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CHAPTER 18

RELIABILITY OF STRUCTURAL SYSTEMS¹

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1. INTRODUCTION

During the last 10 years significant progress has been made in connection with estimation of the reliability of structural systems. Such a reliability analysis of a structural system consists of at least two parts, namely identification of critical (significant) failure modes and estimation of the failure probabilities of the failure modes.

In this paper critical failure modes are identified by the β -unzipping method (see Thoft-Christensen [1], [2] and Thoft-Christensen & Sørensen [3], [4]). The β -unzipping method is quite general in the sense that it can be used for two-dimensional and three-dimensional, framed and trussed structures, for structures with ductile and brittle elements and also in connection with a number of different failure mode definitions.

In the paper the structural system is modelled by a series system, where the elements are parallel systems (failure modes). The failure probability of such a series system is then estimated by

- 1) Estimating the reliability indices for each element in the parallel systems (failure modes),
- 2) Estimating the reliability indices for the parallel systems (failure modes),
- 3) Approximation of the safety margin for each parallel system by an equivalent linear safety margin,
- 4) Estimating the correlation between the parallel systems (failure modes),
- 5) Estimating the reliability index for the series system.

In this paper a brief presentation of the application of the β -unzipping method is given. The method is illustrated by simple examples.

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2. CLASSIFICATION OF FAILURE MODES

Clearly, the definition of failure modes for a structural system is of great importance in estimating the reliability of the structural system. In this section failure modes are classified in a systematic way convenient for the subsequent reliability estimate. A very simple estimate of the reliability of a structural system is based on failure of a single failure element, namely the failure element with the lowest reliability index (highest failure probability) of all failure elements. Failure elements are structural elements or cross-sections where failure can take place. The number of failure elements will usually be considerably higher than the number of structural elements. Such a reliability analysis is in fact not a system reliability analysis, but from a classification point of view it is convenient to call it *system reliability analysis at Level 0*. Let a structure consist of n failure elements and let the reliability index (see e.g. Thoft-Christensen & Baker [5]) for failure element i be β_i , then the system reliability index β_s^0 at level 0 is

$$\beta_s^0 = \min_{i=1, \dots, n} \beta_i \quad (1)$$

Clearly, such an estimate of the system reliability is too optimistic. A more satisfactory estimate is obtained by taking into account the possibility of failure of any failure element by modelling the structural system as a series system with the failure elements as elements (see figure 1). The probability of failure for this series system is

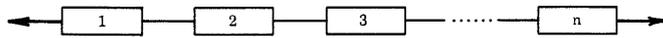


Figure 1. System modelling at level 1.

then estimated on the basis of the reliability indices β_i , $i=1, 2, \dots, n$, and the correlation between the safety margins for the failure elements. This reliability analysis is called *system reliability analysis at level 1*. In general it is only necessary to include some of the failure elements in the series system (namely those with the smallest β -indices) to get a good estimate of the system failure probability P_f^1 and the corresponding generalized reliability index β_s^1 , where

$$\beta_s^1 = -\Phi^{-1}(P_f^1) \quad (2)$$

and where Φ is the standardized normal distribution function. The failure elements included in the reliability analysis are called *critical failure elements*.

The modelling of the system at level 1 is natural for a statically determinate structure, but failure in a single failure element in a structural system will not always result in failure of the total system, because the remaining elements may be able to sustain the external loads due to redistribution of the load effects. This situation is characteristic of statically indeterminate structures. For such structures *system reliability analysis at level 2* or higher levels may be reasonable. At level 2 the systems



Figure 2. System modelling at level 2.

reliability is estimated on the basis of a series system where the elements are parallel systems each with two failure elements - so-called *critical pairs of failure elements* (see figure 2). These critical pairs of failure elements are obtained by modifying the structure by assuming in turn failure in the critical failure elements and adding fictitious loads corresponding to the load-carrying capacity of the elements in failure. Let e.g. element i be a critical failure element, then the structure is

modified by assuming failure in element i and the load-carrying capacity of the failure element is added as fictitious loads if the element is ductile. If the failure element is brittle, no fictitious loads are added. The modified structure is then analysed elastically and new β -values are calculated for all the remaining failure elements. Failure elements with low β -values are then combined with failure element i so that a number of critical pairs of failure elements are defined.

Analyzing modified structures where failure is assumed in critical pairs of failure elements now continues the procedure sketched above. In this way *critical triples of failure elements* are identified and a *reliability analysis at level 3* can be made

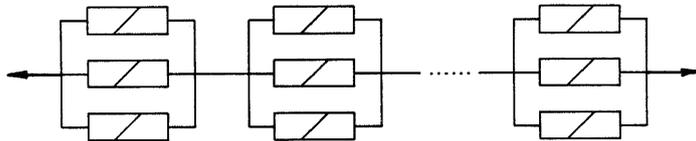


Figure 3. System modelling at level 3.

on the basis of a series system, where the elements are parallel systems each with three failure elements (see figure 3). By continuing in the same way reliability estimates at level 4, 5, etc.

can be performed, but in general analysis beyond level 3 is of minor interest.

Many recent investigations in structural systems theory concern structures, which can be modelled as elastic-plastic structures. In such cases failure of the structure is usually defined as formation of a mechanism. When this failure definition is used it is of great importance to be able to identify the most significant failure modes because the total number of mechanisms is usually much too high to be included in the reliability analysis. The β -unzipping method can be used for this purpose simply by continuing the procedure described above until formation of a mechanism has taken place. However, this will be very expensive due to the great number of reanalyses

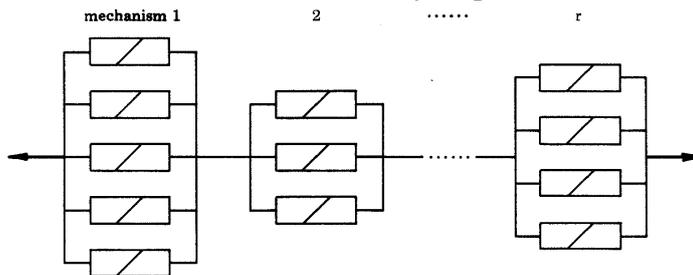


Figure 4. System modelling at mechanism level.

needed. It turns out to be much better to base the unzipping on reliability indices for fundamental mechanisms and linear combination of fundamental mechanisms. When system failure is defined as formation of a mechanism the probability

of failure of the structural system is estimated by modelling the structural system as a series system with the significant mechanisms as elements (see figure 4). Reliability analysis based on the mechanism failure definition is called *systems reliability analysis at mechanism level*.

For real structures a mechanism will often involve a relatively large number of yield hinges and the deflections at the moment of formation of a mechanism can usually not be neglected. Therefore, the failure definition must be combined with some kind of deflection failure definition.

