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Jensen, Kristoffer

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Aesthetics and quality of numbers using the primety measure

Kristoffer Jensen

AD:MT,
Aalborg University Esbjerg,
Niels Bohrsvej 8,
6700 Esbjerg, Denmark
E-mail: krist@create.aau.dk

Abstract: The Farey sequences can be used to create the Euler totient function, by identifying the fractions for number n that did not occur in all Farey sequences up to $n - 1$. This function creates, when divided by $n - 1$, what is here called the primety measure, which is a measure of how close to being a prime number n is. The *quality* of a number sequence can be determined using the primety, which is further generalised to real numbers through the use of real numbered Farey sequences. Ambiguous and interesting results are obtained in this process; for instance it seems that all integers approach full primety when investigated in the real domain. The corresponding numerical sequences are furthermore shown to have interesting artistic properties. In particular the fractal behaviour is interesting. To further investigate fractal ambiguity, common fractals are extended with random elements, which render softer or more varied shapes.

Keywords: Farey sequences; totient function; primety; self-similarity; fractals.

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Biographical notes: Kristoffer Jensen obtained his Masters in Computer Science in 1988 at the Technical University of Lund, Sweden, and his DEA in Signal Processing in 1989 at the ENSEEIHT, Toulouse, France. His PhD was delivered and defended in 1999 at the Department of Computer Science, University of Copenhagen, Denmark, treating signal processing applied to music with a physical and perceptual point-of-view. This mainly involved classification, and modelling of musical sounds. He has been involved in synthesisers for children, state of the art next generation effect processors and signal processing in music informatics. His current research topic is signal processing with musical applications, and related fields, including perception, psychoacoustics, physical models and expression of music. He has chaired more than eight major conferences, been the editor of many books and conference proceedings and published more than 200 papers, and he currently holds a position at the Institute of Architecture, Design and Media Technology, Aalborg University Esbjerg as an Associate Professor.

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1 Introduction

Mathematics is an art form for many. This is true for the shapes that are created through mathematics, such as the fractal shapes, but also for the theoretic development of mathematics.

The quest for (large) prime numbers is one of the important problems in mathematics, with repercussions into fundamental mathematics as well as contemporary society, with the use of prime numbers in coding and security systems. With the unpredictability of prime numbers, only brute force methods are guaranteed to render a result as to whether a number is prime or not. Many mathematicians will find the mathematics involved in prime numbers beautiful, or even artistic in its own right. Byers (2007) states: “Mathematics, as I have been describing it, is an art form. The words ambiguity and metaphor are much more acceptable in the arts than they are in the sciences. But ambiguity and metaphor are the mechanisms through which that ultimate ambiguity, the one that divides the objective from the subjective, the natural world from the mind, is bridged”.

In addition to art in mathematical problem solving, the numbers can represent art in itself. Certainly, the golden ratio, and the accompanying *Fibonacci* numbers are found in many art works (Posamentier and Lehmann, 2007). Another mathematic area of interest with artistic outcomes is fractals. In the musical domain, Larry Austin composed several music pieces based on natural fractal shapes (Dodge and Jerse, 1997), including *Maroon Bells* from 1976, based in part on a mountain range, and *Canadian Coastlines* from 1981, based on actual coastlines.

This work proposes to investigate number sequences, random fractals and the Farey sequence, in particular in the case of real numbers, and show some interesting and aesthetic outcomes of these areas. The paper starts with an overview of the aesthetics of mathematics in Section 2 and proposes random element extension to classical fractals, then the primity measure calculated from the Euler totient function through the Farey sequence is presented in Section 3, alongside the extension to real numbers, and the fractal behaviour of the primity measure and the Farey sequence is presented in Section 4 with examples. Finally a conclusion is given.

2 Mathematics aesthetics

Mathematics is closely related to aesthetics today by the underlying mathematics in many computer programmes used by artists. In addition to this, mathematicians also thought their field of particular importance. For instance Gauss is reported to have said [Burton, (2006), p.548], “Mathematics is the Queen of the sciences, and the theory of numbers is the Queen of mathematics”. Some of the aspects of mathematics that relates to art are the vocabulary of the field, the fractals, and some (if not all) of the particular problems occurring in mathematics.

2.1 Mathematic vocabulary

Mathematics has a vocabulary that most often tries to associate the mathematic term to the naming of that term (Schwartzman, 1994). For instance, *prime*, can be related to the common understanding; *of first class*, as the prime numbers are believed to be belonging to the first most basic category of numbers. Other number theoretic terms related to the prime numbers include *pseudoprime* (non-prime number that divides $(2^n - 2)$). See also *Mersenne prime* (prime of the form $2^p - 1$, where p is a prime), *semiprime* (numbers having two divisors in addition to one and the number itself), *coprime* (two numbers having no common divisor), *lucky* (numbers obtained by an elimination method related to prime numbers), *primorial* (product of prime p and all smaller primes), *star prime* (prime number where the number of digits is also a prime), *superprime* (a prime that can be represented as the sum of two superpowers, n^n), *totient* (the number of smaller relative primes), *twin primes* (primes that are neighbours, with only an even number in between), *square free* (numbers that have no squares in its divisors), etc. In this naming of mathematic properties, not only understanding is sought for, also it seems the names also aim for better remembering of the properties, if not more. It is difficult sometimes to relate the name to the mathematical property. What is *evil* about a number that has an even number of bits in its binary form? What is *odious* about a number that has an odd number of bits in its binary form? (<http://oeis.org/A000069>).

