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S-AMP: Approximate Message Passing for General Matrix Ensembles

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02 November 2014, Hobart, Tasmania

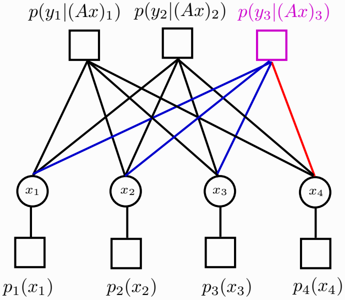
Motivation

- Low complexity and near optimal inference algorithms for
- linear observation models

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\epsilon} \ , \qquad \boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{\epsilon}|0, \sigma^2 \boldsymbol{I}) \ , \qquad \boldsymbol{x} \sim \prod_{k=1}^{K} p_k(x_k)$$

- $\boldsymbol{A}$: $N \times K$, known and drawn from known matrix ensemble
- $N, K \gg 1$

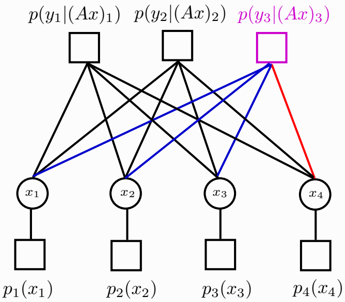
Loopy BP



$$m_{n \to k}(x_k) = \int p(y_n|(Ax)_n) \prod_{l \neq k} m_{l \to n}(x_l) dx_l$$

$$m_{l \to n}(x_l) \cong p(x_l) \prod_{m \neq n} m_{m \to l}(x_l)$$

Loopy BP



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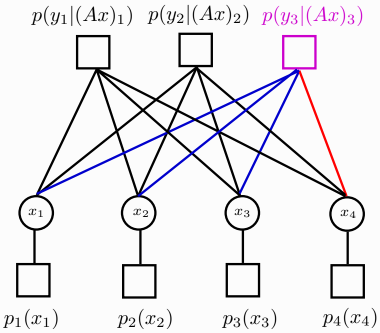
$$m_{l\to n}(x_l) \cong p(x_l) \prod_{m\neq n} m_{m\to l}(x_l)$$

- Local Cavity Argument:

$$h_{n,\setminus k} = \sum_{l\neq k} A_{nl} x_l, \quad x_l \sim m_{l\to n}(x_l)$$

- Due to CLT, $h_{n,\setminus k}$ is approximated by Gaussian.
- This leads Loop BP to Loopy EP

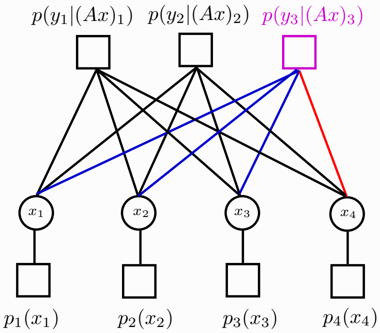
Loopy EP



$$m_{n\to k}(x_k) = \int p(y_n|(Ax)_n) \prod_{l\in\mathcal{K}\setminus k} m_{l\to n}(x_l) dx_l$$

$$m_{l\to n}(x_l) = \exp\left(-\frac{\lambda_{l\to n}}{2} x_l^2 + \gamma_{l\to n} x_l\right)$$

Loopy EP



$$m_{n\to k}(x_k) = \int p(y_n|(Ax)_n) \prod_{l\in\mathcal{K}\setminus k} m_{l\to n}(x_l) dx_l$$

$$m_{l\to n}(x_l) = \exp\left(-\frac{\lambda_{l\to n}}{2} x_l^2 + \gamma_{l\to n} x_l\right)$$

Define

$$q_k(x_k) \cong p_k(x_k) \prod_{n\in\mathcal{N}} m_{n\to k}(x_k)$$

Let $\tilde{q}_k(x_k) \triangleq N(x_k|\mu_k, \sigma_k^2)$ such that

$$\mu_k = \mathbb{E}[x_k|q(x_k)]$$

$$\sigma_k^2 = \mathbb{V}ar[x_k|q(x_k)]$$

Then loopy EP update rule is

$$m_{k\to n}(x_k) = \frac{\tilde{q}_k(x_k)}{\prod_{n\in\mathcal{N}} m_{n\to k}(x_k)}$$

Approximate message passing (AMP) [Donoho et al 2009]

- Assume A_{nk} zero mean-iid, $\overline{A_{nk}^2} = 1/N$, $N, K \to \infty$, $\alpha \equiv N/K$ finite
- Reduces the number of messages to $N + K$ means.