3. RELIABILITY OF FUNDAMENTAL SYSTEMS

In this paper it is assumed that all basic variables (load variables and strength variables) are normally distributed. All geometrical quantities and elasticity coefficients are

assumed deterministic. This assumption significantly facilitates the estimation of the failure probability, but in general basic variables cannot with a satisfactory degree of accuracy be modelled by normally distributed variables. To overcome this problem a number of different transformation methods have been suggested. The most well known method was suggested by Rackwitz & Fiessler [6]. One drawback to these methods is that they increase the computational work considerably due to the fact that they are iterative methods. A simpler (but also less accurate) method called the multiplication factor has been proposed by Thoft-Christensen [2]. The multiplication factor method does not increase the computational work. Without loss of generality it is assumed that all basic variables are standardized, i.e. the mean value is 0 and the variance 1.

The β -unzipping method is in this paper only used for trussed and framed structures but it can easily be modified to other classes of structures. The structure is considered at a fixed point in time, so that only static behaviour has been treated. It is assumed that failure in a structural element (section) is either pure tension/compression or failure in bending. Combined failure criteria have also been used in connection with the (3-unzipping method, but only little experience has been obtained for the time being.

Let the vector $\bar{X} = (X_1, \dots, X_n)$ be the vector of the standardized normally distributed basic variables with the joint probability density function φ_n and let failure of a failure element be determined by a failure function $f: \omega \rightarrow R$, where ω is the n -dimensional basic variable space. Let f be defined in such a way that the space ω is divided into a failure region $\omega_f = \{\bar{x}: f(\bar{x}) \leq 0\}$ and a safe region $\omega_s = \{\bar{x}: f(\bar{x}) > 0\}$ by the failure surface $\partial\omega = \{\bar{x}: f(\bar{x}) = 0\}$ where the vector \bar{x} is a realization of the random vector \bar{X} . Then the probability of failure P_f for the failure element in question is given by

$$P_f = P(f(\bar{X}) \leq 0) = \int_{\omega_f} \varphi_n(\bar{x}) d\bar{x} \quad (2)$$

If the function f is linearized in the so-called design point with distance β to the origin of the coordinate system, then an approximate value for P_f is given by

$$P_f \approx P(\alpha_1 X_1 + \dots + \alpha_n X_n + \beta \leq 0) = P(\alpha_1 X_1 + \dots + \alpha_n X_n \leq -\beta) = \Phi(-\beta) \quad (3)$$

where $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ is the directional cosines of the linearized failure surface. β is the Hasofer-Lind reliability index. Φ is the standardized normal distribution function. The random variable

$$M = \alpha_1 X_1 + \dots + \alpha_n X_n + \beta \quad (4)$$

is the *linearized safety margin* for the failure element.

Next consider a *series system* with k elements. An estimate of the failure probability P_f^s of this series system can be obtained on the basis of the linearised safety margin of the form (4) for the k elements.

$$\begin{aligned} P_f^s &= P\left(\bigcup_{i=1}^k (\bar{\alpha}_i \bar{X} + \beta_i \leq 0)\right) = P\left(\bigcup_{i=1}^k (\bar{\alpha}_i \bar{X} \leq -\beta_i)\right) = 1 - P\left(\bigcap_{i=1}^k (\bar{\alpha}_i \bar{X} > -\beta_i)\right) \\ &= 1 - P\left(\bigcap_{i=1}^k (-\bar{\alpha}_i \bar{X} < \beta_i)\right) = 1 - \Phi_k(\bar{\beta}; \bar{\rho}) \end{aligned} \quad (5)$$

where $\bar{\alpha}_i$ and β_i are the directional cosines and the reliability index for failure element i , $i = 1, \dots, k$ and where $\bar{\beta} = (\beta_1, \dots, \beta_k)$. $\bar{\rho} = \{\rho_{ij}\}$ is the correlation coefficient matrix given by $\rho_{ij} = \bar{\alpha}_i^T \bar{\alpha}_j$ for all $i \neq j$. Φ_k is the standardized k -dimensional normal distribution function.

For a *parallel system* with k elements an estimate of the failure probability p_f can be obtained in the following way

$$P_f^p = P\left(\bigcap_{i=1}^k (\bar{\alpha}_i \bar{X} + \beta_i \leq 0)\right) = P\left(\bigcap_{i=1}^k (-\bar{\alpha}_i \bar{X} < -\beta_i)\right) = \Phi_k(\bar{\beta}; \bar{\rho}) \quad (6)$$

where the same notations as above are used.

It is important to note the approximation behind (5) and (6) namely the linearization of the general non-linear failure surfaces in the distinct design points for the failure elements. The main problem in connection with application of (5) and (6) is numerical calculation of the n -dimensional normal distribution function Φ_n for $n \geq 3$. This problem will be treated later in this paper where a number of methods to get approximate values for Φ_n are mentioned.

4. IDENTIFICATION OF CRITICAL FAILURE MODES

A number of different methods to identify critical failure modes has been suggested (see e.g. Ferregut-Avila [7], Moses [8], Gorman [9], Ma & Ang [10], Klingmiller [11], Murotsu et al. [12] and Kappler [13]). In this paper the β -unzipping method [1] - [4] is used.

At *level 1* the system reliability is defined as the reliability of a series system with n elements - the n failure elements. Therefore, the first step is to calculate β -values for all failure elements and then use equation (5). As mentioned earlier, equation (5) cannot be used directly. However, excellent upper and lower bounds - called Ditlevsen bounds - exist (see Ditlevsen [14] and Kounias [15]). The well-known simple bounds

$$\max_{i=1}^n \Phi(-\beta_i) \leq P_f \leq 1 - \prod_{i=1}^n \Phi(\beta_i) \quad (7)$$

are useful when the safety margins M_i , $i = 1, 2, \dots, n$ are almost perfectly correlated (the lower bound) or when the correlation between any pair of safety margins is very small (the upper bound).

Dunnet & Sobel [16] have shown that

$$P_f = 1 - \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^n \Phi\left(\frac{\beta_i - \sqrt{\rho} t}{\sqrt{1-\rho}}\right) dt \quad (8)$$

if all correlation coefficients ρ_{ij} are equal ($= \rho$) and positive.

When the correlation coefficients ρ_{ij} are unequal, a simple approximation for P_f can be obtained from (8) by putting $\rho = \bar{\rho}$, where $\bar{\rho}$ is the average correlation coefficient (see Thoft-Christensen & Sørensen [17]) defined by

$$\bar{\rho} = \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^n \rho_{ij} \quad (9)$$

Usually for a structure with n failure elements, the estimate of the failure probability of the series system with n elements can be calculated with sufficient accuracy by only including some of the failure elements, namely those with the smallest reliability indices. One way of selecting is to include only failure elements with β -values in an interval $[\beta_{\min}, \beta_{\min} + \Delta\beta_1]$, where β_{\min} is the smallest reliability index of all failure element indices and where $\Delta\beta_1$ is a prescribed positive number. The failure elements chosen to be included in the system reliability analysis at level 1 are called *critical failure elements*. If two or more critical failure elements are perfectly correlated, then only one of them is included in the series system of critical failure elements.

