Aalborg Universitet

Distributed Remote Vector Gaussian Source Coding for Wireless Acoustic Sensor Networks

Zahedi, Adel; Østergaard, Jan; Jensen, Søren Holdt; Naylor, Patrick; Bech, Søren

Link to publication from Aalborg University

Distributed Remote Vector Gaussian Source Coding for Wireless Acoustic Sensor Networks

Adel Zahedi[*], Jan Østergaard[*], Søren Holdt Jensen[*], Patrick Naylor[†],
and Søren Bech[*‡]
[*]Department of Electronic Systems
Aalborg University, 9220 Aalborg, Denmark
Email: {adz, jo, shj, sbe}@es.aau.dk
[†]Electrical and Electronic Engineering Department
London Imperial College, London SW7 2AZ, UK
Email: p.naylor@imperial.ac.uk
[‡]Bang & Olufsen
7600 Struer, Denmark

Abstract

In this paper, we consider the problem of remote vector Gaussian source coding for a wireless acoustic sensor network. Each node receives messages from multiple nodes in the network and decodes these messages using its own measurement of the sound field as side information. The node's measurement and the estimates of the source resulting from decoding the received messages are then jointly encoded and transmitted to a neighboring node in the network. We show that for this distributed source coding scenario, one can encode a so-called conditional sufficient statistic of the sources instead of jointly encoding multiple sources. We focus on the case where node measurements are in form of noisy linearly mixed combinations of the sources and the acoustic channel mixing matrices are invertible. For this problem, we derive the rate-distortion function for vector Gaussian sources and under covariance distortion constraints.

I. INTRODUCTION

A Wireless Acoustic Sensor Network (WASN) is a set of wireless microphones equipped with some communication, signal processing, and possibly memory units, which are randomly scattered in the environment. The communication unit allows a sensor node to communicate with a base station and also with other nodes, and the signal processing unit enables a node to perform local processing. The random structure of the network removes the array-size limitations imposed by classical microphone arrays, thereby providing better performance in sense of high-SNR spatio-temporal sampling of the sound field and spatial diversity. The interested reader is referred to [1] for a review of WASNs.

Due to strict power and bandwidth constraints on wireless microphones, it might not be possible for nodes which are far from the base station to directly deliver their messages. A multi-hop scenario where the message goes along adjacent nodes until reaching the base-station is a commonplace alternative. We assume that a node is supposed to receive messages from multiple neighboring nodes and then combine them with its own measurement of the sound field into a new message to be forwarded through the network. This is illustrated in Fig.1.

In [2], this problem was solved for scalar Gaussian sources in a non-distributed setting i.e., without making use of the availability of the nodes' measurements as side information. It was shown that coding at intermediate nodes can result in significant gains in terms of sum-rate or distortion. In this paper, we consider vector Gaussian sources and also take into account the fact that messages received by a node are correlated with the node's measurement of the sound field. Thus, we consider a distributed scenario, and make use of the destination node's measurement as side information to decrease the required rate for transmission to each node [3], [4], [5].

We derive the rate-distortion (RD) function for an arbitrary node in the network with a distortion constraint defined in form of a covariance matrix, cf. [6], [7]. In Section II, we introduce the notation and formulate the problem, which turns out to involve joint coding of multiple sources. In Section III, we derive a conditional sufficient statistic for multiple Gaussian sources and show that for the above-mentioned distributed source coding (DSC) problem, one can encode a conditional sufficient statistic instead of joint encoding of multiple sources. The RD function for the resulting problem will be derived in Section IV. The paper is concluded in Section V.

II. NOTATION AND PROBLEM FORMULATION

We denote by $I(\cdot;\cdot), h(\cdot)$, and $E[\cdot]$ the information theoretic operations of mutual information, differential entropy, and expectation, respectively. Probability density functions are denoted by $f(\cdot)$ and covariance and cross-covariance matrices are denoted by Σ. We assume that all covariance matrices are of full rank. We denote Markov chains by two-headed arrows; e.g. $\mathbf{u} \leftrightarrow \mathbf{v} \leftrightarrow \mathbf{w}$. We assume that the source generates independent Gaussian vectors of length n denoted by $\mathbf{x} \in \mathbb{R}^n$. While the vectors are independent, the components of each vector are correlated. Node j makes a noisy measurement $\mathbf{y}_j$ of the source given by:

