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## CHAPTER 17

### RELIABILITY ANALYSIS OF ELASTO-PLASTIC STRUCTURES<sup>1</sup>

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#### 1. INTRODUCTION

This paper only deals with framed and trussed structures, which can be modelled as systems with ductile elements. The elements are all assumed to be linear-elastic perfectly plastic. The loading is assumed to be concentrated and time-independent. The strength of the elements and the loads are modelled by normally distributed stochastic variables. This last assumption is not essential, since non-normally distributed variables can be approximated by equivalent normally distributed variables by well-known methods. All geometrical dimensions and stiffness quantities are assumed to be deterministic.

Failure of this type of system is defined either as formation of a mechanism or by failure of a prescribed number of elements. In the first case failure is independent of the order in which the elements fail, but this is not so by the second definition.

The reliability analysis consists of two parts. In the first part significant failure modes are determined. Non- significant failure modes are those that only contribute negligibly to the failure probability of the structure. Significant failure modes are determined by the  $\beta$ -unzipping method by Thoft-Christensen [1]. Two different formulations of this method are described and the two definitions of failure can be used by the first formulation, but only the failure definition based on formation of a mechanism by the second formulation.

The second part of the reliability analysis is an estimate of the failure probability for the structure on the basis of the significant failure modes. The significant failure modes are as usual modelled as elements in a series system (see e.g. Thoft-Christensen & Baker [2]). Several methods to perform this estimate are presented including upper- and lower-bound estimates.

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<sup>1</sup> Proceedings of 11<sup>th</sup> IFIP Conf. On System Modelling and Optimization, Copenhagen, July 1983. Lecture Notes in Control and Information Sciences, Vol. 59, 1984, Springer-Verlag, pp. 556-566.

Upper bounds for the failure probability estimate are obtained if the failure mechanisms are used. Lower bounds can be calculated on the basis of series systems where the elements are the non-failed elements in a non-failed structure (see Augusti & Baratta [3]).

## 2. IDENTIFICATION OF SIGNIFICANT FAILURE MODES

A simple, but also computer-time consuming method to determine significant failure modes is simulation, where realizations of relevant stochastic variables are simulated. Corresponding to each set of realizations failure modes is determined by analysis of the structure (see Ferregut-Avila [4]).

Several authors have suggested heuristic search. An ideal heuristic search method will disclose the significant failure modes in a sequence with decreasing failure probabilities. Moses [5] has suggested an incremental method where one significant failure mode is determined. A drawback by this method is that all loads are fully correlated. Further failure modes can be determined by changing the strength of the elements (increasing the variance) or by simulating realizations of the strength of the elements (see Gorman [6]). Identification of significant failure modes can also be formulated as an optimization problem. For linear structures the safety margins of the mechanisms are linear in load and strength variables. The individual mechanisms are identified by the coefficients to the stochastic variables and the reliability index  $\beta$  for the mechanisms is a non-linear function of these coefficients. Ma & Ang [7] consider identification of significant failure modes as the problem of finding minimum reliability index  $\beta$ . The variables in this optimization problem are the coefficients to the load and strength variables. A local minimum of the reliability index  $\beta$  corresponds to a significant failure mechanism. All mechanisms are linear combinations of a set of fundamental mechanisms. This observation has been used by Ma & Ang [7] in formulating the problem of determining significant failure modes as a non-linear optimization problem. Klingmüller [8] has used a linear programming method to determine significant failure modes. Failure tree methods have also been used for this purpose. Each node (branching point) in the failure tree corresponds to a failure element. Murotsu et al. [9] calculates the failure probabilities for all elements and selects those with the highest failure probabilities. These elements are supposed to fail one by one and additional fictitious loads are applied to the structure corresponding to the yield capacity of the failed elements. By this method a number of failure modes are determined. A failure tree method has also been used by Kappler [10].

The  $\beta$ -unzipping method (Thoft-Christensen [1], Thoft-Christensen & Sørensen [11]) is a failure tree method to identify significant failure modes. Each branch in the failure tree is chosen on the basis of reliability indices for the elements. In this paper two formulations of the  $\beta$ -unzipping method are presented.

It is assumed that the structure can be modelled by  $n$  so-called failure elements with safety margins

$$M_i = \min\{R_i^+ - S_i, R_i^- + S_i\} \quad (1)$$

where  $S_i$  is a stochastic variable describing the loading of the failure element  $i$  and where  $R_i^+$  and  $R_i^-$  are stochastic variables describing the yield capacity in “tension” and “compression”. Often  $R_i^+ = R_i^-$ . By the linear elastic analysis

$$S_i = \sum_{j=1}^k a_{ij} P_j \quad (2)$$

where  $P_j$ ,  $j = 1, 2, \dots, k$  are the loads on the structure and  $a_{ij}$  are the coefficients of influence. The failure criterion for failure element  $i$

$$M_i \leq 0 \quad (3)$$

and the corresponding reliability index  $\beta_i$  can be calculated.

