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CHAPTER 12

RELIABILITY OF STRUCTURAL SYSTEMS WITH CORRELATED ELEMENTS[1]

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Abstract
Calculation of the probability of failure of a system with correlation members is usually a difficult and time-consuming numerical problem. However, for some types of systems with equally correlated elements this calculation can be performed in a simple way. This has suggested two new methods based on so-called average and equivalent correlation coefficients. By using these methods approximate values for the probability of failure can easily be calculated. The accuracy of these methods is illustrated with examples.

INTRODUCTION

Reliability analysis of real structures is usually too difficult to be treated in an exact way. It is therefore necessary to make a number of approximations. These approximations can be related to loads and to resistances of single elements, but can also be obtained in the way the real structure is modelled. It is obvious that mathematical modelling of a real structure will affect the calculated reliability of the structure. Therefore, it is of great importance in estimating the reliability of a structure to be able to use rather sophisticated systems. However, even for a drastically idealized system calculation of the probability of failure may be impossible or very computer-time consuming. Traditionally when designing structures the reliability of a structural system is connected with the safety of some critical members or cross-sections.

In this paper evaluation of the probability of failure of parallel systems and series systems is discussed. Known exact results for systems with members, which are equally correlated, are used as a basis for new methods. The use of so-called average and equivalent correlation coefficients is discussed in detail and illustrated.

[1] Applied Mathematical Modelling, Vol.6, 1982, pp. 171-178.

FUNDAMENTAL SYSTEMS

In some situations it is possible to subdivide a real structure into a number of substructures where these substructures are modelled by so-called series systems or parallel systems. It is therefore of great interest to analyse the behaviour of such systems.

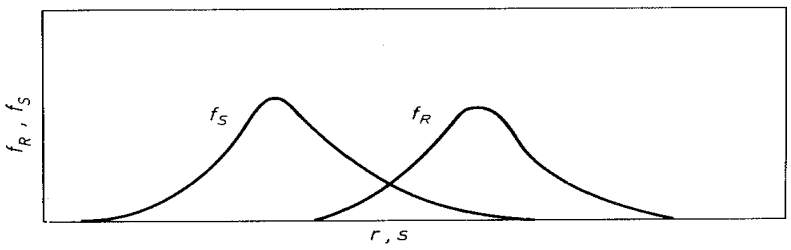

Figure 1.

Series systems and parallel systems are constructed on the basis of single elements. For a single element (structural member, cross-section, etc.) the probability of failure P_f can be determined in a simple way if its strength can be described by a single random variable R, and the loading by a single random variable S (see Figure 1). In such a case failure will occur when $R - S < 0$. Therefore the probability of failure P_f is given by:

$$P_f = P(R-S<0) = \int_{-\infty}^{\infty} F_R(t)\, f_S(t)dt = \int_{-\infty}^{\infty} (1-F_S(t))\, f_R(t)dt \qquad (1)$$

where f_S, F_S and f_R ,F_R are density and distribution functions for S and R. Here, statistical independence between the random variables R and S is assumed. If independence between R and S cannot be assumed the probability of failure is given by:

$$P_f = \int_{\omega_f} f_{RS}(r,s)d\omega \qquad (2)$$

where f_{RS} is the joint density function for R and S, and where the area of integration is the failure domain $\omega_f = \{(r,s) \in R_2 | r-s<0\}$.

It is important to note that evaluation *of* the probability *of* failure, even in the idealized case above for a single member, can only be performed when the density functions f_R and f_S are known. If only the expected values $E[R]$ and $E[S]$ and the variances var[R] and var[S] are known, then the reliability index β, suggested by Coenell [1], can be used as a measure of the reliability. The reliability index β is defined by:

$$\beta = \frac{E[R-S]}{(\mathrm{var}[R-S])^{1/2}} \qquad (3)$$

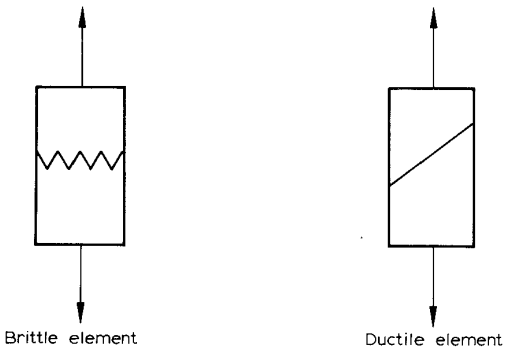

Figure 2.

Returning to the fundamental systems it is of great importance whether the involved elements can be considered perfectly brittle or perfectly ductile. A structural element is called perfectly brittle if it becomes ineffective after failure, which is it loses its load-bearing capacity completely by failure. If an element maintains its load level after failure it is

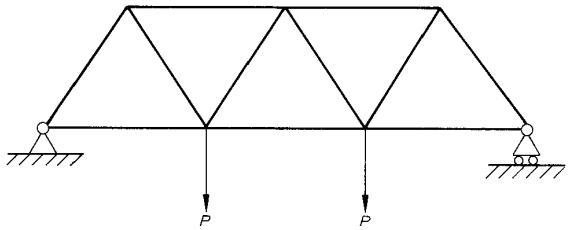

Figure 3.

called perfectly ductile. It has been suggested by Thoft-Christensen [2] that the symbols shown in Figure 2 be used to distinguish these two types of elements.

