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Abstract: In this paper we present a theoretical analysis of the model reduction algorithm for linear switched systems from Shaker and Wisniewski (2011, 2009). This algorithm is based on balanced truncation. More precisely, **(1)** we provide a bound on the approximation error in L_2 norm, **(2)** we provide a system theoretic interpretation of grammians and their singular values, **(3)** we show that the performance of balanced truncation depends only on the input-output map and not on the choice of the state-space representation. In addition, we also show that quadratic stability and LMI estimates of the L_2 gain also depend only on the input-output map.

Keywords: switched systems, model reduction, balanced trunction, realization theory.

1. INTRODUCTION

In this paper we address certain theoretical problems which arise in balanced truncation of continuous-time linear switched systems using balanced truncation. In order to explain the contribution of the paper, we will first present a very informal overview of balanced truncation for switched system, which appeared in Shaker and Wisniewski (2011, 2009). Consider a linear switched system of the form

$$\Sigma : \begin{cases} \dot{x}(t) = A_{q(t)}x(t) + B_{q(t)}u(t), \ x(t_0) = x_0 \\ y(t) = C_{q(t)}x(t). \end{cases} \quad (1)$$

where A_q, B_q, C_q are $n \times n$, $n \times m$ and $p \times n$ matrices respectively, and the switching signal q maps time instances to discrete states in a set Q. Note that the *switching signal q is viewed as an input of the system*. We seek to replace the system Σ by another one of smaller dimension, but which still adequately approximates the input-output behavior of Σ. To this end, define the *observability grammian* of the system above as any positive definite $\mathcal{Q} > 0$ such that

$$\forall q \in Q : A_q^T \mathcal{Q} + \mathcal{Q} A_q + C_q^T C_q < 0. \quad (2)$$

Likewise, define a *controllability grammian* of the system as a strictly positive definite $\mathcal{P} > 0$ which satisfies.

$$\forall q \in Q : A_q \mathcal{P} + \mathcal{P} A_q^T + B_q B_q^T < 0. \quad (3)$$

By applying a suitable state-space isomorphism, the system can be brought into a form where $\mathcal{P} = \mathcal{Q} = \Lambda = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ are diagonal matrices and $\sigma_1 \geq \ldots \geq \sigma_n > 0$. We will call the numbers $\sigma_1 \geq \ldots \geq \sigma_n > 0$ the *singular values* of the pair $(\mathcal{P}, \mathcal{Q})$. It is easy to see that $\sigma_i = \sqrt{\lambda_i(\mathcal{P}\mathcal{Q})}$, where $\lambda_1(\mathcal{P}\mathcal{Q}) \geq \ldots \geq \lambda_n(\mathcal{P}\mathcal{Q})$ are the ordered eigenvalues of $\mathcal{P}\mathcal{Q}$. Following the classical terminology, we will call a state-space representation *balanced*, if $\mathcal{P} = \mathcal{Q} = \Lambda$, where Λ is a diagonal matrix. We reduce the dimension of a balanced state-space representation by discarding the last $n - r + 1$ state-space components. The system matrices $\hat{A}_q, \hat{B}_q, \hat{C}_q$ of the reduced order system $\hat{\Sigma}$

are obtained by partitioning the matrices of the original balanced system as

$$\bar{A}_q = \begin{bmatrix} \hat{A}_q & \star \\ \star & \star \end{bmatrix}, \quad \bar{B}_q = \begin{bmatrix} \hat{B}_q \\ \star \end{bmatrix}, \quad \bar{C}_q^T = \begin{bmatrix} \hat{C}_q^T, \ \star \end{bmatrix}$$

where $\hat{A}_q, \hat{B}_q, \hat{C}_q$ are $r \times r$, $r \times m$ and $p \times r$ matrices respectively. The performance of this procedure has been extensively tested by means of numerical examples in Shaker and Wisniewski (2011, 2009). However, many theoretical questions remain open.

Problem formulation

We would like to find error bounds, and to establish the invariance of the method with respect to state-space representation. More precisely, we seek an answer to the following questions:

(1) **Error bounds**
Can we state an error bound on the distance between the original system Σ and the reduced one $\hat{\Sigma}$, using some metric ? In particular, can we extend the well-known result from the linear case, by proving that

$$||Y^\Sigma - Y^{\hat{\Sigma}}||_{L_2} \leq 2(\sigma_{r+1} + \cdots + \sigma_n) \quad (4)$$

where Y^Σ and $Y^{\hat{\Sigma}}$ are the input-output maps of Σ and $\hat{\Sigma}$ respectively and $||.||_{L_2}$ denotes the L_2 norm of the switched system as defined in Hespanha (2003) ?

(2) **State-space representation invariance of the grammians**
Under which conditions the controllability and observability inequalities (9) and (10) have solutions ? Does the existence of a solution to these inequalities depend on the choice of the state-space representation ? Can we characterize the set of observability/controllability grammians in a way which does not depend on the choice of the state-space representation but only on the input-output map ?

(3) **State-space representation invariance of the singular values**
Do the singular values of the system (i.e., the values $\sigma_1, \ldots, \sigma_n$) depend only on the input-output map of the system or do they also depend on the choice of the state-space representation. Do they have a system theoretic interpretation ?

(4) **State-space representation invariance of the L_2 norm estimates**
Is it possible to estimate the system norm (in our case L_2 norm) in a manner which does not depend on the choice of the basis of the state-space ?

(5) **System theoretic interpretation of the grammians**
What is the relationship between grammians and observability/controllability of the switched systems. Recall that in the linear case, existence of strictly positive observability/controllability grammians implies observability/controllability of the system. Does this extend to the switched case ?

(6) **Preservation of system theoretic properties by the reduced system**
If the original system was reachable, observable, minimal, stable, etc., then will balanced truncation preserve these properties ?

The motivation for the first problem is clear. The motivation for questions 2–4 is the following. The formulation of balanced truncation does not a priori exclude the possibility that the choice the state-space representation might influence existence of grammians and the values of the corresponding singular values. Similarly, the existence of a solution to the LMIs which are used for estimating L_2 gains could, in principle, also depend on the choice of the state-space representation. This would adversely affect the applicability of the method, since the choice of the initial state-space representation is often circumstantial. Question 5 is important for obtaining a deeper theoretical insight and for answering Question 6. Finally, Question 6 is important, because the reduced system is supposed to be used for control design, which is easier if certain important system-theoretic properties remain valid.

Contribution of the paper In this paper, we prove the error-bound (4), and in addition, we show that the existence of grammians, singular values and the estimates of the L_2 norm all depend only on the input-output map and not on the choice of the state-space representation. This is accomplished by using realization theory, i.e., by exploiting the existence and uniqueness of minimal state-space representations Petreczky (2011). Furthermore, the results of the paper show that when dealing with model reduction or with L_2 norms in control, one can restrict attention to *minimal state-space representations* without loss of generality. More precisely, we present the following results.