In the  $\beta$ -unzipping method failure elements are assumed to fail one at a time until a mechanism has been formed. After failure of an element the structure is modified by putting the corresponding strength equal to zero and adding a fictitious loading  $P_{k+i}$  corresponding to the yield capacity of the  $i$ th element in the failure sequence  $\{j_1, j_2, \dots, j_n\}$ . Formation of a mechanism can be unveiled by the fact that the corresponding stiffness matrix is singular. In the computer programme used in this investigation two alternatives can be used for the fictitious loads  $P_{k+i} = f(R_{j_i})$ ,  $i = 1, 2, \dots$ ;  $j_i \in \{1, 2, \dots, n\}$

$$f(R_{j_i}) = R_{j_i} \quad (4)$$

i.e. equal to the yield capacity of the element.

$$f(R_{j_i}) = \gamma_{j_i} R_{j_i} = \frac{r_{j_i}}{\mu_{R_{j_i}}} R_{j_i} \quad (5)$$

where  $r_{j_i}$  is the  $j_i$ -coordinate of the design point and where  $\mu_{R_{j_i}}$  is the expected value at  $R_{j_i}$ . Let element  $i$  be the last element to fail before the mechanism is formed. Then the safety margin  $M_i$  is equal to the safety margin for the mechanism provided all the elements have failed (during the unzipping) in the same manner as the actual mechanism indicates. Experience shows that this is not always the case. Therefore, instead of (1) the following safety margin is used

$$M_i = \min\left\{\left(R_i^+ + \sum_{j=k+1}^{k+m} |a_{ij}| P_j - \sum_{j=1}^k a_{ij} P_j\right), \left(R_i^- + \sum_{j=k+1}^{k+m} |a_{ij}| P_j + \sum_{j=1}^k a_{ij} P_j\right)\right\} \quad (6)$$

where  $m$  is the number of failed elements. By using  $|a_{ij}|$  and not  $a_{ij}$  in (6) the correct safety margin is obtained because then all coefficients to yield forces and yield moments are positive.

In figure 1 part of a typical failure tree determined by the  $\beta$ -unzipping method is shown. In each circle the upper number is equal to the number of the failure element and the lower number to the corresponding  $\beta$ -index. Each branch ending in a box indicates a failure mechanism or failure mode. At each branching point the structure is re-analyzed as described above, and failure elements with reliability indices within a prescribed distance from the lowest index for non-failed elements define the new branches of the failure tree. More detailed description of the automatic generation of the failure tree is given in Thoft-Christensen [1] and Thoft-Christensen & Sørensen [11].

With regard to the safety margin the first parenthesis in (6) is chosen if

$$\sum_{j=1}^k a_{ij} \mu_{P_j} \geq 0$$

and the second parenthesis if

$$\sum_{j=1}^k a_{ij} \mu_{p_i} \leq 0.$$

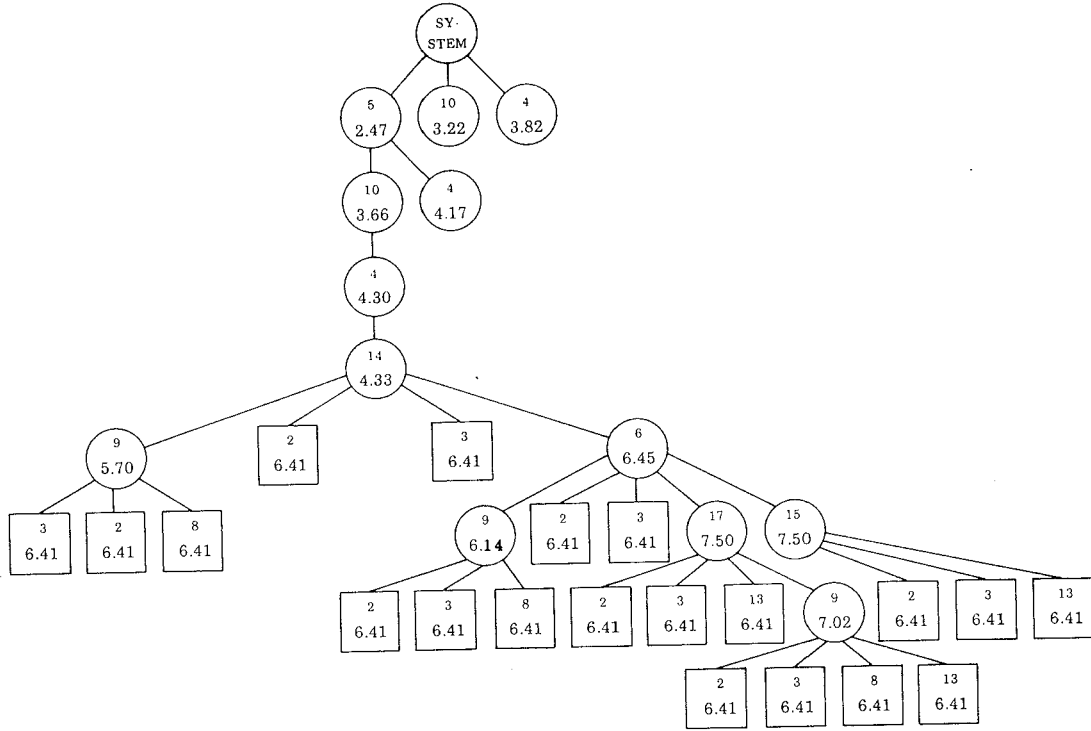


Figure 1. Failure tree for the structure in figure 5.

