single must-transition leaving it. This transition is labeled $(\tau_i, i, a_2, d)$ as per construction of the $Q_4$ relation (see the last summand). Finally this transition targets $n' = n_{\langle q_t,i',\tau'\rangle}$. And thus we again have that $(q', n') \in Q_4$.

(2) We want to show that if $(n, a, n') \in R^{\Diamond}_N$ then there exists a state $q'$ of $M_{\mathrm{ctrl}}$ such that $(q, a, q') \in R^{\Diamond}_{M_{\mathrm{ctrl}}}$ with $(q', n') \in Q_4$. We split the argument into three cases based on the type of state $n$.

— If $n$ is of type 1, so $n = n_{\langle q_s,i,\tau\rangle}$ then by construction of $N$ there is a may $\exists$-transition leaving $n$ targeting $n_{\langle q_s,i,\tau\rangle\exists}$. This is followed by $(q_s, \exists, q_{s\exists}) \in R^{\Diamond}_{M_{\mathrm{ctrl}}}$ and again gives us that $(q', n') \in Q_4$.

— If $n$ is of type 2, $n = n_{\langle q_s,i,\tau\rangle\exists}$ then by the construction of $Q_4$ (see the second summand) $q$ is of the form $q_{s\exists}$. By the construction procedure of $N$ there is a single may $\pi$-transition leaving $n_{\langle q_s,i,\tau\rangle\exists}$ and targeting $n' = n_{\langle q_s,i,\tau\rangle\pi}$.

– If $\mathsf{mode}(q_s) = \mathsf{Univ}$, then there is exactly one transition $(q_{s\exists}, \pi, \langle q_s\tau_i i\rangle) \in R^{\Diamond}_{M_{\mathrm{ctrl}}}$; its target state is related to $n_{\langle q_s,i,\tau\rangle\pi}$ in $Q_4$.

– If $\mathsf{mode}(q_s) = \mathsf{Exst}$ then there can be many may $\pi$-transitions leaving $q_{s\exists}$.

We will choose which one to match with, based on the label of the single transition leaving $n_{\langle q_s,i,\tau\rangle\pi}$. We are, so to speak, looking one step ahead. Since $n_{\langle q_s,i,\tau\rangle\pi}$ says that the head is in position $i$ over a tape containing $\tau$ we choose to match our transition with the transition of $M_{\text{ctrl}}$ targeting the state whose name matches the prefix "$\langle q_s\tau_i i$". Such a state always exists by construction of $M_{\text{ctrl}}$ and it is exactly the state which is related to $n_{\langle q_s,i,\tau\rangle\pi}$ in $Q_4$ (see the last summand).

— For $n$ of type 3, so $n = n_{\langle q_s,i,\tau\rangle\pi}$, we split the argument into two cases based on the mode of $q_s$ in the ATM $T$.

  - First, if $\mathsf{mode}(q_s) = \mathsf{Univ}$ then there are possibly several may-transitions leaving $n_{\langle q_s,i,\tau\rangle\pi}$. Since $N$ has been created from a legal computation tree, we know that any may transition leaving $n_{\langle q_s,i,\tau\rangle\pi}$ and targeting $n' = n_{\langle q_t,i',\tau'\rangle}$ follows the transition relation $\delta$ of $T$. Moreover, by construction of $M_{\text{ctrl}}$, its state $\langle q_s\tau_i i\rangle$ will consequently be able to match this transition arriving in the state $q_t$ related to $n'$ in $Q_4$.
  - Second, if $\mathsf{mode}(q_s) = \mathsf{Exst}$ then there is exactly one may transition leaving $n_{\langle q_s,i,\tau\rangle\pi}$ and exactly one may-transition leaving $\langle q_s\tau_i i a_2 d q_t\rangle$. These transitions have the same label and have respective target states $n_{\langle q_t,i',\tau'\rangle}$ and $q_t$, which are related in $Q_4$.

This concludes the argument that each specification in $\mathcal{M}_w^T$ is refined by $N$.

A.2. *Existence of common implementation implies acceptance*

Let $\mathcal{M}_w^T$ have a common implementation. We need to show that the ATM $T$ accepts input $w$. Given a modal specification

$$U_{\text{new}} = (U_{\text{states}}, R_U, R_U)$$

that is a common implementation of $\mathcal{M}_w^T$ we will construct a computation tree $\mathsf{T}_{\langle M,w\rangle}$ demonstrating that $T$ accepts $w$.

