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An inequality of rearrangements
on the unit circle

by

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AN INEQUALITY OF REARRANGEMENTS ON THE UNIT CIRCLE

CRISTINA DRAGHICI

Abstract. We prove that the integral of the product of two functions over a symmetric set in $\mathbb{S}^1 \times \mathbb{S}^1$, defined as $E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$, where $\sigma_1, \sigma_2$ are diffeomorphisms of $\mathbb{S}^1$ with certain properties and $d$ is the geodesic distance on $\mathbb{S}^1$, increases when we pass to their symmetric decreasing rearrangement. We also give a characterization of these diffeomorphisms $\sigma_1, \sigma_2$ for which the rearrangement inequality holds. As a consequence, we obtain the result for the integral of the function $\Psi(f(x), g(y))$ ($\Psi$ a supermodular function) with a kernel given as $k[d(\sigma_1(x), \sigma_2(y))]$, with $k$ decreasing.

1. Introduction

On a measure space $(X, \mu)$, the Hardy-Littlewood inequality asserts [4]:

$$\int_X f(x)g(x) \, d\mu(x) \leq \int_0^{\mu(X)} f^*(t)g^*(t) \, dt,$$

where $f^*$ and $g^*$ are the decreasing rearrangements of $f$ and $g$, respectively. In what follows, $X = \mathbb{S}^1$, or $X = [-\pi, \pi]$, and the above inequality can be written as:

$$\int_{-\pi}^{\pi} f(x)g(x) \, dx \leq \int_{-\pi}^{\pi} f^\sharp(x)g^\sharp(x) \, dx,$$

with $f^\sharp, g^\sharp$ the symmetric decreasing rearrangements of $f$ and $g$, given by $f^\sharp(x) = f^*(2|x|)$ and $g^\sharp(x) = g^*(2|x|)$.

These inequalities can be proved using the layer-cake formula [10]: Every measurable function $f : X \to \mathbb{R}_+$ can be written as an integral of the characteristic function of its level sets:

$$f(x) = \int_0^{\infty} \chi_{\{f > t\}}(x) \, dt.$$

A more general rearrangement inequality on $X = \mathbb{R}^n$ is the Riesz-Sobolev inequality:

$$\int_{\mathbb{R}^{2n}} f(x)g(y)h(x - y) \, dxdy \leq \int_{\mathbb{R}^{2n}} f^\sharp(x)g^\sharp(x)h^\sharp(x - y) \, dxdy,$$

where $f, g, h$ are non-negative functions which vanish at infinity in a weak sense. The case $n = 1$ is due to Riesz in 1930 (see [12]), and the case $n > 1$ is due to Sobolev in 1938 (see [13]). The proof can be found in the book by Hardy, Littlewood, Pólya [9] which sets the beginning of the systematic study of rearrangement inequalities. A more general version of this inequality in $\mathbb{R}^n$, involving $n$ functions can be found in [5].

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The equivalent of (1.3) for three non-negative functions on the unit circle was proven by Baernstein [1]:

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) g(e^{i\phi}) h(e^{i(\phi-\theta)}) \, d\theta d\phi \leq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f^\sharp(e^{i\theta}) g^\sharp(e^{i\phi}) h^\sharp(e^{i(\phi-\theta)}) \, d\theta d\phi.
\]

The proof of this inequality uses a variational principle applied to the convolution of characteristic functions of sets which does not seem to generalize in higher dimensions.

The Riesz-Sobolev inequality (1.3) is equivalent to the Brunn-Minkowski inequality from convex geometry [8, 11, 7] which states that if $K$ and $L$ are measurable sets in $\mathbb{R}^n$, then their Minkowski (pointwise) sum $K + L$ is related to the measure of the sets $K$ and $L$ by

\[
V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n},
\]

where $V$ denotes the $n$-dimensional volume. An analog of this inequality for $\mathbb{S}^n$ is not known, and, since the proof of rearrangement inequalities in $\mathbb{R}^n$ require it, an analog of the Riesz-Sobolev inequality (1.3) is not known in $\mathbb{S}^n$, for $n > 1$.

However, a partial result in $\mathbb{S}^n$ was proved by Baernstein and Taylor in [2]. They considered a version of the Riesz-Sobolev inequality where one of the functions is symmetric decreasing. They showed that, if $h = K$ is already symmetric decreasing then

\[
\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f(x) g(y) K(x \cdot y) \, d\sigma(x) d\sigma(y) \leq \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f^\sharp(x) g^\sharp(y) K(x \cdot y) \, d\sigma(x) d\sigma(y),
\]

where $d\sigma$ is the surface measure on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, $x \cdot y$ is the usual inner product and $K(t)$ is an increasing function on $[-1, 1]$. Since $x \cdot y = \cos \alpha$, where $\alpha$ is the angle between the vectors $x$ and $y$, we can write $K(x \cdot y) = k(d(x, y))$, with $k$ decreasing. Here $d(x, y)$ is the great circle (geodesic) distance between $x$ and $y$. Their proof is based on the polarization technique. They showed first that the inequality holds for the polarizations of $f$ and $g$ in any hyperplane and then they passed to the limit for the general case. They were led to this version of the Riesz-Sobolev inequality while trying to generalize a 2-dimensional result stating that $u$ is subharmonic implies its star function is also subharmonic.