2.2 Fractal examples

Fractals are objects that are self-similar (i.e., details looks like the whole). Fractals are divided into exact or statistic self-similar objects, full or partial fractals, natural and mathematic fractals, etc. A common natural fractal is the coastline that was used as an example by the ‘father’ of fractal geometry (Mandelbrot, 1967). Common examples of mathematic fractals are the Cantor set, the Koch curve, and the Mandelbrot set. Fractals have two possible definitions (Addison, 1997), both proposed by Mandelbrot, the first saying that the Hausdorff dimension strictly exceeds the topological dimensions, and the second saying that the fractal is a shape made of parts similar to the whole in some way. Often, fractals are created using some kind of feedback mechanism, in which the input of the next step is a function including the output of the current step. The iterative step is in the case of the Koch curve,

$$z = [az, (1-a)z + a] \quad (1)$$

where $z = 1$ is the starting condition, and a is a complex variable, typically, $a = \frac{1}{2} + i\sqrt{3}/6$. In the case of the Mandelbrot set,

$$z = z^2 + c \quad (2)$$

where $z = 0$ is the starting condition, and c is a complex variable that ensures $|z| < 2$. In order to increase the scope of these functions, it is here proposed to add a random element to the iterations, so that,

$$z = (1 + r\rho)f(z) \quad (3)$$

where r is the random weight, ρ is a unit random variable, and $f(z)$ corresponds to the iterative function (1) or (2). The original two-dimensional Koch curve is shown in Figure 1, while the Koch curve with random element ($r = .2$) according to (3) is shown in Figure 2. It is clear that the random element removes some of the order of the Koch curve, while at the same time giving it more complexity.

The original Mandelbrot set is shown in Figure 3 and the Mandelbrot set with a random element (with weight $r = 0.2$) added to the iterations is shown in Figure 4. Again, the random element changes the image, but this time, more by adding random dots to the shape. In this, the self-similar behaviour is lost, in the sense that the shape is not found in the detail.

Figure 1 The two-dimensional Koch curve

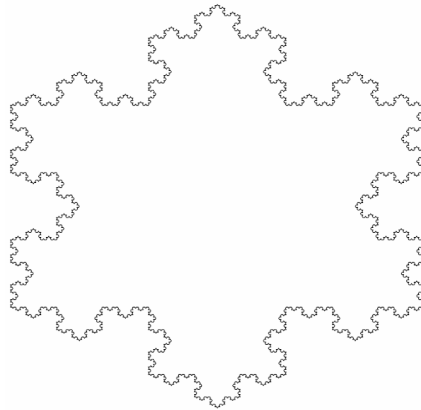


Figure 2 The Koch curve with random element added to the terative step

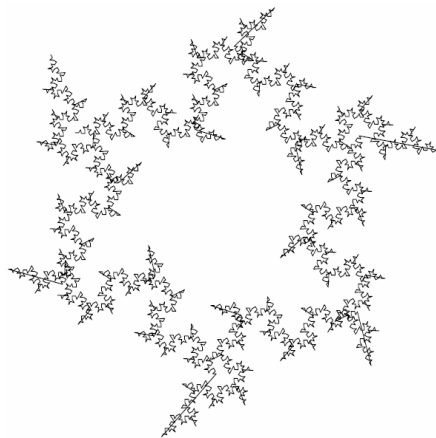
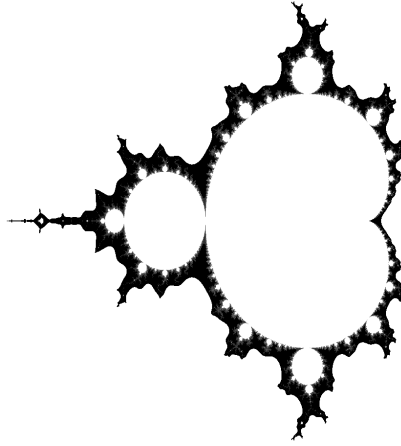
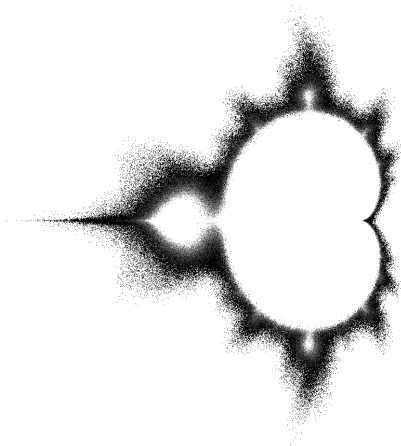


Figure 3 The original Mandelbrot set**Figure 4** The Mandelbrot set with a random element in the terative loop

Both for the Koch and the Mandelbrot set, the random element is a way of extending the possibilities of the fractals, adding eventually (for large r) an ambiguity to what shape and whether the shape has any order in it. Of course, the proposed method of adding randomness to the iterative process is one of many possible methods, each of which renders the randomness differently.

2.3 Primes and real numbers

A prime numbers is a number that has no divisors other than itself (and 1). The prime numbers are so named either because they are the *prime*, i.e., first class numbers, or because they cannot be arranged in a rectangle with both sides having length greater than one. Thus, all numbers are found in the first row (Schwartzman, 1994).