$$\boldsymbol{\mu}^{t+1} = \eta_t \left(\boldsymbol{A}^T \boldsymbol{z}^t + \boldsymbol{\mu}^t \right)$$

$$\boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{\mu}^t + \frac{1}{\alpha} \left\langle \eta'_{t-1}(\boldsymbol{A}^T \boldsymbol{z}^{t-1} + \boldsymbol{\mu}^{t-1}) \right\rangle \boldsymbol{z}^{t-1}$$

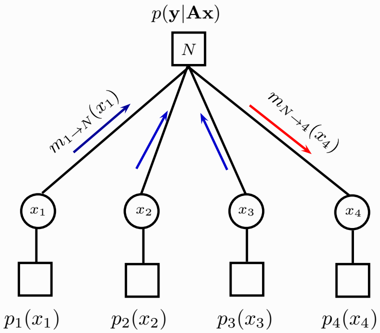
with $\langle \boldsymbol{u} \rangle \triangleq \sum_{k=1}^K u_k / K$.

- $\eta_t(\kappa_k)$ and $\eta'_t(\kappa_k)/\tau$ are (in some cases) the mean and variance of (Krzakala et al 2012)

$$q_k(x_k) \cong p_k(x_k) \mathcal{N}(x_k | \kappa_k, 1/\tau)$$

- Return to τ when discussing EP and S-AMP

EP [Minka 2001], [Opper and Winther 2000]



$$m_{N\to k}(x_k) = \int p(\boldsymbol{y}|\boldsymbol{Ax}) \prod_{l\in\mathcal{K}\setminus k} m_{l\to N}(x_l) dx_l$$

$$m_{k\to N}(x_k) = \exp\left(-\frac{\Lambda_{kk}}{2}x_k^2 + \gamma_k x_k\right)$$

Define

$$q_k(x_k) \cong p_k(x_k) m_{N\to k}(x_k)$$

Let $\tilde{q}_k(x_k) = N(x_k|\mu_k, \sigma_k^2)$ such that

$$\mu_k = \mathbb{E}[x_k|q_k(x_k)]$$
$$\sigma_k^2 = \mathbb{V}ar[x_k|q(x_k)]$$

Then EP update rule is

$$m_{k\to N}(x_k) = \frac{\tilde{q}_k(x_k)}{m_{N\to k}(x_k)}$$

S-AMP

- generalizes AMP for arbitrary (orthogonally invariant) matrix ensembles.

$$\boldsymbol{\mu}^{t+1} = \eta_t \left(\boldsymbol{A}^\dagger \boldsymbol{z}^t + \boldsymbol{\mu}^t \right)$$
$$\boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{\mu}^t + \left(1 - \frac{1}{\mathrm{s}_{\boldsymbol{A}}^{t-1}} \right) \boldsymbol{z}^{t-1}$$
$$\mathrm{s}_{\boldsymbol{A}}^{t-1} \triangleq \mathrm{S}_{\boldsymbol{A}} \left(- \left\langle \eta'_{t-1}(\boldsymbol{A}^\dagger \boldsymbol{z}^{t-1} + \boldsymbol{\mu}^{t-1}) \right\rangle \right)$$

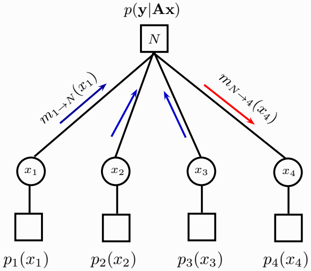
  $\mathrm{S}_{\boldsymbol{A}}$ denotes the S-transform (in free probability theory) of the limiting eigenvalue distribution (LED) of $\boldsymbol{A}^\dagger \boldsymbol{A}$.

- Indeed when the entries of $\boldsymbol{A}$ be iid with zero mean variance $1/N$:

$$\mathrm{S}_{\boldsymbol{A}}(\omega) = \frac{1}{1 + \omega/\alpha}$$

  which yields AMP iteration steps.