At level 2 the system reliability is estimated as the reliability of a series system where the elements are parallel systems each with 2 failure elements (see figure 2) - so-called *critical pairs of failure elements*. Let the structure be modelled by n failure elements and let the number of critical failure elements at level 1 be n_1 . Let the critical failure element l have the lowest reliability index β of all critical failure elements. Failure is then assumed in failure element l and the structure is modified by removing the corresponding failure element and adding a pair of so-called fictitious loads F_l (normal forces or moments). If the removed failure element is brittle, then no fictitious loads are added. However, if the removed failure element l is ductile then the fictitious load F_l is a stochastic load given by $F_l = \gamma_l R_l$, where R_l is the load-carrying capacity of failure element l and where $0 < \gamma_l \leq 1$.

The modified structure with the loads P_1, \dots, P_k and the fictitious load F_l (normal force or moment) is then reanalyzed and influence coefficients a_{ij} with respect to P_1, \dots, P_k and a'_{il} with respect to F_l are calculated. The load effect (force or moment) in the remaining failure elements is then described by a stochastic variable. The load effect in failure element i is called $S_{i|l}$ (load effect in failure element i given failure in failure element l) and

$$S_{i|l} = \sum_{j=1}^k a_{ij} P_j + a'_{il} F_l \quad (10)$$

The corresponding safety margin $M_{i|l}$ then is

$$M_{i|l} = \min(R_i^+ - S_{i|l}, R_i^- + S_{i|l}) \quad (11)$$

where R_i^+ and R_i^- are the stochastic variables describing the (yield) strength capacity in “tension” and “compression” for failure element i . In the following $M_{i|l}$ will be approximated by either $R_i^+ - S_{i|l}$ or $R_i^- + S_{i|l}$ depending on the corresponding reliability indices. The reliability index for failure element i , given failure in failure element l , is

$$\beta_{i|l} = \mu_{M_{i|l}} / \sigma_{M_{i|l}} \quad (12)$$

In this way new reliability indices are calculated for all failure elements (except the one where failure is assumed) and the smallest β -value is called β_{\min} . The failure elements with β -values in the interval $[\beta_{\min}, \beta_{\min} + \Delta\beta_2]$, where $\Delta\beta_2$ is a prescribed positive number, are then in turn combined with failure element l to form a number of parallel systems.

The next step is then to evaluate the failure probability for each critical pair of failure elements. Consider a parallel system with failure elements l and r . During the reliability analysis at level 1 the safety margin M_l for failure element l is determined and the safety margin $M_{r|l}$ for failure element r has the form (11). From these safety margins the reliability indices $\beta_1 = \beta_l$ and $\beta_2 = \beta_{r|l}$ and the correlation coefficient $\rho = \rho_{l,r|l}$ can easily be calculated. The probability of failure for the parallel system then is

$$P_f = \Phi_2(-\beta_1, -\beta_2; \rho) \quad (13)$$

The same procedure is then in turn used for all critical failure elements and further critical pairs of failure elements are identified. In this way the total series system used in the reliability analysis at level 2 is determined (see figure 2). The next step is then to estimate the probability of failure for each critical pair of failure elements (see (13)) and also to determine a safety margin for each critical pair of failure elements. When this is done generalized reliability indices for all parallel systems in figure 2 and correlation coefficients between any pair of parallel systems are calculated. Finally, the probability of failure P_f for the series system (figure 2) is estimated. The so-called equivalent linear safety margin introduced by Gollwitzer & Rackwitz [18] is used as approximations for safety margins for the parallel systems.

An important property by the β -unzipping method is the possibility of using the method when brittle failure elements occur in the structure. When failure occurs in a brittle failure element then the β -unzipping method is used in exactly the same way as presented above, the only difference being that no fictitious loads are introduced. If e.g. brittle failure occurs in a tensile bar in a trussed structure then the bar is simply removed without adding fictitious tensile loads. Likewise, if brittle failure occurs in bending, then a yield hinge is introduced, but no (yielding) fictitious bending moments are added.

The method presented above can easily be generalized to higher levels $N > 2$. At level 3 the estimate of the system reliability is based on so-called *critical triples of failure elements*, i.e. a set of three failure elements. The critical triples of failure elements are identified by the β -unzipping method and each triple forms a parallel system with three failure elements. These parallel systems are then elements in a series system (see figure 3). Finally, the estimate of the reliability of the structural system at level 3 is defined as the reliability of this series system.

Assume that the critical pair of failure elements (l, m) has the lowest reliability index $\beta_{l,m}$ of all critical pairs of failure elements. Failure is then assumed in the failure elements l and m adding for each of them a pair of fictitious loads F_l and F_m (normal forces or moments).

The modified structure with the loads P_1, \dots, P_k and the fictitious loads F_l and F_m are then reanalyzed and influence coefficients with respect to P_1, \dots, P_k and F_l and F_m are calculated. The load effect in each of the remaining failure elements is then described by a stochastic variable $S_{i|l,m}$ (load effect in failure element i given failure in failure elements l and m) and

$$S_{i|l,m} = \sum_{j=1}^k a_{ij} P_j + a'_{il} F_l + a'_{im} F_m \quad (14)$$

The corresponding safety margin $M_{i|l,m}$ then is

$$M_{i|l,m} = \min(R_i^+ - S_{i|l,m}, R_i^- + S_{i|l,m}) \quad (15)$$

where R_i^+ and R_i^- are the stochastic variables describing the load-carrying capacity in “tension” and “compression” for failure element i . In the following $M_{i|l,m}$ will be approximated by either $R_i^- + S_{i|l,m}$ or $R_i^+ - S_{i|l,m}$ depending on the corresponding reliability indices. The reliability index for failure element i , given failure in failure elements l and m , is then given by

$$\beta_{i|l,m} = \mu_{M_{i|l,m}} / \sigma_{M_{i|l,m}} \quad (16)$$

In this way new reliability indices are calculated for all failure elements (except Q and m) and the smallest β -value is called β_{\min} . These failure elements with β -values in the interval $[\beta_{\min}, \beta_{\min} + \Delta\beta_3]$, where $\Delta\beta_3$ is a prescribed positive number, are then in turn combined with failure elements l and m to form a number of parallel systems.

The next step is then to evaluate the failure probability for each of the critical triple of failure elements. Consider the parallel system with failure elements l , m , and r . During the reliability analysis at level 1 the safety margin M_l for failure element l is determined and during the reliability analysis at level 2 the safety margin $M_{m|l}$ for the failure element m is determined. The safety margin $M_{r|l,m}$ for safety element r has the form (15). From these safety margins the reliability indices $\beta_1 = \beta_l$, $\beta_2 = \beta_{m|l}$ and $\beta_3 = \beta_{r|l,m}$ and the correlation matrix $\bar{\rho}$ can easily be calculated. The probability of failure for the parallel system then is

$$P_f = \Phi_3(-\beta_1, -\beta_2, -\beta_3; \bar{\rho}) \quad (17)$$

An equivalent safety margin $M_{i,j,k}$ can be determined by the procedure mentioned above. When the equivalent safety margins are determined for all critical triples of failure elements the correlation between them two and two can easily be calculated. The final step is then to arrange all the critical triples as elements in a series system (see figure 3) and estimate the probability of failure P_f and the generalized reliability index β_s for the series system.

