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Thoft-Christensen, Palle

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## CHAPTER 20

### STRUCTURAL RELIABILITY<sup>1</sup>

P. Thoft-Christensen, Aalborg University

#### Abstract

In this paper a brief presentation of some of the most fundamental concepts in modern structural reliability theory is given. The presentation does not claim to be very precise or exhaustive. The purpose is to show that it is possible to day to evaluate the reliability of a structural system. A much more satisfactory presentation can be found in the extensive literature, see e.g. Thoft-Christensen & Baker [1] or Elishakoff [2]. The method presented belongs to the so-called *level 2* methods, i.e. methods involving certain approximate iterative calculations to obtain an approximate value of the probability of failure of the structural system. In these level 2 methods the joint probability distribution of the relevant variables is often simplified and the failure domain is idealized in such a way that the reliability calculations can be performed without an unreasonable amount of work. If the joint probability distribution function and the true nature of the failure domain are known, then it may, in principle, be possible to calculate the exact probability of failure. In this case we call the method used a *level 3* method. However, this kind of information is usually not at hand for ordinary structures. *Level 1* methods are methods where the structural and loading variables are characterized by nominal, values and where the sufficient degree of reliability is obtained using a number of, for instance, partial coefficients (safety factors).

#### 1. INTRODUCTION

During the last two decades structural reliability theory has grown from being a pure research area to a mature research and engineering field. In structural reliability theory uncertainties in e.g. load intensities and in material properties are treated in a rational way. Structural engineering used to be dominated by deterministic thinking. In the

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<sup>1</sup> Proceedings of "Safety and Reliability in Europe". Pre-Launching Meeting of ESRA, Ispra, Italy, October 9-12, 1984, pp. 82-99.

deterministic approach design is based on specified minimum material properties and specified load intensities and a certain procedure for calculating stresses and deflections is often prescribed in the deterministically based codes. However, it has been recognized for many decades that for example specifications of minimum material properties involve a high degree of uncertainty. Likewise, specification of reasonable load intensities is difficult and uncertain. These types of uncertainty combined with several similar forms of uncertainty result in an uncontrolled risk. It is a fact that total safety of a structure cannot be achieved even when one is ready to offer a lot of money on the structure. Further, it is a serious problem with the deterministic approach that no measure of the safety or reliability of the structure is obtained.

In modern structural reliability theory it is clearly recognized that some risk of structural failure must be accepted. To obtain some measure of this risk - the probability of failure - a probabilistic approach seems to be well suitable. By this approach it is intended to help the structural engineers to design a structure in such a way that at an acceptable level of probability it will not fail at any time during the specified design life.

In this paper a brief presentation of some of the most fundamental concepts in modern structural reliability theory is given. The presentation does not claim to be very precise or exhaustive. The purpose is to show that it is possible to day to evaluate the reliability of a structural system. A much more satisfactory presentation can be found in the extensive literature, see e.g. Thoft-Christensen & Baker [1] or Elishakoff [2]. The method presented belongs to the so-called *level 2* methods, i.e. methods involving certain approximate iterative calculations to obtain an approximate value of the probability of failure of the structural system. In these level 2 methods the joint probability distribution of the relevant variables is often simplified and the failure domain is idealized in such a way that the reliability calculations can be performed without an unreasonable amount of work. If the joint probability distribution function and the true nature of the failure domain are known, then it may in principle be possible to calculate the exact probability of failure. In this case we call the method used a *level 3* method. However, this kind of information is usually not at hand for ordinary structures. *Level 1* methods are methods where the structural and loading variables are characterized by nominal values and where the sufficient degree of reliability is obtained using a number of e.g. partial coefficients (safety factors).

## 2. BASIC VARIABLES AND FAILURE SURFACES

The reliability analysis presented here is based on the assumption that all relevant variables can be modeled by random variables (or stochastic processes). These variables  $X_i$ ,  $i = 1, \dots, n$  called *basic variables* can be material strengths, external loads or geometrical quantities. For a given structure the set of variables  $\bar{x} \in \omega_s$  is a realization of the random vector  $\bar{X} = (X_1, \dots, X_n)$ .  $\bar{x} = (x_1, \dots, x_n)$  is a point in an  $n$ -dimensional *basic variable space*  $\omega$ .

Next it is assumed that a *failure surface* (or limit state surface)  $f(\bar{x}) = 0$  divides the basic variable space  $\omega$  in a failure region  $\omega_f$  and a safe region  $\omega_s$ . The function  $f: \omega \rightarrow R$  is called the *failure function* and is defined in such a way that  $f(\bar{x}) > 0$  when  $\bar{x} \in \omega_s$ , and  $f(\bar{x}) \leq 0$  when  $\bar{x} \in \omega_f$ . The random variable

$$M = f(\bar{X}) \quad (1)$$

is called a *safety margin*.

As an example, consider a single structural element and assume that the load effect is given by a single load variable  $S$  and the strength by a single resistance variable  $R$ . The basic variables  $S$  and  $R$  are given by their distribution functions  $F_S$  and  $F_R$  or alternatively by their distribution functions  $f_S$  and  $f_R$ , see figure 1. In this case the safety margin is  $M = R - S$

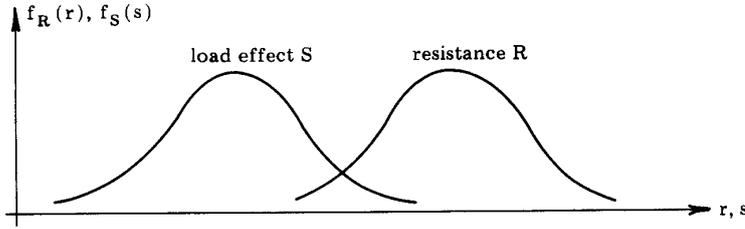


Figure 1. The fundamental case.