In this paper we are interested in the case $n = 1$ of this inequality with $K$ replaced by the characteristic function of a symmetric set which does not depend on the distance between two points, but rather on the distance between their images under two diffeomorphisms $\sigma_1, \sigma_2$ of $\mathbb{S}^1$. We will also obtain a characterization of these diffeomorphisms for which the inequality holds. With the set $E$ defined as

\[
E = \{ (x, y) : d(\sigma_1(x), \sigma_2(y)) \leq \alpha \},
\]

we will show that

\[
\int_E f(x) g(y) \, dx \, dy \leq \int_E f^\sharp(x) g^\sharp(y) \, dx \, dy,
\]

for every $\alpha > 0$. This result implies the main result of this paper, Theorem 3.6:

\[
\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f(x), g(y)) k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy \\
\leq \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^\sharp(x), g^\sharp(y)) k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy,
\]
with \( k \) decreasing and \( \Psi \) the distribution function of a measure \( \mu \).

The paper is organized as follows: We will first prove (1.5) for \( f \) and \( g \) replaced by characteristic functions \( \chi_A, \chi_B \), and \( \sigma_2 \) the identity. Then we will deduce the result (1.5) mentioned above, and we will show that we can replace the product \( f(x)g(y) \) by a function \( \Psi(f(x), g(y)) \) and that we can replace \( \chi_F \) by a decreasing function of the distance between \( \sigma_1(x) \) and \( \sigma_2(y) \), yielding Theorem 3.6.

2. PRELIMINARIES

Recall that a function \( f : I \to \mathbb{R} \), defined on an interval \( I \subset \mathbb{R} \), is called convex if, for every \( 0 < \lambda < 1 \) and every \( a, b \in I \), the following inequality holds:

\[
f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).
\]

A convex function is differentiable almost everywhere on \( I \) and its derivative is increasing.

We denote by \( S^1 \) the unit circle in \( \mathbb{R}^2 \), i.e., \( S^1 = \{z \in \mathbb{C} : |z| = 1 \} \), and by \( S^1_+ \) the upper half unit circle,

\[
S^1_+ = \{e^{i\theta} : 0 \leq \theta \leq \pi \}.
\]

**Definition 2.1.** A function \( \sigma : S^1_+ \to S^1_+ \) is called convex if the function \( \sigma_1 : [0, \pi] \to [0, \pi] \), defined as :

\[
\sigma(e^{i\theta}) = e^{i\sigma_1(\theta)}, \quad 0 \leq \theta \leq \pi,
\]

is convex on \([0, \pi]\).

Let \( f : S^1 \to \mathbb{R}_+ \) be a non-negative measurable function. We define its distribution function:

\[
\lambda_f(t) = |\{ f > t \}|, \quad t \in [0, \infty),
\]

where \( \{ f > t \} := \{ z \in S^1 : f(z) > t \} \) denote the level sets of \( f \), and \( |A| \) is the linear measure on \( S^1 \) of \( A \). Functions which have the same distribution function are called equimeasurable.

We define the symmetric decreasing rearrangement of \( f \) to be the function \( f^\sharp : S^1 \to \mathbb{R}_+ \), given by:

\[
f^\sharp(z) = \inf \{ t : \lambda_f(t) \leq 2d(1, z) \},
\]

where \( d(1, z) \) is the geodesic distance on \( S^1 \) between \( z \) and \( 1 \). It is clear that \( f^\sharp(z) = f^\sharp(\bar{z}) \) and that \( f^\sharp \) decreases as \( d(1, z) \) increases. Also, \( f \) and \( f^\sharp \) are equimeasurable.

If we write \( z = e^{i\theta}, -\pi \leq \theta < \pi \), then \( d(1, z) = d(1, e^{i\theta}) = |\theta| \), and we can think of \( f \) as a function of \( \theta \) via the relation

\[
\tilde{f}(\theta) = f(e^{i\theta}).
\]

For \( \tilde{f} : [-\pi, \pi] \to \mathbb{R}_+ \), one defines its symmetric decreasing rearrangement as:

\[
\tilde{f}^\sharp(\theta) = \inf \{ t : \lambda_{\tilde{f}}(t) \leq 2|\theta| \},
\]

where, as before, \( \lambda_{\tilde{f}}(t) = |\{ \tilde{f} > t \}| \), and thus, there is a one-to-one correspondence between \( f^\sharp \) and \( \tilde{f}^\sharp \), given by

\[
\tilde{f}^\sharp(\theta) = f^\sharp(e^{i\theta}).
\]