○ indicates non-failure of the structure and □ failure of the structure.

It is of interest to note that the general  $\beta$ -unzipping method can also be used when failure of a structure is defined as failure of a number of elements and also when the behaviour of some elements is modelled by brittle elements ( $f(R_{j_i}) = 0$ ).

In the alternative formulation based on fundamental mechanisms only ductile elements can be taken into account and failure is defined as formation of a mechanism. Fundamental mechanisms can be automatically generated by a method suggested by Watwood [12]. The number of fundamental mechanisms is  $m = n - r$ , where  $n$  is the number of potential failure elements (yield hinges) and  $r$  the degree of redundancy. Let the number of real mechanisms be  $m_e > 0$ , then the number of joint mechanisms is  $m_k = m - m_e$ . The safety margin for the fundamental mechanism  $i$  is

$$M_i = \sum_{j=1}^n |a_{ij}| R_j - \sum_{j=1}^k b_{ij} P_j \quad (7)$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are influence matrices. Let  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_{m_e}$  be an ordered set of reliability indices for the  $m_e$  real mechanisms. Further mechanisms can be constructed by linear combinations of fundamental mechanisms. The corresponding failure margins will be like (7). The problem is now to combine the fundamental mechanisms so that the significant mechanisms can be obtained in an efficient way. Here a failure tree

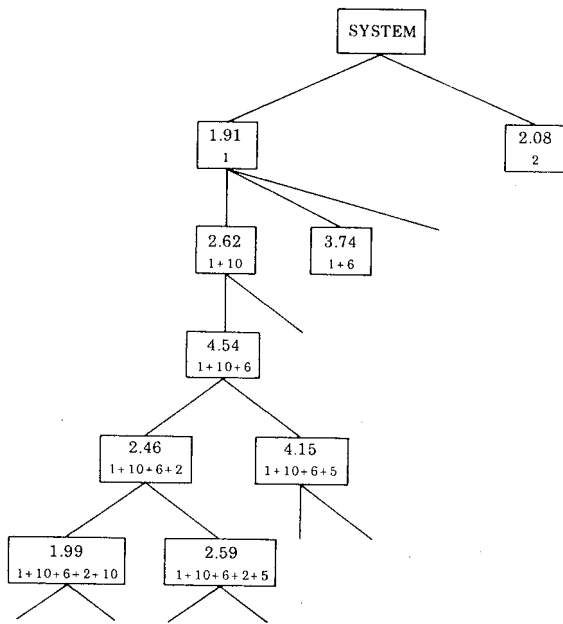


Figure 2. Part of the failure tree for the structure considered in example 1.

2.62
1+10

 indicates that the mechanism obtained by combining the fundamental

formulation can be used. First the real fundamental mechanism 1 is chosen. Then it is combined with all the other fundamental mechanisms and the corresponding reliability indices are calculated. The smallest index  $\beta_{\min}$  among those is determined and mechanisms with indices in the interval  $[\beta_{\min}, \beta_{\min} + \varepsilon_1]$ , where  $\varepsilon_1$  is a chosen constant, define the branches in the failure tree based on the mechanism in question. The combined mechanisms are now starting points for new branches where each branch symbolizes that the branching point at the end of the branch is constructed by combination of a combined mechanism and a fundamental mechanism. Corresponding to each branching point a new  $\beta_{\min}$  value is determined. In general,  $\beta_{\min}$  will decrease when more and more mechanisms are combined. The branching is terminated when the number of levels is

greater than a number  $N$  or when  $\beta_{\min} > \beta_u + \varepsilon_2$  where  $\beta_u$  is the reliability index for the mechanism and where  $\varepsilon_2 > 0$ . In this way each branching point symbolizes a mechanism. This is illustrated in figure 2, where part of the failure tree for the structure in figure 3 is shown. There are in this case 5 real mechanisms and 5 joint mechanisms.

Significant failure mechanisms are determined on the basis of the failure tree. Let the smallest reliability index for a mechanism be  $\beta_0$ . Then the significant mechanisms are defined as those with reliability indices in the interval  $[\beta_0, \beta_0 + \varepsilon_3]$  where  $\varepsilon_3 > 0$ .

### 3. EVALUATION OF THE PROBABILITY OF FAILURE

A measure of the reliability of the structure can be obtained on the basis of the identified significant failure mechanisms by modelling failure of the structure as a series system where the elements are the significant failure mechanisms (see Thoft-Christensen & Baker [2]). The probability of failure then is

$$P_s = P(F_1 \cup F_2 \cup \dots \cup F_n) \quad (8)$$

where  $P(\cdot)$  is the probability measure and  $F_i$ ,  $i = 1, 2, \dots, n$  is the event that failure occurs by failure of mechanism  $i$ . In general  $F_i$  and  $F_j$ ,  $i \neq j$ , will be correlated due to correlation between the failure elements and due to common failure elements in the failure mechanisms. Therefore, estimate of  $P_s$  will involve time-consuming calculation of multi-integrals. However, a number of approximate methods have been suggested (see e.g. Thoft-Christensen & Sørensen [11] or Grimmelt & Schuëller [13]).