As mentioned above, there are two fundamental systems, namely series systems and parallel systems. A system is a series system if it is in a state of failure whenever any of its elements have failed. Such a system is also called a weakest- link system. In the literature series systems are usually illustrated by a statically determinate structure, as shown in Figure 3. Obviously, fracture in any member of such a structure will result in fracture of the whole system. A series system with n elements is generally symbolized as shown in Figure 4. Let the distribution function for the strength of element i be F_{R_i} then the distribution function F_R for the strength of the series systems is given by:

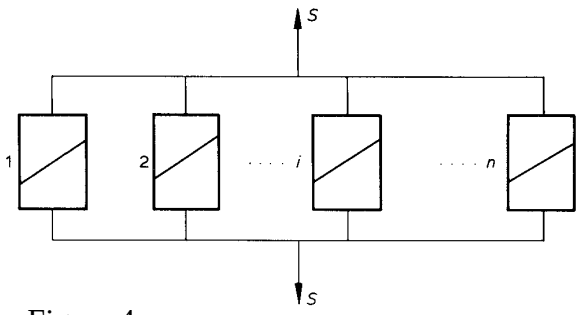

Figure 4.

$$F_R(r) = 1-(1-F_{R_1}(r_1))(1-F_{R_2}(r_2))\ldots(1-F_{R_n}(r_n)) = 1-\prod_{i=1}^{n}(1-F_{R_i}(r_i)) \qquad (4)$$

where r is the external load on the series system resulting in the stress r_i in element i assuming independent strength of the elements.

Using (1), the probability of failure for the series system in Figure 4 can then be determined from:

$$P_f = \int_{-\infty}^{\infty} F_R(r)\, f_S(r)dr = 1-\int_{-\infty}^{\infty} \prod_{i=1}^{n}(1-F_{R_i}(a_i r)\, f_S(r)dr \qquad (5)$$

where $a_i = r_i/r$ and f_S is the density function for the load S on the series system.

It is well known that failure in a single member in a structural system will not always result in failure of the whole system, because the remaining elements by re-distributions may be able to sustain the loads. This situation will often appear for statically indeterminate structures. Failure of such structures will usually require that more than one element fails. Such a set of elements is called a parallel system from a strength point of view and the associated failure state is called a failure mode. A statically indeterminate structure will usually have a great number of failure modes each modelled by a parallel system. Therefore such a structure system will fail when the weakest mode (parallel system) fails. In other words, a parallel system will only fail when all elements in the system fail. On this background it is obvious that the behaviour of such a system depends on whether or not the elements are perfectly ductile or perfectly brittle.

First consider a parallel system with perfectly ductile elements as shown in Figure 5. Due to the assumption of ductile behaviour of the elements the strength R of this system is simply:

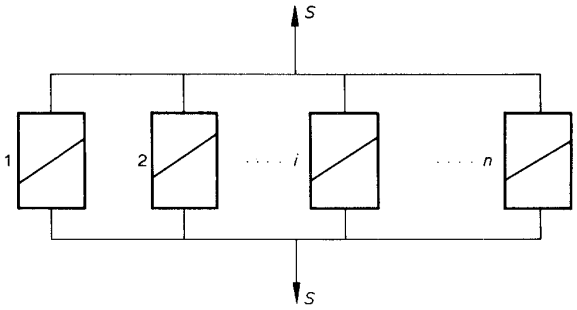

Figure 5.

$$R = \sum_{i=1}^{n} R_i \qquad (6)$$

where the strength of element i is given by the random variable R_i. Note that when R_i, i= 1,…,n are normally distributed, R is also normally distributed. If this is not the case it might according to the central limit theorem be a good approximation to assume R to be normally distributed if the number of elements, n, is not too low.

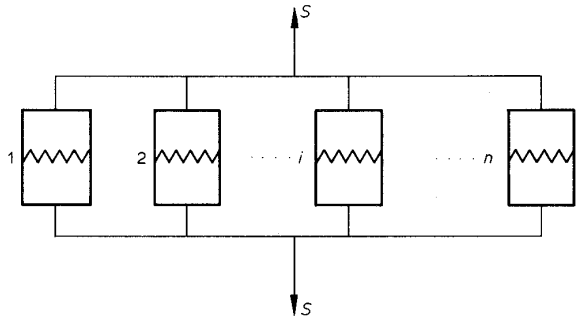

Figure 6.

A parallel system with perfectly brittle elements is shown in Figure 6. Let r_1, r_2, . . . , l_n, where $r_1 < r_2 <. . . < r_n$, be the strength of the n elements. Then the strength r of the system is given by:

$$r=\max(nr_1, (n\text{-}1)r_2,..., 2r_{n-1},r_n) \qquad (7)$$

It has been shown [3] that r under certain conditions approaches a normal distribution.

As mentioned above failure of a statically indeterminate structure can sometimes be evaluated on the basis of a number of failure modes where each failure mode can be modelled by a parallel system. Modelling of the complete structure will then be a series system of parallel systems (see Figure 7). Note that a given element may appear in several failure modes. Also note that correlation in such a system can appear in at least two forms, namely by correlation between single elements and correlation between failure modes.