Whenever necessary, we will think of a function \( f \) defined on \( S^1 \) as a function on \([−\pi, \pi]\). If \( f = \chi_A \) is the characteristic function of a measurable set \( A \subset S^1 \), then
\[ f^t = \chi_{A^t}, \text{ where } A^t \text{ is the open interval on the unit circle centered at } 1, \text{ having the same linear measure as } A. \]

Next, we introduce the Hardy-Littlewood-Pólya preorder relation \(<\) for non-negative functions defined on the interval \([-\pi, \pi]\). We say that (see [3, 4]):

\[ f < F \quad \text{iff} \quad \int_{-t}^t f^t(s) \, ds \leq \int_{-t}^t F^t(s) \, ds, \text{ for all } 0 \leq t \leq \pi. \]

This is equivalent to

\[ \int_{-\pi}^\pi f^t(s) h^t(s) \, ds \leq \int_{-\pi}^\pi F^t(s) h^t(s) \, ds, \]

for every positive symmetric decreasing function \(h^t\) defined on \([-\pi, \pi]\). To see this, write \(h^t(s) = \int_0^\infty \chi_{\{h^t > \tau\}}(s) \, d\tau\) (this is the layer cake formula (1.2)), and, using Fubini’s formula and the fact that \(\{h^t > t\} = (-l(t), l(t))\) is a symmetric interval,

\[ \int_{-\pi}^\pi f^t(s) h^t(s) \, ds = \int_0^\infty \left[ \int_{-l(t)}^{l(t)} f^t(s) \, ds \right] \, dt \leq \int_0^\infty \left[ \int_{-l(t)}^{l(t)} F^t(s) \, ds \right] \, dt = \int_{-\pi}^\pi F^t(s) h^t(s) \, ds. \]

Yet another equivalent characterization is:

\[ f < F \Leftrightarrow \int_E f(s) \, ds \leq \int_E F(s) \, ds, \text{ for every } E \subset [-\pi, \pi]. \]

The next result is well-known and it follows from the proof of the equality case in the Hardy-Littlewood inequality, presented by Lieb and Loss in [10, pp.82]. We will include a proof here for consistency.

**Lemma 2.2.** Let \(f : [-\pi, \pi] \to \mathbb{R}_+\) be a measurable function such that

\[ (2.1) \quad \int_{-t}^t f(x) \, dx \geq \int_{-t}^t f^t(x) \, dx, \quad \text{for every } 0 \leq t \leq \pi. \]

Then \(f = f^t\) a.e. on \([-\pi, \pi]\).

**Proof.** From (1.1) applied to \(\chi_{(-t,t)}\) and \(f\), it follows that we must have equality in (2.1), i.e.,

\[ (2.2) \quad \int_{-t}^t f(x) \, dx = \int_{-t}^t f^t(x) \, dx. \]

We will use the layer-cake formula to write \(f(x) = \int_0^\infty \chi_{\{f > \tau\}}(x) \, d\tau\), and similarly for \(f^t(x)\).

Using (1.1), we obtain:

\[ (2.3) \quad \int_{-t}^t \chi_{\{f > \tau\}}(x) \, dx \leq \int_{-t}^t \chi_{\{f^t > \tau\}}(x) \, dx, \quad \text{for every } \tau \geq 0. \]

Fubini’s theorem and (2.2) imply that:

\[ \int_{-t}^t f(x) \, dx = \int_0^\infty \left[ \int_{-t}^t \chi_{\{f > \tau\}}(x) \, dx \right] \, d\tau = \int_0^\infty \left[ \int_{-t}^t \chi_{\{f^t > \tau\}}(x) \, dx \right] \, d\tau = \int_{-t}^t f^t(x) \, dx. \]
From this equality and (2.3) it follows that, for a fixed $t$, there exists a set of measure zero $S_t$, such that
\[
\int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx = \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx, \quad \text{for every } s \in (0, \infty) \setminus S_t.
\]

Next, we choose $T_N$ a countable dense set in $[0, \pi]$ and we denote by $S_{T_N} = \cup_{t \in T_N} S_t$. Then:
\[
(2.4) \quad \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx = \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx, \quad \text{for every } t \in T_N \text{ and } s \in (0, \infty) \setminus S_{T_N}.
\]

Since for every fixed $s$, $t \rightarrow \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx$ is a continuous function of $t$, in fact (2.4) holds for every $0 \leq t \leq \pi$. Thus,
\[
\int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx = \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx, \quad \text{for all } 0 \leq t \leq \pi \text{ and a.e. } s \in (0, \infty).
\]

Now, let $t$ be such that $\{f^2 > s\} = (-t, t)$. Then, it follows that $\{f > s\} = (-t, t) = \{f^2 > s\}$ a.e., and thus, $f = f^2$ by the layer cake formula.