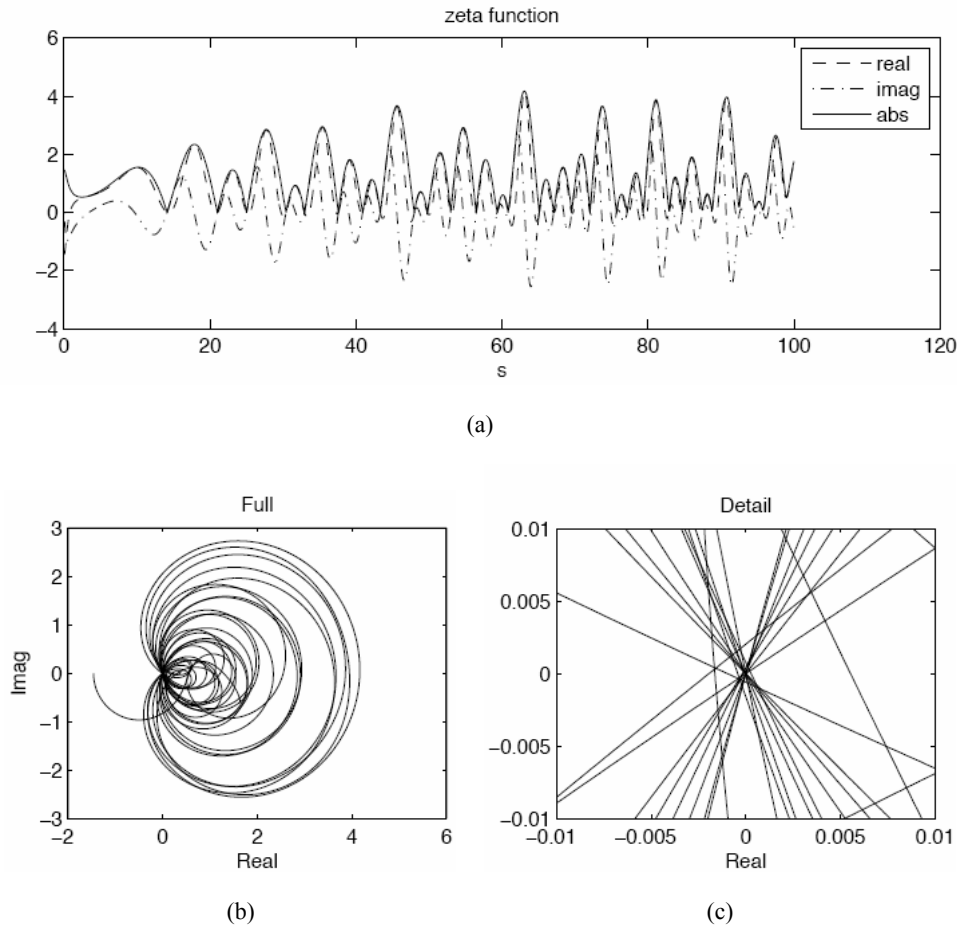
The main problem in the study of prime numbers has been, since the time of Riemann until today, the Riemann hypothesis. The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (4)$$

and the Riemann hypothesis is that the zeta function, properly rewritten, has all *non-trivial* zeros on the line $\text{real}(s) = 1/2$. The function has to be rewritten so as to be valid for $s < 1$, and the *trivial* zeros are found on the *real* line $s = -2k$. The zeta function is commonly plotted with the real part on the x-axis and the imaginary part on the y-axis. An example of this for numbers up to 100 can be seen in Figure 5(b).

The interesting part of the zeta function is when this is equal to zero. This can be seen when both the real and the imaginary part is zero, or when the absolute value is zero, or when the two-dimensional curve obtained when plotting the real vs. the imaginary of the zeta function crosses the zero position. As can be seen in Figure 5(c), when going into details, some crossings pass close to zero, although not through it. This may be because the function does not cross zero here, or because of numerical and precision issues.

Figure 5 Zeta function, (a) real and imaginary, (b) complex full and (c) detail



3 Primety development

The prime numbers are among the most important numbers, according to the mathematical community, and many theorems are based on the Riemann hypothesis. In this section, the notion of prime number is further developed with the introduction of the primety measure, a measure of how much prime a number is. This is done using the Farey sequence. Calculated using the integers, the Farey sequence renders a number of new fractions that is equal to the Euler totient function, while if the Farey sequence is calculated using real numbers, a more ambiguous result is obtained.

The Farey sequence (Sylvester, 1883) for a number n consists of $[0/n, 1/2, 1/3, 2/3, \dots, 1/n, \dots, n-1/n]$. For $n = 0$, it is the empty set $[\]$, for $n = 1$, it is $[0]$, for $n = 2$, it is $[0, 1/2]$, for $n = 3$ it is $[0, 1/3, 1/2, 2/3]$, etc. If all unique elements up to $n-1$ are retained, then for $n = 1$, or 2 there is one new element and for $n = 3$ or 4 there are two new elements, etc. The number of new element is $[1, 1, 2, 2, 4, 2, 6, 4, 6, 4, \dots]$ and it corresponds (Sylvester, 1883) to the Euler totient function $\phi(n)$. $\phi(n)$ has the property that $\phi(p) = (p-1)$ for p primes, and that $\phi(n) \leq (n-1)$, equal only in the case of n prime.