and the probability of failure is

$$P_f = P(R - S \leq 0) = \int_{-\infty}^{\infty} F_R(t) f_S(t) dt \quad (2)$$

By definition the reliability is

$$R = 1 - P_f \quad (3)$$

In the general case the basic variables  $\bar{X} = (X_1, \dots, X_n)$  are correlated. However, by a linear transformation a set of uncorrelated basic variables  $\bar{Y} = (Y_1, \dots, Y_n)$  can be obtained. The uncorrelated basic variable  $\bar{Y}$  is then normalized so that a set of uncorrelated basic variables  $\bar{Z} = (Z_1, \dots, Z_n)$ , where the expected value  $E[Z_i]$  and the variance  $\text{Var}[Z_i] = 1, i = 1, \dots, n$ , are obtained. By these

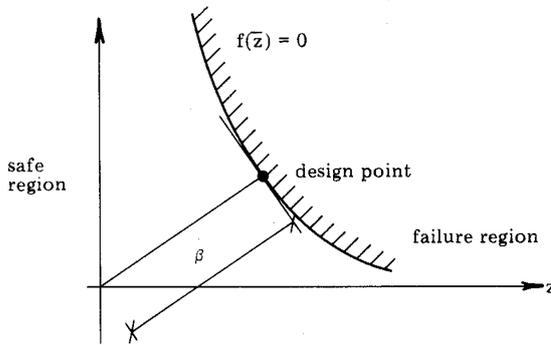


Figure 2. Definition of the reliability index  $\beta$ .

transformations the failure surface  $f_1(\bar{x}) = 0$  in the  $x$ -space is transformed into a failure surface  $\delta\omega$  given by  $f(\bar{z}) = 0$  in the corresponding  $z$ -space. In the  $n$ -dimensional  $z$ -space the so-called *reliability index*  $\beta$  is defined as the shortest distance from the origin to the failure surface (see figure 2)

$$\beta = \min \left( \sum_{i=1}^n z_i^2 \right)^{1/2} \quad (4)$$

In the special cases where the failure surface is linear and all basic variables are normally distributed it is easy to show that there is a one-to-one relation between the failure probability  $P_f$  and reliability index  $\beta$ , namely

$$P_f = \Phi(-\beta) \Leftrightarrow \beta = -\Phi^{-1}(P_f) \quad (5)$$

where  $\Phi$  is the standardized normal distribution function. In general, the failure surface is non-linear, and the basic variables non-normal. The equations (5) are then invalid, but a so-called *generalized reliability index*  $\beta_g$  can be defined by

$$\beta_g = -\Phi^{-1}(P_f) \quad (6)$$

The estimate of the reliability of a structural system is greatly facilitated if the basic variables can be assumed normally distributed. The only information needed will then be the expected values  $E[X_i] = \mu_{X_i}$ , the standard deviations  $\sigma_{X_i}$  and the coefficients of correlation  $\rho_{X_i X_j}$  between any pair of basic variables  $X_i$  and  $X_j$ . However, in general basic variables cannot be modeled with a satisfactory degree of accuracy by normally distributed variables.

*Resistance variables* are frequently modeled by the lognormal distribution, which has the advantage of precluding non-positive values. A number of other distributions such as the Weibull distribution (type III extreme value distribution) have the same advantage.

*Load variables* can in some cases be satisfactorily modeled by normally distributed variables (permanent loads) but often they are better modeled by an extreme value distribution, e.g. the Gumbel distribution (type 1 extreme value distribution).

To overcome the problem that basic variables are usually modeled by non-normal probability distributions a number of different so-called distribution transformations have been suggested, see e.g. Thoft-Christensen & Baker [1], and Thoft-Christensen [3]. It is beyond the scope of this paper to discuss these transformation methods. It will therefore be assumed that all basic variables are normally distributed.

### 3. DEFINITIONS OF STRUCTURAL FAILURE

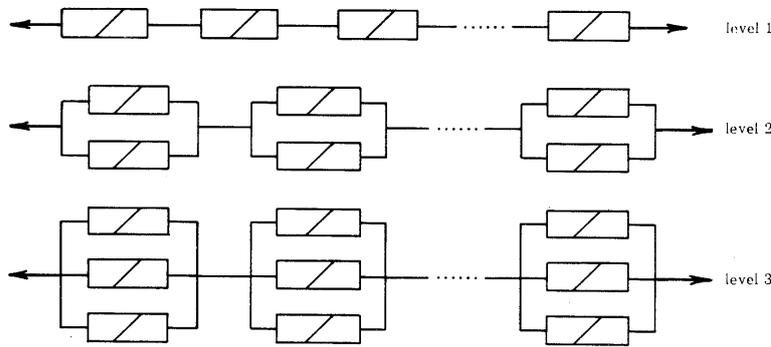
In section 2 a single element and a single failure function (failure mode) are dealt with. It is of course much more complicated to evaluate the reliability of a real structure, since failure of a single element does not always imply failure of the complete structure due to redundancy. In such a case it is useful to consider the structure from a systems point of view, see e.g. Thoft-Christensen [4]. The following presentation is to some extent based on reference [4].

Let a structure consist of  $q$  failure elements, i.e. elements or points (cross-sections) where failure can take place, and let the reliability index for failure element  $i$  be  $\beta_i$ . An estimate of the structural systems reliability is obtained by taking into account the possibility of failure of any failure element by modeling the structural system as a series system with the failure elements as elements. The probability of failure for this series system is then estimated on the basis of the reliability indices  $\beta_i$ ,  $i = 1, 2, \dots, q$ , and the correlation between the safety margins for the failure elements. This reliability analysis is called *systems reliability analysis at level 1*. In general, it is only necessary to include some of the failure elements in the series system (namely those with the smallest  $\beta$ -indices) to get a good estimate of the systems failure probability. The failure elements included in the reliability analysis are called *critical failure elements*.