Upper and lower bounds for  $P$  have been suggested by Ditlevsen [14] and Kounias [15]. Calculation of these bounds (here called Ditlevsen bounds) requires estimate of the probability of the intersection of  $F_i$  and  $F_j$ , i.e.  $P(F_i \cap F_j)$ . In this paper  $P(F_i \cap F_j)$  is calculated by numerical integration. Ang & Ma [16] have suggested an approximate method called the PNET-method, to evaluate  $P_s$ . This method is based on a grouping of all elements according to mutual correlation.

In the following, safety margins are assumed normally distributed and linear. Then from (8)

$$P_s = 1 - \int_{-\infty}^{\beta_1} \int_{-\infty}^{\beta_2} \dots \int_{-\infty}^{\beta_n} \varphi_n(\mathbf{x}; \boldsymbol{\rho}) dx_1 dx_2 \dots dx_n = 1 - \Phi_n(\boldsymbol{\beta}; \boldsymbol{\rho}) \quad (9)$$

where  $\varphi_n(\mathbf{x}; \boldsymbol{\rho})$  is the  $n$ -dimensional density function for  $n$  standardized stochastic variables and  $\boldsymbol{\rho}$  is the matrix of correlation coefficients. When all correlation coefficients are equal,  $\rho_{ij} = \rho \geq 0$ , then (see Dunnet & Sobel [18]) |

$$P_s = 1 - \int_{-\infty}^{\infty} \varphi(u) \prod_{i=1}^n \Phi\left(\frac{\beta_i - \sqrt{\rho} u}{\sqrt{1-\rho}}\right) du \quad (10)$$

Plackett [17] has shown that

$$P_s = 1 - \Phi_n(\boldsymbol{\beta}; \boldsymbol{\kappa}) - \sum_{i < j} \int_{\kappa_{ij}}^{\rho_{ij}} \frac{\partial \Phi_n(\boldsymbol{\beta}; t\boldsymbol{\rho} + (1-t)\boldsymbol{\kappa})}{\partial \lambda_{ij}} d\lambda_{ij} \quad (11)$$

where  $\boldsymbol{\rho}$  and  $\boldsymbol{\kappa}$  are regular matrices of correlation and

$$\lambda_{ij} = t\rho_{ij} + (1-t)\kappa_{ij}, \quad 0 \leq t \leq 1 \quad (12)$$

and where e.g.

$$\frac{\partial \Phi_n(\boldsymbol{\beta}; \boldsymbol{\lambda})}{\partial \lambda_{12}} = \varphi_2(\beta_1, \beta_2; \lambda_{12}) \Phi_{n-2}(\beta_3 - \mu_{21}, \beta_4 - \mu_{22}, \dots, \beta_n - \mu_{2, n-2}; \mathbf{m}_{22}) \quad (13)$$

$\mu_2$  and  $\mathbf{m}_{22}$  are the conditional expected value vector and covariance matrix for  $X_3, X_4, \dots, X_n$  given  $X_1 = \beta_1$  and  $X_2 = \beta_2$ .

A simple approximate value for  $P_s$  can be obtained from (10) even if the correlation coefficients are unequal, namely by putting  $\rho = \bar{\rho}$ , where  $\bar{\rho}$  is the average correlation coefficient defined by

$$\bar{\rho} = \frac{1}{n(n-1)} \sum_{i \neq j} \rho_{ij} \quad (14)$$

This corresponds to neglecting the last term in (11) and putting  $\kappa_{ij} = \bar{\rho}$

$$P_s \cong 1 - \Phi_n(\boldsymbol{\beta}; \{\bar{\rho}\}) \quad (15)$$

where  $\{\bar{\rho}\}$  is a correlation matrix with  $\rho_{ij} = \bar{\rho}$ ,  $i \neq j$ .

From equation (13) it is seen that

$$0 \leq \frac{\partial \Phi_n(\boldsymbol{\beta}; \boldsymbol{\rho})}{\partial \rho_{ij}} \leq \varphi_2(\beta_i, \beta_j; \rho_{ij}) \quad (16)$$

Let  $\rho_{\min} = \min_{i < j} \{\rho_{ij}\}$  and  $\rho_{\max} = \max_{i < j} \{\rho_{ij}\}$  then from (11), (13), and (16) the following bounds for  $\Phi_n$  can be derived ( $\Phi_n$  is an increasing function of  $\rho_{ij}$ ):

$$\Phi_n(\boldsymbol{\beta}; \boldsymbol{\rho}) \leq \min\{\Phi_n(\boldsymbol{\beta}; \{\rho_{\max}\}), [\Phi_n(\boldsymbol{\beta}; \{\rho_{\min}\}) + \sum_{i < j} (\Phi_2(\beta_i, \beta_j; \rho_{ij}) - \Phi_2(\beta_i, \beta_j; \rho_{\min}))]\} \quad (17)$$

$$\Phi_n(\boldsymbol{\beta}; \boldsymbol{\rho}) \geq \max\{\Phi_n(\boldsymbol{\beta}; \{\rho_{\min}\}), [\Phi_n(\boldsymbol{\beta}; \{\rho_{\max}\}) + \sum_{i < j} (\Phi_2(\beta_i, \beta_j; \rho_{ij}) - \Phi_2(\beta_i, \beta_j; \rho_{\max}))]\} \quad (18)$$