Therefore,

$$P(n) = \phi(n) / n - 1 \quad (5)$$

is here set as the primety function, i.e., a numerical indication of how close to being prime a number is. The primety values are shown in Figure 6 for the numbers 1 to 100. The prime numbers, for which $P(n) = 1$, are denoted with a ring. It is now possible to investigate specific number categories with respect to their primety value. For instance, the *pseudoprimes* all have different primety values, although always above 0.5 for numbers up to 10^6 . As a comparison, the minimum primety value in that range is below 0.2. The *lucky* numbers also have primety values above 0.5 in that range and, of course, also many primety values equal to one. As a comparison, the *even* numbers have primety values below 0.5 (except for low integers) and *odd* numbers have primety value above 0.36. Other integer sequences, such as *evil* and *odious*, have any primety value. If the *quality* of an integer sequence is supposed to be linked to its primety value, then the mean of the primety values of this sequence would constitute a good measure. As such, of course, prime numbers has mean 1, *lucky* numbers mean 0.92, the *pseudoprimes* 0.87, the *odd* numbers 0.81, *Fibonacci* numbers 0.69, the *odious* numbers 0.62, and the *evil* numbers 0.60. Finally the *even* numbers have mean primety of 0.41. The *natural* numbers have, as a comparison, mean primety of 0.61. Thus, most investigated sequences have higher than average *quality*, as measured by the mean of the primety of the sequence.

Now, if we use real numbers in the construction of the Farey sequence, $[0/r, 1/r, \dots, r/r, \dots, 1/(r+\delta), \dots, (r)/(r+\delta), \dots]$, and then find all new unique rations for each incremental δ , it is possible to show that the primety function for real numbers,

$$P(r) = \phi(r) / r - 1 \quad (6)$$

where $\phi(r)$ is calculated as the new unique element in the Farey sequence for increasing real numbers (r) . $P(r)$ follows a simple rule, as is shown in Figure 7.

Figure 6 The primity for numbers 1 to 100

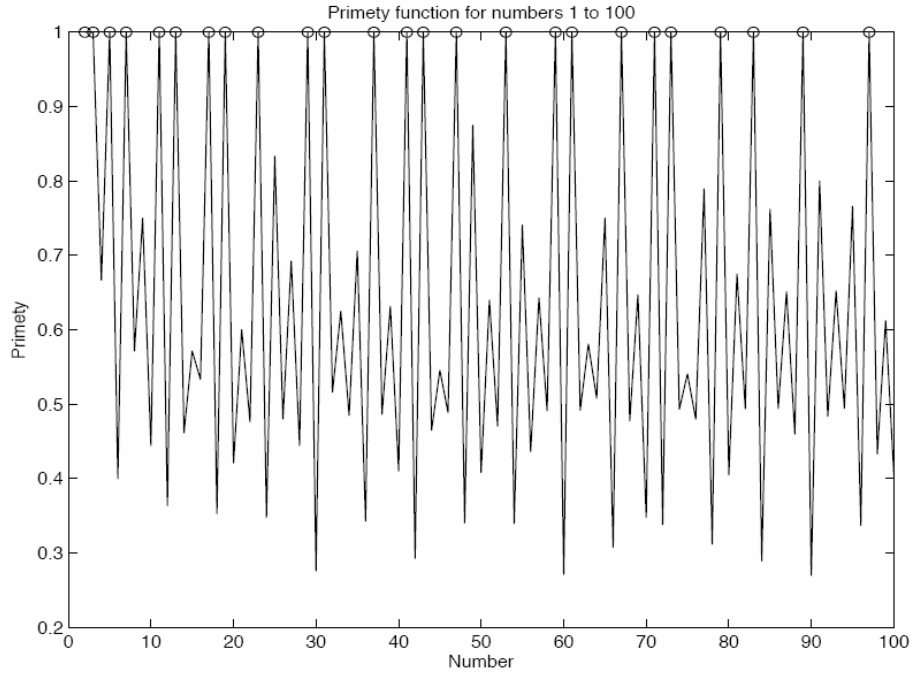
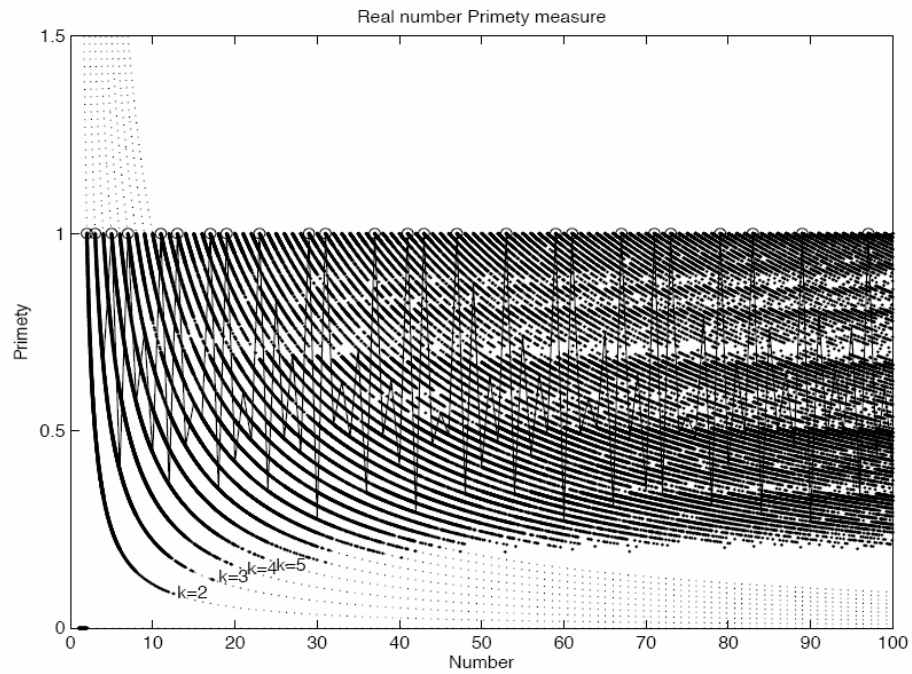