The modeling of the system at level 1 is natural for a statically determinate structure, but failure in a single failure element in a structural system will not always result in failure of the total system, because the remaining elements may be able to sustain the external loads due to redistribution of the load effects. This situation is characteristic of statically indeterminate structures. For such structures *systems reliability analysis at level 2* or higher levels may be reasonable. At level 2 the systems reliability is estimated on the basis of a series system, where the elements are parallel systems each with two failure elements - so-called *critical pairs of failure elements*. These critical pairs of failure elements are obtained by modifying the structure by

assuming in turn failure in the critical failure elements and adding fictitious load corresponding to the load-carrying capacity of the elements in failure.

Analyzing modified structures where failure is assumed in critical pairs of failure elements now continues the procedure sketched above. In this way *critical triples of failure elements are identified and a reliability analysis at level 3* can be made on the basis of a series system, where the elements are parallel systems each with



three failure elements. By continuing in the same way reliability estimates at level 4, 5, etc. can be performed, but in general analysis beyond level 3 is of minor interest. System modeling at levels 1, 2, and 3 is shown in figure 3.

Figure 3. System modeling at levels 1, 2, and 3.

The behavior of some structures is elastic-plastic. In such cases failure of the structure is usually defined as formation of a mechanism. The probability of failure of the structural system is then estimated by modeling the structural system as a series system with the (significant) mechanisms as elements (see figure 4). Reliability analysis based on the mechanism failure definition is called *systems reliability analysis at mechanism level*.

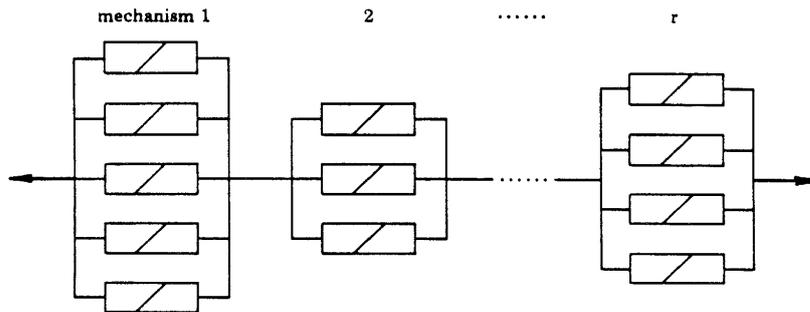


Figure 4. System modeling at mechanism level.

In this section 3, system modeling at a number of different levels has been introduced. The notation system level must not be confused with the reliability level treated in section 1.

#### 4. CALCULATION OF FAILURE PROBABILITIES OF FUNDAMENTAL SYSTEMS

In this section it is shown how approximate failure probabilities for critical failure elements, series systems and parallel systems can be calculated by linearizing the safety margins.

First consider a *critical failure element* with the failure function  $f(\bar{z})$ . It is assumed that  $\bar{Z}$  is normally distributed, so the probability of failure is

$$P_f = P(f(\bar{Z}) \leq 0) = \int_{\omega_f} \varphi_n(\bar{z}) d\bar{z} \tag{7}$$

where  $\varphi_n$  is the standardized n-dimensional normal density function.

If the function  $f$  is linearised in the so-called design point with distance  $\beta$  to the origin of the coordinate system, then an approximate value for  $P_f$  is given by (compare with (5))

$$\begin{aligned} P_f &\approx P(\alpha_1 Z_1 + \dots + \alpha_n Z_n + \beta \leq 0) \\ &= P(\alpha_1 Z_1 + \dots + \alpha_n Z_n \leq -\beta) = \Phi(-\beta) \end{aligned} \quad (8)$$

where  $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$  is the directional cosines of the linearised failure surface.  $\beta$  is the reliability index.  $\Phi$  is the standardized normal distribution function.

Next consider a *series system* with  $k$  elements. An estimate of the failure probability  $p_f^s$  of this series system can be obtained on the basis of linearised safety margins for the  $k$  elements

$$\begin{aligned} p_f^s &= P\left(\bigcup_{i=1}^k (\bar{\alpha}_i \bar{Z} + \beta_i \leq 0)\right) = P\left(\bigcup_{i=1}^k (\bar{\alpha}_i \bar{Z} \leq -\beta_i)\right) \\ &= 1 - P\left(\bigcap_{i=1}^k (\bar{\alpha}_i \bar{Z} > -\beta_i)\right) = 1 - P\left(\bigcap_{i=1}^k (-\bar{\alpha}_i \bar{Z} < \beta_i)\right) \\ &= 1 - \Phi_k(\bar{\beta}; \bar{\rho}) \end{aligned} \quad (9)$$

where  $\bar{\alpha}_i$  and  $\beta_i$  are the directional cosines and the reliability index for failure element  $i$ ,  $i = 1, \dots, k$  and where  $\bar{\beta} = (\beta_1, \dots, \beta_k)$ .  $\bar{\rho} = \{\rho_{ij}\}$  is the correlation coefficient matrix given by  $\rho_{ij} = \bar{\alpha}_i^T \bar{\alpha}_j$  for all  $i \neq j$ .  $\Phi_k$  is the standardized  $k$ -dimensional normal distribution function.

For a *parallel system* with  $k$  elements an estimate of the failure probability  $p_f^p$  can be obtained in the following way

$$p_f^p = P\left(\bigcap_{i=1}^k (\bar{\alpha}_i \bar{Z} + \beta_i \leq 0)\right) = P\left(\bigcap_{i=1}^k (\bar{\alpha}_i \bar{Z} \leq -\beta_i)\right) = \Phi_k(-\bar{\beta}; \bar{\rho}) \quad (10)$$

where the same notation as above is used.

One serious problem in connection with application of (9) and (10) is numerical calculation of the  $k$ -dimensional normal distribution function  $\Phi_k$  or  $k \geq 3$ . However, a more serious problem is how to identify the failure modes (parallel systems) in figures 3 and 4. The last-mentioned problem is treated in the next sections.