The β -unzipping method can be used in exactly the same way as described in the preceding text to estimate the system reliability at levels $N > 3$. However, a definition of failure modes based on a fixed number of failure elements greater than 3 will hardly be of practical interest.

The application of the β -unzipping method presented above can also be used when failure is defined as formation of a mechanism. However, it is much more efficient to use the β -unzipping method in connection with fundamental mechanisms. Experience has shown that such a procedure is less computer time consuming than unzipping based on failure elements.

If unzipping is based on failure elements, then formation of a mechanism can be unveiled by the fact that the corresponding stiffness matrix is singular. Therefore, the unzipping is simply continued until the determinant of the stiffness matrix is zero. By this procedure a number of mechanisms with different numbers of failure elements will

be identified. The number of failure elements in a mechanism will often be quite high so that several re-analyses of the structure are necessary.

As emphasized above it is more efficient to use the β -unzipping method in connection with fundamental mechanisms. Consider an elasto-plastic structure and let the number of potential failure elements (e.g. yield hinges) be n . It is then known from the theory of plasticity that the number of fundamental mechanisms is $m = n - r$, where r is the degree of redundancy. All other mechanisms can then be formed by linear combinations of the fundamental mechanisms. Some of the fundamental mechanisms are so-called joint mechanisms. They are important in the formation of new mechanisms by linear combinations of fundamental mechanisms, but they are not real failure mechanisms. Real failure mechanisms are by definition mechanisms, which are not joint mechanisms.

Let the number of loads be k . The safety margin for fundamental mechanism i can then be written

$$M_i = \sum_{j=1}^n |a_{ij}| R_j - \sum_{j=1}^k b_{ij} P_j \quad (18)$$

where a_{ij} and b_{ij} are the influence coefficients. R_j is the yield strength of failure element j and P_j is load number j . a_{ij} is the rotation of yield hinge j corresponding to the yield mechanism i and b_{ij} is the corresponding displacement of load j . The numerical value of a_{ij} is used in the first summation at the right-hand side of (18) to make sure that all terms in this summation are non-negative.

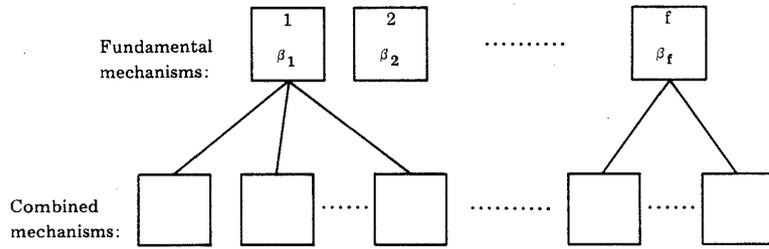
The total number of mechanisms for a structure is usually too high to include all possible mechanisms in the estimate of the system reliability. It is also unnecessary to include all mechanisms because the majority of them will in general have a relatively small probability of occurrence. Only the most critical or most significant failure modes should be included. The problem is then how the most significant mechanisms (failure modes) can be identified. In this section it is shown how the β -unzipping method can be used for this purpose. It is not possible to prove that the β -unzipping method identifies all significant mechanisms, but experience with structures where all mechanisms can be taken into account seems to confirm that the β -unzipping method gives reasonably good results. Note that since some mechanisms are excluded the estimate of the probability of failure by the β -unzipping method is a lower bound for the correct probability of failure. The corresponding generalized reliability index determined by the β -unzipping method is therefore an upper bound of the correct generalized reliability index. However, the difference between these two indices is usually negligible.

The first step is to identify all fundamental mechanisms and calculate the corresponding reliability indices. Fundamental mechanisms can be automatically generated by a method suggested by Watwood [19], but when the structure is not too complicated the fundamental mechanisms can be identified manually.

The next step is then to select a number of fundamental mechanisms as starting points for the unzipping. By the β -unzipping method this is done on the basis of the reliability index β_{\min} for the real fundamental mechanism that has the smallest reliability index and on the basis of a preselected constant ε_1 (e.g. $\varepsilon_1 = 0.50$). Only real fundamental mechanisms with β -indices in the interval $[\beta_{\min}; \beta_{\min} + \varepsilon_1]$ are used as starting mechanisms in the β -unzipping method. Let $\beta_1 \leq \beta_2 \leq \dots \beta_f$ be an ordered set

of reliability indices for f real fundamental mechanisms 1, 2, ..., f , selected by this simple procedure.

The f fundamental mechanisms selected as described above are now in turn combined linearly with all m (real and joint) mechanisms to form new mechanisms. First the fundamental mechanism 1 is combined with the fundamental mechanisms 2, 3, ..., m and reliability indices ($\beta_{1,2}, \dots, \beta_{1,m}$) for the new mechanisms are calculated. The smallest reliability index is determined, and the new mechanisms with reliability



indices within a distance ε_2 from the smallest reliability index are selected for further investigation. The same procedure is then used on the basis of the fundamental mechanisms 2, ..., f and

Figure 5. Construction of new mechanisms.

a failure tree as the one shown in figure 5 is constructed. Let the safety margins M_i and M_j of two fundamental mechanisms i and j combined as described above (see (18)) be

$$M_i = \sum_{r=1}^n |a_{ir}| R_r - \sum_{s=1}^k b_{is} P_s \quad (19)$$

$$M_j = \sum_{r=1}^n |a_{jr}| R_r - \sum_{s=1}^k b_{js} P_s \quad (20)$$

The combined mechanism $i \pm j$ then has the safety margin

$$M_{i \pm j} = \sum_{r=1}^n |a_{ir} \pm a_{jr}| R_r - \sum_{s=1}^k (b_{is} \pm b_{js}) P_s \quad (21)$$

where + or - is chosen dependent on which sign will result in the smallest reliability index. From the linear safety margin (21) the reliability index $\beta_{i \pm j}$ for the combined mechanism can easily be calculated.

More mechanisms can be identified on the basis of the combined mechanisms in the second row of the failure tree in figure 5 by adding or subtracting fundamental mechanisms. Note that in some cases it is necessary to improve the technique by modifying (21), namely when a new mechanism requires not only a combination with $1 \times$ but a combination with $k \times$ a new fundamental mechanism. The modified version is

$$M_{i \pm k} = \sum_{r=1}^n |a_{ir} \pm k a_{jr}| R_r - \sum_{s=1}^k (b_{is} \pm k b_{js}) P_s \quad (22)$$

where k is chosen equal to e.g. -1, 1, -2, +2, -3 or +3 dependent on which value of k will result in the smallest reliability index. By (22) it is easy to calculate the reliability index $\beta_{i \pm k}$ for the combined mechanism $i + kj$.

By repeating this simple procedure the failure tree for the structure in question can be constructed. The maximum number of rows in the failure tree must be chosen and can typically be $m + 2$, where m is the number of fundamental mechanisms. A satisfactory estimate of the system reliability index can usually be obtained by using the same ε_2 -value for all rows in the failure tree.