(1) The error bound (4) holds.
(2) If a system admits an observability (controllability) grammian, then any minimal linear switched system which describes the same input-output map will admit an observability (controllability) grammian.

There is a one-to-one correspondence between controllability (resp. observability) grammians of minimal systems which describe the same input-output map. This correspondence preserves the singular values. That is, the *existence of grammians is a property of the input-output map and not of the state-space representation. For minimal state-space representations, the singular values are functions of the input-output map and not of the state-space representation.* We also relate th largest singular value to the Hankel-norm of the system.

As a byproduct, we also show that *if an input-output map can be realized by a quadratically stable system[1], then any minimal realization of this map will be quadratically stable.*

(3) We show how to estimate the L_2 norm using LMIs in such a way that the obtained estimate does not depend on the choice of the state-space representation.

(4) For minimal systems, if controllability and observability grammians exist, then they are necessarily strictly positive definite.

(5) Balanced truncation preserves quadratic stability. However, it does not necessarily preserve minimality. The fact that balanced truncation does not preserve minimality is a further indication of that the method might be very conservative.

Related work To the best of our knowledge, the results of the paper are new. A rich literature covers the subject of model reduction for switched systems, Shaker and Wisniewski (2011, 2009); Birouche et al. (2010); Monshizadeh et al. (2011); Zhang et al. (2008); Mazzi et al. (2008); Habets and van Schuppen (2002); Zhang et al. (2009); Zhang and Shi (2008); Gao et al. (2006); Kotsalis and Rantzer (2010). In particular, balanced truncation,was explored in Shaker and Wisniewski (2011, 2009); Kotsalis and Rantzer (2010); Gao et al. (2006); Monshizadeh et al. (2011). The procedure dealt with in this paper was already described in Shaker and Wisniewski (2011, 2009). Error bounds were dealt with in Kotsalis and Rantzer (2010), however there the authors worked with discrete-time stochastic systems, while here we study continuous-time deterministic ones. Induced L_2 norms for switched systems was addressed in Hespanha (2003); Margaliot and Hespanha (2008); Hirata and Hespanha (2009), but those papers did not focus on the invariance of the computed estimates with respect to the choice of the state-space representation.

Outline In §2, we present the formal definition and system theoretic properties of linear switched systems. In §3, we present a brief overview of realization theory of linear switched systems. In §4, we present the formal definition of L_2 norms and grammians and show the relationship between these concepts along with conditions which guarantee their existence. In §5 we show the state-space representation invariance of quadratic stability, estimates of the L_2 norm, existence of grammians and the singular values of the system. In §6 we discuss the system theoretic interpretation of grammians and their singular values. Finally, in §7, we present the proof of the error bound for balanced truncation and we discuss which system theoretic properties are preserved by balanced truncation.

Notation
Denote by $\mathbb{N}$ the set of natural numbers, including 0. Denote by $T = \mathbb{R}_+$ the set of nonnegative reals. We denote

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[1] i.e. a system with a common quadratic Lyapunov function

by $||x||_2$ the Eucledian norm of a vector $x \in \mathbb{R}^n$. We denote by $\mathbb{R}^{k \times l}$ the set of all $k \times l$ matrices with real entries. If $A \in \mathbb{R}^{n \times n}$ is symmetric, then we denote the fact that A is strictly positive definite, strictly negative definite, positive semi-definite and negative semi-definite by $A > 0$, $A < 0$, $A \geq 0$ and $A \leq 0$ respectively. We denote by $\text{diag}(a_1, a_2, \ldots, a_n)$ the $n \times n$ diagonal matrix, diagonal entries of which are $a_1, \ldots, a_n \in \mathbb{R}$.

We use the standard notation of automata theory Eilenberg (1974). For a finite set X, called the *alphabet*, denote by X^* the set of finite sequences (also called strings or words) of elements of X. The length of a word w is denoted by $|w|$, i.e., $|w| = k$. We denote by ϵ the *empty sequence (word)*. In addition, we define $X^+ = X^* \setminus \{\epsilon\}$.

We say that a map $f : T \to \mathbb{R}^n$ is *piecewise-continuous*, if f has finitely many points of discontinuity on any compact subinterval of T, and at any point of discontinuity the left-hand and right-hand side limits of f exist and are finite. We denote by $PC(T, \mathbb{R}^n)$ the set of all piecewise-continuous functions of the above form. We denote by $AC(I, \mathbb{R}^n)$ the set of all absolutely continuous maps $f : I \to \mathbb{R}^n$. We denote by $L_2(T, \mathbb{R}^n)$ the set of all Lebesque measurable maps $f : T \to \mathbb{R}^n$ for which $\int_0^\infty ||f(s)||_2^2 ds < +\infty$. For $f \in L_2(T, \mathbb{R}^n)$, we denote by $||f||_2$ the standard norm of f, i.e. $||f||_2 = \sqrt{\int_0^\infty ||f(s)||_2^2 ds}$.

2. LINEAR SWITCHED SYSTEMS

Below we present the formal definition of linear switched systems and their system theoretic properties. The presentation is based on Petreczky (2011, 2006).

Let $Q = \{1, \ldots, D\}$, $0 < D \in \mathbb{N}$.

Definition 1. (Linear switched systems). A linear switched system with external switching (abbreviated as LSS) is a tuple

$$\Sigma = (n, Q, \{(A_q, B_q, C_q) \mid q \in Q\}),$$

where for each $q \in Q$, $(A_q, B_q, C_q) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$. A *solution* of the switched system (with external switching) Σ with initial state $x_0 \in \mathbb{R}^n$ relative to the pair $(u, q) \in PC(T, \mathbb{R}^m) \times PC(T, Q)$ is by definition a pair $(x, y) \in AC(T, \mathbb{R}^n) \times PC(T, \mathbb{R}^p)$ which solves the Cauchy problem

$$\dot{x}(t) = A_{q(t)} x(t) + B_{q(t)} u(t), \ x(t_0) = x_0$$
$$y(t) = C_{q(t)} x(t).$$

We shall call u the control input, q the switching signal, x the state trajectory and y the output trajectory. In the sequel, we use the following notation: the state space $X = \mathbb{R}^n$, the output space $Y = \mathbb{R}^p$, and the input space $U = \mathbb{R}^m$. The elements of the set Q will be called the discrete modes, and Q will be called the set of discrete modes.

Definition 2. (Input-to-state and input-output maps). For an initial condition $x_0 \in X$, we define the maps $X_{x_0}^\Sigma : PC(T, U) \times PC(T, Q) \to AC(T, X)$ and $Y_{x_0}^\Sigma : PC(T, U) \times PC(T, Q) \to PC(T, Y)$ by $X_{x_0}^\Sigma(u, q)(t) = x(t)$ and $Y_{x_0}^\Sigma(u, q)(t) = y(t)$, where (x, y) is the solution of the switched system Σ at x_0 relative to (u, q). The map $Y_{x_0}^\Sigma$

is called the *input-output map induced by the initial state* x_0.