The following result shows that $\int_{-t}^{t} f^2(x) \, dx$ is attained as a supremum. A proof can be found in [4, Theorem 7.5, pp.82].

**Theorem 2.3.** (J. V. Ryff) For every measurable function $f$ as in Lemma 2.2, there exists a measure preserving transformation $T$ such that $f = f^2 \circ T$. This guarantees, for every $t$, the existence of a set $A \subset [-\pi, \pi]$ of measure $2t$ such that $\int_A f(x) \, dx = \int_{-t}^{t} f^2(x) \, dx$.

### 3. Main results: inequalities on the circle

**Notation.** As before, $d$ is the geodesic distance, also called the arclength, on the unit circle $S^1$. We have:
\[
(3.1) \quad d(u, v) = d(\bar{v}, 1), \quad \text{for all } u, v \in S^1,
\]
where $\bar{v}$ denotes the complex conjugate of $v$.

We define, for $\alpha > 0$, the function:
\[
\chi_\alpha(u, v) = \begin{cases} 1, & \text{if } d(u, v) \leq \alpha, \\ 0, & \text{otherwise} \end{cases}
\]
and we observe that $\chi_\alpha(u, v) = \chi_\alpha(u \bar{v}, 1)$, by (3.1).

We introduce a new function, which we call again $\chi_\alpha : S^1 \rightarrow \mathbb{R}_+$, given by $\chi_\alpha(z) := \chi_\alpha(z, 1)$, which is the characteristic function of the closed interval on $S^1$ of linear length $2\alpha$, centered at 1.

We will make use, in what follows, of the relation:
\[
(3.2) \quad \chi_\alpha(u \bar{v}) = \chi_\alpha(u, v), \quad \text{for all } u, v \in S^1.
\]

Given two positive measurable functions $f, g : S^1 \rightarrow \mathbb{R}_+$, their convolution, $f * g$, is defined to be the function:
\[
(f * g)(z_0) = \int_{S^1} f(z_0 \bar{z}) g(z) \, dz = \int_{-\pi}^{\pi} f(e^{i(\theta_0 - \theta)}) g(e^{i\theta}) \, d\theta,
\]
with \( z_0 = e^{i\theta_0} \) and \( dz \) represents the arclength element on \( S^1 \), usually denoted by \( |dz| \).

Given three positive functions \( f, g, h \) defined on \( S^1 \), we can write
\[
(3.3) \quad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{i(\theta-t)})g(e^{i\theta})h(e^{i\theta}) \, dt \, d\theta = (f \ast g \ast h^-)(1),
\]
where \( h^-(z) = h(\bar{z}) \), i.e., \( h^{-}(e^{i\theta}) = h(e^{-i\theta}) \).

**Theorem 3.1.** Let \( \sigma : S^1 \to S^1 \) be a \( C^1 \) diffeomorphism such that \( \sigma(1) = 1 \) and \( \sigma(-1) = -1 \). Additionally, we assume that \( \sigma(S^1) \subseteq S^1 \) and \( \sigma(S^1) \subseteq S^1 \). Let \( d \) be the geodesic distance on the unit circle, \( \alpha \) be a positive real number, and we define the set \( E = \{ (x,y) \in S^1 \times S^1 : d(\sigma(x),y) \leq \alpha \} \). For \( A, B \subset S^1 \) measurable sets, let
\[
I_\alpha(A,B) = \int_{S^1 \times S^1} \chi_A(u)\chi_B(v)\chi_E(u,v) \, du \, dv.
\]

Then, for any \( A, B \) measurable subsets of \( S^1 \), and \( \alpha > 0 \),
\[
(3.4) \quad I_\alpha(A,B) \leq I_\alpha(A^2,B^2),
\]
if and only if, \( \sigma \) is symmetric (i.e. \( \overline{\sigma(z)} = \sigma(\bar{z}) \), for every \( z \in S^1 \)) and convex on \( S^1 \).

**Proof.** Sufficiency. We define \( \sigma_1 : [-\pi,\pi) \to [-\pi,\pi) \) by \( e^{\sigma_1(\theta)} := \sigma(e^{i\theta}) \) and we assume that \( \sigma_1 \) is convex on \((0,\pi)\). Using change of variables, \((\sigma(x),y) = (u,v)\), the integral \( I_\alpha \) becomes:
\[
I_\alpha(A,B) = \int_{S^1 \times S^1} \chi_{\sigma(A)}(u)\chi_B(v)\chi_\alpha(u,v)(\sigma^{-1})'(u) \, du \, dv.
\]

With \( \chi_\alpha(u,v) = \chi_\alpha(u\bar{v}) \), as in (3.2), the above expression becomes:
\[
(3.5) \quad I_\alpha(A,B) = \int_{S^1 \times S^1} \chi_{\sigma(A)}(u)\chi_B(v)\chi_\alpha(u\bar{v})\psi(u) \, du \, dv,
\]
where \( \psi(e^{i\theta}) = \tau_1(\theta) \) and \( \tau_1 \) is defined by \( \sigma^{-1}(e^{i\theta}) = e^{\tau_1(\theta)} \), and is the inverse of \( \sigma_1 \).