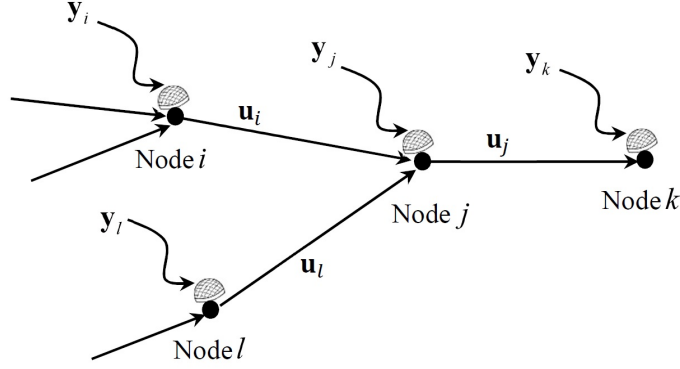$$\mathbf{y}_j = \mathbf{A}_j \mathbf{x} + \mathbf{n}_j, \tag{1}$$

Figure 1: A schematic graph of the WASN

where $\mathbf{n}_j$ is the additive Gaussian noise at node j and the matrix $\mathbf{A}_j$ models the mixing effect of the acoustic channel. The noise is assumed to be independent of $\mathbf{x}$. Node j also receives messages from nodes $i, l, ...$ denoted by $\mathbf{u}_i, \mathbf{u}_l, ...$, from which it can make estimations $\hat{\mathbf{x}}_i, \hat{\mathbf{x}}_l, ...$ of the source by decoding the messages with $\mathbf{y}_j$ as the decoder side information. The problem is to find the minimum rate R_j to jointly encode $\mathbf{y}_j, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}_l, ...$ into a message $\mathbf{u}_j$ to be sent to node k, while satisfying the given distortion constraint $\mathbf{D}_j$ and considering $\mathbf{y}_k$ as side information. This is illustrated in Figs.1 and 2.

Note that one could further decrease the network sum-rate by taking into account the correlation between the messages which are sent to a common node. However, we leave out this possibility in this paper.

Throughout this work, we assume that the acoustic mixing matrices are fixed and known. We also assume that joint statistics of the source and the noise variances are available. Finally, although the model in (1) is appropriate for acoustic networks, we do not consider real acoustic signals in this work. Instead, we consider Gaussian sources for simplicity of mathematical analysis. The case of real audio measurements is the focus of our future work.

III. CONDITIONAL SUFFICIENT STATISTICS

Assume that $\mathbf{y}$ is a random vector or a collection of random vectors with probability density function $f(\mathbf{y})$.

Definition 3.1: A sufficient statistic for the estimation of $\mathbf{x}$ from $\mathbf{y}$ is a function $T(\mathbf{y})$ of $\mathbf{y}$ for which $f(\mathbf{y}|T(\mathbf{y}), \mathbf{x})$ is not a function of $\mathbf{x}$. In other words, $\mathbf{y} \leftrightarrow T(\mathbf{y}) \leftrightarrow \mathbf{x}$.

Theorem 3.1 (Neyman-Fisher Factorization Theorem [8]): $T(\mathbf{y})$ is a sufficient statistic of $\mathbf{y}$ for estimating $\mathbf{x}$ if and only if $f(\mathbf{y}|\mathbf{x})$ can be factorized as:

$$f(\mathbf{y}|\mathbf{x}) = p(\mathbf{y}) q(T(\mathbf{y}), \mathbf{x}), \qquad (2)$$

where p (depending on $\mathbf{y}$ only) and q (depending on $\mathbf{x}$ and $T(\mathbf{y})$ but not on $\mathbf{y}$ directly) are nonnegative functions.
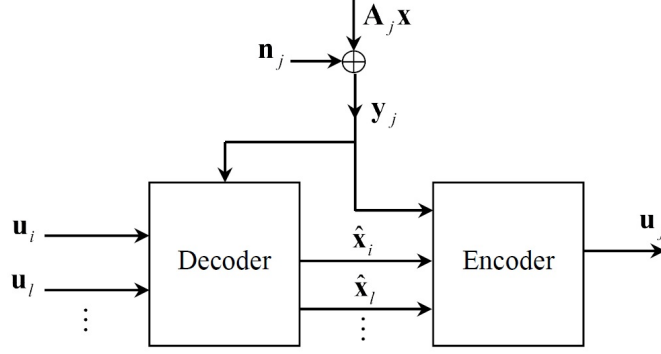
Figure 2: Node j of the network

The Factorization Theorem enables us to find a sufficient statistic for e.g. random vectors in Gaussian noise as shown in the following lemma[1].