The potential input-output maps of a linear switched system are maps of the form

$$f : PC(T, U) \times PC(T, Q) \to PC(T, Y). \tag{6}$$

Below, we define when such a map is realized by a linear switched system. To this end, we will fix a designated initial state for LSSs. Since in the sequel we will mostly deal with exponentially stable LSSs, we will set the initial state to be zero. Note that while this choice seems natural for the current paper, other choices might be more appropriate in other circumstances. Many of the results of this paper can be extended to the case of non-zero initial conditions.

Definition 3. (Realization). The *input-output map* Y^Σ of a LSS Σ is the input-output map $Y^\Sigma = Y_0^\Sigma$ induced by the zero initial state. The LSS Σ is said to be a *realization* of an input-output map f of the form (6), if $Y^\Sigma = f$.

The LSSs Σ_1 and Σ_2 are *equivalent*, if $Y^{\Sigma_1} = Y^{\Sigma_2}$.

It is clear that any LSS is a realization of its own input-output map induced by the zero initial state. In the sequel, we will need the notions of minimality, reachability and observability of LSSs. Below we recall these notions.

Definition 4. (Dimension and minimality). The dimension $\dim \Sigma$ of a LSS Σ of the form (5) is the dimension of its state-space $\mathcal{X}$. The LSS Σ_m is said to be a *minimal* realization of an input-output map f, if Σ_m is a realization of f and if for any other LSS Σ which is a realization of f, $\dim \Sigma_m \leq \dim \Sigma$. We say that Σ_m is a minimal LSS , if it is a minimal realization of its input-output map Y^{Σ_m}.

Definition 5. (Observability). An LSS Σ is said to be *observable*, if for any two states $x_1 \neq x_2 \in \mathcal{X}$, the input-output maps induced by x_1 and x_2 are different, i.e. $Y_{x_1}^\Sigma \neq Y_{x_2}^\Sigma$.

Definition 6. (Reachability). The LSS Σ is said to be *reachable* if every state is reachable from the zero initial state, i.e., if $\{X_0^\Sigma(u, q)(t) \mid u \in PC(T, U), q \in PC(T, Q), t \in T\} = X$.

3. OVERVIEW OF REALIZATION THEORY

In this section, we recall from Petreczky (2011, 2007, 2006) the main results on realization theory of LSSs.

Definition 7. (Isomorphism) Consider a LSS

$$\Sigma_1 = (n, Q, \{(A_q, B_q, C_q) \mid q \in Q\}),$$

and a LSS Σ_2

$$\Sigma_2 = (n, Q, \{(A_q^a, B_q^a, C_q^a) \mid q \in Q\})$$

A non-singular matrix $S \in \mathbb{R}^{n \times n}$ is said to be an *isomorphism* from Σ_1 to Σ_2, denoted by $S : \Sigma_1 \to \Sigma_2$, if

$$\forall q \in Q : A_q^a S = S A_q, B_q^a = S B_q, C_q^a S = C_q.$$

Theorem 1. (Minimality, Petreczky (2011, 2007)). An LSS realization Σ of f is minimal if and only if it is reachable and observable. All minimal LSS realizations of f are isomorphic.

Observability and reachability of an LSS Σ can be characterized by linear-algebraic conditions. In order to present these conditions, we need the following notation.

Notation 1. Consider a LSS $\Sigma = (n, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$. For a sequence $w \in Q^*$, we write

$$A_w = \begin{cases} I_n & \text{if } w = \epsilon, \\ A_{q_k} \cdots A_{q_2} A_{q_1} & \text{if } w = q_1 q_2 \cdots q_k, \end{cases}$$

where I_n denotes the $n \times n$ identity matrix, and ϵ is the empty sequence.

Denote by M the cardinality of the set of all words $w \in Q^*$ of length at most n, i.e., $M = |\{w \in Q^* \mid |w| \le n\}|$. Fix an ordering $\{v_1, \ldots, v_M\}$ of the set $\{w \in Q^* \mid w \le n\}$.

Theorem 2. (Sun and Ge (2005); Petreczky (2006)).
Reachability: The LSS Σ is reachable if and only if rank $\mathcal{R}(\Sigma) = n$, where

$$\mathcal{R}(\Sigma) = \begin{bmatrix} A_{v_1}\widetilde{B}, & A_{v_2}\widetilde{B}, & \ldots, & A_{v_M}\widetilde{B} \end{bmatrix} \in \mathbb{R}^{n \times m|Q|M}$$

with $\widetilde{B} = [B_1, B_2, \ldots, B_D] \in \mathbb{R}^{n \times |Q|m}$.
Observability: The LSS Σ is observable if and only if rank $\mathcal{O}(\Sigma) = n$, where

$$\mathcal{O}(\Sigma) = \left[(\widetilde{C}A_{v_1})^T, (\widetilde{C}A_{v_2})^T, \ldots, (\widetilde{C}A_{v_M})^T \right]^T \in \mathbb{R}^{p|Q|M \times n}.$$

where $\widetilde{C} = \begin{bmatrix} C_1^T & C_2^T, & \ldots, & C_D^T \end{bmatrix}^T \in \mathbb{R}^{p|Q| \times n}$.

The matrix $\mathcal{R}(\Sigma)$ (resp. $\mathcal{O}(\Sigma)$) will be called a *controllability matrix* (resp. *observability matrix*) of Σ.

Remark 1. If a linear subsystem of a LSS Σ is observable (reachable), then Σ is observable (resp. reachable). Hence, by Theorem 1, if a linear subsystem of Σ is minimal, then Σ itself is minimal.

Remark 2. Note that observability (reachability) of a LSS does not imply observability (reachability) of any of its linear subsystems. In fact, it is easy to construct a counter example, see Petreczky (2011). Together with Theorem 1, which states that minimal realizations are unique up to isomorphism, this implies that *there exist input-output maps which can be realized by a LSS, but which cannot be realized by a LSS where all (or some) of the linear subsystems are observable (or reachable).*

We can formulate the following algorithms for reachability/observability reduction and minimization.