It follows from (12) - (16) that  $P_s$  can also be approximated by

$$P_s \cong 1 - \Phi_n(\boldsymbol{\beta}; \{\bar{\rho}\}) - \sum_{i < j} \{\Phi_2(\beta_i, \beta_j; \rho_{ij}) - \Phi_2(\beta_i, \beta_j; \bar{\rho})\} \quad (19)$$

The approximation (15) is suggested by Thoft-Christensen & Sørensen [19] (see also Ditlevsen [20]). An approximation based on the so-called equivalent correlation coefficient  $\bar{\bar{\rho}}$  indirectly defined by

$$P_s \cong 1 - \Phi_n(\boldsymbol{\beta}; \{\bar{\bar{\rho}}\}) - \Phi_2(\beta_i, \beta_j; \rho_{\max}) + \Phi_2(\beta_i, \beta_j; \bar{\bar{\rho}}) = 1 - \Phi_n(\boldsymbol{\beta}; \{\bar{\bar{\rho}}\}) \quad (20)$$

has also been suggested by Thoft-Christensen & Sørensen [19]. This approximation can be derived from (19) by neglecting all terms in the summation except the one with the maximum correlation coefficient.

Let  $n_s$  and  $n_r$  be the number of kinematically admissible mechanisms and the number of identified significant failure mechanisms, respectively. Then it follows from the upper-bound theorem of plasticity (see Augusti & Baratta [3]) that  $P_s^l$  given by

$$P_s^l = P\left(\bigcup_{i=1}^{n_r} F_i\right) \quad (21)$$

is a lower bound of  $P_s$ . This can easily be seen from (8) when  $n$  is substituted by  $n_s$  and  $n_r$ . An upper bound of  $P_s$  can be derived from the lower-bound theorem of plasticity (see [3])

$$P_s^u = P\left(\bigcap_{i=1}^{n_m} \bigcup_{j=1}^{n_{u,i}} F'_{ij}\right) \quad (22)$$

where  $n_m$  is a number of statically admissible stress distributions,  $n_{u,i}$  is the number of non-failed failure elements in the structure  $i$  corresponding to a statically admissible stress distribution, and  $F'_{ij}$  is the event that failure element  $j$  in such a structure fails. If the numerical signs in (6) are removed, then  $F'_{ij}$  is equal to the event that  $M_i \leq 0$  and (22) can be written

$$P_s^u = P\left(\bigcap_{i=1}^{n_m} \bigcup_{j=1}^{n_{u,i}} (F_{ij}^C \cup F_{ij}^T)\right) = P\left(\bigcap_{i=1}^{n_m} F_i^n\right) \quad (23)$$

where  $F_{ij}^C$  and  $F_{ij}^T$  correspond to the linear safety margins in the first and second parenthesis in (6). Note that (23) corresponds to a parallel system with elements, which are series systems with  $2n_{u,i}$  elements. It is often useful to use the following upper bound of  $P_s^u$

$$P_s^u \leq \min_i P(F_i^n) \quad (24)$$

Finally

$$P\left(\bigcup_{i=1}^{n_r} F_i\right) \leq P_s \leq \min_i P\left(\bigcup_{j=1}^{n_{u,i}} (F_{ij}^C \cup F_{ij}^T)\right) \quad (25)$$

The bounds in (25) can be estimated by the approximations and bounds for series systems shown earlier in this chapter.

The safety margins corresponding to the right hand side of (25) can only be determined if the first formulation of the  $\beta$ -unzipping method is used.

In the next chapters some of the methods shown above will be used in two examples. The results will be presented by the so-called generalized  $\beta$ -index,  $\beta_G$ , defined by (see Ditlevsen [21])

$$\beta_G = -\Phi^{-1}(P_s) \quad (26)$$

#### 4. EXAMPLE 1

Consider the frame structure in figure 3 with corresponding expected values and coefficients of variation for the stochastic variables in table 1. Yield moments in the same line are considered fully correlated and yield moments in different lines are mutually independent.

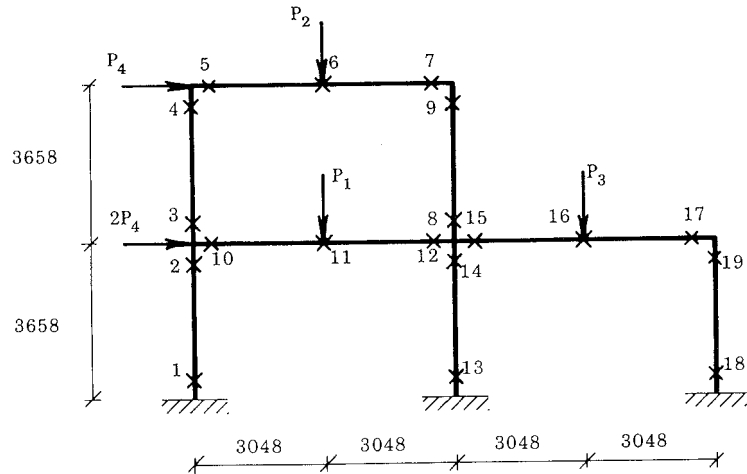


Figure 3. Geometry, loading and potential yield hinges (x) for the structure analysed in example 1.