Lemma 3.1: If $\mathbf{y}_j$; $j = 1, 2, ..., N$ are measurements of a random vector $\mathbf{x}$ in mutually independent Gaussian noises as in (1), then $T(\mathbf{y}_1, ..., \mathbf{y}_N)$ is a sufficient statistic of $\mathbf{y}_j$ for estimating $\mathbf{x}$, where

$$T(\mathbf{y}_1, ..., \mathbf{y}_N) = \sum_{j=1}^{N} \mathbf{A}_j^T \mathbf{\Sigma}_j^{-1} \mathbf{y}_j, \tag{3}$$

and $\mathbf{\Sigma}_j$ is the covariance matrix of the noise $\mathbf{n}_j$.

Proof: We prove the lemma for $N = 2$. The proof for the general case is similar.

Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are independent, the joint conditional density function of $\mathbf{y}_1$ and $\mathbf{y}_2$ can be written as:

$$f(\mathbf{y}_1, \mathbf{y}_2|\mathbf{x}) = \alpha \exp\left(-\frac{1}{2}\varphi\right), \tag{4}$$

where α is independent of $\mathbf{x}$, $\mathbf{y}_1$ and $\mathbf{y}_2$, and φ is defined as:

$$\varphi = (\mathbf{y}_1 - \mathbf{x})^T \mathbf{\Sigma}_1^{-1} (\mathbf{y}_1 - \mathbf{x}) + (\mathbf{y}_2 - \mathbf{x})^T \mathbf{\Sigma}_2^{-1} (\mathbf{y}_2 - \mathbf{x}). \tag{5}$$

Expanding φ, rearranging the terms, and substituting in (4), leads to

$$f(\mathbf{y}_1, \mathbf{y}_2|\mathbf{x}) = \alpha \exp\left(-\frac{\mathbf{y}_1^T \mathbf{\Sigma}_1^{-1} \mathbf{y}_1 + \mathbf{y}_2^T \mathbf{\Sigma}_2^{-1} \mathbf{y}_2}{2}\right.$$

---

[1]To the best of our knowledge, no proof for this lemma appears in the literature. We provide a proof here for completeness.

$$\times \exp\left(-\frac{\mathbf{x}^T\left(\boldsymbol{\Sigma}_1^{-1} + \boldsymbol{\Sigma}_2^{-1}\right)\mathbf{x}}{2}\right)\exp\left(\frac{T^T\mathbf{x} + \mathbf{x}^T T}{2}\right), \tag{6}$$

which shows that T is a sufficient statistic according to the Factorization Theorem.

∎

To make the notion of sufficient statistics applicable to our DSC problem, we need to introduce a conditional version:

Definition 3.2: A conditional sufficient statistic for the estimation of $\mathbf{x}$ from $\mathbf{y}$ given $\mathbf{z}$ is a function $T(\mathbf{y})$ of $\mathbf{y}$ for which $f(\mathbf{y}|\mathbf{T}(\mathbf{y}), \mathbf{x}, \mathbf{z})$ is not a function of $\mathbf{x}$.

Lemma 3.2: If $\mathbf{y} \leftrightarrow \mathbf{x} \leftrightarrow \mathbf{z}$, then any sufficient statistic $T(\mathbf{y})$ of $\mathbf{y}$ for estimating $\mathbf{x}$ is also a conditional sufficient statistic of $\mathbf{y}$ for estimating $\mathbf{x}$ given $\mathbf{z}$.

Proof:

$$
\begin{aligned}
f\left(\mathbf{y}|T\left(\mathbf{y}\right), \mathbf{x}, \mathbf{z}\right) \quad &= \frac{f\left(\mathbf{y}|\mathbf{x}, \mathbf{z}\right)}{f\left(T\left(\mathbf{y}\right)|\mathbf{x}, \mathbf{z}\right)} \\
&= \frac{f\left(\mathbf{y}|\mathbf{x}\right)}{f\left(T\left(\mathbf{y}\right)|\mathbf{x}\right)} \\
&= f\left(\mathbf{y}|T\left(\mathbf{y}\right), \mathbf{x}\right) \\
&= f\left(\mathbf{y}|T\left(\mathbf{y}\right)\right)
\end{aligned}
\tag{7}
$$

which is independent of $\mathbf{x}$.