Thus, we can write using convolution and (3.3):
\[
I_\alpha(A,B) = [(\chi_{\sigma(A)} \cdot \psi) \ast \chi_\alpha \ast \chi_B](1),
\]
where we used the fact that \( \chi_\alpha \) is a symmetric function.

It was proved in [1] (see also (1.4)) by Baernstein that, for any three positive measurable functions \( f, g, h \) on \( S^1 \), the following inequality holds:
\[
(3.6) \quad (f \ast g \ast h^-)(1) \leq (f^2 \ast g^2 \ast h^2)(1).
\]

One can replace \( h^- \) in the inequality above by \( h \) since they are equimeasurable functions. Thus, based on (3.6) and the fact that \( \chi_\alpha \) is symmetric decreasing, we conclude that:
\[
(3.7) \quad I_\alpha(A,B) \leq [(\chi_{\sigma(A)} \cdot \psi)^2 \ast \chi_\alpha \ast \chi_B](1).
\]

**Fact:** If \( F \) is a positive symmetric decreasing function and if \( f \prec F \) in the sense of Hardy-Littlewood-Pólya (i.e. \( \sup_{|G|=2\pi} \int_G f \leq \int_{\theta}^\theta F \)), then \( f^2 \) in inequality (3.6) can be replaced by \( F \). Indeed, \( f \prec F \) is equivalent to \( \int \int f^2(z)g^2(z) \, dz \leq \int \int F(z)g^2(z) \, dz \),
for all positive symmetric decreasing functions $g^\sharp$. Now, since $g^\sharp \ast h^\sharp$ is symmetric decreasing and since the convolution $(f^\sharp \ast g^\sharp \ast h^\sharp)(1)$ can be written as the integral of the product $f^\sharp(z)(g^\sharp \ast h^\sharp)(z)$, we conclude that:

$$(f^\sharp \ast g^\sharp \ast h^\sharp)(1) \leq (F \ast g^\sharp \ast h^\sharp)(1).$$

Therefore, using (3.7) and the Fact, we can prove (3.4) if we show that $\chi_{\sigma(A)} \psi < \chi_{\sigma(A)} \psi$, i.e.

$$\int_E \chi_{\sigma(A)} \psi \leq \int_{E^\sharp} \chi_{\sigma(A)} \psi.$$

Let $E' = \sigma^{-1}(E)$, and $E'' = \sigma^{-1}(E^\sharp)$. With these notations, inequality (3.8) becomes:

$$\int_{A \cap E'} dx \leq \int_{A \cap E''} dx,$$

or equivalently, $|A \cap E'| \leq |A^\sharp \cap E''|$, which is true if $|E'| \leq |E''|$, since $E''$ is symmetric. Since $\psi$ is symmetric decreasing, we have that $\int_{E} \psi(u) du \leq \int_{E'} \psi(u) du$, which is equivalent to $\int_{I^{-1}(E)} dx \leq \int_{I^{-1}(E')} dx$, using change of variables. The latter inequality simply states that $|E'| \leq |E''|$, and the proof of the sufficiency is now complete.

**Necessity.** Dividing (3.5) by $2\alpha$, and letting $\alpha$ tend to zero, we obtain:

$$I_0(A, B) = \int_{S^1} \chi_{\sigma(A)}(u) \chi_B(u) \psi(u) du,$$

and inequality (3.4) implies that:

$$I_0(A, B) \leq I_0(A^\sharp, B^\sharp).$$

With the notation $\tau = \sigma^{-1}$, $\psi$ the Jacobian of $\tau$, and $x = \tau(u)$, $I_0$ becomes:

$$I_0(A, B) = \int_{S^1} \chi_A(x) \chi_{\tau(B)}(x) dx = |A \cap \tau(B)|.$$

First, we will show that the symmetry condition is necessary. Suppose $\tau$ is not symmetric. Then, there exists a point $x = e^{i\theta}$ in $S^1_+$ such that $\tau(x) \neq \bar{\tau}(x)$. If we consider $A = \tau(\{e^{it} : |t| < \theta\})$ and $B = \{e^{it} : |t| < \theta\}$, then we have: $|A \cap \tau(B)| = |\tau(B)| > |A^\sharp \cap \tau(B^\sharp)|$, since $\tau(B^\sharp)$ is not symmetric and $|A| = |\tau(B)|$. But this contradicts (3.9) and therefore (3.4).