The 12 most dominant failure mechanisms are shown in table 2 (from Ma & Ang [7]). By the first formulation of the  $\beta$ -unzipping method and by (4) the mechanisms shown in table 3 are determined. The calculations are performed on a CDC Cyber 170-720 computer. 10 of the most dominant failure mechanisms are determined after 2000 sec. computer time, but the remaining mechanisms 10 and 11 are closely correlated with mechanisms 3 and 1 and will therefore have a negligible influence on the reliability of the structure.

Variables	Expected values	Coefficients of variation
$P_1$	$169 \times 10^3$	0.15
$P_2$	$89 \times 10^3$	0.25
$P_3$	$116 \times 10^3$	0.25
$P_4$	$31 \times 10^3$	0.25
$R_1, R_2, R_{13}, R_{14}, R_{18}, R_{19}$	$95 \times 10^6$	0.15
$R_3, R_4, R_8, R_9$	$95 \times 10^6$	0.15
$R_5, R_6, R_7$	$122 \times 10^6$	0.15
$R_{10}, R_{11}, R_{12}$	$204 \times 10^6$	0.15
$R_{15}, R_{16}, R_{17}$	$163 \times 10^6$	0.15

Table 1.

No.	Failure elements										$\beta$
1	1	6	8	9	11	12	13	14	18	19	1.88
2	1	2	13	14	18	19					1.91
3	1	6	9	11	12	13	16	18	19		1.94
4	1	3	11	12	13	14	18	19			1.99
5	4	6	9								1.99
6	15	16	19								1.99
7	1	6	7	8	11	12	13	14	18	19	2.08
8	10	11	12								2.08
9	1	6	7	11	12	13	16	18	19		2.10
10	1	4	9	11	12	13	16	18	19		2.17
11	1	4	8	9	11	12	13	14	18	19	2.18
12	5	6	7								2.18

Table 2. The 12 most significant failure mechanisms for the structure in Figure 3.

Run No.	Significant failure mechanisms										Computer time (sec.)
1	8	12									125
2	7	8	12								150
3	7	8	9	12							300
4	2	4	7	8	9	12					750
5	2	4	5	6	7	8	9	12			1000
6	1	2	3	4	5	6	7	8	9	12	2000

Table 3. Failure mechanisms by the  $\beta$ -unzipping method.

By the second formulation of the  $\beta$ -unzipping method based on the set of fundamental mechanisms shown in fig. 4 all 12 significant failure mechanisms were identified in only 17 sec.  $N_{\max} = 20$ ,  $\varepsilon_1 = 0.5$ ,  $\varepsilon_2 = 2$  and  $\varepsilon_3 = 0.3$  were used. Although the choice of fundamental mechanisms and of  $N_{\max}$ ,  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\varepsilon_3$  values may be of

importance for this result it seems to indicate that the formulation based on fundamental mechanisms is superior to the more direct first formulation. The main reason for the reduced computer time is of course that only one analysis of the structure is necessary by the second formulation, whereas the first formulation requires such an analysis at each branching point.

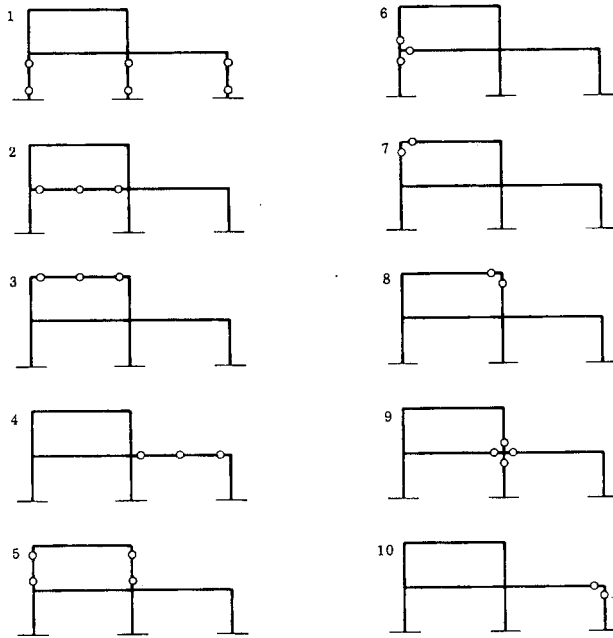


Figure 4. Set of fundamental mechanisms used in example 1.

Run No.	Ditlevsen bounds	$\beta_G(\bar{\rho})$	$\beta_G(\bar{\bar{\rho}})$	PNET
1	1.83 - 1.83	1.79	1.83	1.83
2	1.69 - 1.69	1.67	1.69	1.62
3	1.62 - 1.68	1.58	1.66	1.62
4	1.39 - 1.52	1.35	1.42	1.50
5	1.17 - 1.30	1.14	1.20	1.29
6	1.15 - 1.36	1.08	1.16	1.15

Table 4. Generalized reliability index  $\beta_G$  for the structure in figure 3.

The generalized reliability index  $\beta_G$  will in this example be estimated on the basis of some of the methods presented in chapter 3. The mechanisms in table 3 are used as elements in a series system and the results are shown in table 4.

Ma & Ang [7] have calculated the Ditlevsen bounds on the basis of all different real mechanisms with the result  $0.53 \leq \beta_G \leq 1.47$  and by Monte-Carlo simulation determined the estimate  $\beta_G = 1.20$ .