Procedure 1. (Petreczky (2011, 2006)).
Reachability reduction:
Assume rank $\mathcal{R}(\Sigma) = n^{\mathrm{r}}$ and choose a basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ such that $b_1, \ldots, b_{n^{\mathrm{r}}}$ span Im $\mathcal{R}(\Sigma)$. In the new basis, $A_q, B_q, C_q, q \in Q$ become as follows

$$A_q = \begin{bmatrix} A_q^{\mathrm{r}} & A_q' \\ 0 & A_q' \end{bmatrix}, C_q = \begin{bmatrix} C_q^{\mathrm{r}}, & C_q' \end{bmatrix}, B_q = \begin{bmatrix} B_q^{\mathrm{r}} \\ 0 \end{bmatrix}, \quad (7)$$

where $A_q^R \in \mathbb{R}^{n^{\mathrm{r}} \times n^{\mathrm{r}}}, B_q^{\mathrm{r}} \in \mathbb{R}^{n^{\mathrm{r}} \times m}$, and $C_q^{\mathrm{r}} \in \mathbb{R}^{p \times n^{\mathrm{r}}}$. As a consequence, $\Sigma_{\mathrm{r}} = (n^{\mathrm{r}}, Q, \{(A_q^{\mathrm{r}}, B_q^{\mathrm{r}}, C_q^{\mathrm{r}}) \mid q \in Q\})$ is reachable, and has the same input-output map as Σ.

Intuitively, Σ_{r} is obtained from Σ by restricting the dynamics and the output map of Σ to the space Im $\mathcal{R}(\Sigma)$.

Procedure 2. (Petreczky (2011, 2006)).
Observability reduction:
Assume that $\ker \mathcal{O}(\Sigma) = n - n^{\mathrm{o}}$ and let $b_1, \ldots, b_n$ be a basis in $\mathbb{R}^n$ such that $b_{n^{\mathrm{o}}+1}, \ldots, b_n$ span $\ker O(\Sigma)$. In this new basis, A_q, B_q, and C_q can be rewritten as

$$A_q = \begin{bmatrix} A_q^{\mathrm{o}} & 0 \\ A_q' & A_q'' \end{bmatrix}, C_q = [C_q^{\mathrm{o}}, 0], B_q = \begin{bmatrix} B_q^{\mathrm{o}} \\ B_q' \end{bmatrix},$$

where $A_q^{\mathrm{o}} \in \mathbb{R}^{n^{\mathrm{o}} \times n^{\mathrm{o}}}, B_q^{\mathrm{o}} \in \mathbb{R}^{n^{\mathrm{o}} \times m}, C_q^{\mathrm{o}} \in \mathbb{R}^{p \times n^{\mathrm{o}}}$ and $x_0^{\mathrm{o}} \in \mathbb{R}^{n^{\mathrm{o}}}$. Then the LSS $\Sigma_{\mathrm{o}} = (n^{\mathrm{o}}, Q, \{(A_q^{\mathrm{o}}, B_q^{\mathrm{o}}, C_q^{\mathrm{o}}) \mid q \in Q\})$ is observable and its input-output map is the same as that of Σ. If Σ is reachable, then so is Σ_{o}.

Intuitively, Σ_{o} is obtained from Σ by merging any two states x_1, x_2 of Σ, for which $\mathcal{O}(\Sigma)x_1 = \mathcal{O}(\Sigma)x_2$.

Procedure 3. (Petreczky (2011, 2006)). **Minimization**:
Transform Σ to a reachable LSS Σ_{r} by Procedure 1. Subsequently, transform Σ_{r} to an observable LSS $\Sigma_{\mathrm{m}} = (\Sigma_{\mathrm{r}})_{\mathrm{o}}$ using Procedure 2. Then Σ_{m} is a minimal LSS which us equivalent to Σ.

4. STABILITY, GRAMMIANS AND L_2 NORMS

In this section, we briefly review the definition of controllability/observability grammians, L_2 norms and quadratic stability for LSSs. We also recall the basic relationships between these concepts.

Definition 8. (Quadratic stability). A LSS
$$\Sigma = (n, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$
is said to be quadratically stable, if and only if there exists a positive definite matrix $P = P^T > 0$ such that

$$\forall q \in Q : A_q^T P + P A_q < 0. \quad (8)$$

It is well-known Liberzon (2003) that quadratic stability implies exponential stability for all switching signals. For our purposes quadratic stability is convenient, as it implies the existence of an L_2 gain and controllability/observability grammians.

Definition 9. (*Controllability/observability grammians*)
An *observability grammian* of Σ is a strictly positive definite solution $\mathcal{Q} > 0$ of the following inequality

$$\forall q \in Q : A_q^T \mathcal{Q} + \mathcal{Q} A_q + C_q^T C_q < 0. \quad (9)$$

A *controllability grammian* of Σ is a strictly positive definite $\mathcal{P} > 0$ of the following inequality

$$\forall q \in Q : A_q \mathcal{P} + \mathcal{P} A_q^T + B_q B_q^T < 0. \quad (10)$$

We will call the eigenvalues of $\mathcal{P}\mathcal{Q}$ the *singular values* of the pair of grammians $(\mathcal{P}, \mathcal{Q})$.

It is easy to see that quadratic stability implies the existence of controllability and observability grammians, using techniques from Boyd et al. (1994). Existence of a controllability or observability grammian trivially implies quadratic stability.

Next, we define the L_2 gain for LSSs.

Definition 10. (Hespanha (2003)). We say that Y^Σ has a L_2 gain $\gamma > 0$, if for all $u \in L_2(T, U) \cap PC(T, U)$,

$$\sup_{q \in PC(T,Q)} \int_0^\infty ||Y^\Sigma(u, q)(s)||_2^2 ds \le \gamma^2 ||u||_2^2. \quad (11)$$

If Y^Σ has a finite L_2 gain, then we define the L_2 norm of Y^Σ, denoted by $||Y^\Sigma||_{L_2}$ as the infimum of all $\gamma > 0$ such that (11) holds. If Y^Σ does not have a finite L_2 gain, then we set $||Y^\Sigma||_{L_2} = +\infty$.

Lemma 1. If for Σ, the matrix inequality (10) admits a solution $\mathcal{P} > 0$, then there exists a solution $R > 0$ to

$$A_q^T R + R A_q + C_q^T C_q + \gamma R B_q B_q^T R < 0. \quad (12)$$

In particular, if Σ is quadratically stable, then (12) has a positive definite solution R. If a positive definite solution

to (12) exists, then the L_2 norm of Y^Σ is not greater than γ.

Lemma 1 seems to be folklore, for the sake of completeness we briefly sketch its proof.

Proof. [Sketch of the proof of Lemma 1] The proof that existence of a solution $R > 0$ to (12) implies that $||Y^\Sigma||_{L_2}$ exists and $||Y^\Sigma||_{L_2} \le \gamma$ follows from (Hirata and Hespanha, 2009, Theorem 1) by taking $V(x) = x^T R x$.

It is left to show that (12) holds. Consider the solution $\mathcal{P}$ of (10) and multiply the equation (10) from the left and the right by $\mathcal{P}^{-1}$. Subsequently, we can find suitably small $\gamma > 0$ such that by setting $R = \frac{1}{\gamma}\mathcal{P}^{-1}$, the inequality (12) holds.

