AALBORG UNIVERSITY
DENMARK

Spaces of distributions with mixed Lebesgue norms

Cleanthous, Galatia; Georgiadis, Athanasios; Nielsen, Morten

Published in:
Proceedings of the 15th Panhellenic Conference of Mathematical Analysis

Publication date:
2016

Document Version
Publisher's PDF, also known as Version of record

[Link to publication from Aalborg University](#)

Citation for published version (APA):
Cleanthous, G., Georgiadis, A., & Nielsen, M. (2016). Spaces of distributions with mixed Lebesgue norms. In *Proceedings of the 15th Panhellenic Conference of Mathematical Analysis* (pp. 29-38).

SPACES OF DISTRIBUTIONS WITH MIXED LEBESGUE NORMS

G. CLEANTHOUS, A. G. GEORGIADIS, AND M. NIELSEN

ABSTRACT. We consider smoothness spaces of distributions on $\mathbb{R}^n$ with mixed Lebesgue norms, where different level of integrability is used for every coordinate. In this note we will state our recent results in this area and we will present some new properties of mixed-norm Besov and Triebel-Lizorkin spaces.

1. INTRODUCTION

The theory of spaces of functions and distributions forms an integral part of functional analysis. Here we aim to present some recent and some new properties of smoothness spaces. Besov and Triebel-Lizorkin spaces form two closely related families of smoothness spaces with numerous applications in approximation theory and functional analysis, see [12, 27, 30]. The construction of the above mentioned spaces is based on a dyadic decomposition of the frequency space, and their proven usefulness for applications relies to a large degree on the fact that universal and stable discrete decomposition systems exist for the two families of spaces.

The significance of these spaces can be partially understood by the fact that several spaces of functional analysis, with their own history, are recovered for specific values of the parameters in the definitions of Besov and Triebel-Lizorkin spaces. Some examples are Lebesgue, Hardy, Sobolev and Lipschitz spaces.

The study of Besov and Triebel-Lizorkin spaces has been expanded significantly since the introduction of the so called φ-transform by Frazier and Jawerth in their seminal papers [10–12]. As solid bases for introduction in the study of these spaces we refer the reader to the books of Peetre [27], Triebel [30] and the booklet of Frazier, Jawerth and Weiss [13].

The influence of [10–12] on mathematical analysis has been impressive. Any citation database will show a huge number of citations to the above papers. Moreover these papers have guided researchers with specialities in distribution spaces, wavelets, and approximation theory. Some related works on $\mathbb{R}^n$ are [4–6, 22, 24]. For decompositions on other settings such as on the ball, on the sphere and the interval, see for example [20, 21, 23, 26, 28].

In this paper we present some recent and some new results for Besov and Triebel-Lizorkin spaces in a mixed-norm setting. The content of the article has been presented by the first named author during the fifteenth Panhellenic conference of

mathematical analysis which took place in Heraklion between 27 and 29 of May of 2016.

Recently, there has been significant interest in the study of inhomogeneous Besov and Triebel-Lizorkin spaces with mixed Lebesgue norms, see [15–19].

In [7] we introduced and studied homogeneous mixed-norm Besov spaces $\dot{B}^s_{\vec{p}q}$, for $s \in \mathbb{R}$, $\vec{p} = (p_1, \dots, p_n) \in (0, \infty)^n$ and $q \in (0, \infty]$. The homogeneous spaces are defined over the class $\mathcal{S}'/\mathcal{P}$ of tempered distributions modulo the polynomials. Homogeneous mixed-norm Triebel-Lizorkin spaces $\dot{F}^s_{\vec{p}q}$, are introduced in the recent preprint [14].

Here we present some first properties on $\dot{B}^s_{\vec{p}q}$ spaces proven in [7] and we offer some new results as well. Namely we will prove the connection between inhomogeneous and homogeneous mixed-norm Besov and Triebel-Lizorkin spaces.