## 5. IDENTIFICATION OF CRITICAL FAILURE MODES AT LEVELS 1, 2, AND 3

A number of different methods to identify critical failure modes have been suggested in the literature. In this paper the  $\beta$ -unzipping method [3] - [8] is used. Only the main features of the  $\beta$ -unzipping method are presented here. A very detailed description is given in Thoft-Christensen [3].

At *level 1* the systems reliability is defined as the reliability of a series system with e.g.  $n$  elements - the  $n$  critical failure elements. Therefore, the first step is to calculate  $\beta$ -values for all failure elements and then use equation (9). As mentioned earlier, equation (9) cannot be used directly. However, excellent upper and lower bounds and good approximations exist. One way of selecting the critical failure elements is to select the failure elements with  $\beta$ -values in the interval  $[\beta_{\min}; \beta_{\min} + \Delta\beta_1]$

where  $\beta_{\min}$  is the smallest reliability index and where  $\Delta\beta_1$  is a prescribed positive number.

At *level 2* the systems reliability is estimated as the reliability of a series system where the elements are parallel systems each with 2 failure elements (see figure 3) - so-called *critical pairs of failure elements*. Let the structure be modeled by  $n$  failure elements. Let the critical failure element  $l$  have the lowest reliability index  $\beta$  of all critical failure elements. Failure is then assumed in failure element  $l$ , and the structure is modified by removing the corresponding failure element and adding a pair of so-called fictitious loads  $F_l$ . If the removed failure element is brittle, then no fictitious loads are added. However, if the removed failure element  $l$  is ductile then the fictitious load  $F_l$  is a stochastic load given by  $F_l = \gamma_l R_l$  where  $R_l$  is the load-carrying capacity of failure element  $l$  and where  $0 < \gamma_l \leq 1$ .

The modified structure with the external loads and the fictitious load  $F_l$  are then reanalysed and new reliability indices are calculated for all failure elements (except the one where failure is assumed) and the smallest  $\beta$ -value is called  $\beta_{\min}$ . The failure elements with  $\beta$ -values in the interval  $[\beta_{\min}; \beta_{\min} + \Delta\beta_2]$  where  $\Delta\beta_2$  is a prescribed positive number are then in turn combined with failure element  $l$  to form a number of parallel systems. Consider a parallel system with failure elements  $l$  and  $r$ . During the reliability analysis at level 1 the safety margin and the reliability index  $\beta_l$  for element  $l$  were determined. By the reanalysis of the structure the safety margin and the reliability index  $\beta_r$ , are determined. From these safety margins the correlation coefficient can easily be calculated. Then it follows from (10) that the probability of failure of this parallel system is

$$P_f = \Phi_2(-\beta_l, -\beta_r; \rho) \quad (11)$$

The same procedure is then in turn used for all critical failure elements and further critical pairs of failure elements are identified. In this way the total series system used in the reliability analysis at level 2 is determined (see figure 3). The next step is then to estimate the probability of failure for each critical pair of failure elements (see (11)) and also to determine a safety margin for each critical pair of failure elements. When this is done generalized reliability indices for all parallel systems in figure 3 and correlation coefficients between any pair of parallel system are calculated. Finally, the probability of failure for the series system (figure 3) is estimated. The so-called equivalent linear safety margin introduced by Gollwitzer & Rackwitz [9] is used as approximations for safety margins for the parallel systems.

The method presented above can easily be generalized to higher levels  $N > 2$ . At *level 3* the estimate of the systems reliability is based on so-called *critical triples of failure elements*, i.e. a set of three failure elements. The critical triples of failure elements are identified by the  $\beta$ -unzipping method and each triple forms a parallel system with three failure elements. These parallel systems are then elements in a series system (see figure 3). Finally, the estimate of the reliability of the structural system at level 3 is defined as the reliability of this series system.

Assume that the critical pair of failure elements  $(l, m)$  has the lowest reliability index  $\beta_{l,m}$  of all critical pairs of failure elements. Failure is then assumed in the failure elements  $l$  and  $m$ , and the structure is modified by adding for each of them a pair of fictitious loads  $F_l$  and  $F_m$ . The modified structure with the external loads and the fictitious loads  $F_l$  and  $F_m$  is then reanalysed and new reliability indices are calculated for all failure elements (except  $l$  and  $m$ ) and the smallest  $\beta$ -value is called  $\beta_{\min}$ . Failure elements with  $\beta$ -values in the interval  $[\beta_{\min}; \beta_{\min} + \Delta\beta_3]$ , where  $\Delta\beta_3$  is a prescribed

positive number, are then in turn combined with failure elements  $l$  and  $m$  to form a number of parallel systems.

The next step is then to evaluate the failure probability for each of the critical triple of failure elements. Consider the parallel system with failure elements  $l$ ,  $m$ , and  $r$ . During the reliability analysis at level 1 the safety margin for failure element  $l$  is determined and during the reliability analysis at level 2 the safety margin for the failure element  $m$  is determined. The safety margin for safety element  $r$  is determined during the reanalysis of the structure. From these safety margins the reliability indices  $\beta_l$ ,  $\beta_m$ , and  $\beta_r$  and the correlation matrix  $\bar{\rho}$  can easily be calculated. The probability of failure for the parallel system then is

$$P_f = \Phi_3(\beta_l, \beta_m, \beta_r; \bar{\rho}) \quad (12)$$

## 6. IDENTIFICATION OF CRITICAL FAILURE MODES AT MECHANISM LEVEL

When failure of a structure is defined as formation of a mechanism the  $\beta$ -unzipping method is used in connection with fundamental mechanisms. Consider an elasto-plastic structure and let the number of potential failure elements (e.g. yield hinges) be  $n$ . It is then known from the theory of plasticity that the number of fundamental mechanisms is  $m = n - r$ , where  $r$  is the degree of redundancy. All other mechanisms can then be formed by linear combinations of the fundamental mechanisms.