5. STATE-SPACE REPRESENTATION INVARIANCE

In the previous section, we defined quadratic stability, L_2 gains, and grammians. The concepts were defined in terms of LMIs. We will show that the existence of a solution to those LMIs is a property of the input-output map. Furthermore, for equivalent minimal systems, the set of solutions are isomorphic. In order to present this result formally, we will introduce the following notation.

Definition 11. For a LSS
$$\Sigma = (n, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$
define the following subsets of the set of $n \times n$ strictly positive definite matrices

- $\mathbf{S}(\Sigma)$ is the set of all $P > 0$ which satisfy (8).
- $\mathbf{O}(\Sigma)$ is the set of all $\mathcal{Q} > 0$ for which (9) holds.
- $\mathbf{C}(\Sigma)$ is the set of all $\mathcal{P} > 0$ for which (10) holds.
- For $\gamma > 0$, let $\mathbf{G}_\gamma(\Sigma)$ be the set of all $R > 0$ which satisfy (12).

We can now state the following result.

Theorem 3. Let $\mathbf{K}$ be any symbol from $\{\mathbf{S}, \mathbf{O}, \mathbf{C}, \mathbf{G}_\gamma\}$.

(1) If the LSS Σ is such that $\mathbf{K}(\Sigma) \ne \emptyset$, then for any minimal LSS Σ_m which is equivalent to Σ, $\mathbf{K}(\Sigma_m) \ne \emptyset$.

(2) Let Σ_1 and Σ_2 be two LSSs of dimension n and let $\mathcal{S} : \Sigma_1 \to \Sigma_2$ be an isomorphism between them. If $\mathbf{K} \in \{\mathbf{S}, \mathbf{O}, \mathbf{G}_\gamma\}$ then define $M = \mathcal{S}^{-1} \in \mathbb{R}^{n \times n}$, if $\mathbf{K} = \mathbf{C}$, then define $M = \mathcal{S}^T$. Then
$$P \in \mathbf{K}(\Sigma_1) \iff M^T P M \in \mathbf{K}(\Sigma_2). \quad (13)$$
In particular, for any two minimal and equivalent LSSs Σ_1 and Σ_2, there exists a nonsingular matrix M such that (13) holds.

The theorem above means that quadratic stability and existence of controllability/observability grammians is preserved by minimization. In fact, if one of these properties holds for a state-space representation, then it holds for any state-space representation.

Corollary 1. For minimal LSSs, the singular values of grammians do not depend on the choice of state-space representation. Indeed, assume that Σ_1 is a minimal LSS, and consider the singular values $\sigma_1, \ldots, \sigma_n$ for a choice of grammians $(\mathcal{P}_1, \mathcal{Q}_1)$ of Σ_1. Then for any minimal LSS Σ_2 which is equivalent to Σ_1 there exists a pair of grammians

$(\mathcal{P}_2, \mathcal{Q}_2)$ of Σ_2 such that the singular values of $(\mathcal{P}_2, \mathcal{Q}_2)$ are also $\sigma_1, \ldots, \sigma_n$.

Corollary 2. For a LSS Σ, define $\gamma(\Sigma) = \inf\{\gamma > 0 \mid \mathbf{G}_\gamma(\Sigma) \ne \emptyset\}$. Then clearly the L_2 norm of the input-output map of Σ is at most $\gamma(\Sigma)$. From Theorem 3, we obtain that

- for any minimal LSS Σ_m which is equivalent to Σ, $\gamma(\Sigma_m) \le \gamma(\Sigma)$, and
- If $\Sigma_i, i = 1, 2$, are two minimal and equivalent LSSs, then $\gamma(\Sigma_1) = \gamma(\Sigma_2)$.

As a consequence, the number $\gamma(\Sigma)$, where Σ is minimal, depends only on the input-output map of Y^Σ. Note that $\gamma(\Sigma)$ can be computed by solving a classical optimization problem.

Proof. [Proof of Theorem 3] The proof of the second part of the theorem follows by an easy computation and by recalling that if Σ_1 and Σ_2 are two equivalent and minimal LSSs, then they are related by an LSS isomorphism.

Hence, it is enough to show that if $\mathbf{K}(\Sigma) \ne \emptyset$ and we apply Procedures 1–2 to obtain a minimal LSS Σ_m, then $\mathbf{K}(\Sigma_m) \ne \emptyset$.

To this end, define for all $q \in Q$, $P > 0$, $\mathbf{S}(q, \Sigma, P) = A_q^T P + P A_q$, $\mathbf{O}(q, \Sigma, P) = A_q^T P + P A_q + C_q^T C_q$, $\mathbf{C}(q, \Sigma, P) = A_q^T P + P A_q + P B_q B_q^T P$ and $\mathbf{G}_\gamma(q, \Sigma, P) = A_q^T P + P A_q + C_q^T C_q + \gamma P B_q B_q^T P$. For any choice of the symbol $\mathbf{K} \in \{\mathbf{S}, \mathbf{O}, \mathbf{G}_\gamma\}$, $P \in \mathbf{K}(\Sigma)$ if and only if for all $x \in \mathbb{R}^n$, $\forall q \in Q : x^T \mathbf{K}(q, \Sigma, P) x < 0$. By multiplying (10) by $\mathcal{P}^{-1}$ from left and right, we get that $\mathcal{P} \in \mathbf{C}(\Sigma)$ if and only if for any $x \in \mathbb{R}^n$, $\forall q \in Q : x^T \mathbf{C}(q, \Sigma, \mathcal{P}^{-1}) x < 0$.

First, we show that the application of Procedure 1 preserves the non-emptiness of $\mathbf{K}(\Sigma)$. Recall the partitioning of A_q from (7) and consider the corresponding partitioning of P
$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$
A simple computation reveals that for any $x_r \in \mathbb{R}^{n_r}$
$$\begin{bmatrix} x_r \\ 0 \end{bmatrix}^T \mathbf{K}(q, \Sigma, P) \begin{bmatrix} x_r \\ 0 \end{bmatrix} = x_r^T \mathbf{K}(q, \Sigma_r, P_{11}) x_r$$

Since x_r and $q \in Q$ are arbitrary, we obtain that $P \in \mathbf{K}(\Sigma)$ if and only if $P_{11} \in \mathbf{K}(\Sigma_r)$ for $\mathbf{K} \ne \mathbf{C}$ and $P^{-1} \in \mathbf{C}(\Sigma)$ if and only if $P_{11}^{-1} \in \mathbf{C}(\Sigma_r)$. Thus P_{11} is positive definite, if P is positive definite. As a consequence, we obtain that $\mathbf{K}(\Sigma_r) \ne \emptyset$.