∎

Lemma 3.3: Given side information $\mathbf{z}$ at the decoder, there is no loss in terms of rate and distortion by encoding a conditional sufficient statistic of $\mathbf{y}_1, ..., \mathbf{y}_N$ given $\mathbf{z}$ instead of joint encoding of these sources.

Proof: The proof follows along the lines of the proof for the unconditional case presented in [9], and is therefore omitted.

∎

Combining the results of this section, we have the following theorem for the problem formulated in Section II:

Theorem 3.2: Given side information $\mathbf{y}_k$ at the decoder, the RD function for the problem of joint encoding of multiple sources $\mathbf{y}_j, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}_l, ...$ coincides with the RD function for the DSC problem of encoding the single source:

$$T_j = \mathbf{A}_j^T\boldsymbol{\Sigma}_j^{-1}\mathbf{y}_j + \mathbf{D}_i^{-1}\hat{\mathbf{x}}_i + \mathbf{D}_l^{-1}\hat{\mathbf{x}}_l + ... \tag{8}$$

where $\mathbf{D}_i$ is the distortion matrix for node i.

Proof: Using the backward channel model we have $\mathbf{x} = \hat{\mathbf{x}}_i + \mathbf{q}_i$ where the covariance matrix of $\mathbf{q}_i$ is $\mathbf{D}_i$. This means that $\hat{\mathbf{x}}_i$ can be written as:

$$\hat{\mathbf{x}}_i = \mathbf{H}_i \mathbf{x} + \eta_i, \tag{9}$$

where

$$\mathbf{H}_i = (\boldsymbol{\Sigma}_x - \mathbf{D}_i) \, \boldsymbol{\Sigma}_x^{-1}, \tag{10}$$

$$\boldsymbol{\Sigma}_{\eta_i} = (\boldsymbol{\Sigma}_x - \mathbf{D}_i) \, \boldsymbol{\Sigma}_x^{-1} \mathbf{D}_i. \tag{11}$$

Substituting (10) and (11) in (9) and using (3) yields (8). Since T_j is a sufficient statistic of $\mathbf{y}_j, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}_l, ...$ and $\mathbf{y}_j, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}_l, ... \leftrightarrow \mathbf{x} \leftrightarrow \mathbf{y}_k$, it follows from Lemma 3.2 that it is also a conditional sufficient statistic given $\mathbf{y}_k$. From Lemma 3.3, one can then replace $\mathbf{y}_j, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}_l, ...$ by T_j in (8) and get the same RD function.

∎

At this point, we have shown that the above problem with multiple sources can be converted into a single source DSC problem for Gaussian sources with covariance distortion constraint. This problem is illustrated in Fig.3. For the case of mean-squared error distortion constraint, the RD function was found in [10], while the case of covariance distortion was not treated in that work. In the next section, we derive the RD function for the covariance distortion constraint under some mild technical assumptions.

IV.   RATE-DISTORTION FUNCTION

For the sake of simplicity of derivations, we write $\mathbf{x}$ and T_j in terms of their linear estimations based on the known Gaussian vectors in Fig.3. In particular, we have that

$$T_j = \mathbf{A}\mathbf{y}_k + \mathbf{B}\mathbf{u}_j + v_1, \tag{12}$$

$$\mathbf{x} = \mathbf{C}T_j + \mathbf{G}\mathbf{y}_k + v_2 = (\mathbf{G} + \mathbf{C}\mathbf{A}) \, \mathbf{y}_k + \mathbf{C}\mathbf{B}\mathbf{u}_j + \mathbf{C}v_1 + v_2, \tag{13}$$

$$T_j = \boldsymbol{\Gamma}\mathbf{y}_k + v_3, \tag{14}$$

where v_1, v_2, v_3 are estimation errors with covariance matrices $\boldsymbol{\Sigma}_{T_j|\mathbf{y}_k,\mathbf{u}_j}, \boldsymbol{\Sigma}_{\mathbf{x}|T_j,\mathbf{y}_k}, \boldsymbol{\Sigma}_{T_j|\mathbf{y}_k}$, respectively, and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{G}$ and $\boldsymbol{\Gamma}$ depend only on the covariance and cross-covariance matrices of $\mathbf{x}, T_j, \mathbf{y}_k$ and $\mathbf{u}_j$. (See Appendix for the mathematical statement for $\mathbf{C}$ as an example.) We will show that if the mixing matrices $\mathbf{A}_j$ in (1) are invertible[2], then for $\boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k} \prec \mathbf{D}_j \preceq \boldsymbol{\Sigma}_{x|\mathbf{y}_k}$ the RD function for node j is given by:

$$R_j \left(\mathbf{D}_j \right) = \frac{1}{2} \log \left(\frac{\left| \boldsymbol{\Sigma}_{x|\mathbf{y}_k} - \boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k} \right|}{\left| \mathbf{D}_j - \boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k} \right|} \right). \tag{15}$$

---

[2]The mixing matrices are actually non-square and thus non-invertible. In this work we consider a square approximation. The general non-square case will be considered in our future work.
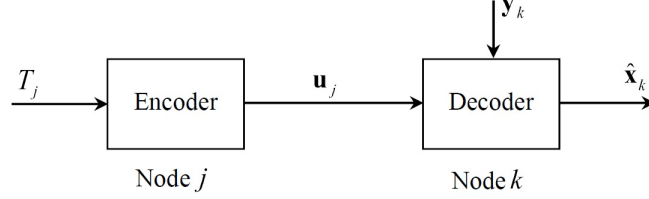
Figure 3: The simplified DSC problem

A. Lower Bound

First we need the following lemma:

Lemma 4.1: If the mixing matrices $\mathbf{A}_j, j = 1, 2, ...,$ are invertible, the matrix $\mathbf{C}$ in (13) is also invertible.

Proof: See the appendix. ∎

From the first equality in (13) we have:

$$\mathbf{C}\boldsymbol{\Sigma}_{T_j|\mathbf{y}_k}\mathbf{C}^T = \boldsymbol{\Sigma}_{x|\mathbf{y}_k} - \boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k}. \tag{16}$$

From the second equaltiy in (13) we can write:

$$\mathbf{x} - \hat{\mathbf{x}}_k = \mathbf{C}\upsilon_1 + \upsilon_2. \tag{17}$$

Assume now that the sequence of independent vectors $\mathbf{x}_i$ generated by the source is encoded in block vectors $\mathbf{X}$ each containing N vectors. From the distortion constraint and (17) we can write:

$$\mathbf{D}_j \succeq \frac{1}{N}\sum_{i=1}^{N} E\left[\left(\mathbf{x}_i - \hat{\mathbf{x}}_k^i\right)\left(\mathbf{x}_i - \hat{\mathbf{x}}_k^i\right)^T\right] = \mathbf{C}\boldsymbol{\Sigma}_{\upsilon_1}\mathbf{C}^T + \boldsymbol{\Sigma}_{\upsilon_2} \tag{18}$$

or equivalently:

$$\mathbf{C}\boldsymbol{\Sigma}_{T_j|\mathbf{u}_j,\mathbf{y}_k}\mathbf{C}^T \preceq \mathbf{D}_j - \boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k}. \tag{19}$$

Denoting the blocks of N vectors $T_j^i, \mathbf{y}_k^i, \mathbf{u}_j^i; \; i = 1, ..., N$ by $\mathbf{T}_j, \mathbf{Y}_k, \mathbf{U}_j$, respectively, we have:

$$
\begin{aligned}
NR_j\left(D_j\right) \quad &\geq I\left(\mathbf{T}_j; \mathbf{U}_j|\mathbf{Y}_k\right) \\
&= h\left(\mathbf{T}_j|\mathbf{Y}_k\right) - h\left(\mathbf{T}_j|\mathbf{U}_j, \mathbf{Y}_k\right) \\
&= \sum_{i=1}^{N} h\left(T_j^i|\mathbf{y}_k^i\right) - h\left(T_j^i|\mathbf{T}_j^{i-1}, \mathbf{u}_j^i, \mathbf{y}_k^i\right)
\end{aligned}
$$

$$\geq \sum_{i=1}^{N} h\left(T_j^i | \mathbf{y}_k^i\right) - h\left(T_j^i | \mathbf{u}_j^i, \mathbf{y}_k^i\right) \tag{20}$$

$$= \frac{N}{2} \log\left(\frac{|\boldsymbol{\Sigma}_{T_j|\mathbf{y}_k}|}{|\boldsymbol{\Sigma}_{T_j|\mathbf{u}_j,\mathbf{y}_k}|}\right)$$