On the basis of the safety margins determined by the first formulation of the IJ-unzipping method one gets  $-1.17 \leq \beta_G \leq 1.30$ . These bounds are determined as Ditlevsen bounds for the series systems in the right and left hand side of (25). The lower bound is for a structure with failure in elements 1,6,11,12,15,18, and 19. Computer time is 2000 seconds.

The upper bound 1.30 for  $\beta_G$  calculated by (25) is close to the simulated estimate 1.20, but the lower bound is useless. It follows from table 4 that the approximate estimate  $\beta_G(\bar{\rho}) = 1.16$  and

$\beta_G(PNET) = 1.15$  only differ by 3% from the simulated estimate.

## 5. EXAMPLE 2

Consider the structure in figure 5 and the data in table 5. All stochastic variables are considered independent. This structure has been investigated by Grimmelt & Schuëller [13]. By the first formulation of the  $\beta$ -unzipping method and (6) all significant failure mechanisms (see table 6) were identified in 154 seconds. The number of fundamental mechanisms is  $n - r = 17 - 9 = 8$ . By the second formulation of the  $\beta$ -unzipping method based on the set of fundamental mechanisms in figure 6 and the same data as in

example 1 all significant failure mechanisms were identified in 8 seconds. This result confirms the conclusion in example 1.

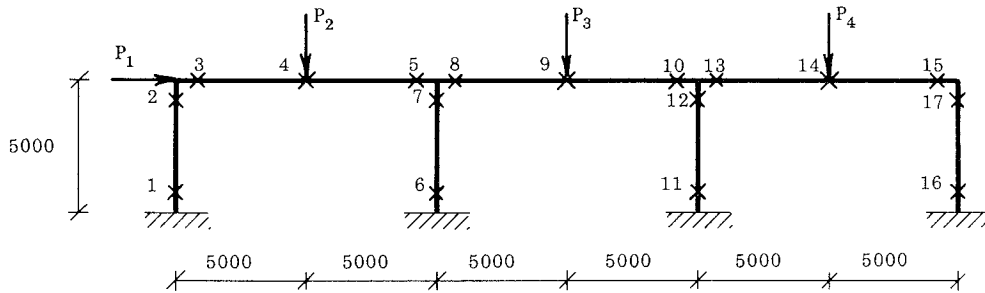


Figure 5. Geometry, loading and potential yield hinges for the structure analysed in example 2.

On the basis of the significant mechanisms in table 6 the Ditlevsen bounds are  $6.17 \leq \beta_G \leq 6.17$ . Further,  $\beta_G(\bar{\rho}) = 6.16$  and  $\beta_G(PNET) = 6.24$ . These values are close to the results by Grimmelt & Schuëller [13] calculated on the basis of all possible mechanisms. It can therefore be concluded that the 5 mechanisms in table 6 really are the most dominant.

Variables	Expected values	Coefficient of variation
$P_1$	$50 \cdot 10^3$	0.1
$P_2, P_3, P_4$	$40 \cdot 10^3$	0.1
$R_1, R_2, \dots, R_{17}$	$101 \cdot 10^6$	0.1

Table 5.

No.	Failure elements			$\beta$
1	2	4	5	6.41
2	3	4	5	6.41
3	8	9	10	6.41
4	13	14	15	6.41
5	13	14	17	6.41

Table 6. Significant failure mechanisms.

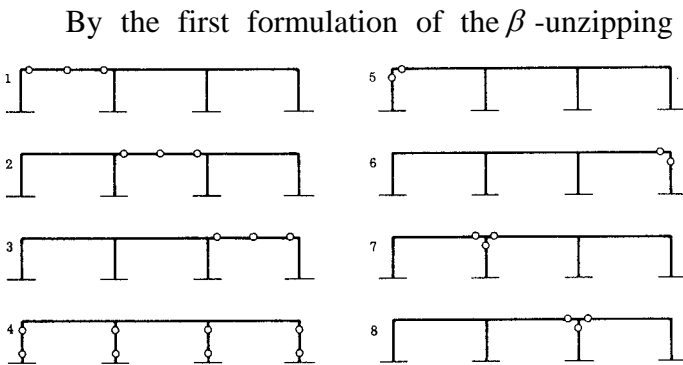


Figure 6. Set of fundamental mechanisms used in example 2,

By the first formulation of the  $\beta$ -unzipping method the following bounds corresponding to the right hand side of (25) can be calculated:  $3.94 \leq \beta_G \leq 3.94$ . They correspond to a structure with failure in elements 6, 9, 18, 19 or 6, 9, 19, 20. Then by (25) one gets  $3.94 \leq \beta_G \leq 6.17$ . Again only the upper bound is close to the correct value.

## 6. CONCLUSIONS

In the first part of this paper two formulations of the so-called  $\beta$ -unzipping method to identify significant failure modes were presented. Both methods are failure tree methods. The efficiency of the two methods is evaluated on the basis of two examples. The conclusion is that the second formulation seems to be superior when it can be used. It is based on fundamental mechanisms and is therefore only applicable to failure definitions based on mechanisms. The first formulation is more general. It can be used for ductile and brittle elements and failure of the structure need not be defined as formation of a mechanism.