Next, we show that Procedure 2 preserves non-emptiness of $\mathbf{K}(\Sigma)$. We will use the duality between observability and reachability. Define the dual system $\Sigma^T = (n, Q, \{(A_q^T, C_q^T, B_q^T) \mid q \in Q\})$. The following properties of the dual Σ^T hold

(1) For $P \in \mathbf{S}(\Sigma) \iff P^{-1} \in \mathbf{S}(\Sigma^T)$, $P \in \mathbf{G}_\gamma(\Sigma) \iff \frac{1}{\gamma}P^{-1} \in \mathbf{G}_\gamma(\Sigma^T)$, $\mathbf{C}(\Sigma) = \mathbf{O}(\Sigma^T)$, and $\mathbf{O}(\Sigma) = \mathbf{C}(\Sigma^T)$.

(2) If Σ_{rt} is the result of applying Procedure 1 to Σ^T, then $\Sigma_{rt}^T = \Sigma_o$, where Σ_o is the result of application of Procedure 2 to Σ.

As a consequence, $\mathbf{K}(\Sigma) \ne \emptyset$ if and only if $\mathbf{K}(\Sigma^T) \ne \emptyset$. Since Procedure 1 preserves non-emptiness of $\mathbf{K}(\Sigma^T)$, we

have that $\mathbf{K}(\Sigma_{\mathrm{rt}}) \neq \emptyset$, which implies that $\mathbf{K}(\Sigma_{\mathrm{rt}}^T) = \mathbf{K}(\Sigma_{\mathrm{o}}) \neq \emptyset$.

6. SYSTEM-THEORETIC INTERPRETATION OF GRAMMIANS AND THEIR SINGULAR VALUES

In this section, we try to provide a system theoretic interpretation of grammians and their singular value, by linking them to observability, reachability and Hankel-norms.

Theorem 4. If Σ is observable (resp. reachable), then any positive semi-definite solution to (9) (resp. (10)) is strictly positive definite.

The proof of Theorem 4 is based on the following results, which are interesting on their own right.

Lemma 2. Assume that $\mathcal{Q} \geq 0$ is a solution to (9). Then for all $q \in PC(T,Q)$, $t > 0$

$$x^T \mathcal{Q} x \geq \int_0^t ||Y_x^\Sigma(0,q)(s)||_2^2 ds$$

Lemma 3. Assume that $\mathcal{P} > 0$ is a solution to (10). Then for all $q \in PC(T,Q)$, $u \in PC(T,U)$, and $t > 0$

$$x^T \mathcal{P}^{-1} x \leq \int_0^t ||u(s)||_2^2 ds,$$

where $x = X_0^\Sigma(u,q)(t)$.

Proof. [Proof of Theorem 4] We prove the statement for the observability by contradiction. Assume that there exists $x \in \mathbb{R}^n \setminus \{0\}$ such that $x^T \mathcal{Q} x = 0$. By Lemma 2, this implies that for all $q \in PC(T,Q)$, the map $Y_x^\Sigma(0,q) = 0$ and thus $Y_x^\Sigma(0,q) = Y_0^\Sigma(0,q)$ for all q. Note that $Y_x^\Sigma(u,q) = Y_x^\Sigma(0,q) + Y_0^\Sigma(u,q)$, and hence we get that $Y_x^\Sigma(u,q) = Y_0^\Sigma(u,q)$ for all q and u, which contradicts the observability of Σ.
The statement for controllability grammian follows by duality.

Notice that unlike in the linear case, an LSS may fail to be observable (resp. reachable), even if (9) (resp. (10)) has a strictly positive definite solution, see Example 1 of Section 7. This is due to the fact that in (9) and (10), we require inequalities instead of equalities. As we shall see in Section 7, an side effect of this phenomenon is that the reduced order model obtained by balanced truncation may fail to be minimal.

As a consequence, one is tempted to ask the question what happens if in (9) (resp. in (10)) we require that for some or all $q \in Q$, $A_q^T \mathcal{Q} + \mathcal{Q} A_q + C_q^T C_q = 0$ (resp. $\mathcal{P} A_q^T + A_q \mathcal{P} + B_q B_q^T = 0$) holds with equality. In this case, the existence of a strictly positive definite solution to the equations will imply observability (resp. controllability) of some (or all) linear subsystems. However, in Remark 2 we already explained that for a large class of input-output maps, including those which are realizable by quadratically stable LSSs, such state-space representation do not exist. Hence, by replacing inequalities by equalities we necessarily restrict applicability of the model reduction approach.

Finally, we present the interpretation of the largest singular value of a grammian pair $(\mathcal{P}, \mathcal{Q})$ in terms of the Hankel-norm of the input-output map.

Definition 12. (Hankel-norm). Let Σ be a LSS and define the *Hankel-norm* $||Y^\Sigma||_H$ as follows. Let $\mathbf{HG}(\Sigma)$ be the set of all $\gamma > 0$ such that for all $u \in PC(T,U) \cap L_2(T,U)$,

$$\sup_{(q,t) \in PC(T,Q) \times T} \int_t^\infty ||Y_0^\Sigma(u \#_t 0, q)(s)||_2^2 ds \leq \gamma^2 ||u||_2^2,$$

where

$$u \#_t 0(s) = \begin{cases} u(s) & \text{for } s \in [0,t] \\ 0 & \text{for } s > t. \end{cases}$$

If $\mathbf{HG}(\Sigma) = \emptyset$, then define $||Y^\Sigma||_H = +\infty$, and if $\mathbf{HG}(\Sigma) \neq \emptyset$, then define $||Y^\Sigma||_H = \inf \mathbf{HG}(\Sigma)$.

Intuitively, the Hankel-norm of Y^Σ gives us the maximum output energy of the system, if we first feed in a continuous input u with unit energy and from some time t we stop feeding in continuous input and we let the system to develop autonomously.

Theorem 5. Assume that Σ is a LSS , $\mathcal{P} > 0$ is a controllability grammian and $\mathcal{Q} > 0$ is an observability grammian of Σ. The largest singular value σ_{max} of $(\mathcal{P}, \mathcal{Q})$ satisfies

$$||Y^\Sigma||_H \leq \sigma_{max}.$$

Proof. [Proof of Theorem 5] Pick a switching signal q an input u and a time instance t such that $u(s) = 0$ for all $s > t$ and $||u||_{L_2} \leq 1$. Denote by x and y the corresponding state and output trajectories. By combining Lemma 2 and Lemma 3, we obtain that $x(t)^T \mathcal{P}^{-1} x(t) \leq 1$ and

$$x^T(t) \mathcal{Q} x(t) \geq \int_t^\infty ||y(s)||_2^2 ds.$$

Since u, q and t are arbitrary, we then obtain that

$$\sup_{x^T \mathcal{P}^{-1} x \leq 1} x^T \mathcal{Q} x \geq ||Y^\Sigma||_H$$

We proceed to prove that

$$\sqrt{\lambda_{max}(\mathcal{P}\mathcal{Q})} = \sup_{x^T \mathcal{P}^{-1} x \leq 1} x^T \mathcal{Q} x$$

For Let $\mathcal{P}^{-1} = S^T S$, and define $\hat{\mathcal{Q}} = (S^{-1})^T \mathcal{Q} S^{-1}$. It follows that

$$\{Sx \mid x^T \mathcal{P}^{-1} x \leq 1\} = \{v \mid v^T v \leq 1\}.$$

Hence,

$$\sup_{x^T \mathcal{P}^{-1} x \leq 1} x^T \mathcal{Q} x = \sup_{v^T v \leq 1} v^T \hat{\mathcal{Q}} v = \sqrt{\lambda_{\max}(\hat{\mathcal{Q}})},$$

where λ_{max} is the maximal eiegenvalues of $\hat{\mathcal{Q}}$. But $\hat{\mathcal{Q}} = S\mathcal{P}\mathcal{Q}S^{-1}$, hence the eigenvalues of $\hat{\mathcal{Q}}$ and $\mathcal{P}\mathcal{Q}$ coincide.