RELIABILITY BOUNDS FOR SYSTEMS

Calculation of the exact probability of failure for a complicated system as shown in Figure 7 is usually not possible and a numerical determination is often rather time-consuming. It is therefore of great interest to investigate the possibility of deriving upper and lower bounds for the exact probability of failure of a given system. Of course the practical value of such bounds depends on how narrow they are. In this section only systems with positive correlated elements will be considered.

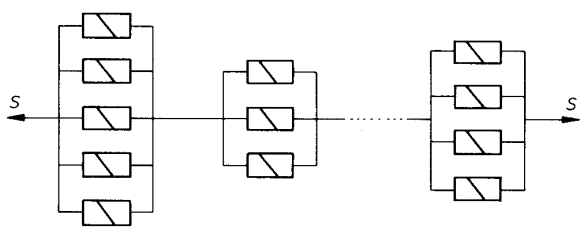

Figure 7.

If a system has a great number of elements it is convenient to use Boolean variables to describe the state of the elements [4] – [6]. For each element it is assumed that it can exist in two states, namely "failure" or "non-failure". Therefore, a Boolean variable S_i is associated with each element:

$$S_i = \begin{cases} 1 & \text{if element is in a non} - \text{failure state} \\ 0 & \text{if element is in a failure state} \end{cases} \tag{8}$$

It is useful to define a set of Boolean variables F_i by:

$$F_i = 1 - S_i \tag{9}$$

Consider now a system with n elements. The state of such a system is determined by the vector $\boldsymbol{S} = (S_1, S_2, \ldots, S_n)$ and can be described by a so-called system function $S_S(\boldsymbol{S})$. The system function is also a Boolean variable:

$$S_S(\boldsymbol{S}) = \begin{cases} 1 \text{ if element is in a non} - \text{failure state} \\ 0 \text{ if element is in a failure state} \end{cases} \tag{10}$$

For a series system one obtains by analogy with (4) the following system function:

$S_S(\boldsymbol{S}) = \prod_{i=1}^{n} S_i$ or by definition

$$S_S(\boldsymbol{F}) = 1 - S_S(\boldsymbol{S}) = 1 - \prod_{i=1}^{n} (1 - F_i) \tag{11}$$

where $\boldsymbol{F} = (F_1, F_2, \ldots, F_n)$.

For a parallel system the system function is given by:

$$S_S(\boldsymbol{S}) = 1 - \prod_{i=1}^{n} (1 - S_i) \quad \text{or} \quad S_S(\boldsymbol{F}) = \prod_{i=1}^{n} F_i \tag{12}$$

From the definition of S_i and F_i, i = 1,…, n it follows for a single element i that the expected values $E[S_i]$ and $E[F_i]$ are associated with the probability of failure $P(F_i = 1)$ in the following way:

$$\begin{aligned} E[S_i] &= 1 - P(F_i = 1) \\ E[F_i] &= P(F_i = 1) \end{aligned} \tag{13}$$

The probability of failure for the system is then by analogy:

$$P_f = E[S_S(\boldsymbol{F})] \tag{14}$$

Determination of the probability of failure for a system on the basis of (14) will imply calculation of n-dimensional multi-integrals. The exact value of P_f can therefore only be determined in some simplified cases. As a consequence of this it is of great interest to find approximations for P_f in the form of upper and lower bounds. Such bounds have been derived, [l], [5], [7], [8].

Very simple bounds can be derived for a series system with positive correlation between the strength of the elements on the basis of (11). When:

$$P\left(\prod_{j=1}^{i+1} S_j = 1\right) \geq P(S_{i+1} = 1) P\left(\prod_{j=1}^{i} S_j = 1\right) \tag{15}$$

for all $1 \leq i \leq n-1$it is easy to show the following bounds:

$$\max_{i=1,n} P(F_i = 1) \le P_f \le 1 - \prod_{i=1}^{n} (1 - P(F_i = 1)) \tag{16}$$

where the lower bound corresponds to perfect dependence between all elements and the upper bound to no dependence between any pair of elements. For a parallel system one obtains the bounds:

$$\prod_{i=1}^{n} (1 - P(F_i = 1)) \le P_f \le \min_{i=1,n} P(F_i = 1) \tag{17}$$

The above-mentioned authors have derived better bounds. The closed bounds are apparently those derived by Ditlevsen [5]. As his bounds are used later in this paper a brief derivation will be given here. It follows from (11) that:

$$\begin{aligned} S_S(\boldsymbol{F}) &= 1 - S_1 S_2 ... S_n = 1 - S_1 S_2 ... S_{n-1} + S_1 S_2 ... S_{n-1} F_n \\ &= F_1 + S_1 F_2 + S_1 S_2 F_3 + ... + S_1 S_2 ... S_{n-1} F_n \end{aligned} \tag{18}$$