The total number of mechanisms for a structure is usually too high to include all possible mechanisms in the estimate of the systems reliability. It is also unnecessary to include all mechanisms because the majority of them will in general have a relatively small probability of occurrence. Only the most critical or most significant failure modes should be included. The problem is then how the most significant mechanisms (failure modes) can be identified. In this section it is shown how the  $\beta$ -unzipping method can be used for this purpose. It is not possible to prove that the  $\beta$ -unzipping method identifies all significant mechanisms, but experience with structures where all mechanisms can be taken into account seems to confirm that the  $\beta$ -unzipping method gives reasonably good results. Note that since some mechanisms are excluded the estimate of the probability of failure by the  $\beta$ -unzipping method is a lower bound for the probability of failure.

The first step is to identify all fundamental mechanisms and calculate the corresponding reliability indices. The next step is then to select a number of fundamental mechanisms as starting points for the unzipping. By the  $\beta$ -unzipping method this is done on the basis of the reliability index  $\beta_{\min}$  for the real fundamental mechanism that has the smallest reliability index and on the basis of a preselected constant  $\varepsilon_1$  (e.g.  $\varepsilon_1 = 0.50$ ). Only real fundamental mechanisms with  $\beta$ -indices in the interval  $[\beta_{\min}; \beta_{\min} + \varepsilon_1]$  are used as starting mechanisms in the  $\beta$ -unzipping method. Let  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_f$  be an ordered set of reliability indices for  $f$  real fundamental mechanisms  $1, 2, \dots, f$ , selected by this simple procedure.

The  $f$  fundamental mechanisms selected as described above are now in turn combined linearly with all  $m$  (real and joint) mechanisms to form new mechanisms. First the fundamental mechanism 1 is combined with the fundamental mechanisms 2, 3, . . . ,  $m$  and reliability indices  $\beta_{1,2}, \dots, \beta_{1,m}$  for the new mechanisms are calculated. The smallest reliability index is determined, and the new mechanisms with reliability

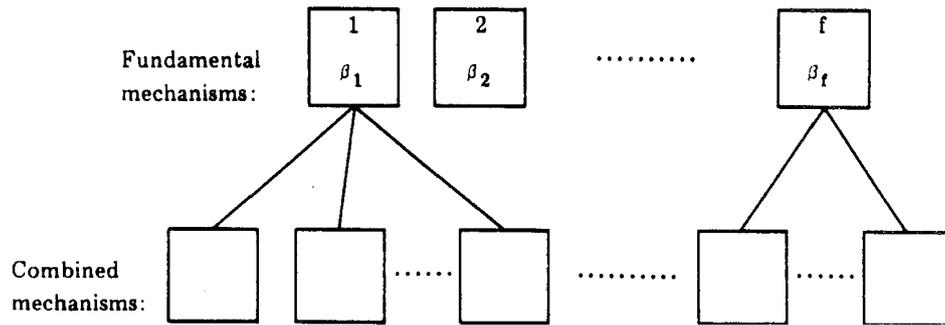


Figure 5. Construction of new mechanisms.

indices within a distance  $\varepsilon_2$  from the smallest reliability index are selected for further investigation. The same procedure is then used on the basis of the fundamental mechanisms 2, ..., f and a failure tree as the one shown in figure 5 is constructed.

More mechanisms can be identified on the basis of the combined mechanisms in the second row of the failure tree in figure 5 by adding or subtracting fundamental mechanisms. By repeating this simple procedure the failure tree for the structure in question can be constructed. The maximum number of rows in the failure tree must be chosen and can typically be  $m + 2$ , where  $m$  is the number of fundamental mechanisms. A satisfactory estimate of the systems reliability index can usually be obtained by using the same  $\varepsilon_2$ -value for all rows in the failure tree.

The final step in the application of the  $\beta$ -unzipping method in evaluating the reliability of an elasto-plastic structure at mechanism level is to select the significant mechanisms from the mechanisms identified in the failure tree. This selection can, in accordance with the selection criteria used in making the failure tree, e.g. be made first, identifying the smallest  $\beta$ -value,  $\beta_{\min}$  of all mechanisms in the failure tree and then selecting a constant  $\varepsilon_3$ . The significant mechanisms are then by definition those with  $\varepsilon$ -values in the interval  $[\beta_{\min}; \beta_{\min} + \varepsilon_3]$ . The probability of failure of the structure is then estimated by modeling the structural system as a series system with the significant mechanisms as elements (see figure 4).

The systems reliability index  $\beta_s$  is by definition equal to the generalized reliability index, i.e.

$$\beta_s = -\Phi^{-1}(P_f) \quad (13)$$

where  $P_f$  is the probability of failure of the (structural) system.

## 7. EXAMPLE

Consider the plane frame shown in figure 6. It is part of an offshore platform. It has 19 structural elements which are all tubular beam elements made of steel with the modulus of elasticity  $E = 0.21 \times 10^9$  kN/m<sup>2</sup>. The cross-sectional areas  $A_i$ , moment of inertia  $I_i$ , expected value of moment capacity  $E[M_i]$  and expected value of load-bearing capacity in tension/compression  $E[R_i]$ ,  $i = 1, 2, \dots, 19$  are shown in table 1.

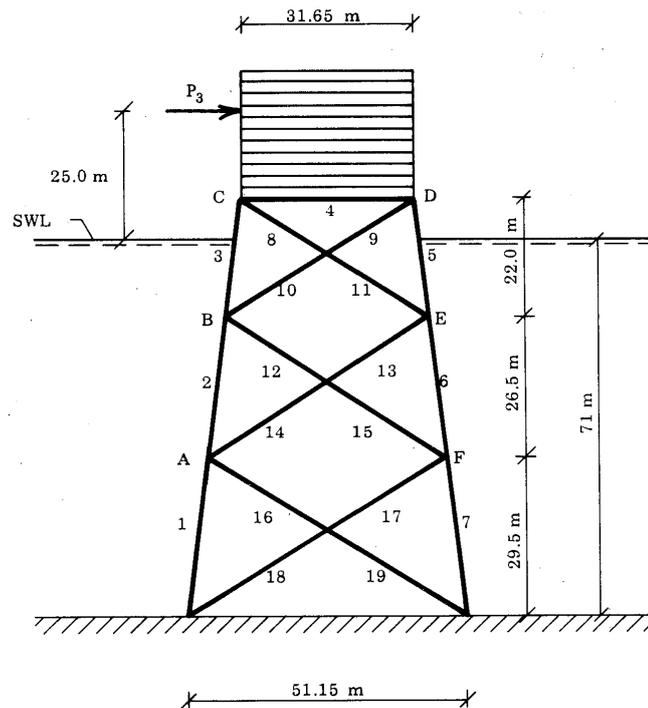


Figure 6. Offshore platform data.