$$= \frac{N}{2} \log\left(\frac{|\mathbf{C}\boldsymbol{\Sigma}_{T_j|\mathbf{y}_k}\mathbf{C}^T|}{|\mathbf{C}\boldsymbol{\Sigma}_{T_j|\mathbf{u}_j,\mathbf{y}_k}\mathbf{C}^T|}\right) \tag{21}$$

$$\geq \frac{N}{2} \log\left(\frac{|\boldsymbol{\Sigma}_{x|\mathbf{y}_k} - \boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k}|}{|\mathbf{D}_j - \boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k}|}\right), \tag{22}$$

where the block vector $\mathbf{T}_j^{i-1}$ contains $i-1$ vectors $T_j^1, ..., T_j^{i-1}$, (20) is because conditioning reduces the entropy, (21) is because $\mathbf{C}$ is invertible, and (22) is the result of applying (16) and (19) to (21).

B. Upper Bound

The lower bound derived in the previous part can be used as a guideline for the best possible performance for any coding scheme. The question is then, given the source T_j, how to encode it into a discrete source $\mathbf{u}_j$, so that for the given distortion constraint, the required rate for transmission of $\mathbf{u}_j$ achieves the lower bound?

Let us assume that T_j is quantized to $\mathbf{u}_j$, in a way that satisfies the distortion constraint. From the results of DSC, due to the availability of the side information $\mathbf{y}_k$ at the decoder, it is possible to noiselessly encode $\mathbf{u}_j$ in blocks of sufficiently large length N with a rate arbitrarily close to $H(\mathbf{u}_j|\mathbf{y}_k) = I\left(T_j; \mathbf{u}_j|\mathbf{y}_k\right)$ [3], [4], [5]. The remaining task is to design a scheme for which $I\left(T_j; \mathbf{u}_j|\mathbf{y}_k\right)$ achieves (15). This is possible by using the following scheme:

$$\mathbf{u}_j = \mathbf{U}\mathbf{C}T_j + \nu_j, \tag{23}$$

where we have denoted the eigenvalue decomposition of $\left(\boldsymbol{\Sigma}_{x|\mathbf{y}_k} - \boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k}\right)$ by $\mathbf{U}^T\boldsymbol{\Lambda}\mathbf{U}$, and the covariance $\boldsymbol{\Sigma}_{\nu_j}$ of the coding noise ν_j is defined as [3]:

$$\boldsymbol{\Sigma}_{\nu_j} = \mathbf{U}\left(\boldsymbol{\Sigma}_{x|\mathbf{y}_k} - \boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k}\right)\left(\boldsymbol{\Sigma}_{x|\mathbf{y}_k} - \mathbf{D}_j\right)^{-1}\left(\mathbf{D}_j - \boldsymbol{\Sigma}_{x|T_j,\mathbf{y}_k}\right)\mathbf{U}^T, \tag{24}$$

To verify this, one can write $I\left(T_j; \mathbf{u}_j|\mathbf{y}_k\right)$ as $h\left(\mathbf{u}_j|\mathbf{y}_k\right) - h\left(\mathbf{u}_j|T_j, \mathbf{y}_k\right)$, substitute (23)–(24), and utilize (14) and (16).

It is worth noting that the RD function (15) generalizes the RD function of [11], which treated the scalar case.

---

[3] Note that it is a symmetric positive definite matrix.

V. Conclusions, Discussions, and Future Work

We showed that the rate-distortion function for a distributed source coding problem with multiple sources at the encoder is identical to the rate-distortion function for the distributed encoding of a so-called conditional sufficient statistic of the sources. We derived a conditional sufficient statistic for the case that additive noises are Gaussian and mutually independent, and calculated the rate-distortion function in case that the sources are vector Gaussian and the distortion constraint is defined as a covariance matrix. Since vector sources were considered in order to take the memory into account, and the covariance constraint on the distortion is a more flexible fidelity criterion compared to mean-squared error, these results can be applied to the problem of source coding for audio signals in presence of reverberation, which will be the scope of our future work.