Therefore;

$$P_f = E[F_1] + E[S_1 F_2] + ... + E[S_1 S_2 ... S_{n-1} F_n] \tag{19}$$

It is easy to see that:

$$1 - (F_1 + F_2 + ... + F_{n-1}) \le S_1 S_2 ... S_{n-1} \le 1 - F_i \tag{20}$$

where $i < n$. By inserting (20) into (19) and bearing in mind that all probabilities are nonnegative one obtained the following lower and upper bounds:

$$P_f \ge P(F_1 = 1) + \sum_{i=2}^{n} \max \left[P(F_i = 1) - \sum_{j=2}^{i-1} P((F_i = 1) \bigcap (F_j = 1)), 0 \right] \tag{21}$$

$$P_f \le \sum_{i=1}^{n} P(F_1 = 1) - \sum_{i=1}^{n} \max_{j<i} P((F_i = 1) \bigcap (F_j = 1)) \tag{22}$$

The upper bound (22) has been derived by Vanmarcke [8]. A lower bound similar to (21) has been suggested by Spaethe [9]. The numbering of the elements may influence the bounds (21) and (22). Therefore, to obtain the best bounds one has to choose from the different possible numberings of the elements. The bounds (21) and (22) will be used later in evaluating a new simplified method to estimate the probability of failure for a series system.

SYSTEMS WITH EQUALLY CORRELATED ELEMENTS

The previous section showed bounds for the probability of failure of systems. In some cases, however, it is possible to calculate the exact probability of failure. Grigoriu & Turkstra 10 has done this for series systems and for parallel systems with ductile elements on the assumption that the strength of the elements can be modelled by normally distributed random variables, R_i, $i = 1,..., n$, which are equally correlated with a common correlation coefficient p. Further, the loads are deterministic and constant in time and all elements have the same reliability index β_e.

It is for some structures relevant to assume equally correlated elements. For a great number of structures such an assumption cannot be justified. It is, however, as

shown by Grigoriu and Turkstra [10] a great advantage to use this assumption because then the exact probability of failure can be calculated. Bearing this in mind it seems worth- while to investigate the possibility of using a kind of 'average' coefficient of correlation in the general case where the correlation is unequal. Such an investigation is performed later in the paper for parallel systems and series systems, respectively. The rest of this section gives a brief presentation of the work by Grigoriu and Turkstra [10] so that the main results can be used in the following sections.

For a series system with *n* elements it can be shown (see Appendix 1) that the probability of failure on the assumptions mentioned above is given by:

$$P_f(\rho) = 1 - \int_{-\infty}^{\infty} \left[\Phi\left(\frac{\beta_e + \sqrt{\rho}t}{\sqrt{1-\rho}} \right) \right]^n \phi(t)dt \qquad (23)$$

where Φ and ϕ denote the distribution and density function for the standard Gaussian random variable. The variation with ρ of the probability of failure P_f is shown in Figure 8 for n = 1, 2, 5, and 10 and β_e = 3.0. The upper and lower bounds (16) corresponding to ρ = 0 and ρ = 1 are given by:

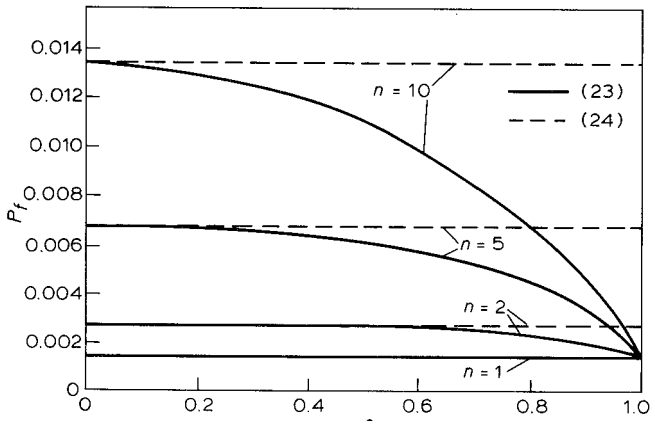

Figure 8.

$$\rho = 0: \quad P_f = 1 - \left[\Phi(\beta_e)\right]^n \qquad (24)$$

$$\rho = 1: \quad P_f = 1 - \Phi(\beta_e) \qquad (25)$$

Note that P_f, as expected, decreases with the correlation coefficient ρ and is increased with the number of elements *n*.

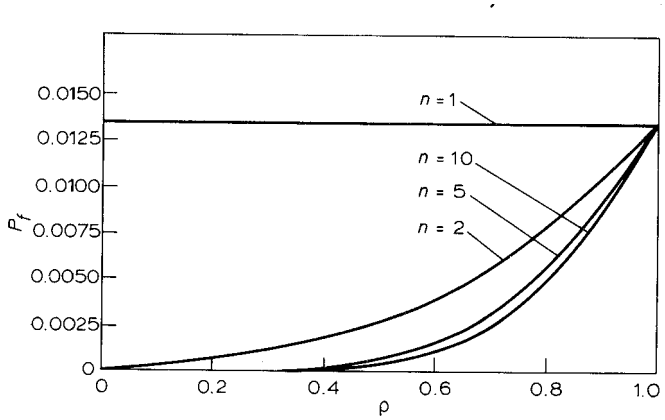

Figure 9.

Grigoriu and Turkstra [l0] consider a parallel system with *n* ductile elements having identical Gaussian member resistances. Further, the elements are assumed to be equally correlated with the common positive correlation coefficient ρ and the load is deterministic and constant in time. As shown in Appendix 2 the reliability index β_S for the system is then given by the simple expression:

$$\beta_S = \beta_e \sqrt{\frac{n}{1 + \rho(n-1)}} \qquad (26)$$

where β_e is the common reliability index for all elements.