Structural elements	A, m <sup>2</sup>	I, m <sup>4</sup>	E[M], kNm	E[R], kN
1, 7	0.2476	0.1201	40630	84180
2, 3, 5, 6	0.1847	0.0887	30330	62800
4	~ ∞	0.0300	15000	~ ∞
8, 9	0.1083	0.0132	8770	36820
10, 11	0.0766	0.0091	6190	26050
12, 13, 14, 15	0.1008	0.0144	8940	34270
16, 17, 18, 19	0.1291	0.0301	14720	43900

Table 1. Cross-sectional area  $A$ , moment of inertia  $I$ , expected value of moment capacity  $E[M]$ , and expected value of load-bearing capacity in tension / compression  $E[R]$ .

All load-bearing capacities  $M_i$  and  $R_i$  are assumed fully correlated and the coefficients of variation are  $V[M_i] = V[R_i] = 0.15$  for  $i = 1, 2, \dots, 19$ . Each structural element is supposed to have three failure elements, namely failure in tension/compression and bending failure at each end of the beam.

The loading of the frame consists of three parts, namely wave load, wind load, and dead load (eigenweight and topside loading). The wave load  $L_i$  of beam  $i$  is uniformly distributed and working perpendicularly to the beam. It can be written  $L_i = a_i p_1 + b_i p_2$ , where  $a_i p_1$  is the inertia part and  $b_i p_2$  is the drag part of the wave load. The coefficients  $a_i$  and  $b_i$ ,  $i = 1, 2, \dots, 19$  are shown in table 2. It is assumed that,

$$E[P_1] = 1.00 \text{ kN/m}, \quad V[P_1] = 0.28 \quad (14)$$

$$E[P_2] = 1.00 \text{ kN/m}, \quad V[P_2] = 0.40 \quad (15)$$

$P_1$  and  $P_2$  are fully correlated.

Beam, $i$	$a_i$	$b_i$
1	-1.244	8.212
2	0.100	11.960
3	1.549	20.907
4	6.643	-0.767
5	8.286	12.821
6	7.114	8.309
7	7.164	5.379
8	3.408	-6.541
9	4.160	4.885
10	3.163	4.713
11	1.751	-4.756
12	2.347	-4.001
13	3.411	2.913
14	1.989	3.076
15	0.689	-2.771
16	1.410	-3.055
17	2.782	2.388
18	0.566	2.708
19	-0.911	-2.175

Table 2. Loading coefficients.

At the intersection point A, B, C, D, E, F (see figure 6) the wave load consists of a horizontal part  $H_i$  (positive to the right) and a vertical part  $V_i$  (positive downwards),  $i = A, \dots, F$ .  $H_i$  and  $V_i$  can be written

$$H_i = c_i P_1 + d_i P_2 \quad (16)$$

$$V_i = e_i P_1 + f_i P_2 \quad (17)$$

where  $c_i P_1$  and  $e_i P_1$  are the inertia parts and  $d_i P_2$  and  $f_i P_2$  are the drag parts of the loading. The coefficients  $c_i$ ,  $d_i$ ,  $e_i$  and  $f_i$ ,  $i = A, \dots, F$  are shown in table 3. It is assumed that

$$E[P_1] = 1.00 \text{ kN}, \quad V[P_1] = 0.28 \quad (18)$$

$$E[P_2] = 1.00 \text{ kN}, \quad V[P_2] = 0.40 \quad (19)$$

Intersection point	$c_i$	$d_i$	$e_i$	$f_i$
A	-17.6	189.0	90.9	0.4
B	-7.8	228.6	127.9	0.3
C	-1.3	417.2	148.2	0.0
D	54.2	315.7	103.7	-42.0
E	80.8	164.2	107.2	-38.2
F	103.9	128.7	70.4	-13.7

Table 3. Loading parameters.

At the left hand side of the topside the lowest 6.0 m are loaded by the load 29.831  $P_2$  is the load at the lowest 6.0 m. The lowest 4.3 m of the right hand side is loaded by the load 11.175  $P_1 + 22.267 P_2$ , where  $P_1$  and  $P_2$  are defined earlier. The wind load  $P_3$  is shown in figure 6.  $E[P_3] = 1000 \text{ kN}$  and  $V[P_3] = 0.4$ . The dead load is two vertical single loads  $P_4$  at the intersection points C and D (positive downwards) with  $E[P_4] = 30377 \text{ kN}$  and  $V[P_4] = 0.05$ . It is assumed that  $P_1$ ,  $P_2$ ,  $P_3$  are fully correlated and that  $P_4$  is uncorrelated with  $P_1$ ,  $P_2$ ,  $P_3$ .

As mentioned earlier, each structural element has 3 failure elements so that the total number of failure elements for this structure is  $3 \times 19 = 57$ . The first step in the reliability analysis is calculation of reliability indices  $\beta_i$ ,  $i = 1, \dots, 57$  for all failure elements. It turns out that failure element 53 (see figure 7) in structural element 6 has the lowest  $\beta$ -index, namely  $\beta_{53} = 2.93$ . With  $\Delta\beta_1 = 2.0$ , 8 critical failure elements are identified. They are indicated in figure 7, and their  $\beta$ -indices are given in table 4.