Appendix

Assume that the Gaussian vector $\mathbf{x}$ is estimated using two other Gaussian vectors $\mathbf{y}$ and $\mathbf{z}$ as:

$$\mathbf{x} = \mathbf{Cy} + \mathbf{Gz} + \mathbf{n}, \tag{25}$$

where the covariance matrix of the estimation error $\mathbf{n}$ is $\Sigma_n = \Sigma_{x|yz}$, and $\mathbf{y}$ and $\mathbf{z}$ are related to $\mathbf{x}$ via $\mathbf{y} = \mathbf{Ax} + \mathbf{n_1}$ and $\mathbf{z} = \mathbf{Bx} + \mathbf{n_2}$, and $\mathbf{A}$ and $\mathbf{B}$ are invertible. Therefore we have:

$$\Sigma_{xy} = \Sigma_x \mathbf{A}^T, \tag{26}$$

$$\Sigma_y = \mathbf{A}\Sigma_x \mathbf{A}^T + \Sigma_{n_1}, \tag{27}$$

$$\Sigma_{yz} = \mathbf{A}\Sigma_x \mathbf{B}^T, \tag{28}$$

$$\Sigma_{xz} = \Sigma_x \mathbf{B}^T. \tag{29}$$

From results in linear estimation theory, we obtain the following expression for $\mathbf{C}$ [12]:

$$\mathbf{C}\Sigma_y = \Sigma_{xy} + \left(\Sigma_{xy}\Sigma_y^{-1}\Sigma_{yz} - \Sigma_{xz}\right)\Delta^{-1}\Sigma_{yz}^T, \tag{30}$$

$$\Delta = \Sigma_z - \Sigma_{yz}^T\Sigma_y^{-1}\Sigma_{yz}. \tag{31}$$

The goal is to prove that $\mathbf{C}$ or equivalently $\mathbf{C}\Sigma_y$ is invertible. Substituting (26)–(29) in (30) yields:

$$
\begin{aligned}
\mathbf{C}\Sigma_y \quad &= \Sigma_x\mathbf{A}^T + \left(\Sigma_x\mathbf{A}^T\Sigma_y^{-1}\mathbf{A}\Sigma_x\mathbf{B}^T - \Sigma_x\mathbf{B}^T\right)\Delta^{-1}\Sigma_{yz}^T \\
&= \Sigma_x\mathbf{A}^T + \Sigma_x\left(\mathbf{A}^T\Sigma_y^{-1}\mathbf{A}\Sigma_x - \mathbf{I}\right)\mathbf{B}^T\Delta^{-1}\Sigma_{yz}^T \\
&= \Sigma_x\mathbf{A}^T + \Sigma_x\mathbf{A}^T\left(\Sigma_y^{-1} - \mathbf{A}^{-T}\Sigma_x^{-1}\mathbf{A}^{-1}\right)\mathbf{A}\Sigma_x\mathbf{B}^T\Delta^{-1}\Sigma_{yz}^T \\
&= \Sigma_x\mathbf{A}^T + \Sigma_x\mathbf{A}^T\left[\left(\mathbf{A}\Sigma_x\mathbf{A}^T + \Sigma_{n_1}\right)^{-1} - \left(\mathbf{A}\Sigma_x\mathbf{A}^T\right)^{-1}\right]\Sigma_{yz}\Delta^{-1}\Sigma_{yz}^T \\
&= \Sigma_x\mathbf{A}^T\left(\mathbf{I} + \mathbf{H}^{-1}\Sigma_{yz}\Delta^{-1}\Sigma_{yz}^T\right),
\end{aligned}
\tag{32}
$$

where $\mathbf{H}$ is defined as:

$$\mathbf{H}^{-1} = \left(\mathbf{A}\boldsymbol{\Sigma}_x\mathbf{A}^T + \boldsymbol{\Sigma}_{n_1}\right)^{-1} - \left(\mathbf{A}\boldsymbol{\Sigma}_x\mathbf{A}^T\right)^{-1}. \tag{33}$$