Suppose now that $\tau_1$ is symmetric, but not concave (or, equivalently, $\sigma_1$ is symmetric, but $\sigma_1$ is not convex on $(0, \pi)$). Then, there exist $e^{ib}, e^{ic} \in S^1_+$ with $b, c \in (0, \pi)$ such that:

$$\frac{\tau_1(b) + \tau_1(c)}{2} > \tau_1\left(\frac{b + c}{2}\right).$$

Without loss of generality we can assume that $b > c$ and let us denote by $a = \frac{b + c}{2}$. Letting $B = \{e^{it} : -c < t < b\}$, it follows that $B^\sharp = \{e^{it} : -a < t < a\}$. We calculate $|\tau(B)| = \tau_1(b) - \tau_1(-c) = \tau_1(b) + \tau_1(c)$ and $|\tau(B^\sharp)| = 2\tau_1(a)$.

From (3.11) we obtain that $|\tau(B)| > |\tau(B^\sharp)|$ which shows that $I_0(S^1, B) > I_0(S^1, B^\sharp)$ and contradicts (3.4). Therefore, $\tau$ must also be concave. □
Theorem 3.2. Suppose we have two functions \( \sigma_1, \sigma_2 \) satisfying the conditions of \( \sigma \) in Theorem 3.1 and define \( E = \{(x, y) \in S^1 \times S^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha \} \), for \( \alpha \in \mathbb{R}_+ \). Let

\[
I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_A(x)\chi_B(y)\chi_E(x, y)dx dy.
\]

Then, for any \( A, B \) subsets of \( S^1 \) and \( \alpha > 0 \),

\[
(3.12) \quad I_\alpha(A, B) \leq I_\alpha(A^1, B^2),
\]

if and only if \( \sigma_1, \sigma_2 \) are symmetric and convex on \( S^1_+ \).

Proof. Sufficiency. Very similar to Theorem 3.1. Using change of variables, \( (\sigma_1(x), \sigma_2(y)) = (u, v) \), the integral becomes:

\[
I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_{\sigma_1(A)}(u)\chi_{\sigma_2(B)}(v)\chi_\alpha(u\bar{v})\psi_1(u)\psi_2(v)dudv,
\]

where \( \psi_1, \psi_2 \) are defined similarly to \( \psi \) in Theorem 3.1 (see (3.5)). Using convolution, this integral can be written as:

\[
I_\alpha(A, B) = [\left(\chi_{\sigma_1(A)} \cdot \psi_1\right) * \chi_\alpha * \left(\chi_{\sigma_2(B)} \cdot \psi_2\right)(1)].
\]

We have already proven that \( \chi_{\sigma_1(A)} \psi_1 \prec \chi_{\sigma_1(A)} \psi_1 \) and \( \chi_{\sigma_2(B)} \psi_2 \prec \chi_{\sigma_2(B)} \psi_2 \), from which it follows that \( I_\alpha(A, B) \leq I_\alpha(A^1, B^2) \).

Necessity. Using change of variable \( v = \sigma_2(y) \), \( I_\alpha \) becomes:

\[
I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_A(x)\chi_{\{(x, v) \in S^1 \times S^1 : d(\sigma_1(x), v) \leq \alpha \}}\chi_{\sigma_2(B)}(v)\psi_2(v)dx dv.
\]

Dividing by \( \alpha \) and letting \( \alpha \to 0 \), we obtain:

\[
I_0(A, B) = \int_{S^1} \chi_A(x)\chi_{\sigma_2(B)}(\sigma_1(x))\psi_2(\sigma_1(x))dx.
\]

Inequality (3.12) of the theorem implies the following inequality:

\[
(3.13) \quad I_0(A, B) \leq I_0(A^1, B^2),
\]

for all subsets \( A \) and \( B \) of \( S^1 \).

Now let \( B = S^1 \) in the above identity. Then:

\[
I_0(A, S^1) = \int_{S^1} \chi_A(x)\psi_2(\sigma_1(x))dx \leq \int_{S^1} \chi_{\sigma_1}(x)\psi_2(\sigma_1(x))dx,
\]

or equivalently,

\[
\int_A \psi_2(\sigma_1(x)) dx \leq \int_{\sigma_1^{-1}A} \psi_2(\sigma_1(x)) dx,
\]

for every measurable set \( A \subset S^1 \). Since the inequality is true for every measurable set \( A \), we conclude by Lemma 2.2 and Theorem 2.3 that \( \psi_2 \circ \sigma_1 \) is symmetric (i.e., \( \psi_2(\sigma_1(z)) = \psi_2(\sigma_1(\bar{z})) \)) and decreasing, which implies that \( \psi_2 \) is decreasing on \( S^1_+ \). Likewise, \( \psi_1 \circ \sigma_2 \) is symmetric and decreasing on \( S^1_+ \), implying that \( \psi_1 \) is decreasing on \( S^1_+ \). Thus, \( \sigma_1^{-1} \) and \( \sigma_2^{-1} \) are concave on \( S^1_+ \) and therefore, \( \sigma_1 \) and \( \sigma_2 \) are convex on \( S^1_+ \).
Next, we denote by $\tau = \sigma_1^{-1} \circ \sigma_2$. With this notation, $I_0$ becomes:

$$I_0(A, B) = \int_{S^1} \chi_A(x) \chi_{\tau(B)}(\sigma_1(x)) [\psi_2 \circ \sigma_1](x) \, dx$$

$$= \int_{S^1} \chi_A(x) \chi_{\tau(B)}(x) [\psi_2 \circ \sigma_1](x) \, dx = \int_{A \cap \tau(B)} [\psi_2 \circ \sigma_1](x) \, dx.$$  

We will show that $\tau$ is symmetric, i.e., $\tau(\bar{x}) = \tau(x)$, for every $x \in S^1$. Suppose this is not the case. Then there exists $x = e^{i\theta}$, with $\theta \in (0, \pi)$, such that $\tau(x) \neq \tau(\bar{x})$.

Let $B = \{e^{i\theta} : |t| < \theta\} = B^2$ and $A = \tau(B) \neq A^2$. Then, we have that $A^2 \cap \tau(B^2) \subset A \cap \tau(B) = A$ and $|A \cap \tau(B)| > |A^2 \cap \tau(B^2)|$. Since $\psi_2 \circ \sigma_1$ is positive, it follows that $I_0(A, B) > I_0(A^2, B^2)$, which contradicts (3.13). Thus, $\sigma_1^{-1} \circ \sigma_2$ is symmetric.

We have shown before that $\psi_1 \circ \sigma_2$ is also symmetric.

**Claim:** $\sigma_1^{-1} \circ \sigma_2$ and $\psi_1 \circ \sigma_2$ symmetric imply $\sigma_2$ is symmetric.

**Proof of claim:** We define $f_2$ on the interval $[-\pi, \pi]$ as follows:

$$\sigma_2(e^{i\theta}) = e^{if_2(\theta)}.$$  

Since $\psi_1 \circ \sigma_2$ is symmetric and $[\psi_1 \circ \sigma_2](e^{i\theta}) = \psi_1(e^{if_2(\theta)}) = \tau_1(f_2(\theta))$, as in (3.5), it follows that $\tau_1 \circ f_2$ is even.

Since $[\sigma_1^{-1} \circ \sigma_2](e^{i\theta}) = e^{i\pi (f_2(\theta))}$ is symmetric, it follows that $\tau_1 \circ f_2$ is odd.

Now, $(\tau_1 \circ f_2)' = (\tau_1 \circ f_2 \circ f_2)'$ is even and $\tau_1 \circ f_2$ is also even (as we have previously shown) and nonzero, so that $f_2'$ is even and thus $f_2$ is odd. Therefore $\sigma_2$ is symmetric and the proof of the claim is now complete.

Following exactly the same steps, we can show that $\sigma_1$ is symmetric. We have shown that $\sigma_1, \sigma_2$ are symmetric and convex on $S^1_+$.

**Corollary 3.3.** With $\sigma$, $\alpha$ and $E = \{(x, y) \in S^1 : d(\sigma(x), y) \leq \alpha\}$, as in Theorem 3.1, we have the following result: For every $f, g : S^1 \to \mathbb{R}_+$ positive measurable functions, and every $\alpha > 0$,

$$\int_E f(x) g(y) \, dx \, dy \leq \int_E f^\alpha(x) g^\alpha(y) \, dx \, dy,$$

if and only if, $\sigma$ is symmetric, and convex on $S^1_+$.

To sketch the proof, we write $f$ and $g$ as the integrals of their level sets, using the layer-cake representation formula (1.2):

$$f(x) = \int_0^\infty \chi_{\{f > t\}}(x) \, dt \quad \text{and} \quad g(y) = \int_0^\infty \chi_{\{g > t\}}(y) \, dt,$$

and we notice that $\{f > t\}^g = \{f^g > t\}$ and $\{g > t\}^f = \{g^f > t\}$ so that inequality (3.14) reduces to the case where $f$ and $g$ are characteristic functions, and thus, Theorem 3.1 applies.

**Corollary 3.4.** Let $\sigma_1$, $\sigma_2$ and $E = \{(x, y) \in S^1 \times S^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$ be as in Theorem 3.2. For every $f, g : S^1 \to \mathbb{R}_+$ positive measurable functions, and every $\alpha > 0$,

$$\int_E f(x) g(y) \, dx \, dy \leq \int_E f^\alpha(x) g^\alpha(y) \, dx \, dy,$$

if and only if, $\sigma_1$ and $\sigma_2$ are symmetric, and convex on $S^1_+$. 
The proof of Corollary 3.4 is indeed very similar to the proof of Corollary 3.3, in which one represents \( f \) and \( g \) as integrals of the characteristic functions of their level sets.