EP→S-AMP: Start with EP Update Rule



$$m_{N\to k}(x_k) = \int p(\boldsymbol{y}|\boldsymbol{Ax}) \prod_{l\in\mathcal{K}\setminus k} m_{l\to N}(x_l)dx_l$$

$$m_{k\to N}(x_k) = \exp\left(-\frac{\Lambda_{kk}}{2}x_k^2 + \gamma_k x_k\right)$$

$$q_k(x_k) \cong p_k(x_k)m_{N\to k}(x_k)$$

$$m_{N\to k}(x_k) = \tilde{q}_k(x_k)/m_{k\to N}(x_k)$$

- Let $\boldsymbol{J} = \boldsymbol{A}^\dagger\boldsymbol{A}/\sigma^2$ and $\boldsymbol{\theta} = \boldsymbol{A}^\dagger\boldsymbol{y}/\sigma^2$. Define

$$\boldsymbol{\Sigma} = (\boldsymbol{\Lambda} + \boldsymbol{J})^{-1} \quad \boldsymbol{\mu} = \boldsymbol{\Sigma}(\boldsymbol{\theta} + \gamma)$$

- Then we have

$$m_{N\to k}(x_k) = \exp\left\{-\frac{1}{2}\left(\frac{1}{\Sigma_{kk}} - \Lambda_{kk}\right)x_k^2 + \left(\frac{\mu_k}{\Sigma_{kk}} - \gamma_k\right)x_k\right\}$$

EP $\rightarrow$ S-AMP: Use AMP Notations

- Let $\tau_k = \frac{1}{\Sigma_{kk}} - \Lambda_{kk}$ and $\kappa_k = (\frac{\mu_k}{\Sigma_{kk}} - \gamma_k)/\tau_k$.
- Hence we can write

$$m_{N\rightarrow k}(x_k) \cong N(x_k|\kappa_k, 1/\tau_k)$$

- Write $q_k(x_k)$ in the form of

$$q_k(x_k) = \frac{p_k(x_k)N(x_k|\kappa_k, 1/\tau_k)}{Z(\kappa_k, \tau_k)}$$

- Define

$$\eta(\kappa_k;\tau_k) \triangleq \kappa_k + \frac{1}{\tau_k}\frac{\partial \log Z(\kappa_k, \tau_k)}{\partial \kappa_k}$$

$$\eta'(\kappa_k;\tau_k) \triangleq \frac{\partial \eta(\kappa_k;\tau_k)}{\partial \kappa_k}$$

where $\eta(\kappa_k;\tau_k)$ and $\eta'(\kappa_k;\tau_k)/\tau_k$ are respectively the mean and the variance of $q_k(x_k)$ [Krzakala et.al. 2012].

EP → S-AMP: Move to ADATAP [Opper and Winther 2001]

- Note that

$$\boldsymbol{\mu} = (\boldsymbol{\Lambda} + \boldsymbol{J})^{-1}(\gamma + \boldsymbol{\theta}) \iff (\boldsymbol{\Lambda} + \boldsymbol{J})\boldsymbol{\mu} = \gamma + \boldsymbol{\theta}$$

EP $\rightarrow$ S-AMP: Move to ADATAP [Opper and Winther 2001]

- Note that

$$\boldsymbol{\mu} = (\boldsymbol{\Lambda} + \boldsymbol{J})^{-1}(\gamma + \boldsymbol{\theta}) \iff (\boldsymbol{\Lambda} + \boldsymbol{J})\boldsymbol{\mu} = \gamma + \boldsymbol{\theta}$$

- Putting everything together leads EP to

$$\mu_k = \eta(\kappa_k; \tau_k)$$
$$\kappa_k = \frac{1}{\tau_k \sigma^2} \sum_{n \in \mathcal{N}} A_{nk} \left(y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l \right) + \mu_k$$
$$\tau_k = \frac{1}{\Sigma_{kk}} - \Lambda_{kk}, \quad \Lambda_{kk} = \frac{\tau_k}{\eta'(\kappa_k; \tau_k)} - \tau_k$$

exactly coincides ADATAP for the linear observation models.

EP→ S-AMP: Apply Adaptive Damping

Define

$$z_{n,k} \triangleq \frac{1}{\tau_k \sigma^2} \left(y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l \right).$$

Using this definition we “devise" the following identity:

$$z_{n,k} = y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l + (1 - \sigma^2 \tau_k) z_{n,k}.$$

EP→ S-AMP: Apply Adaptive Damping

Define

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Using this definition we “devise" the following identity:

$$z_{n,k} = y_n - \sum_{l \in \mathcal{K}} A_{nl}\mu_l + (1 - \sigma^2\tau_k)z_{n,k}.$$

Doing so leads to

$$\mu_k = \eta\left(\sum_{n \in \mathcal{N}} A_{nk}z_{n,k} + \mu_k; \tau_k\right)$$

$$z_{n,k} = y_n - \sum_{l \in \mathcal{K}} A_{nl}\mu_l + (1 - \sigma^2\tau_k)z_{n,k}$$

$$\tau_k = \frac{1}{\Sigma_{kk}} - \Lambda_{kk}, \quad \Lambda_{kk} = \frac{\tau_k}{\eta'(\kappa_k; \tau_k)} - \tau_k$$

This equations can be thought as a finite size interpretation of AMP.