Figure 7 The primity values for real numbers between 1 and 100



Notes: The integer numbers are shown in the figure, and the primes are denoted with 'o'.
Equation (7) is shown, for low numbers ($2 \leq k \leq 10$), in dashed.

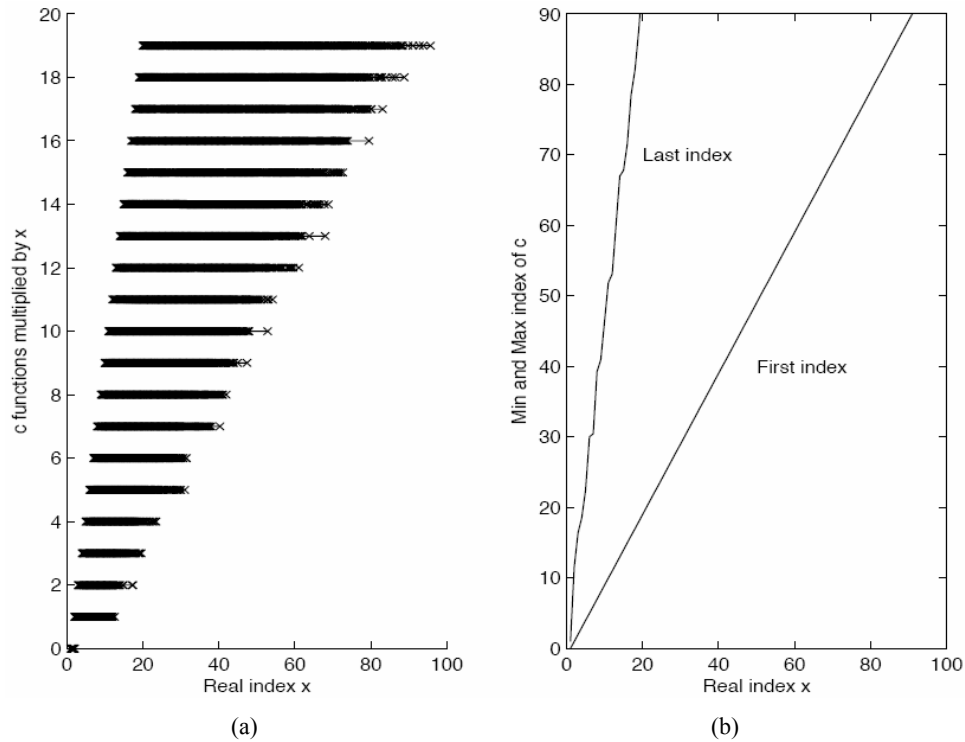
The real primety shares many properties with the integer primety, but has some remarkable differences too, in particular that the integer primety approaches 1 for all integers in the real case, but only for the primes in the integer case. Among the common properties is that the cumulative sum divided by the index of the primety approximates $\pi^2/6$ (Sylvester, 1883) both for integer and for real primety values.

Another common property is that the primety values are always found on one of the curves;

$$c_k = k/x, \quad (7)$$

where k is an integer, and $x > 0$ is an arbitrary value. $P(r)$ jumps between the different c_k , as r is increasing, so it is not possible to predict $P(r)$ from it. Nonetheless, this seems like a promising area of further research. For instance, equation (7) gives an absolute minimum for the value of $P(r)$. The maximum value seems to be equal to one for all values of x , with the exceptions of $r = 1$. For $r < 1$, $P(r) = 0$. All integer values approaches full prime number primety value; $P(r) \rightarrow 1$ as $r \rightarrow n^-$. It is not clear how this relates to the fact that many of these integer numbers have primety values less than 1.

Figure 8 (a) Transformed real value primety function (b) with corresponding min and max index values