In the second part of the paper the estimate of the failure probability is investigated. A lower bound for the probability of failure is calculated on the basis of the identified significant failure mechanisms. These failure mechanisms are elements in a series system. A number of different methods to estimate the failure probability of series systems are discussed, namely Ditlevsen bounds, the PNET-method, and approximate methods based on the average and the equivalent correlation coefficient. New bounds and a new approximation are suggested. The reliability of two different structures is estimated by some of these methods and it can be concluded that the upper bound for the failure probability  $P_s$  (lower bound for the generalized index  $\beta_G$  is a bad estimate for the correct value. However, the lower bound for  $P_s$  (upper bound for  $\beta_G$  is close to the correct value. For both structures approximate estimates of  $\beta_G$  based on the equivalent correlation coefficient  $\bar{\rho}$  and the PNET-method are good estimates.

## 7. REFERENCES

- [1] Thoft-Christensen, P.: *The  $\beta$ -unzipping Method*. Institute of Building Technology and Structural Engineering, Aalborg University Centre, Report 8207, Aalborg 1982.
- [2] Thoft-Christensen, P. & Baker, M. J.: *Structural Reliability and Its Applications*. Springer Verlag, Berlin - Heidelberg - New York, 1982.
- [3] Augusti, G. & Baratta, A.: *Limit Analysis of Structures with Stochastic Strength Variations*. Journal of Structural Mechanics, Vol. 1, No.1, 1972, pp. 43-62.
- [4] Ferregut-Avila, C. M.: *Reliability Analysis of Elasto-Plastic Structures*. NATO Advanced Study Institute, P. Thoft-Christensen (ed.), Martinus Nijhoff, The Netherlands, 1983, pp. 445-451.
- [5] Moses, F.: *Structural System Reliability and Optimization*. Computers & Structures, Vol. 7, 1977, pp. 283-290.
- [6] Gorman, M. R.: *Reliability of Structural Systems*. Report No. 79-2, Case Western Reserve University, Ohio, Ph.D. Report, 1979.
- [7] Ma, H.-F. & Ang, A. H.-S.: *Reliability Analysis of Ductile Structural Systems*. Civil Eng. Studies, Structural Research Series No. 494, University of Illinois, Urbana, August 1981.
- [8] Klingmüller, O.: *Anwendung der Traglastberechnung für die Beurteilung der Sicherheit von Konstruktionen*. Forschungsberichte aus dem Fachbereich Bauwesen. Gesamthochschule Essen, Heft 9, Sept. 1979.
- [9] Murotsu, Y., Okada, H., Yonezawa, M. & Taguchi, K.: *Reliability Assessment of Redundant Structure*. ICOSSAR '81, Trondheim, Norway, 1981, pp. 315-329.

- [10] Kappler, H.: *Beitrag zur Zuverliissigkeitstheorie von Tragwerken unter BerUcksichtigung nichtlineare Verhaltens*. Dissertation, Technische Universität München, 1980.
- [11] Thoft-Christensen, P. & Sørensen, J. D.: *Calculation of Failure Probabilities of Ductile Structures by the  $\beta$ -unzipping Method*. Institute of Building Technology and Structural Engineering, Aalborg University Centre, Report 8208, Aalborg 1982.
- [12] Watwood, V. B.: *Mechanism Generation for Limit Analysis of Frames*. ASCE, Journal of the Structural Division, Vol. 105, NO. ST1, Jan. 1979, pp. 1-15.
- [13] Grimmelt, M. & Schuëller, G. I.: *Benchmark Study on Methods to Determine Collapse Failure Probabilities of Redundant Structures*. SFB 96, Heft 51, Technische Universität München, 1981.
- [14] Ditlevsen, O.: *Narrow Reliability Bounds for Structural Systems*. Journal of Structural Mechanics, Vol. 7, No.4, 1979, pp. 453-472.
- [15] Kounias, E. G.: *Bounds for the Probability of a Union, with Applications*. The Annals of Mathematical Statistics, Vol. 39, 1968, pp. 2154-2158.
- [16] Ang, A. H.-S. & Ma, H.-F.: *On the Reliability Analysis of Framed Structures*. Proc. ASCE Specialty Conference on Probabilistic Mechanics and Structural Reliability, Tucson 1979, pp. 106-111.
- [17] Plackett, R. K.: *A Reduction Formula for Normal Multivariate Integrals*. Biometrika, Vol. 41, 1954, pp. 351-360.
- [18] Dunnet, C. W. & Sobel, M.: *Approximations to the Probability Integral and Certain Percentage Points of a Multivariate Analogue of Student's t-Distribution*. Biometrika, Vol. 42, 1955, pp. 258-260.
- [19] Thoft-Christensen, P. & Sørensen, J. D.: *Reliability of Structural Systems with Correlated Elements*. Applied Mathematical Modelling, Vol. 6, 1982, pp. 171-178.
- [20] Ditlevsen, O.: *Taylor Expansion of Series System Reliability*. DCAMM Report No. 235, Lyngby 1982.
- [21] Ditlevsen, O.: *Generalized Second Moment Reliability Index*. Journal of Structural Mechanics, Vol. 7, 1979, pp. 435-451.