EP$\rightarrow$ S-AMP: Invoke Self Averaging Ansatz [Opper and Winther 2001]

- We can recover self-averaging matrix ensembles $\tau_k \rightarrow \tau$:

$$\Sigma_{kk} = \left[(\boldsymbol{\Lambda} + \boldsymbol{J})^{-1} \right]_{kk} = \frac{\partial}{\partial \Lambda_{kk}} \ln \det(\boldsymbol{\Lambda} + \boldsymbol{J})$$

- by using

$$\frac{1}{K} \ln \det(\boldsymbol{\Lambda} + \boldsymbol{J}) \rightarrow \frac{1}{K} \mathbb{E}_{\boldsymbol{J}} \left[\ln \det(\boldsymbol{\Lambda} + \boldsymbol{J}) \right] \quad \text{for} \quad K \rightarrow \infty$$

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- Doing so leads τ_k to τ that is the solution of

$$\sigma^2 \tau = \frac{1}{\sigma^2} \mathrm{R}_{\boldsymbol{A}} \left(- \frac{< \eta'(\boldsymbol{A}^\dagger \boldsymbol{z} + \boldsymbol{\mu}; \tau) >}{\sigma^2 \tau} \right)$$

$\mathrm{R}_{\boldsymbol{A}}$ is the R-transform (in free probability theory) of the LED of $\boldsymbol{A}^\dagger \boldsymbol{A}$ and

$$\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{\mu} + (1 - \sigma^2 \tau)\boldsymbol{z}$$

EP$\to$ S-AMP: Move to S-transform

- Recall that

$$\sigma^2\tau = \mathrm{R}_{\boldsymbol{A}}\left(\frac{- < \eta'(\boldsymbol{A}^\dagger \boldsymbol{z} + \boldsymbol{\mu}; \tau) >}{\sigma^2\tau}\right)$$

- By invoking the fact [Haagerup and Larsen 2001]

$$\mathrm{S}_{\boldsymbol{A}}(\omega) = \frac{1}{\mathrm{R}_{\boldsymbol{A}}(\omega \mathrm{S}_{\boldsymbol{A}}(\omega))}$$

- we have

$$\sigma^2\tau = \frac{1}{\mathrm{S}_{\boldsymbol{A}}(- < \eta'(\boldsymbol{A}^\dagger \boldsymbol{z} + \boldsymbol{\mu}; \tau) >)}$$

EP→ S-AMP: Move to S-transform

- Recall that

$$\sigma^2\tau = \mathrm{R}_{\boldsymbol{A}}\left(\frac{-<\eta'(\boldsymbol{A}^\dagger \boldsymbol{z} + \boldsymbol{\mu};\tau)>}{\sigma^2\tau}\right)$$

- By invoking the fact [Haagerup and Larsen 2001]

$$\mathrm{S}_{\boldsymbol{A}}(\omega) = \frac{1}{\mathrm{R}_{\boldsymbol{A}}(\omega \mathrm{S}_{\boldsymbol{A}}(\omega))}$$

- we have

$$\sigma^2\tau = \frac{1}{\mathrm{S}_{\boldsymbol{A}}(-<\eta'(\boldsymbol{A}^\dagger \boldsymbol{z} + \boldsymbol{\mu};\tau)>)}$$

- this completes the mapping at "fixed points":

$$\boldsymbol{\mu} = \eta\left(\boldsymbol{A}^\dagger \boldsymbol{z} + \boldsymbol{\mu};\tau\right)$$

$$\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{\mu} + (1 - \frac{1}{\mathrm{s}_{\boldsymbol{A}}})\boldsymbol{z}$$

$$\mathrm{s}_{\boldsymbol{A}} = \mathrm{S}_{\boldsymbol{A}}\left(-\left\langle \eta'(\boldsymbol{A}^\dagger \boldsymbol{z} + \boldsymbol{\mu};\tau)\right\rangle\right)$$

What is S-AMP?