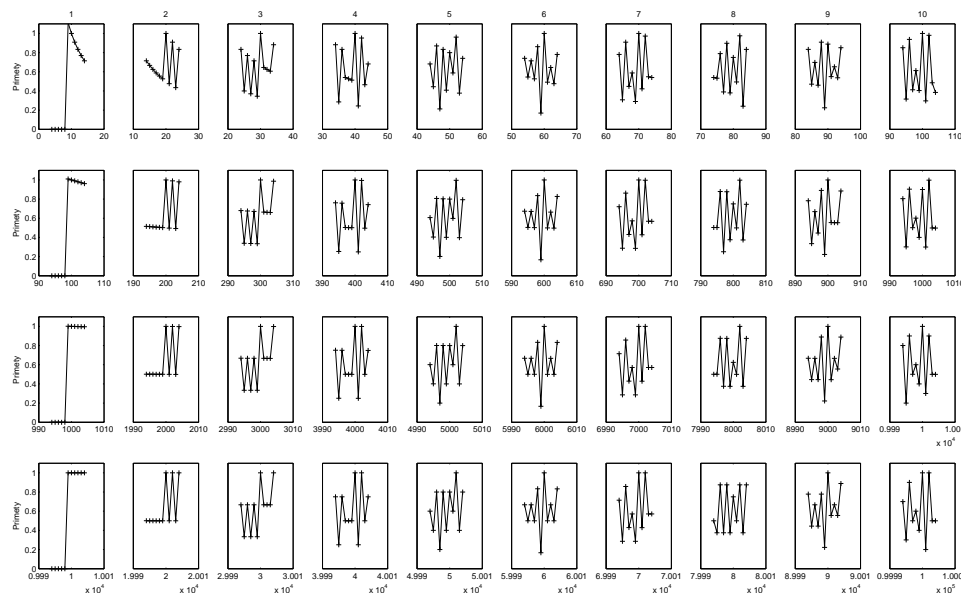


This is investigated further by transforming $P(r)$, so that the shape of it, according to equation (3) is visible. By identifying the values of $P(r)$ that correspond to each ck , and then multiplying the found values by x , a constant function is obtained. The result is shown in Figure 8. It is clear that $P(r') \cdot r'$ where r' corresponds to the values where $P(r) = c_k$, is a constant function equal to k . The min and max index values shown in the right plot of Figure 8 indicates that the minimum index is equal to k , and the maximum index is fluctuating slowly.

The first index corresponds to $P(r) \rightarrow 1$ as $r \rightarrow n$. The last index fluctuates slowly, in a non-predictable way, it seems. The primety function for real numbers has the appearance of bringing new findings to the quest for the understanding of prime numbers, but more work is necessary in order to assert this further. Hopefully this will allow more inspiring and artistically ambiguous mathematic development and further the art of mathematics and mathematic art. Some initial investigations of the latter are made in the next section.

Another approach is to investigate the behaviour of $P(r)$ in the vicinity of integers. $P(r)$ has been calculated for $0 \leq r \leq 10$ and for different precisions of r . The five primety values before and after $r = i$, $i = \{1 \dots 10\}$, are shown in Figure 9 for precision 0.1, 0.01, 0.001 and 0.0001.

Figure 9 Behaviour of $P(r)$ in the vicinity of integer values for different precisions



Note: The columns denote the first ten integer numbers with increasingly narrower context going down the rows.

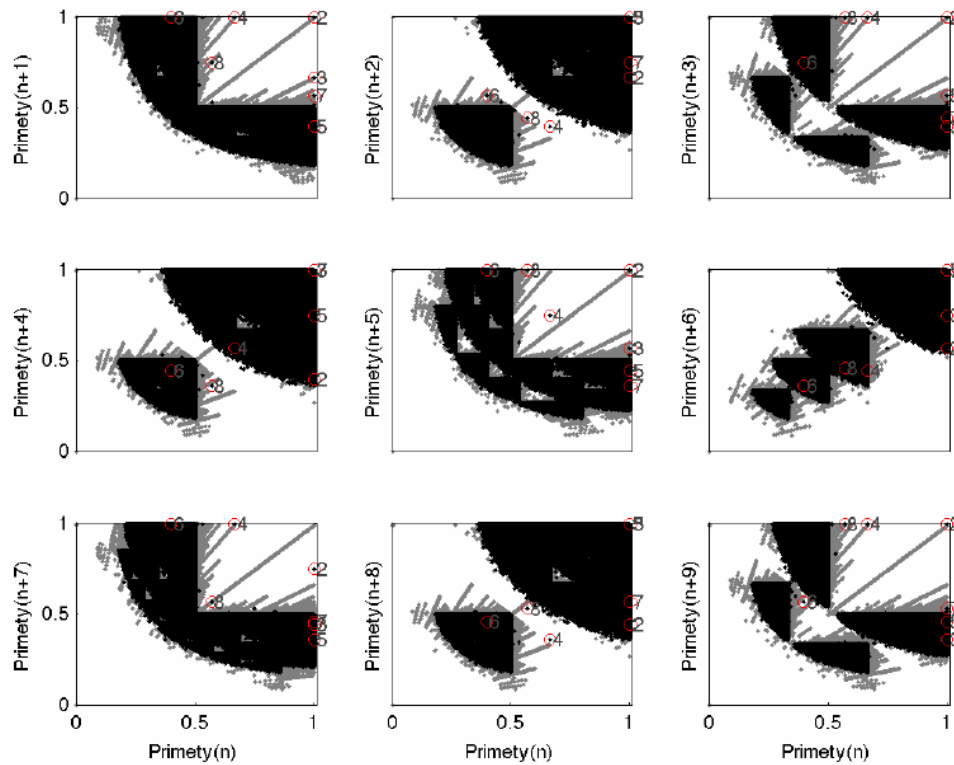
All primety values in the vicinity of each integer from 1 to 10 expose the same behaviour, independent of precisions. This can be seen in each column in Figure 9. The precision is increasing downwards, which makes the same interval (± 5) span less of the actual distance for higher precisions. Thus the higher rows expose more falling behaviour, as they travel further along the $1/x$ curve of (7). The real primety value is not necessarily approaching one for $r = i$, as can be seen in the figure, for 5, for instance, where the

maximum is found two steps after $r = i$. While the primety seems independent of the precision of the calculation of the real primety, how is it with the actual values of primety in the vicinity of integers? An analysis of $P(r)$ for high precision at the vicinity of integers from 1 to 100 shows a simple rule; the primety is repeating every three-integer values. Thus, as can also be seen in Figure 9, means $P(r)$ is similar when r is close to 1, 4, 7, ... or close to 2, 5, ... or close to 3, 6, ... The next repeating is found for each 21 integer. It does not seem possible to estimate classical prime numbers from the behaviour of the real primety.