During the identification of new mechanisms it will often occur that a mechanism already identified will turn up again. If this is the case, then the

corresponding branch of the failure tree is terminated just one step earlier so that the same mechanism does not occur more than once in the failure tree.

The final step is the application of the β -unzipping method in evaluating the reliability of an elasto-plastic structure at mechanism level is to select the significant mechanisms from the mechanisms identified in the failure tree. This selection can, in accordance with the selection-criteria used in making the failure tree, e.g. be made by first identifying the smallest β -value β_{\min} of all mechanisms in the failure tree and then selecting a constant ε_3 . The significant mechanisms are then by definition those with β -values in the interval $[\beta_{\min}; \beta_{\min} + \varepsilon_3]$. The probability of failure of the structure is then estimated by modelling the structural system as a series system with the significant mechanisms as elements (see figure 4).

5. EXAMPLE 1

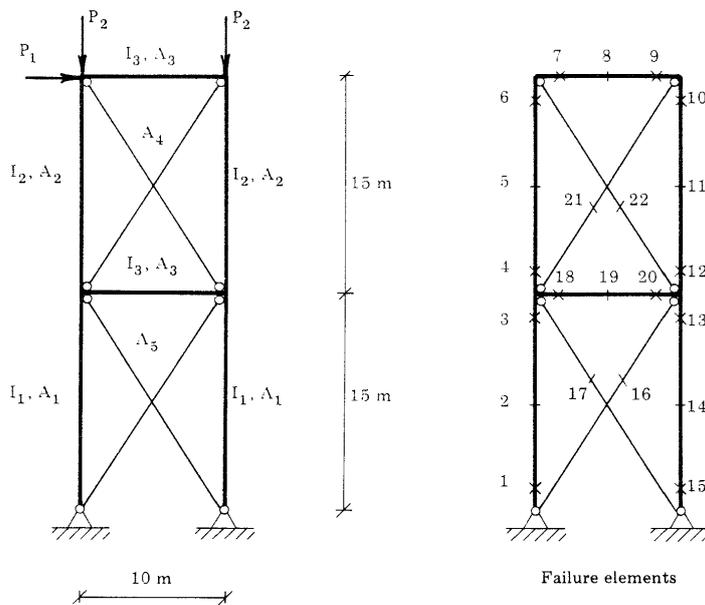


Figure 6. Two-storied brace frame.

	$A \cdot 10^3 \text{ m}^2$	$I \cdot 10^6 \text{ m}^4$	$E[M], \text{ kNm}$	$E[R], \text{ kN}$
1	4.59	57.9	135	1239
2	2.97	29.4	76	802
3	4.59	4.25	9.8	1239
4	2.6	0	0	702
5	2.8	0	0	756

Table 1. Section data and expected strength values.

Consider the two-storied braced frame in figure 6. The geometry and the loading are shown in the figure. This example is taken from [2] where all detailed calculations are shown. The area A and the moment of inertia I for each structural member are shown in table 1. In the same table the expected values of the yield moment M and the tensile strength capacity R for all structural members are also stated. The compression strength capacity of one structural member is assumed to be one half of the tensile

strength capacity. The expected values of the loading are

$$E[P_1] = 100 \text{ kN}$$

$$E[P_2] = 350 \text{ kN}$$

For the sake of simplicity the coefficient of variation for any load or strength is assumed to be $V[\bullet] = 0.1$. All elements are assumed to be perfectly ductile and made of a material having the same modulus of elasticity $E = 0.21 \times 10^9 \text{ kN/m}^2$.

The failure elements are shown in figure 6. \times indicates a potential yield hinge and $|$ failure in tension/compression. The total number of failure elements is 22, namely $2 \times 6 = 12$ yield hinges in 6 beams and 10 tension/compression failure

Failure element	2	3	4	5	6	7	8	9	10	11
β -value	8.13	9.96	9.92	4.97	9.98	9.84	9.79	9.86	9.98	1.81
Failure element	12	13	14	16	17	18	19	20	21	22
β -value	9.94	9.97	1.80	9.16	4.67	9.91	8.91	9.91	8.04	3.34

Table 2. Reliability indices at level 0 (and level 1).

Further, the loads P_1 and P_2 are uncorrelated.

The β -values for all failure elements are shown in table 2. Failure element 14 has the lowest reliability index $\beta_{14} = 1.80$ of all failure elements. Therefore, at level 0 the system reliability index is $\beta_s^0 = 1.80$.

Let $\Delta\beta_1 = 0$. It then follows from table 2 that the critical failure elements are 14, 11, 22, and 17. The corresponding correlation matrix (between the safety margins in the same order) is

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.24 & 0.20 & 0.17 \\ 0.24 & 1.00 & 0.21 & 0.16 \\ 0.20 & 0.21 & 1.00 & 0.14 \\ 0.17 & 0.16 & 0.14 & 1.00 \end{bmatrix} \quad (23)$$

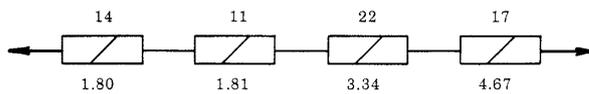


Figure 7. Series system used in estimating the system reliability at level 1.

The Ditlevsen bounds for the system probability of failure P_f^1 (see figure 7) gives

$$0.06843 \leq P_f^1 \leq 0.06849$$

Therefore, a (good) estimate for the system reliability index at level 1 is $\beta_s^1 = 1.49$. It follows from (23) that the coefficients of correlation are rather small. Therefore, the simple upper bound in (7) can be expected to give a good approximation. One gets $\beta_s^1 = 1.48$. A third estimate can be obtained by (8) and (9). The result is $\beta_s^1 = 1.49$.

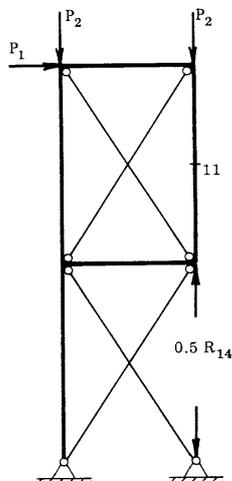


Figure 8. Modified structure when failure takes place in failure element 14.

the unzipping with failure element 14.

At level 2 it is initially assumed that the ductile failure element 14 fails (in compression) and a fictitious load equal to $0.5 \times R_{14}$ is added (see figure 8). This modified structure is then analysed elastically and new reliability indices are calculated for all the remaining failure elements (see table 3). Failure element 11 has the lowest β -value 1.87. With $\Delta\beta_2 = 1.00$ failure element 11 is the only failure element with a β -value in the interval $[1.87, 1.87 + \Delta\beta_2]$. Therefore, in this case only one critical pair of failure elements is obtained by initiating

possibilities in the 10 structural elements. The following pairs of failure elements (1, 3), (4, 6), (7, 9), (10, 12), (13, 15) and (18, 20) are assumed fully correlated. All other pairs of failure elements are uncorrelated.