7. MODEL REDUCTION FOR LINEAR SWITCHED SYSTEMS IN CONTINUOUS-TIME

In this section, we state formally the procedure for model reduction by balanced truncation, and we prove a bound of the approximation error.

Procedure 4. **Balanced truncation, Shaker and Wisniewski (2011)** Consider a LSS $\Sigma = (n, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$.

(1) Find a positive definite solution $\mathcal{Q} > 0$ to (9).
(2) Find a positive definite solution $\mathcal{P} > 0$ to (10).
(3) Find U such that $\mathcal{P} = UU^T$ and find K such that $U^T \mathcal{Q} U = K \Lambda K^T$, where Λ is diagonal with the

diagonal elements taken in decreasing order. Define the transformation
$$\mathcal{S} = \Lambda^{1/2} K^T U^{-1}$$

(4) Replace Σ with $\Sigma_{\text{bal}} = (n, Q, (\bar{A}_q = \mathcal{S} A_q \mathcal{S}^{-1}, \bar{B}_q = \mathcal{S} B_q, \bar{C}_q = C_q \mathcal{S}^{-1})_{q \in Q})$.

(5) The transformed system Σ_{bal} is balanced, i.e. for all $q \in Q$,
$$\begin{aligned} \bar{A}_q^T \Lambda + \Lambda \bar{A}_q^T + \bar{C}_q^T \bar{C}_q &< 0 \\ \bar{A}_q \Lambda + \Lambda \bar{A}_q^T + \bar{B}_q \bar{B}_q^T &< 0 \end{aligned} \tag{14}$$
Indeed, $\mathcal{S} P \mathcal{S}^T = \Lambda$ and $(\mathcal{S}^{-1})^T Q \mathcal{S}^{-1} = \Lambda$ and $\mathcal{S} P \mathcal{S}^T$ and $(\mathcal{S}^{-1})^T Q \mathcal{S}^{-1}$ satisfy (10) and (9) for Σ_{bal}.

(6) Assume that $\Lambda = \text{diag}(\sigma_1, \dots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Choose $r < n$ and let $\Lambda_1 = \text{diag}(\sigma_1, \dots, \sigma_r)$. Choose $\hat{A}_q \in \mathbb{R}^{r \times r}$, $\hat{B}_q \in \mathbb{R}^{r \times m}$ and $\hat{C}_q \in \mathbb{R}^{p \times r}$ so that
$$\bar{A}_q = \begin{bmatrix} \hat{A}_q & A_{q,12} \\ A_{q,21} & A_{q,22} \end{bmatrix}, \quad \bar{B}_q = \begin{bmatrix} \hat{B}_q \\ B_{q,2} \end{bmatrix}, \quad \bar{C}_q^T = \begin{bmatrix} \hat{C}_q^T \\ C_{q,2}^T \end{bmatrix} \tag{15}$$
Return as a reduced model $\hat{\Sigma} = (r, Q, \{(\hat{A}_q, \hat{B}_q, \hat{C}_q) \mid q \in Q\})$.

In the following, we will state an error bound for the difference between the input-output maps of Σ and $\hat{\Sigma}$. To this end, we will use the following simple fact.

Lemma 4. (Shaker and Wisniewski (2011)). The LSS $\hat{\Sigma}$ returned by Procedure 4 is balanced and quadratically stable.

One may wonder if the system $\hat{\Sigma}$ returned by Procedure 4 is minimal, at least when Σ was minimal. The answer is negative, as demonstrated by Example 1. The fact that the reduced system need not even be minimal already indicates that Procedure 4 might be too conservative.

Example 1. Assume $Q = \{1\}$ consists of one element,
$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1\ 1\ 0]. \text{ Then}$$
(A, B, C) is balanced according to our definition with $\Lambda = \text{diag}(2, 1, 0.5)$. However, $\hat{A} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{C} = [1\ 1]$, which is clearly not minimal.

Theorem 6. (Error bound). For the system $\hat{\Sigma}$ returned by Procedure 4,
$$\|Y^{\Sigma} - Y^{\hat{\Sigma}}\|_{L_2} \leq 2 \sum_{k=r+1}^{n} \sigma_k. \tag{16}$$

Proof. [Proof of Theorem 6] The proof of Theorem 6 is based in the following lemma.

Lemma 5. For $r = n-1$, (16) is true.

Proof. [Proof of Lemma 5] The proof is inspired by the PhD thesis Sandberg (2004). Without loss of generality, we assume that Σ is already balanced and hence $\Sigma_{\text{bal}} = \Sigma$. Assume that the balanced observability and controllability grammians are of the following form.
$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \beta \end{bmatrix}.$$
We will also use the notation of the partitioning in (15).