Since $U_{\text{new}}$ is a common implementation of $\mathcal{M}_w^T$ we have $3+n$ refinement relations:

$$Q_{\text{ctrl}},\ Q_{\text{head}},\ Q_{\text{exist}},\ Q_1,\ \ldots,\ Q_n$$

— each demonstrating for one of the corresponding specifications $S \in \mathcal{M}_w^T$ that $S \prec U_{\text{new}}$.

The construction of $\mathsf{T}_{\langle M,w\rangle}$ is inductive. Along with the construction, we argue that the nodes of the tree preserve the following property (**IH**):

(1) For every configuration $\langle q,i,\tau\rangle$ of $\mathsf{T}_{\langle M,w\rangle}$ there exists a state $u_x \in U_{\text{states}}$ such that (IH1) $(u_x,x_1) \in Q_{\text{exist}}$ and (IH2) $(u_x,q) \in Q_{\text{ctrl}}$ and (IH3) $(u_x,p_i) \in Q_{\text{head}}$, and finally (IH4) $(u_x,p_{\langle k,\tau_k\rangle}) \in Q_k$ for each $k = 1..n$. (We follow the conventions of Section 4 here. So $q$ is a name of $T$'s state that also uniquely identifies a state of $M_{\text{ctrl}}$. Specifically we mean that $q$ represents a label without any special suffixes. Label $p_i$ is referring to a particular state of $M_{\text{head}}$, the one representing position $i$. Similarly $p_{\langle k,\tau_k\rangle}$ denotes the state of $M_k$ that represents that the $k$th symbol of $\tau$ is stored in the $k$th cell of the tape.)

(2) Moreover, (IH5) if a configuration $\langle q', i', \tau' \rangle$ is a successor of $\langle q, i, \tau \rangle$ in $\mathsf{T}_{\langle M,w \rangle}$ then it also is a successor of $\langle q, i, \tau \rangle$ in the ATM $T$, and conversely (IH6) the tree $\mathsf{T}_{\langle M,w \rangle}$ has all the successors of $\langle q, i, \tau \rangle$ that $T$ has for universal states and at least one of them for all existential states.

Only after discussing the construction of $\mathsf{T}_{\langle M,w \rangle}$, and after arguing that the above inductive property holds for it, we shall address the problem of whether $\mathsf{T}_{\langle M,w \rangle}$ actually is an accepting computation tree of $T$, witnessing acceptance of $w$.

**Root (base case):** The root of $\mathsf{T}_{\langle M,w \rangle}$ is selected to be the configuration $\langle q_0, 1, w \rangle$, where $q_0$ is the initial control state of $T$. We need to show that $\langle q_0, 1, w \rangle$ exhibits property IH.

Observe that $U_{\text{new}}$ has a distinct initial state $u_0$. Take $u_x$ to be this $u_0$. Since $M_{\text{ctrl}} \prec U_{\text{new}}$ there is a pair $(u_0, q_0) \in Q_{\text{ctrl}}$, fulfilling condition IH1. Since $M_{\text{head}} \prec U_{\text{new}}$ and $p_1$ is the initial state of $M_{\text{head}}$, we know that $(u_0, p_1) \in Q_{\text{head}}$, fulfilling condition IH3. Since $w$ is the initial content of the tape, and consequently $p_{\langle k, w_k \rangle}$ is an initial state of $M_k$, the refinement $M_k \prec U_{\text{new}}$ gives us that $(u_0, p_{\langle k, w_k \rangle}) \in Q_k$, so IH4 holds for $\langle q_0, 1, w \rangle$. Since $M_{\text{exist}} \prec_{U_{\text{new}}}$ we get that $(u_0, x_1) \in Q_{\text{exist}}$ finishing the base case (IH2).

We shall argue that IH5 and IH6 hold for the root node, when we discuss adding successors below.

**Non-root nodes (inductive step):** Given a configuration $\langle q, i, \tau \rangle$ for which properties IH1-IH4 hold, we will now construct the next level of $\mathsf{T}_{\langle M,w \rangle}$ in such a way that IH5-IH6 hold for $\langle q, i, \tau \rangle$ and IH1-IH4 hold for all its successors.

Before we split into two cases based on modes of states, we shall describe the part of the proof which these two have in common. The induction hypothesis allows us to assume existence of a specific state $u_x$ of $U_{\text{states}}$ and the respective refinement relations. Since state $u_x$ is related to a state without a $\pi$ or $\exists$ subscript in $M_{\text{ctrl}}$, that $u_x$ must implement an $\exists$ transition to a new state, let us call this state $u_{x\exists}$. Because $(u_x, x_1) \in Q_{\text{exist}}$ we know that $(u_{x\exists}, x_2) \in Q_{\text{exist}}$ and thus $u_{x\exists}$ must implement a $\pi$ transition to a new state, say $u_{x\pi}$. Since all $\pi$ and $\exists$ transitions in $M_{\text{head}}$ and $M_1$ up to $M_n$ are loops we know that $u_{x\pi}$ is related to the same states as $u_x$ in these specifications.