Failure element	2	3	4	5	6	7	8	9	10	11
β -value	9.67	10.02	9.99	5.01	9.99	9.92	9.97	9.93	9.99	1.87
Failure element	12	13	16	17	18	19	20	21	22	
β -value	10.00	10.03	5.48	3.41	10.24	8.87	10.25	9.08	4.46	

Table 3.

parallel system is

$$P_f = \Phi_2(-1.80, -1.87; 0.28) = 0.00347 \quad (24)$$

and the corresponding generalized index

$$\beta_{14,11} = 2.70 \quad (25)$$

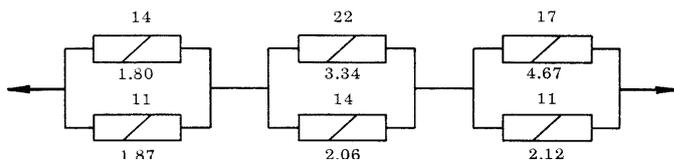


Figure 9. Series system used to estimate the reliability at level 2.

The same procedure can be performed with the three other critical failure elements 11, 22 and 17. The final result is shown in figure 9 and the corresponding failure modes in figure 10. Generalized reliability indices and approximate equivalent safety margins for each parallel system in figure 9 are calculated. Then the correlation matrix $\bar{\rho}$ can be calculated

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.56 & 0.45 \\ 0.56 & 1.00 & 0.26 \\ 0.45 & 0.26 & 1.00 \end{bmatrix} \quad (26)$$

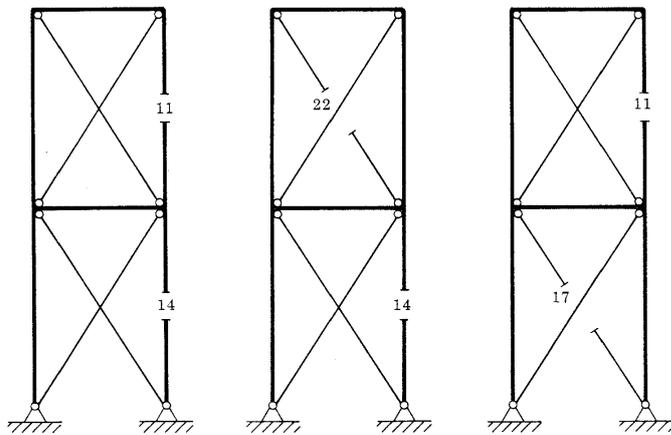


Figure 10. Failure modes at level 2.

The Ditlevsen bounds for the failure probability of the series system, see figure 9, are

$$0.3488 \times 10^{-2} \leq P_f^2 \leq 0.3488 \times 10^{-2}$$

Therefore, estimate of the system reliability at level 2 is $\beta_s^2 = 2.70$.

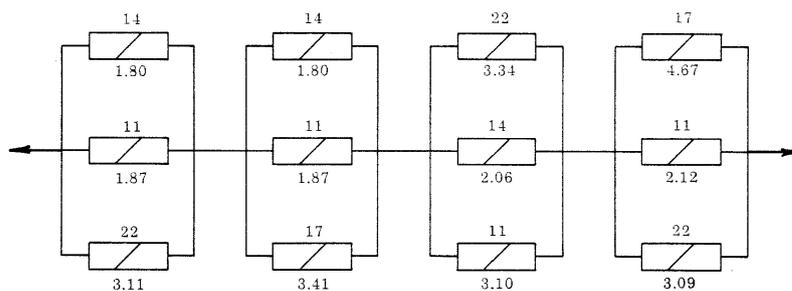


Figure 11. Reliability modelling at level 3 of the two-storied braced frame.

At level 3 (with $\Delta\beta_3 = 1.00$), four critical triples of failure elements are identified (see figure 11) and an estimate of the system reliability at level 3 is $\beta_s^3 = 3.30$.

It is of interest to note that the estimates of the system

reliability index at levels 1, 2, and 3 are very different.

Level	1	2	3
Reliability index	1.49	2.70	3.30

6. EXAMPLE 2

Consider the same structure as in example 1, but now the failure elements 2, 5, 11, and 14 are assumed brittle (see figure 6). All other data are unchanged. By a linear elastic analysis the same reliability indices for all (brittle and ductile) failure elements as in table 2 are calculated. Therefore, the critical failure elements are 14 and 11, and the estimate of the system reliability at level 1 is unchanged, i.e. $\beta_S^1 = 1.49$.

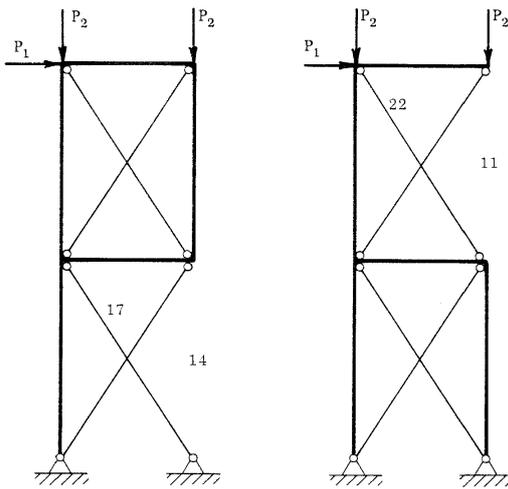


Figure 12. Modified structures.

It can be mentioned that $\beta_{16|14} = -3.74$ so that failure element 16 also fails instantly after failure element 14.

Again, by assuming brittle failure in failure element 11 (see figure 12, right) only one critical pair of failure elements is identified, namely the pair of failure elements 11 and 22, where

$$\beta_{22|11} = -4.33 \quad (28)$$

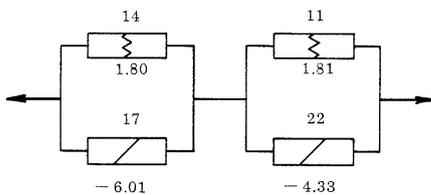


Figure 13. Modelling at level 2.

This failure mode is not a mechanism. The series system used in calculating an estimate of the system reliability at level 2 is shown in figure 13. Due to the small reliability indices (27) and (28) the strength variables 17 and 22 do not affect the safety margins for the two parallel systems in figure 13 significantly. Therefore, the reliability index at level 2 is unchanged from level 1, namely $\beta_S^2 = 1.49$.

As expected this value is much lower than the value 2.70 (see example 1) calculated for the structure with only ductile failure elements. This fact stresses the importance of the reliability modelling of the structure.

It is of interest to note that the β -unzipping method was capable of disclosing that the structure cannot survive failure in failure element 14. Therefore, when brittle failure occurs it is often reasonable to define failure of the structure as failure of just one failure element. This is equivalent to estimating the reliability of the structure at level 1.

7. EXAMPLE 3

In this example it is shown how the system reliability at mechanism level can be estimated in an efficient way. Consider the simple framed structure in figure 14 with corresponding expected values and coefficients of variation for the basic variables in table 4.