Due to the assumption of Gaussian member resistances it follows from (6) that the strength of the parallel system is also Gaussian so that the probability of failure is given by:

$$P_f = \Phi(-\beta_S) \tag{27}$$

The variation with ρ of the probability of failure P_f is shown in Figure 9 for n = 1,2,5 and 10 and β_e = 3.0. Bounds for P_f can be derived from (26) and (27) by inserting ρ = 0 and ρ = 1. One obtains:

$$\Phi(-\beta_e\sqrt{n}) \le P_f \le \Phi(-\beta_e) \tag{28}$$

Note from Figure 9 that P_f increases with the correlation coefficient ρ and decreases with the number of elements.

PARALLEL SYSTEMS WITH DUCTILE ELEMENTS

In a real structure modelled by a system with n elements the correlation coefficient between the elements will usually not be equal. This may be the case if, for example, different contractors produce the elements. In this section the influence on the probability of failure of abandoning the assumption of equal correlation coefficients is investigated for parallel and series systems. As in the previous section the following assumptions are made:

(1) The loads are deterministic and constant with time,

(2) The strength R_i, i = 1,2, . . . , n for the members are Gaussian $N(\mu_i, \sigma_i)$,

where $\mu_i = E[R_i]$ and $\sigma_i^2 = \mathrm{var}[R_i]$,

(3) All elements are designed to have a common reliability index β_e.

Note that the only difference from the assumptions of Grigoriu and Turkstra [10] is connected to the correlation between the elements. The correlation coefficient between element i and j will be denoted ρ_{ij} and the correlation matrix is defined by the matrix:

$$\underline{\underline{C}} = [\rho_{ij}] = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix} \tag{29}$$

For a parallel system with n ductile elements the probability of failure can easily be calculated by generalization of (26), when $\mu_i = \mu$ and $\sigma_i = \sigma$. Following the same procedure as in Appendix 2 the reliability index β_S for the system is now:

$$\begin{aligned} \beta_S &= \frac{E[R]-Q}{\mathrm{var}[R]} = (n\mu - (n\mu - n\beta_e\sigma))(n\sigma^2 + \sigma^2 \sum_{i,j=1, i\neq j}^{n} \rho_{ij})^{-½} \\ &= \beta_e \sqrt{\frac{n}{1+\bar{\rho}(n-1)}} \end{aligned} \tag{30}$$

where

$$\bar{\rho} = \frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^{n} \rho_{ij} \tag{31}$$

By comparing (26) and (30) it is seen that for systems with nonequal correlation coefficients the reliability index β_S can be calculated by the simple expression (26) by inserting for ρ the average correlation coefficient $\bar{\rho}$ defined by (31). $\bar{\rho}$ is the average of all $\rho_{ij}, i \neq j$.

This result is interesting because the probability of failure for such a system can now easily be calculated for any correlation matrix (29). One only needs to make a set of curves as in Figure 9 once.

SERIES SYSTEM

In this section series systems with *n* elements are considered with the same three assumptions as in the previous section. An exact determination of the probability of failure will, as mentioned earlier, imply extensive numerical calculations of n-dimensional integrals. It is therefore of interest to investigate the possibility of using an approximate calculation. Suggesting new methods for determination of approximate values for the probability of failure P_f for such a series system will do this. Use will be made of the fact that it is relatively easy to compute P_f when the correlation coefficients $\rho_{ij}, i \neq j$ are equal (see earlier section). The method suggested here is thus based on the same correlation coefficients (equivalent correlation coefficient) for any pair of elements and defining this equivalent correlation coefficient so that the resulting value of P_f is close to the exact value. Evaluation of this method is done by comparing the results with lower and upper bonds (21) and (22) derived by Ditlevsen [5].

The equivalent correlation coefficient $\bar{\bar{\rho}}$ is chosen so that:

(1) $\bar{\bar{\rho}}$ is the common ρ for a system with equal correlation coefficient $\rho_{ij}, i \neq j$

(2) $\bar{\bar{\rho}}$ is in the interval [0; 1].

In the last section it was shown that use of the average correlation coefficient $\bar{\rho}$ gives the correct value of P_f for a parallel system. It is therefore reasonable to investigate whether the use of $\bar{\rho}$ as an equivalent correlation coefficient gives good results. To investigate this a series system with the following simple correlation matrix $\bar{\bar{C}}$ is considered.

$$\bar{\bar{C}} = \begin{bmatrix} 1 & \rho_{12} & 0.2 & \cdots & 0.2 \\ \rho_{12} & 1 & 0.2 & \cdots & 0.2 \\ 0.2 & 0.2 & 1 & \cdots & 0.2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.2 & 0.2 & 0.2 & \cdots & 1 \end{bmatrix} \tag{32}$$

The probability of failure can now be calculated from (23) with ρ equal to the average correlation coefficient ρ defined by (31). The result is shown in. Figure 10 for *n* = 2, 3, 5 and 10 and β_e = 3.0 as the curves②. The lower and upper bounds (21) and (22) by Ditlevsen [5] are calculated for some values of ρ_{12} and are shown in Figure 10 as intervals ④The curves① are upper bounds corresponding to independent elements

(equation (16)). The curves ③ are approximations based on an equivalent correlation coefficient defined later in this section.