Structural element	6	5	3	2	7	1	3	5
Failure element	53	56	8	5	50	2	9	55
$\beta_i$	2.93	3.30	3.53	3.57	3.74	4.58	4.69	4.77
Critical failure type	c	c	c	c	c	c	b	b

Table 4. Critical failure elements. c = failure in compression.  
b = failure in bending.

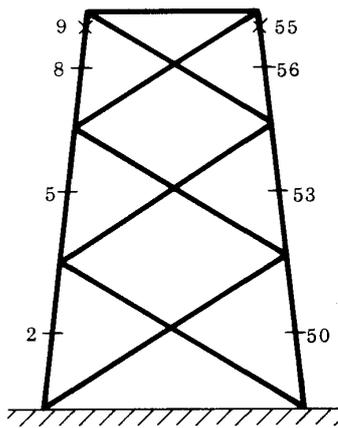


Figure 7. Critical failure elements. X = failure in bending. I = failure in compression.

The critical failure elements given in table 4 supplemented with the correlation coefficients between the corresponding safety margins are used in estimating the systems reliability at level 1. It turns out that some of the safety margins are almost fully correlated ( $\rho > 0.98$ ) so that the number of critical failure elements can be reduced when estimating the failure probability of the system at level 1. Only 4 critical failure elements (nos. 53, 5, 9, 55) have mutual correlation coefficients smaller than or equal to 0.98.

The correlation matrix (between the safety margin in the same order) is

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.97 & 0.92 & 0.94 \\ 0.97 & 1.00 & 0.81 & 0.83 \\ 0.92 & 0.81 & 1.00 & 0.54 \\ 0.94 & 0.83 & 0.54 & 1.00 \end{bmatrix}$$

The so-called Ditlevsen bounds for the systems probability of failure at level 1 (see figure 8) are identical and equal to

$$P_f^1 = 0.001696 \quad (20)$$

The corresponding systems reliability index at level 1 is

$$\beta_s^1 = 2.93 \quad (21)$$

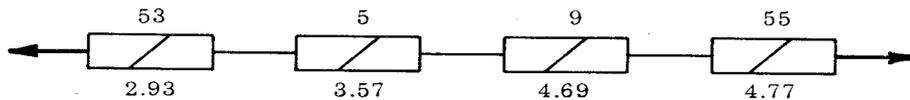


Figure 8. Series system used in estimating the systems reliability at level 1.

At level 2 it is initially assumed that the ductile failure element 53 fails (compression failure in structural element 6). The modified structure is then analyzed and new reliability indices are calculated for all the remaining failure elements. With  $\Delta\beta_2 = 0.25$  a number of critical pairs of failure elements are identified. This procedure is then performed with all critical failure elements. It turns out that 22 critical pairs of failure elements are identified but only seven of them are not fully correlated

( $\rho \leq 0.98$ ). The seven critical pairs of failure elements are shown in figure 9 with generalized  $\beta$  -indices for the parallel systems based on approximate equivalent safety margins.

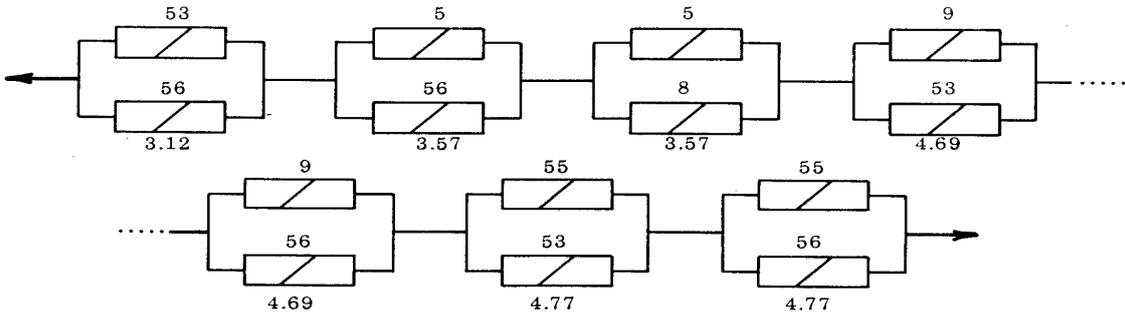


Figure 9. Series system used in estimating the systems reliability at level 2.

The correlation matrix is

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.98 & 0.98 & 0.90 & 0.90 & 0.92 & 0.92 \\ 0.98 & 1.00 & 0.50 & 0.40 & 0.40 & 0.42 & 0.42 \\ 0.98 & 0.50 & 1.00 & 0.81 & 0.81 & 0.83 & 0.83 \\ 0.90 & 0.40 & 0.81 & 1.00 & 0.57 & 0.57 & 0.57 \\ 0.90 & 0.40 & 0.81 & 0.57 & 1.00 & 0.60 & 0.60 \\ 0.92 & 0.42 & 0.83 & 0.57 & 0.60 & 1.00 & 0.26 \\ 0.92 & 0.42 & 0.83 & 0.57 & 0.60 & 0.26 & 1.00 \end{bmatrix}$$

The three most significant failure modes at level 2 are shown in figure 10. It follows from the correlation matrix that they are highly correlated. It is therefore to be expected that the systems reliability index  $\beta_s^2$  at level 2 is equal to the reliability index for the most serious failure mode. The Ditlevsen bounds for the generalized reliability index are identical and equal to

$$\beta_s^2 = 3.12 \tag{22}$$

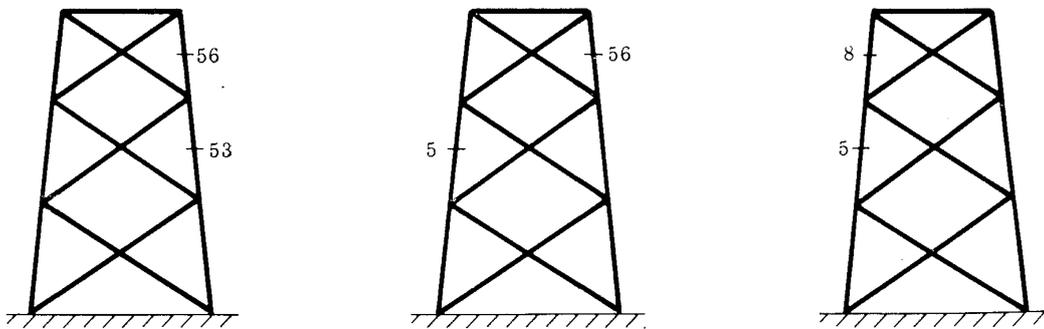


Figure 10. The three most significant failure modes at level 2.