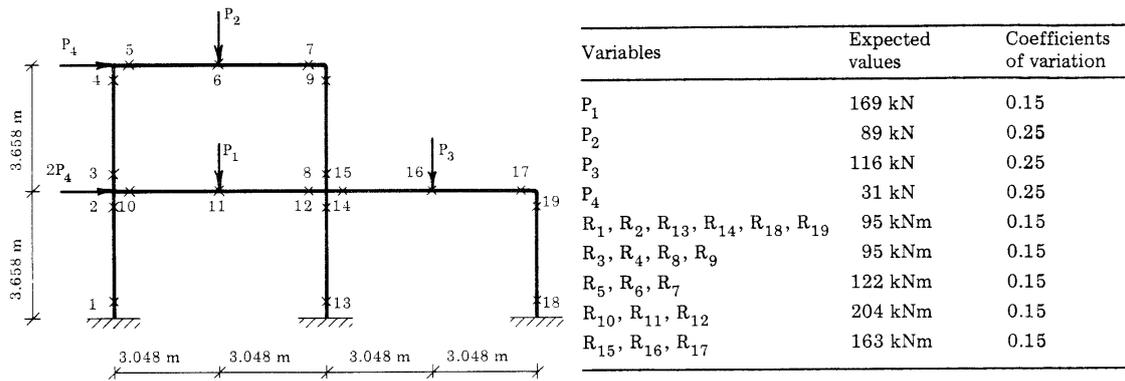


Figure 14. Geometry, loading and potential yield hinges (x).

Table 4

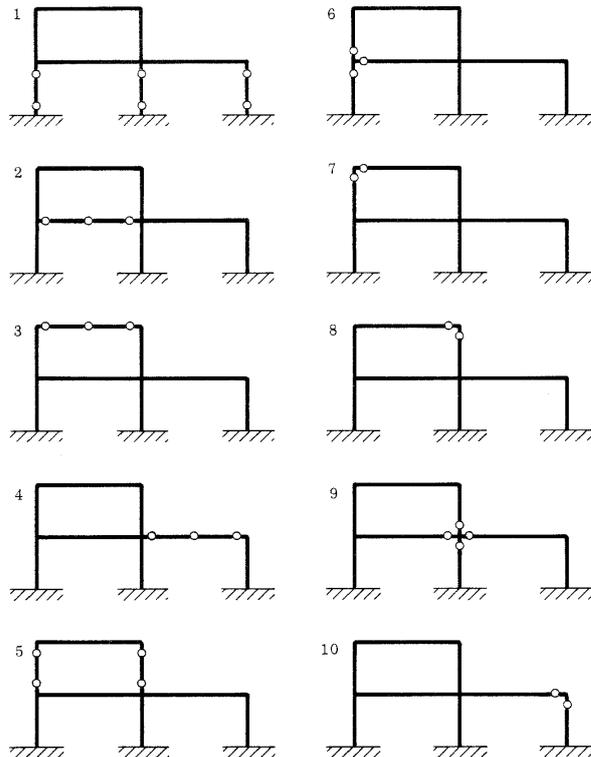


Figure 15. Set of fundamental mechanisms.

The load variables are $P_i, i = 1, \dots, 4$ and the yield moments are $R_i, i = 1, \dots, 19$. Yield moments in the same line are considered fully correlated and the yield moments in different lines are mutually independent. The number of potential yield hinges is $n = 19$ and the degree of redundancy is $r = 9$. Therefore, the number of fundamental mechanisms is $n - r = 10$.

One possible set of fundamental mechanisms is shown in figure 15. The safety margins M_i for the fundamental mechanisms can be written

$$M_i = \sum_{j=1}^{19} |a_{ij}| R_j - \sum_{i=1}^4 b_{ij} P_j, \quad (29)$$

$$i = 1, \dots, 10$$

where the influence coefficients a_{ij} and b_{ij} are determined by considering the mechanisms in the deformed state.

i	1	2	3	4	5	6	7	8	9	10
β_i	1.91	2.08	2.17	2.26	4.19	10.75	9.36	9.36	12.65	9.12

Table 5. Reliability indices for the fundamental mechanisms.

The reliability indices β_i , $i = 1, \dots, 10$ for the 10 fundamental mechanisms can be calculated from the safety margins taking into account the correlation

between the yield moments. The result is shown in table 5.

With $\varepsilon_1 = 0.50$ the fundamental mechanisms 1, 2, 3, and 4 are selected as starting mechanisms in the β -unzipping and combined in turn with the remaining fundamental mechanisms. As an example consider the combination 1 + 6 of mechanisms 1 and 6. The linear safety margin M_{1+6} is obtained from the linear safety margins M_1 and M_6 by addition taking into account the signs of the coefficients. The corresponding reliability index is $\beta_{1+6} = 3.74$.

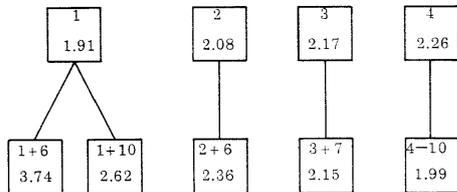


Figure 16. The first two rows in the failure tree.

With $\varepsilon_2 = 1.20$ the following new mechanisms 1 + 6, 1 + 10, 2 + 6, 3 + 7, and 4 - 10 are identified by this procedure. The failure tree at this stage is shown in figure 16. It contains 4 + 5 = 9 mechanisms. The reliability indices and the fundamental mechanisms involved are shown in the same figure.

This procedure is now continued as explained earlier by adding or subtracting fundamental mechanisms. If the procedure is continued 8 times (up to 10 fundamental mechanisms in one mechanism) and if the significant mechanisms are selected by $\varepsilon_3 = 0.31$, then the system modelling at mechanism level will be a series system where the elements are 12 parallel systems. These 12 parallel systems (significant mechanisms) and corresponding reliability indices are shown in table 6. The correlation matrix is

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.65 & 0.89 & 0.44 & 0.04 & 0.91 & 0.59 & 0.97 & 0.87 & 0.81 & 0.36 & 0.92 \\ 0.65 & 1.00 & 0.55 & 0.00 & 0.09 & 0.67 & 0.00 & 0.63 & 0.54 & 0.58 & 0.00 & 0.71 \\ 0.89 & 0.55 & 1.00 & 0.35 & 0.45 & 0.83 & 0.61 & 0.88 & 0.98 & 0.95 & 0.31 & 0.83 \\ 0.44 & 0.00 & 0.35 & 1.00 & 0.00 & 0.03 & 0.00 & 0.45 & 0.37 & 0.04 & 0.89 & 0.08 \\ 0.04 & 0.09 & 0.45 & 0.00 & 1.00 & 0.05 & 0.00 & 0.04 & 0.44 & 0.48 & 0.00 & 0.05 \\ 0.91 & 0.67 & 0.83 & 0.03 & 0.05 & 1.00 & 0.73 & 0.87 & 0.80 & 0.88 & 0.00 & 0.98 \\ 0.59 & 0.00 & 0.61 & 0.00 & 0.00 & 0.73 & 1.00 & 0.58 & 0.60 & 0.65 & 0.00 & 0.64 \\ 0.97 & 0.63 & 0.88 & 0.45 & 0.04 & 0.87 & 0.58 & 1.00 & 0.90 & 0.77 & 0.48 & 0.87 \\ 0.87 & 0.54 & 0.98 & 0.37 & 0.44 & 0.80 & 0.60 & 0.90 & 1.00 & 0.90 & 0.41 & 0.78 \\ 0.81 & 0.58 & 0.95 & 0.04 & 0.48 & 0.88 & 0.65 & 0.77 & 0.90 & 1.00 & 0.00 & 0.88 \\ 0.36 & 0.00 & 0.31 & 0.89 & 0.00 & 0.00 & 0.00 & 0.48 & 0.41 & 0.00 & 1.00 & 0.00 \\ 0.92 & 0.71 & 0.83 & 0.08 & 0.05 & 0.98 & 0.64 & 0.87 & 0.78 & 0.88 & 0.00 & 1.00 \end{bmatrix}$$