For the remainder of the proof, we do a case analysis on the mode of $q$:

— If $\mathsf{mode}(q) = \mathsf{Exst}$ then we know that $(M_{\text{ctrl}}, q)$ has to implement an $\exists$-transition followed by at least one $\pi$-transition reaching a state of the form $\langle q\tau_i i a' d q' \rangle$. Also because $u_{x\exists}$ is related to $q_\exists$, it must be possible to choose $u_{x\pi}$ above such that $(u_{x\pi}, \langle q\tau_i i a' d q' \rangle) \in Q_{\text{ctrl}}$, but then we know that $u_{x\pi}$ can take a transition labeled $(\tau_i, i, a', d)$ to some state $u'_x$ related to $q'$ in $Q_{\text{ctrl}}$.
So if we extend $\mathsf{T}_{\langle M,w \rangle}$ at $\langle q, i, \tau \rangle$ with a new child $\langle q', i', \tau[\tau_i \mapsto a'] \rangle$ then the new execution step will follow the semantics of the ATM $T$ satisfying conditions IH1–IH6 — provided that $i' = i + 1$ if $d = \mathsf{r}$, and $i' = i - 1$ otherwise.
The argument that IH5-IH6 hold is direct — we have added a successor as required out of all those available in the semantics of $T$. The arguments that IH1-IH4 hold are more involved, but standard — for each of them a unique successor in $M_{\text{exist}}$, $M_{\text{ctrl}}$,

$M_{\text{head}}$ and $M_k$'s can be pointed out by following the transition labeled $(\tau_i, i, a', d)$, and then shown to witness fullfilment of the condition for $u'_x$ by inductive hypothesis (from refinement of $u_x$).

— If $\mathsf{mode}(q) = \mathsf{Univ}$ then, since $(U_{\text{new}}, u_x)$ is a refinement of $(M_{\text{ctrl}}, q)$ and $(M_{\text{exist}}, x_1)$, we get that it is possible to choose $u_{x\pi}$ above so that it refines a state of $M_{\text{ctrl}}$ which has a label of the form $(M_{\text{ctrl}}, \langle q\tau_i i\rangle)$. The refinement relation with $M_{\text{head}}$ and $M_i$ ensures that this state is the only successor of $q$ in $M_{\text{ctrl}}$ that can be implemented, implying that $u_{x\pi}$ must implement all the transitions corresponding to the transition relation $\delta$ of $T$. Thus we can extend $\mathsf{T}_{\langle M,w\rangle}$ with new children $\langle q', i', \tau'[\tau_i \mapsto a']\rangle$ for all $(q', i', \tau')$ such that $(M_{\text{ctrl}}, q')$ can be reached from $(M_{\text{ctrl}}, \langle q\tau_i i\rangle)$ in one step with a transition labeled $(\tau', i, a', d)$. Also $i' = i + 1$ if $d = \mathsf{r}$, and $i' = i - 1$ otherwise. Again, it is not hard to see that all newly added successors maintain the inductive hypothesis.

For all of these target states we now have to prove that the induction hypothesis holds. As all of the target states are reached by a transition in $M_{\text{ctrl}}$ we know that there exists a state $u_y \in U_{\text{states}}$ such that $(u_y, q') \in Q_{\text{ctrl}}$. Because of the label on the transition we also know that $(u_y, p_l) \in Q_{\text{head}}$ for $l = i+1$ if $d = \mathsf{r}$, and $l = i-1$ if $d = \mathsf{l}$. This is also ensured to be done in such a way that the tape cell specifications $M_1$ to $M_n$ again match the content of the tape. We also know, because of all the transitions of type $(\_,\_,\_,\_)$ going from $x_3$ to $x_1$ in $M_{\text{exist}}$, that $(u_y, x_1) \in Q_{\text{exist}}$. This finishes the proof of the inductive step.

In this way we can recursively construct a pruned computation tree $\mathsf{T}_{\langle M,w\rangle}$. The constructed tree is finite, because we have argued that it follows the semantics of the ATM $T$, and $T$ repeats no configuration along a single computation path. Moreover $\mathsf{T}_{\langle M,w\rangle}$ is accepting as it never is stuck in a rejecting (existential) state.