4 Fractal behaviour

This section will show some of the fractal behaviours of the primety function, and the related Farey sequence. First, we calculate the primety for all numbers up to a large number, and show the scatterplot, i.e., the following primety value as a function of the current value, as seen in Figure 10.

Figure 10 Next primety values as function of current primety value for integers (black) and real numbers (grey) (see online version for colours)

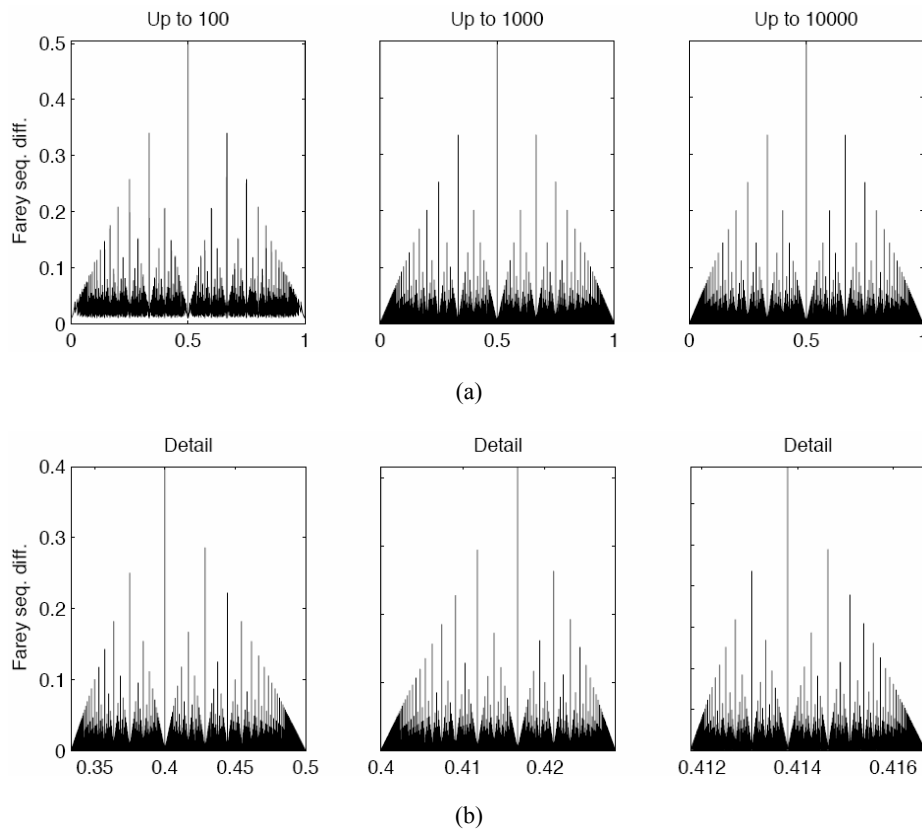


Notes: Context going from next primety (upper left corner) to nine values after (lower right corner). The first eight integer values are denoted with a ring, and the corresponding value of n .

Some initial conclusions can be made from Figure 10. First, except for very low numbers, all integer primety values are found above the half-circle $-P(n) - 1)^2 + (P(n + 1) - 1)^2 = z$. Furthermore, except for a few low numbers, all high primety values $-P(n) > 0.5$ gives $P(n + 1) < 0.5$ and vice-versa. There is a systematic shift from high to low values. Thus, all even numbered primety values are found below 0.5 and all odd numbered values are found above 0.5. Furthermore, a majority of $P(n)$ are found at or close to small integer fractions. Inside, the hole visible above $P(n) = 1/3$, $P(n + 1) = 2/3$, a shape very similar to the full plot is visible. If it is a fractal shape, it is a very slow one, i.e., the fractals only show up in detail for high precision. Each scatterplot reveals interesting and informative aspects of the primety values.

For the real numbered primety values, $P(r)$, even though the scatterplots place the real primety values in the vicinity of the integer values, the situation is different, as it seems the scatterplot reveals lines; $P(n + k) = (l/m) \bullet P(n)$, where l and m are low integers. For instance, (for the next value scatterplot), for the line leading to the '2', $l = m = 1$, for '3' $l = 2, m = 3$, for '4', $l = 3, m = 2$, etc.