The probability of failure P_f for the series system with the 12 significant mechanisms as elements can then be estimated by the usual techniques. The Ditlevsen bounds are

$$0.08646 \leq P_f \leq 0.1277$$

No.	Significant mechanisms	β
1	1 + 6 + 2 + 5 + 7 + 3 - 8	1.88
2	1	1.91
3	1 + 6 + 2 + 5 + 7 + 3 + 4 - 8 - 10	1.94
4	3 + 7 - 8	1.98
5	4 - 10	1.99
6	1 + 6 + 2	1.99
7	2	2.08
8	1 + 6 + 2 + 5 + 7 + 3 + 8	2.09
9	1 + 6 + 2 + 5 + 7 + 3 + 8 + 9 + 4 - 10	2.11
10	1 + 6 + 2 + 5 + 9 + 4 - 10	2.17
11	3	2.17
12	1 + 6 + 2 + 5	2.18

Table 6

If the average value of the lower and upper bounds is used, the estimate of the reliability index β_s at mechanism level is $\beta_s = 1.25$. Hohenbichler [20] has derived an approximate method to calculate estimates for the system failure probability Pf (and the corresponding reliability index β_s). The estimate of β_s is $\beta_s = 1.21$. It can finally be noted that Monte Carlo simulation gives $\beta_s = 1.20$.

8. STRUCTURAL OPTIMIZATION AND SYSTEM RELIABILITY

In *classical structural optimization* for trussed and framed structures the *design variables* are usually the cross-sectional areas A_i , $i = 1, \dots, k$, where k is the number of structural elements, so that each structural member is characterized by only one number. This is fully satisfactory for trussed structures. However, when bending occurs in the structural members, the plastic resistance moments W_i , $i = 1, \dots, k$, and the second moments of area I_i , $i = 1, \dots, k$ are important. To maintain the great advantage of having only one design variable for each structural member it is assumed that

$$W_i = k_1 A_i^{3/2} \quad \text{and} \quad I_i = k_2 A_i^2 \quad (30)$$

where k_1 and k_2 are constants. A natural choice of *objective function* for a structure is the cost of the structure, not only the cost of the structural material used, but also fabrication, transportation, etc. It is often assumed that the total cost of a structure with a satisfactory degree of accuracy is proportional to the weight of the structure. For structures where only one material (steel, concrete, etc.) is used the weight is proportional to

$$W = \sum_{i=1}^k l_i A_i \quad (31)$$

where l_i is the length of structural member i .

The *constraints* in classical structural optimization signify that the stresses should everywhere be smaller than some prescribed values and likewise that the displacements should be smaller than some prescribed values. Further that all cross-sectional areas are greater than or equal to zero. The stresses and the displacements will for statically indeterminate structures depend on the areas. Therefore, the constraints can be given by

$$g_j(\bar{A}) \geq 0 \quad , \quad j = 1, \dots, m \quad (32)$$

$$A_i \geq 0 \quad , \quad i = 1, \dots, k \quad (33)$$

where $\bar{A} = (A_1, \dots, A_k)$ and g_j , $j = 1, \dots, m$, are functions expressing stresses and displacements.

The classical optimization problem can be formulated in the following way. Determine the cross-sectional areas A_i , $i = 1, \dots, k$ so that the weight function $W(\bar{A})$ is minimum under the constraints (32) and (33).

In classical structural optimization structural reliability is not taken into account. If a structure has to be designed from a reliability point of view at least two different structural optimization problems can be formulated. The first one is concerned with *element reliability*. In this case the constraints (32) are replaced by

$$\beta_i(\bar{A}) \geq \beta_i^0, \quad i = 1, \dots, n \quad (34)$$

where β_i is the reliability index for failure element i and β_i^0 is the corresponding target value. n is the number of failure elements.

The second optimization problem is related to *system reliability*. In this case the constraints (32) are replaced by only one constraint, namely

$$\beta_s(\bar{A}) \geq \beta_s^0 \quad (35)$$

where β_s is the system reliability index and β_s^0 is the corresponding target value. In some situations it might be useful to combine the constraints (34) and (35). Note that use of (35) instead of (32) reduces the number of constraints drastically. So in this sense the optimization problem is simpler. However, calculation of the system reliability index β_s is now the main problem.

The structural system reliability optimization problem can be formulated in the following way. Determine the cross-sectional areas A_i , $i = 1, \dots, n$ for all failure elements so that the weight function $W(\bar{A})$ is minimum under the constraints (35) and (33). Clearly, the well-known optimization methods can also be used for this optimization problem and the β -unzipping method can be used in calculating the system reliability index β_s . From a computational point of view, calculation of β_s is expensive. When solving the optimization problem it is therefore important to reduce the number of recalculations of β_s .

The following optimization procedure has been used with some success although a number of problems have still not been completely solved:

- 1) Choose some starting values for the cross-sectional areas.
- 2) Identify significant failure modes by the β -unzipping method and calculate equivalent safety margins for the failure modes.
- 3) Find a suboptimal design by an optimization method, e.g. SUMT (Sequential Unconstrained Minimization Technique) without updating the system of failure modes and the directional cosinus of the equivalent safety margins. The system reliability index β_s is calculated (evaluated) during this sub optimal design by a simple approximate method.
- 4) Check the convergence by a suitable stopping rule. If the stopping rule is satisfied, go to 5), if not, go to 2).
- 5) Repeat 2), 3), and 4), but use a more accurate method to evaluate β_s . When the stopping rule is satisfied, the optimization procedure is stopped.

This procedure has been used in a number of examples with different system levels. Due to the updating of significant failure modes some convergence problems occur, namely when new failure modes are introduced. This can be avoided if updating takes place at each iterative step. However, such a procedure is much too expensive. Research in this field is still going on, so only some tentative conclusions can be made by now. One conclusion is that the penalty function methods seem to be suitable for this kind of problem. Further, that the updating problem mentioned above seems to be negligible at level 1 and at mechanism level. By choosing the different β -unzipping parameters in an appropriate way the convergence can be improved. It seems also to be an improvement to use A_i^{-1} as design variables. A more detailed description of this procedure is given with examples in Thoft-Christensen & Sørensen [21].

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