Fix an input u and a switching signal q and denote by $x(t)$ the corresponding state trajectory of Σ and by $\hat{x}(t)$ the corresponding state trajectory of the reduced order model $\hat{\Sigma}$. Consider the decomposition $x(t) = (x_1(t), x_2(t))$ where $x_1(t) \in \mathbb{R}^{n-1}$, and define
$$z(t) = A_{q(t),21} \hat{x}(t) + B_{q(t),2} u(t).$$
With this notation, consider the following vectors
$$X_c(t) = \begin{bmatrix} x_1(t) + \hat{x}(t) \\ x_2(t) \end{bmatrix}, \quad X_o(t) = \begin{bmatrix} x_1(t) - \hat{x}(t) \\ x_2(t) \end{bmatrix}.$$
An easy calculation reveals that
$$\begin{aligned} \dot{X}_c(t) &= A_{q(t)} X_c(t) - \begin{bmatrix} 0 \\ z(t) \end{bmatrix} + 2 B_{q(t)} u(t) \\ \dot{X}_o(t) &= A_{q(t)} X_o(t) + \begin{bmatrix} 0 \\ z(t) \end{bmatrix} \end{aligned}.$$
It follows that for all $0 < P = \begin{bmatrix} P_1 & 0 \\ 0 & \gamma \end{bmatrix}$, $0 < \gamma \in \mathbb{R}$, $P_1 \in \mathbb{R}^{(n-1) \times (n-1)}$,
$$\begin{aligned} \frac{d}{dt}(X_c(t)^T P X_c(t)) = 2 X_c(t)^T A_{q(t)}^T P X_c(t) + \\ 4 u^T(t) B_{q(t)}^T P X_c(t) - 2\gamma z(t)^T x_2(t) \end{aligned} \tag{17}$$
Similarly, the derivative of $X_o^T(t) P X_o(t)$ satisfies
$$\begin{aligned} \frac{d}{dt}(X_o(t)^T P X_o(t)) = \\ 2 X_o(t)^T A_{q(t)}^T P X_o(t) + 2\gamma z(t)^T x_2(t) \end{aligned} \tag{18}$$
Notice that (10) for $\mathcal{P} = \Lambda$ can be rewritten as
$$\forall q \in Q : A_q^T \Lambda^{-1} + A_q \Lambda^{-1} + \Lambda^{-1} B_q B_q^T \Lambda^{-1} < 0$$
by multiplying the original equation by Λ^{-1} from the left and from the right. In other words, for all $0 \neq x \in \mathbb{R}^n$,
$$\forall q \in Q : 2 x^T A_q^T \Lambda^{-1} x < -x^T \Lambda^{-1} B_q B_q^T \Lambda^{-1} x \tag{19}$$
Similarly, (9) is equivalent to
$$\forall q \in Q : 2 x^T A_q^T \Lambda x < -x^T C_q^T C_q x, \tag{20}$$
for all $0 \neq x \in \mathbb{R}^n$. Applying (19) to (17) with $P = \Lambda^{-1}$ and completing the squares yields
$$\frac{d}{dt}(X_c(t)^T \Lambda^{-1} X_c(t)) \leq -X_c(t)^T \Lambda^{-1} B_{q(t)} B_{q(t)}^T \Lambda^{-1} X_c(t) +$$
$$+ 4 u^T(t) B_{q(t)}^T \Lambda_1^{-1} X_c(t) - 2\frac{1}{\beta} z(t)^T x_2(t) \leq$$
$$- \|B_{q(t)}^T \Lambda^{-1} X_c(t) + 2u(t)\|^2 + 4\|u(t)\|^2 - 2\frac{1}{\beta} z(t)^T x_2(t) \leq$$
$$4\|u(t)\|^2 - 2\frac{1}{\beta} z(t)^T x_2(t).$$
By noticing that $X_c(0) = 0$, we get that
$$X_c(t)^T \Lambda^{-1} X_c(t) = \int_0^t \frac{d}{ds}(X_c(s)^T \Lambda^{-1} X_c(s)) ds \leq$$
$$4 \int_0^t \|u(s)\|^2 ds - 2 \int_0^t \frac{1}{\beta} z(s)^T x_2(s) ds$$
Since $X_c(t)^T \Lambda^{-1} X_c(t) \geq 0$,
$$\int_0^{\infty} z(s)^T x_2(s) ds \leq 2\beta \|u\|_2^2 \tag{21}$$
Similarly, applying (20) to (18) with $P = \Lambda$ implies
$$\frac{d}{dt}(X_o(t)^T P X_o(t)) \leq$$
$$- X_o(t)^T C_{q(t)}^T C_{q(t)} X_o(t) + 2\beta z(t)^T x_2(t)$$

which together with $X_o(0) = 0$ implies that

$$X_o(t)^T \Lambda X_o(t) = \int_0^t \frac{d}{ds}(X_o(s)^T \Lambda X_o(s))ds \leq$$

$$-\int_0^t X_o(s)^T C_{q(s)}^T C_{q(s)} X_o(s)ds + 2\beta \int_0^t z(s)^T x_2(s)ds. \tag{22}$$

Since $X_o(t)^T \Lambda X_o(t) \geq 0$, we get

$$\int_0^t X_o(s)^T C_{q(s)}^T C_{q(s)} X_o(s)ds \leq 2\beta \int_0^t z(s)^T x_2(s)ds.$$

(21) and taking the limit as $t \rightarrow +\infty$ yields

$$\int_0^\infty X_o(s)^T C_{q(s)}^T C_{q(s)} X_o(s)ds \leq 4\beta^2 ||u||_2^2. \tag{23}$$

Notice that $C_{q(t)} X_o(t) = y(t) - \hat{y}(t)$, where $y(t)$ is the output trajectory of Σ and $\hat{y}(t)$ is the output trajectory of $\hat{\Sigma}$. Hence, (21) is equivalent to

$$||y - \hat{y}||_2^2 \leq 4\beta^2 ||u||_2^2$$

From this the statement of the lemma follows.

The proof of Theorem 6 is based on Lemma 5 and goes as follows. Suppose that $\hat{\Sigma}_1$ is the reduced system obtained by removing the singular value σ_n. It is easy to see that $\hat{\Sigma}_1$ is again balanced with grammian Λ_1. We can again apply the model reduction procedure to $\hat{\Sigma}_1$, remove its smallest singular value σ_2 and obtain $\hat{\Sigma}_2$. Suppose that the balanced system $\hat{\Sigma}_i$ with grammian $\Lambda_i = \text{diag}(\sigma_1, \ldots, \sigma_{n-i})$ is given. Define $\hat{\Sigma}_{i+1}$ as the system which is obtained from $\hat{\Sigma}_i$ by applying the balanced truncation to the last state, i.e., to the state which corresponds to σ_{n-i}. In this way, we obtain systems $\hat{\Sigma}_1, \ldots, \hat{\Sigma}_{n-r}$ such that $\dim \hat{\Sigma}_i = n - i$ and $||Y_{i-1}^{\hat{\Sigma}} - Y_i^{\hat{\Sigma}}||_{L_2} \leq 2\sigma_{n-i+1}$, where $\hat{\Sigma}_0 = \Sigma$. Notice that $\hat{\Sigma}_{n-r} = \hat{\Sigma}$

$$||\Sigma - \hat{\Sigma}||_{L_2} \leq \sum_{i=1}^{n-r} ||\hat{\Sigma}_{i-1} - \hat{\Sigma}_i||_{L_2} \leq 2 \sum_{k=r+1}^{n} \sigma_k,$$

i.e., the error bound holds.

8. CONCLUSIONS

We have made the first steps towards the theoretical analysis of balanced truncation of linear switched systems. A great deal of questions still remain open. For example, it is not clear if error bounds could be derived for norms which are different than the L_2 norm. In fact, it is not at all evident that the L_2 norm is the natural choice for all the applications of switched systems. Another open question is how to extend balanced truncation to systems which do not admit a common Lyapunov function. Model reduction of unstable systems also remains a challenge, especially when it comes to deriving error bounds. Finally, the problem of model reduction for switched systems with autonomous switching is still open.

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