Figure 11 (a) Difference of Farey sequence for three sizes, and (b) three different details of the large size



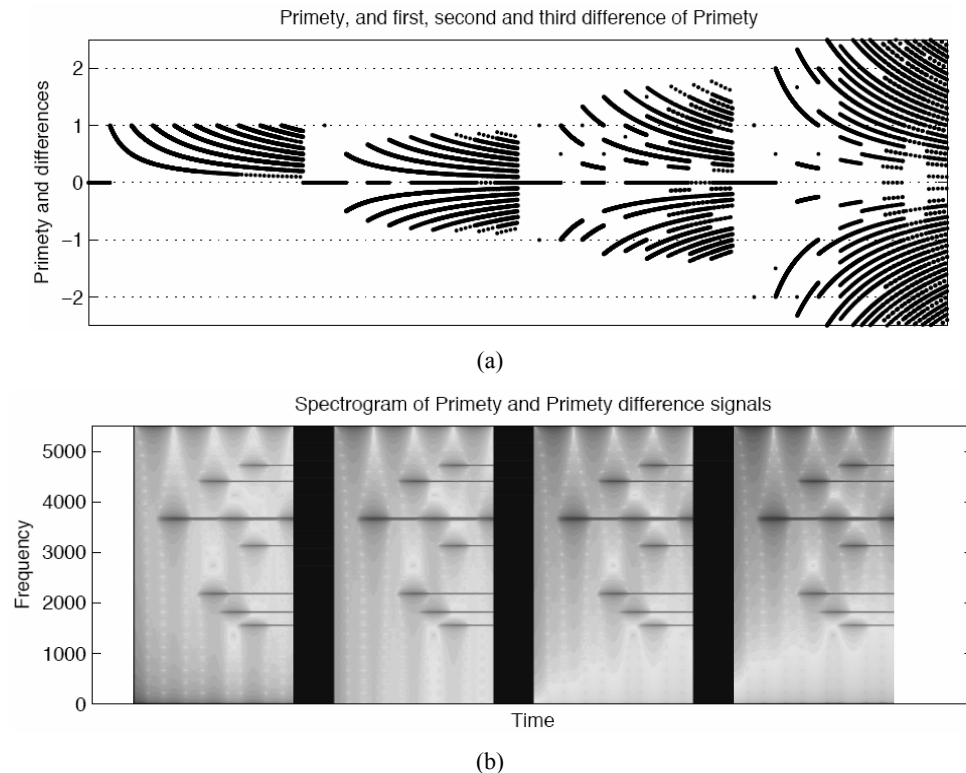
Notes: The different sizes show how the Farey difference signal always retains the same shape, and the different details show how the outer shape is copied in the details.

More fractal behaviour is coming from the Farey sequences. The difference signal of the integer Farey sequence has a characteristic shape, which is independent on the size of the Farey sequence. Furthermore, this same shape is exactly replicated inside the larger shape (assuming n is large). These findings are shown in Figure 11 for three different sizes, $n = 100$ (left), $n = 1,000$, and $n = 10,000$ (right). All three sizes have the same shape, albeit the small sequence (left) is coarser. The peaks of the differences are found on small integer ratios, ($1/2$, $1/3$, $3/8$, $2/5$, $5/12$, etc.).

Figure 11 shows the fractal behaviour of the Farey difference signal, both for different sizes [Figure 11(a)], and for different details [Figure 11(b)].

In addition to this, the real signal primety function $P(r)$ exposes an interesting behaviour. In Figure 12, the real values primety function is shown for $0 < r < 10$, along with the first to third difference signals [Figure 12(a)] and spectrograms [Figure 12(b)]. It is interesting to observe how the difference signals look similar to the primety signal itself, although with larger values for each difference, and also clearly changing shape at each integer value for higher difference signals. Treated as an audio signal, this function exposes a characteristic toned and rhythmic sound that eventually blends into a noisy signal. The sound is very similar across the primety difference signals.

Figure 12 (a) Real value primety function and first three difference signals, and (b) spectrograms



It is certain that nature contains vast amounts of at least statistic self-similarities fractals (Mandelbrot, 1967). Many artists have, consciously or not, integrated fractal shapes in their art. Among audio signals to potentially be used in music, as shown above, fractals

have a long tradition for quasi-artistic qualities. While the figures shown here are not necessary polished enough, or do not contain the aesthetic or other qualities to be used in artwork, it is still possibly to envision such a use.

5 Conclusions

This work has strived to increase the ambiguity of fractals and number sequences. Firstly, this is done by augmenting the classical fractals with random behaviour, thus rendering more complex or noisy shapes.

The Farey sequence of real numbers, along with the introduction of the primety value permits exposing a certain number of laws that govern prime numbers. Among these are the fact that the primety values are always found on functions of the shape $c_k = k / x$, which enables to assert a (low) lowest possible value of the primety function and thus of the Euler totient function. Furthermore, all even primety values are (asymptotically) lower than 0.5, and all odd values are higher than 0.36. The mean of the primety for a given number sequence can be used as a measure of the *quality* of the sequence. Thus, *lucky* numbers have high *quality*, while *even* number have low *quality*.

Three interesting visual results from this research are;

- 1 the k / x shape of the primety function
- 2 the almost fractal scatterplot of the primety function, and the complete fractal shape of the Farey difference signal.

Some interesting visual and sonic results are furthermore obtained from these signals.

Fractals and related mathematic results, as presented here, are placed in a double art context, first, because the quest for the mathematic results leads to ambiguity, “one of art’s most potent aesthetic functions” (Byers, 2007), secondly, because the end result, graphics, plots, music, represents art in itself.

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