It appears from Figure 10 that the bounds (21) and (22) are very close for n = 3 and n = 5. For n = 10 some gap is found. As P_f for ρ_{12} = 0.2 is correct using $\rho = \bar{\rho}$ the lower bound (21) is apparently very close to the correct value for ρ_{12} in the neighbourhood of 0.2.

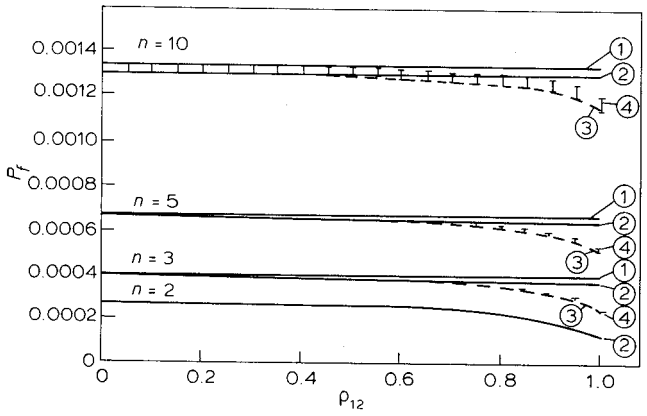

Figure 10

For ρ_{12} > 0.7 it is clear from Figure 10 that using $\bar{\rho}$ as an equivalent correlation coefficient is on the safe side. The curves 2 do not decrease as fast as the correct values of P_f for ρ_{12} approaching 1.0. To obtain a better agreement a modification of $P(\bar{\rho})$ must be used.

A much better agreement can in this case be obtained by using an equivalent correlation coefficient $\bar{\bar{\rho}}$ indirectly defined by:

$$P_f(\bar{\bar{\rho}}) = P_f(\bar{\rho}) - (P_{f,2}(\bar{\rho}) - P_{f,2}(\rho_{\max})) \tag{33}$$

where $P_{f,2}(\rho_{\max})$ is the probability of failure calculated from (23) with n = 2 and $\rho = \rho_{\max} = \max \rho_{ij}, i \neq j$. Note that $P_f(\bar{\bar{\rho}})$ gives the correct value when all $\rho_{ij}, i \neq j$ are equal. $P_f(\bar{\bar{\rho}})$ is shown in Figure 10 as the curves 3 and it can be concluded that in this example the values of $P_f(\bar{\bar{\rho}})$ are close to the (lower) bounds, but a little on the unsafe side.

It is convenient to use approximate values for P_f based on $P_f(\bar{\rho})$ or $P_f(\bar{\bar{\rho}})$ because one only needs a table giving P_f as a function of n and ρ as given by (23). Then for any correlation matrix $\bar{\bar{C}}$ approximate values can easily be calculated by hand.

The example shown in Figure 10 seems to indicate that $P_f(\bar{\rho})$ is an upper bound for the exact probability of failure. It has not been possible to show this in general. The statement that $P_f(\bar{\rho})$ is an upper bound is correct for n = 3 where the exact value can be calculated and compared with $P_f(\bar{\rho})$. A complete systematic investigation of the statement by numerical testing is very computer-time consuming. Therefore evaluation of $P_f(\bar{\rho})$ and $P_f(\bar{\bar{\rho}})$ is performed by generating randomly a great number of correlation matrices $\bar{\bar{C}}$. The single elements in $\bar{\bar{C}}$ are assumed to be uniformly distributed in the interval [0; 1]. Nonpositive definite matrices were of course rejected.

The results of these extensive investigations are shown in Table 1 where the upper bound (22) is denoted P_f^u and the lower bound (21) P_f^l.

From Table 1 it can be concluded that in none of almost 20000 computer tests with *n* equal to 3, 5 or 10 and β_e equal to 2.5, 3.0 or 3.5, was $P_f(\bar{\rho}) < P_f^l$. Therefore, the statement that $P_f(\bar{\rho})$ is an upper bound for P_f cannot be rejected by these experiments. Note that the relative number of cases where $P_f(\bar{\rho}) < P_f^u$ is increasing with the number of elements *n*, but decreasing with the element reliability index β_e.