From (31) and (26)–(29) we have:

$$\begin{aligned}
\boldsymbol{\Delta} \quad &= \mathbf{B}\boldsymbol{\Sigma}_x\mathbf{B}^T + \boldsymbol{\Sigma}_{n_2} - \mathbf{B}\boldsymbol{\Sigma}_x\mathbf{A}^T\left(\mathbf{A}\boldsymbol{\Sigma}_x\mathbf{A}^T + \boldsymbol{\Sigma}_{n_1}\right)^{-1}\mathbf{A}\boldsymbol{\Sigma}_x\mathbf{B}^T \\
&= \mathbf{B}\boldsymbol{\Sigma}_x\left[\mathbf{I} - \mathbf{A}^T\left(\mathbf{A}\boldsymbol{\Sigma}_x\mathbf{A}^T + \boldsymbol{\Sigma}_{n_1}\right)^{-1}\mathbf{A}\boldsymbol{\Sigma}_x\right]\mathbf{B}^T + \boldsymbol{\Sigma}_{n_2} \\
&= \mathbf{B}\boldsymbol{\Sigma}_x\mathbf{A}^T\left[\mathbf{A}^{-T}\boldsymbol{\Sigma}_x^{-1}\mathbf{A}^{-1} - \left(\mathbf{A}\boldsymbol{\Sigma}_x\mathbf{A}^T + \boldsymbol{\Sigma}_{n_1}\right)^{-1}\right]\mathbf{A}\boldsymbol{\Sigma}_x\mathbf{B}^T + \boldsymbol{\Sigma}_{n_2} \\
&= \boldsymbol{\Sigma}_{yz}^T\left(-\mathbf{H}^{-1}\right)\boldsymbol{\Sigma}_{yz} + \boldsymbol{\Sigma}_{n_2}. \tag{34}
\end{aligned}$$

Substituting (34) in (32) and defining $\mathbf{P}$ as:

$$\mathbf{P}^{-1} = \left(\boldsymbol{\Sigma}_{yz}^T\mathbf{H}^{-1}\boldsymbol{\Sigma}_{yz}\right)^{-1} - \left(\boldsymbol{\Sigma}_{yz}^T\mathbf{H}^{-1}\boldsymbol{\Sigma}_{yz} - \boldsymbol{\Sigma}_{n_2}\right)^{-1}, \tag{35}$$

we have:

$$\mathbf{C}\boldsymbol{\Sigma}_y = \boldsymbol{\Sigma}_x\mathbf{A}^T\mathbf{H}^{-1}\boldsymbol{\Sigma}_{yz}\mathbf{P}^{-1}\boldsymbol{\Sigma}_{yz}^T, \tag{36}$$

which is clearly invertible.

REFERENCES

[1] A. Bertrand, *Applications and trends in wireless acoustic sensor networks: a signal processing perspective*, in Proc. IEEE Symposium on Communications and Vehicular Technology (SCVT), Nov. 2011.

[2] J. Østergaard and M. S. Derpich, *Sequential Remote Source Coding in Wireless Acoustic Sensor Networks*, European Signal Processing Conference (Eusipco), pp. 1269-1273, Aug. 2012.

[3] D. Slepian and J.Wolf, *Noiseless coding of correlated information sources*, IEEE Transactions on Information Theory, vol. 19, no. 4, pp.4711-480, Jul. 1973.

[4] A. Wyner and J. Ziv, *The rate-distortion function for source coding with side information at the decoder*, IEEE Transactions on Information Theory, vol. 22, no. 1, pp.1-10, Jan. 1976.

[5] A. Wyner, *The rate-distortion function for source coding with side information at the decoder- II: General sources*, Information and Control, vol. 38, no. 1, pp.60-80, Jul. 1978.

[6] M. S. Rahman and A. Wagner, *Rate region of the Gaussian scalar-help-vector source-coding problem*, IEEE Transactions on Information Theory, vol. 58, no. 1, pp.172-188, Jan. 2012.

[7] G. Zhang, *On the rate region of the vector Gaussian one-helper distributed source-coding problem*, in Proc. Data Compression Conference, 2011.

[8] S. Kay, *Fundamentals of Statistical Signal Processing, vol. I: Estimation Theory*, Prentice Hall, 1st edition, 1993.

[9] K. Eswaran and M. Gastpar, *Rate loss in the CEO problem*, in 39th Annual Conference on Information Sciences and Systems, Baltimore, MD, 2005.

[10] C. Tian and J. Chen, *Remote vector Gaussian source coding with decoder side information under mutual information and distortion constraints*, IEEE Transactions on Information Theory, vol. 55, no. 10, pp.4676-4680, Oct. 2009.

[11] S. C. Draper and G. W. Wornell, *Side information aware coding strategies for estimation under communication constraints*, IEEE Journal on Selected Areas in Communications, vol. 22, no. 6, pp. 1-11, Aug. 2004.

[12] T. Kailath, A. Sayed, and B. Hassibi, *Linear Estimation*, Prentice-Hall, 2000.