- In summary

$$\boldsymbol{\mu}^{t+1} = \eta_t\left(\boldsymbol{A}^\dagger \boldsymbol{z}^t + \boldsymbol{\mu}^t\right)$$

$$\boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{\mu}^t + \left(1 - \frac{1}{\mathrm{s}_{\boldsymbol{A}}^{t-1}}\right)\boldsymbol{z}^{t-1}$$

$$\mathrm{s}_{\boldsymbol{A}}^{t-1} \triangleq \mathrm{S}_{\boldsymbol{A}^\dagger \boldsymbol{A}}\left(-\left\langle \eta'_{t-1}(\boldsymbol{A}^\dagger \boldsymbol{z}^{t-1} + \boldsymbol{\mu}^{t-1})\right\rangle\right)$$

where $\eta_t(x^t) = \eta(x^t; \tau^t)$ and

$$\tau^t = \frac{1}{\sigma^2 \mathrm{S}_{\boldsymbol{A}}\left(-\left\langle \eta'(\boldsymbol{A}^\dagger \boldsymbol{z}^t + \boldsymbol{\mu}^t; \tau^t)\right\rangle\right)}$$

- Oops, S-AMP includes a hard fixed point equation.
- As a matter of fact we don't know what is the best update rule for τ^t

A Variant of S-AMP

- By making analogy with the state evolution formula [Bayati and Montari 2011]

$$\boldsymbol{\mu}^{t+1} = \eta\left(\boldsymbol{A}^{\dagger}\boldsymbol{z}^{t} + \boldsymbol{\mu}^{t}; \tilde{\tau}^{t}\right)$$

$$\boldsymbol{z}^{t} = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{\mu}^{t} + \left(1 - \frac{1}{\mathrm{s}_{\boldsymbol{A}}^{t-1}}\right)\boldsymbol{z}^{t-1}$$

$$\mathrm{s}_{\boldsymbol{A}}^{t-1} \triangleq \mathrm{S}_{\boldsymbol{A}}\left(-\left\langle \eta'(\boldsymbol{A}^{\dagger}\boldsymbol{z}^{t-1} + \boldsymbol{\mu}^{t-1}; \tilde{\tau}^{t-1})\right\rangle\right)$$

where $\tilde{\tau}^{t}$ is updated by using the solution

$$\tilde{\tau}^{t} = \frac{1}{\sigma^2 \mathrm{S}_{\boldsymbol{A}}\left(-\frac{\tilde{\tau}^{t}}{\tilde{\tau}^{t-1}}\left\langle \eta'(\boldsymbol{A}^{\dagger}\boldsymbol{z}^{t-1} + \boldsymbol{\mu}^{t-1}; \tilde{\tau}^{t-1}\right\rangle\right)}$$

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- i.e.

$$\tilde{\tau}^{t} = \frac{1}{\sigma^{2}}\mathrm{R}_{\boldsymbol{A}}\left(-\frac{\left\langle \eta'(\boldsymbol{A}^{\dagger}\boldsymbol{z}^{t-1} + \boldsymbol{\mu}^{t-1}; \tilde{\tau}^{t-1})\right\rangle}{\sigma^{2}\tilde{\tau}^{t-1}}\right)$$

Application: Row Orthogonal Ensembles in Compressed Sensing

- A random row orthogonal ensemble defined as

$$\boldsymbol{A} = \alpha^{-\frac{1}{2}} \boldsymbol{P}_\alpha \boldsymbol{O}, \quad \alpha \leq 1$$

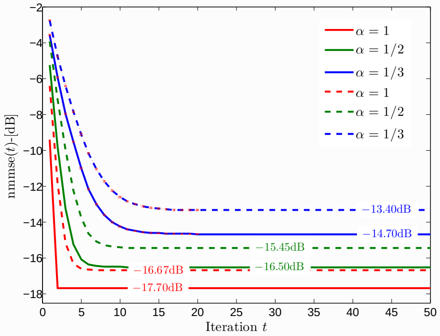
  where $\boldsymbol{P}_\alpha$ is the $N \times K$ matrix with entries $[\boldsymbol{P}_\alpha]_{ij} = \delta_{ij}, \forall ij$.
- In this case we have

$$\mathrm{S}_{\boldsymbol{A}}(z) = \frac{1+z}{1+z/\alpha}$$

$$\mathrm{R}_{\boldsymbol{A}}(z) = \frac{z - \alpha + \sqrt{(\alpha - z)^2 + 4\alpha^2 z}}{2\alpha z}$$

Simulation Results

- Let $p_k(x_k) = (1-\rho)\delta(x_k) + \rho N(x_k|0,1)$, with $\rho \in (0,1)$.
- For the closed-forms of $\eta_t(\cdot)$ and $\eta_t'(\cdot)$, see [Krzakala et.al. 2012].



- S-AMP for the row orthogonal matrix ensemble (solid curves) and the iid zero-mean ensemble (dashed curves).
- Confidence intervals (CIs) are also shown for $\alpha = 1/3$.
- We set $\sigma^2 = -20$ dB and $\rho = 0.1$, and $K = 1200$.
- The numbers in the plot are the predictions of replica theory [Kabashima and Vekapera 2014]