Table 1

Numbers of elements *n*	3			5			10		
Reliability index β	2.5	3.0	3.5	2.5	3.0	3.5	2.5	3.0	3.5
$P_f(\bar{\rho}) < P_f^u$, number in %	25.0	17.2	11.5	26.5	8.1	2.8	66.5	15.9	0
$P_f(\bar{\rho}) < P_f^l$, number in %	0	0	0	0	0	0	0	0	0
$P_f(\bar{\bar{\rho}}) < P_f^u$, number in %	97.7	95.4	92.8	94.9	76.0	52.9	96.7	65.3	11.7
$P_f(\bar{\bar{\rho}}) < P_f^l$, number in %	79.6	82.2	82.1	1.8	2.1	1.8	0	0	0
max $(P_f^l/P_f(\bar{\rho}))$	1.08	1.06	1.05	1.01	1.01	1.005			
$E[P_f(\bar{\rho})/P_f(0)]$	0.905	0.938	0.961	0.882	0.931	0.964	0.820	0.896	0.950
$\mathrm{Var}[P_f(\bar{\rho})/P_f(0)]$	0.005	0.004	0.003	0.002	0.001	0.001	0.001	0.000	0.000
$E[P_f(\bar{\bar{\rho}})/P_f(0)]$	0.041	0.880	0.910	0.826	0.882	0.922	0.778	0.858	0.917
$\mathrm{Var}[P_f(\bar{\bar{\rho}})/P_f(0)]$	0.009	0.008	0.006	0.003	0.002	0.002	0.001	0.001	0.000
$E[P_f(\bar{\bar{\rho}})/P_f(\bar{\rho})]$	0.928	0.938	0.947	0.936	0.947	0.957	0.949	0.958	0.965
$\mathrm{Var}[P_f(\bar{\bar{\rho}})/P_f(\bar{\rho})]$	0.003	0.003	0.003	0.001	0.001	0.001	0.001	0.001	0.000
Number of cases	3600	3600	3600	2588	2624	2659	154	154	154

Table 1 shows that $P_f(\bar{\bar{\rho}})$ is not an upper bound for P_f. However, for increasing *n* the relative number of cases where $P_f(\bar{\bar{\rho}}) < P_f$ is decreasing fast. Note that the maxi- mum difference between P_f and $P_f(\bar{\bar{\rho}})$ is only worth mentioning for *n* = 3.

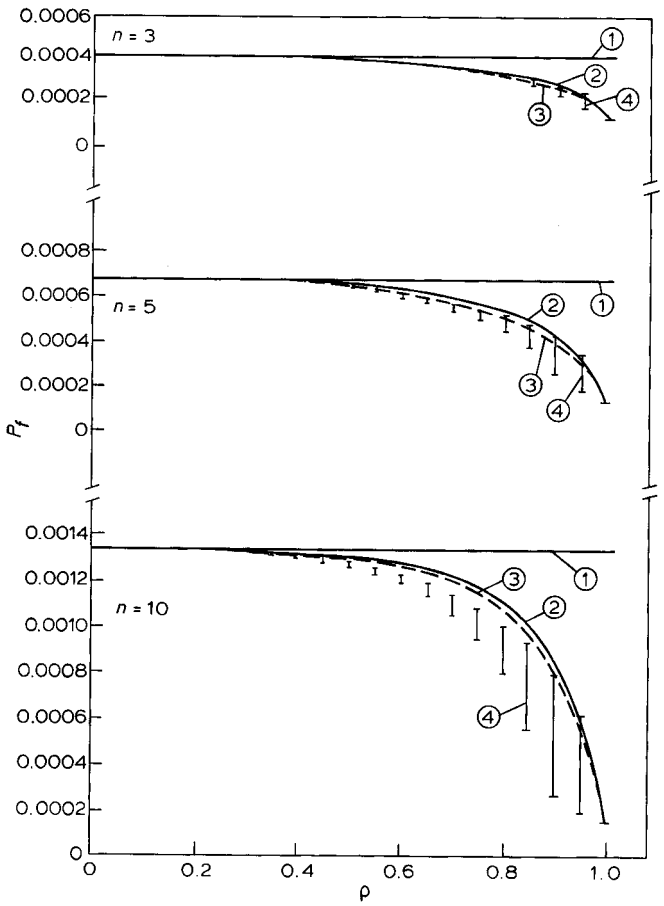

Figure 11.

Finally, in Table 1 the improvement by using $P_f(\bar{\rho})$ or $P_f(\bar{\bar{\rho}})$ instead of $P_f(0)$ is shown. $P_f(0)$ is the primitive upper bound corresponding to inde-pendent elements. The im- provement is increasing with *n* and decreasing with β_e. For *n* = 10 and β_e = 2.5 the probability of failure is averagely 18% less by using $P_f(\bar{\bar{\rho}})$ instead of P_f (0). The probability of failure is on average 5% less by using $P_f(\bar{\bar{\rho}})$ instead of $P_f(\bar{\rho})$.

A final illustration of the use of the different methods to evaluate the

probability of failure of series systems is made for a system with the following correlation matrix:

$$\overline{\overline{C}} = \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^n \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho^2 & \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \vdots & \vdots & \vdots & & & \vdots \\ \rho^n & \rho^{n-1} & \rho^{n-2} & & \cdots & 1 \end{bmatrix} \tag{34}$$

The results for β_e = 3.0 are shown in Figure 11 for *n* = 3, 5 and 10. The upper bounds $P_f(0)$ are the curves ①. $P_f(\bar{\rho})$ are the curves ② and $P_f(\overline{\overline{\rho}})$ the curves ③. The lower and upper bounds (21) and (22) are the intervals ④. In this case the curves for $P_f(\bar{\rho})$ and $P_f(\overline{\overline{\rho}})$ are rather close and they differ only slightly from the bounds (21) and (22). Note the great improvement obtained by using $P_f(\bar{\rho})$ or $P_f(\overline{\overline{\rho}})$ instead of $P_f(0)$ corresponding to independent elements.