The next theorem is a generalization of the previous results, where one replaces the product by a function \( \Psi \) defined as follows:

\[
\Psi : \mathbb{R}_+^2 \rightarrow \mathbb{R}, \text{ i.e., } \Psi|_{\{x_1=0\}} = \Psi|_{\{x_2=0\}} = 0, \text{ and }
\Psi(x_1, x_2) + \Psi(y_1, y_2) \leq \Psi(x_1 \wedge x_2, y_1 \wedge y_2) + \Psi(x_1 \vee x_2, y_1 \vee y_2).
\]

If \( \Psi \) is twice continuously differentiable, then the above inequality is equivalent to \( \partial_{12} \Psi \geq 0 \).

Crowe, Zweibel and Rosenbloom [6] noticed that a continuous such \( \Psi \) is the distribution function of a Borel measure \( \mu \) on \( \mathbb{R}_+^2 \), i.e.,

\[
(3.17) \quad \Psi(s, t) = \mu([0, s) \times [0, t]),
\]

and using Fubini’s theorem:

\[
(3.18) \quad \int S_1 \int \Psi(f(x), g(y)) \, dx \, dy = \int S_1 \left[ \int \chi_{(f>\tau)}(x) \chi_{(g>\tau)}(y) \, dx \, dy \right] \, d\mu(s, t).
\]

We are now ready to state our next result.

**Theorem 3.5.** With \( \sigma_1, \sigma_2 \) and \( E = \{(x, y) \in S^1 \times S^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\} \) as in Theorem 3.2, and \( \Psi \) the distribution function of a Borel measure \( \mu \) on \( \mathbb{R}_+^2 \) as in (3.16), the following inequality holds for every \( \alpha > 0 \):

\[
\int_E \Psi(f(x), g(y)) \, dx \, dy \leq \int_E \Psi(f^\sharp(x), g^\sharp(y)) \, dx \, dy,
\]

if and only if, \( \sigma_1 \) and \( \sigma_2 \) are symmetric on \( S^1 \), and convex on \( S^1_\downarrow \).

Again, we can reduce \( \Psi(f(x), g(y)) \) to a product of characteristic functions, using (3.17), and the result follows from Theorem 3.2.

The next theorem shows that we can replace the characteristic function of the set \( E \) by a decreasing function of the distance between \( \sigma_1(x) \) and \( \sigma_2(y) \), call it \( k[d(\sigma_1(x), \sigma_2(y))] \).

**Theorem 3.6.** Let \( \sigma_1, \sigma_2 \) be as in Theorem 3.2 and let \( k : [0, \infty) \rightarrow [0, \infty) \) be a decreasing function, and \( \Psi \) the distribution function of a Borel measure \( \mu \) on \( \mathbb{R}_+^2 \) as in (3.16). Then, the following inequality holds for every decreasing function \( k \),

\[
\int S^1 \int S^1 \Psi(f(x), g(y))k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy
\]

\[
\leq \int S^1 \int S^1 \Psi(f^\sharp(x), g^\sharp(y))k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy,
\]

if and only if, \( \sigma_1 \) and \( \sigma_2 \) are symmetric on \( S^1 \), and convex on \( S^1_\downarrow \).

**Proof.** Using (1.2), we can write:

\[
k(\tau) = \int_0^\infty \chi_{(k>\tau)}(\tau) \, d\tau = \int_0^\infty \chi_{[0,\infty)}(\tau) \, d\tau,
\]

and substituting \( d(\sigma_1(x), \sigma_2(y)) \) for \( \tau \) in the above formula, we have

\[
k[d(\sigma_1(x), \sigma_2(y))] = \int_0^\infty \chi_{[0,\infty)}[d(\sigma_1(x), \sigma_2(y))] \, d\tau.
\]
We define the set $E_{l(t)}$ as follows:

$$E_{l(t)} = \{(x, y) : d(\sigma_1(x), \sigma_2(y)) \leq l(t)\}.$$ 

Then

$$\chi_{[0, l(t)]} [d(\sigma_1(x), \sigma_2(y))] = 1 \iff (x, y) \in E_{l(t)}.$$ 

Using this fact, (3.18), Fubini’s theorem and Theorem 3.5 we obtain the conclusion of Theorem 3.6 by:

$$\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f(x), g(y)) k[d(\sigma_1(x), \sigma_2(y))] \, dxdy = \int_0^{\infty} \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f(x), g(y)) \chi_{E_{l(t)}}(x, y) \, dxdy \, dt$$

$$\leq \int_0^{\infty} \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^*(x), g^*(y)) \chi_{E_{l(t)}}(x, y) \, dxdy \, dt$$

$$= \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^*(x), g^*(y)) k[d(\sigma_1(x), \sigma_2(y))] \, dxdy.\quad \square$$

REFERENCES


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