At level 3 (with  $\Delta\beta_3 = 0.1$ ) six not fully correlated triples of failure elements are identified (see figure 11).

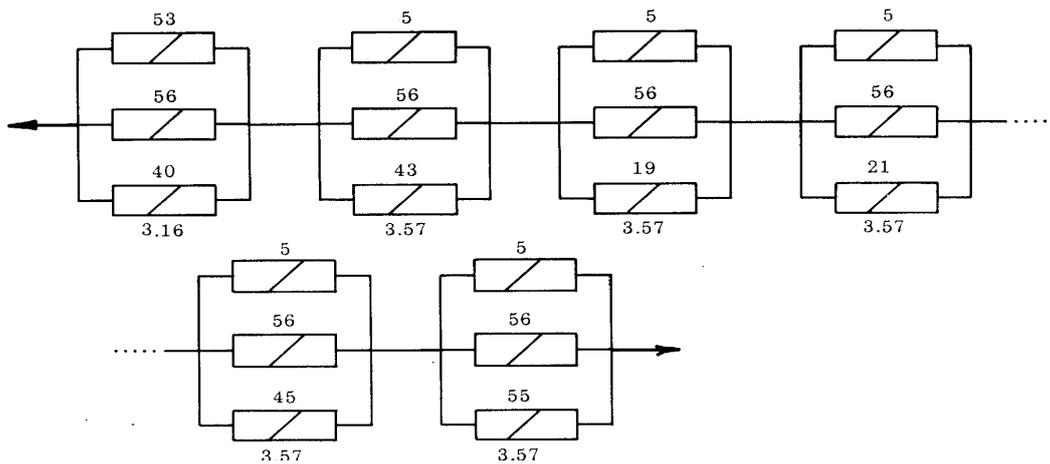


Figure 11. Series system used in estimating the systems reliability at level 3.

The correlation matrix is

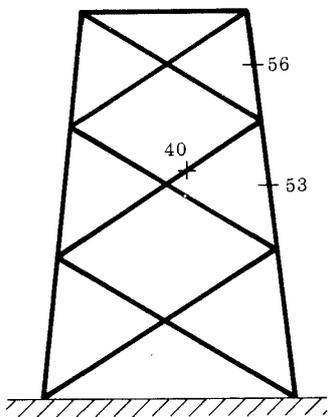


Figure 12. The most significant failure mode at level 3. I = failure in compression. x = failure in bending.

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.98 & 0.98 & 0.98 & 0.98 & 0.98 \\ 0.98 & 1.00 & 0.18 & 0.18 & 0.18 & 0.18 \\ 0.98 & 0.18 & 1.00 & 0.46 & 0.46 & 0.46 \\ 0.98 & 0.18 & 0.46 & 1.00 & 0.27 & 0.27 \\ 0.98 & 0.18 & 0.46 & 0.27 & 1.00 & 0.32 \\ 0.98 & 0.18 & 0.46 & 0.27 & 0.32 & 1.00 \end{bmatrix}$$

It follows from the correlation matrix that the most significant failure mode at level 3 (see figure 12) is almost fully correlated ( $\rho = 0.98$ ) with the remaining failure modes. It can therefore be expected that the systems reliability index at level 3  $\beta_s^3$  is equal to the reliability index  $\beta = 3.16$  for the failure mode with the failure elements 53, 56, and 40. This expectation is confirmed by calculation of the Ditlevsen bounds. Therefore,

$$\beta_s^3 = 3.16 \tag{23}$$

It is of interest to note that the estimates of the systems reliability index at levels 1 and 2 are very different, but that only a small increase takes place from level 2 to level 3 (see table 5). This is due to the fact that after failure in two structural elements the remaining structural elements can only sustain a small increase in the loading. It is therefore to be expected that an estimate of systems reliability indices at higher levels than level 3 will only result in slightly higher  $\beta$ -values than 3.16.

Reliability systems level	1	2	3
Systems reliability index	2.93	3.12	3.16

Table 5. Comparison of systems reliability indices at different levels.

The numerical calculations in this example are made in cooperation with J. D. Sørensen and G. Sigurdsson. A CDC-Cyber 170-730 Computer was used for this purpose.

## 8. CONCLUSIONS

In this paper an engineering area is presented where modern structural reliability theory has been used with some success, namely static loading of elasto-plastic framed and trussed structures. The so-called  $\beta$ -unzipping method to identify significant failure modes is used, but several other methods exist. Brittle behavior and stability problems can also be treated by this method. Recent research results seem to indicate that this method is convenient in optimum design, where the constraints are reliability conditions. However, it is not yet clear how to include dynamic behavior and fatigue problems. The systems reliability index calculated for a given structure has to be considered a relative measure of the reliability of the structure and not an absolute measure.

Modern structural reliability theory has been used with success in many other important areas such as dynamic response analysis, quality control, gross errors, hydrodynamics, code theory, soil mechanics, dams, concrete structures, offshore engineering, fatigue, ship design, etc. More details concerning these and many more areas can be found in the vast literature, see e.g. references [1], [2], [10], [11], and [12], and also a recently published bibliography [13] containing approximately 1500 references.

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