With regard to the bounds (21) and (22) it can be concluded from Figure 11 that they may be rather wide especially for correlation coefficients greater than 0.7 and when *n* increases. For small correlation coefficients the bounds are narrow and they are therefore in such cases of great importance in evaluating the probability of failure.

CONCLUSIONS

This paper presents two new methods to calculate the probability of failure for some series systems and some parallel systems with ductile elements taking into account the correlation between the elements. It is shown by a number of examples that correlation between member resistances is an essential parameter in determination of the probability of failure for systems.

The first new method is based on the so-called average correlation coefficient. It is shown for parallel systems that under some assumptions, exact values for the probability of failure can be obtained in this way by very simple calculations. For series systems use of the average correlation coefficient seems to be on the safe side although it has not been possible to give a formal proof of this.

In some situations, especially for highly correlated systems, the value of the probability of failure may be too far on the safe side when using the average correlation coefficient. Therefore, an equivalent correlation coefficient is introduced. This new correlation coefficient is almost as simple to use as the average correlation coefficient and extensive numerical testing shows that it gives values for the probability of failure close to well-known bounds for this probability.

It is safer for a series system to ignore the correlation between the elements. But some improvements may be obtained by using the two methods presented in this paper compared with assuming no correlation.

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APPENDIX 1

Derivation of equation (23)

Following Stuart [11] formula (23) will be derived in this appendix. The strength of element i is described by the random variable R_i which is assumed to be Gaussian $N(\mu_i, \sigma_i)$. If $\tilde{R}_i = (R_i - \mu_i)/\sigma_i$, then the probability of failure for a series system with n elements is given by:

$$P_f = 1 - P(\min_{1 \le i \le n} R_i \ge Q) = 1 - P(\max_{1 \le i \le n} \tilde{R}_i \le \beta_e) \tag{A1}$$

where β_e is the common reliability index for all elements and Q the load effect for all members.

Let X_i, $i = 0, 1, \ldots, n$, be random variables with:

$$E[X_i] = 0 \tag{A2}$$

$$E[X_i^2] = 1 \tag{A3}$$

$$E[X_i X_j] = 0, \; i \neq j \tag{A4}$$

Define random variables $\bar{R}_i$, $i = 1, \ldots, n$, by:

$$\bar{R}_i = X_i - a_i X_0 \tag{A5}$$

then from (A2)-(A4):

$$E[\bar{R}_i] = 0 \tag{A6}$$

$$E[\bar{R}_i^2] = 1 + a_i^2 \tag{A7}$$

$$E[R_i R_j] = a_i a_j, \; i \neq j \tag{A8}$$

Therefore, the coefficients of correlation for $\bar{R}_i$ and $\bar{R}_j$ are:

$$\rho_{ij} = \begin{cases} \dfrac{a_i a_j}{\sqrt{1 + a_i^2}\sqrt{1 + a_j^2}} & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \tag{A9}$$

By assuming $\rho_{ij} = \rho, i \neq j, 0 \leq \rho \leq 1$, the coefficients $a_i = a$ can be determined:

$$a = \sqrt{\frac{\rho}{1-\rho}} \tag{A10}$$

If X_i, i = 0, 1,…,n are Gaussian then they are independent. From (A7) the variance of $\overline{R}_i$ is equal to $1+a^2$. Therefore, the probability of $\tilde{R}_i \leq \beta_e$ is equal to the probability of $\bar{R}_i \leq \beta_e\sqrt{1+a^2}$:

$$\begin{aligned} P_f(\rho) &= 1 - P(\max_{1\leq i\leq n} \bar{R}_i \leq \beta_e\sqrt{1+a^2}) = 1 - P(\max_{1\leq i\leq n} X_i \leq aX_0 + \beta_e\sqrt{1+a^2}) \\ &= 1 - \int_{-\infty}^{\infty} [\Phi(at + \beta_e\sqrt{1+a^2})]^n \phi(t)dt = 1 - \int_{-\infty}^{\infty} \left[\Phi\left(\frac{\beta_e + \sqrt{\rho}t}{\sqrt{1-\rho}}\right)\right]^n \phi(t)dt \end{aligned} \tag{A11}$$

APPENDIX 2

Derivation of equation (26)

A parallel system with n ductile elements is considered. The strength R of the system is equal to the sum of the resistances R_i, i = 1,2,..., n, of the elements. Then:

$$E[R] = \sum_{i=1}^{n} \mu_i \tag{A12}$$

$$\text{var}[R] = \sum_{i=1}^{n} \sigma_i^2 + \rho \sum_{i,j=1, i\neq j}^{n} \sigma_i\sigma_j \tag{A13}$$

where μ_i = $E[R_i]$, σ_i^2 = var[R_i] and ρ is the common correlation coefficient. Let the load of the system be Q and the common reliability index β_e. Further, if members are identically distributed, then μ_i = μ and σ_i = σ for i = 1,2,…, n. Then the reliability index β_S for the system is:

$$\beta_S = \frac{E[R] - Q}{(\text{var}[R])^{1/2}} = \frac{n\mu - (n\mu - n\beta_e\sigma)}{\sqrt{n\sigma^2 + n(n-1)\sigma^2\rho}}$$

or:

$$\beta_S = \beta_e\sqrt{\frac{n}{1+\rho(n-1)}} \tag{A14}$$