Notation: Through the article, positive constants will denoted by c and they may vary at every occurrence. The Fourier transform of a (proper) function f will be stated by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$. The set of positive integers will be denote by $\mathbb{N} := \{1, 2, \dots\}$. For two quasi-normed spaces X, Y we will denoted by $X \hookrightarrow Y$ a continuous embedding.

2. Preliminaries

In this section we present some background needed for the development of mixed norm Besov and Triebel-Lizorkin spaces.

2.1. **Schwartz functions and distributions.** Let us recall some basic facts about Schwartz functions and distributions. We denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. A function $\varphi \in \mathcal{C}^\infty$ belongs to $\mathcal{S}$, when for every $k \in \mathbb{N} \cup \{0\}$ and every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$,

$$(2.1) \qquad \mathcal{P}_{k,\alpha}(\varphi) := \sup_{x \in \mathbb{R}^n} (1 + |x|)^k |D^\alpha \varphi(x)| < \infty.$$

The dual $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}$ is the space of tempered distributions.
We will further denote

$$\mathcal{S}_\infty := \mathcal{S}_\infty(\mathbb{R}^n) = \Big\{ \psi \in \mathcal{S} : \int_{\mathbb{R}^n} x^\alpha \psi(x)dx = 0, \ \forall \alpha \in (\mathbb{N} \cup \{0\})^n \Big\}.$$

We note that $\mathcal{S}_\infty$ is a Fréchet space, because it is closed in $\mathcal{S}$ and its dual is $\mathcal{S}'_\infty = \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ the family of polynomials on $\mathbb{R}^n$.

We will define inhomogeneous mixed-norm Besov spaces for elements of $\mathcal{S}'$ and the homogeneous ones for tempered distributions modulo polynomials $\mathcal{S}'/\mathcal{P}$.

2.2. **Mixed norm Lebesgue spaces.** In our setting, the integrability will be measured in terms of the mixed Lebesgue norms which we present immediately.

Let $\vec{p} = (p_1, \dots, p_n) \in (0, \infty)^n$ and $f : \mathbb{R}^n \to \mathbb{C}$. We say that $f \in L_{\vec{p}} = L_{\vec{p}}(\mathbb{R}^n)$ if
(2.2)

$$\|f\|_{\vec{p}} := \|f\|_{L_{\vec{p}}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \cdots dx_n \right)^{\frac{1}{p_n}} < \infty,$$

The quasi-norm $\|\cdot\|_{\vec{p}}$, is actually a norm when $\min(p_1, \dots, p_n) \geq 1$ and turns $(L_{\vec{p}}, \|\cdot\|_{\vec{p}})$ into a Banach space. Note that when $\vec{p} = (p, \dots, p)$, then $L_{\vec{p}}$ coincides

with L_p. More properties of $L_{\vec{p}}$, can be found for example in [1–3, 9, 25, 29]. For smoothness spaces with mixed Lebesgue norms we refer the reader to [15–17, 25] and their references.

3. Inhomogeneous mixed-norm Besov and Triebel-Lizorkin spaces

Inhomogeneous mixed-norm Besov and Triebel-Lizorkin spaces have been extensively studied the last years, see for example [15, 18, 19] and the references therein. Let us recall their definitions.

Let a function $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$(3.3) \qquad \operatorname{supp} \widehat{\phi_0} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\},$$

and

$$(3.4) \qquad |\widehat{\phi_0}(\xi)| \geq c > 0 \quad \text{if} \ \ |\xi| \leq 2^{3/4}.$$

Let also $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$(3.5) \qquad \operatorname{supp} \widehat{\phi} \subseteq \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\},$$

and

$$(3.6) \qquad |\widehat{\phi}(\xi)| \geq c > 0 \quad \text{if} \ \ 2^{-3/4} \leq |\xi| \leq 2^{3/4}.$$

We set $\phi_\nu(x) := 2^{\nu n}\phi(2^\nu x), \ \ \forall \nu \in \mathbb{N}$.

Definition 3.1. *Let $s \in \mathbb{R}$, $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n$, $q \in (0, \infty]$ and ϕ_0, ϕ as above.*

(i) The inhomogeneous mixed-norm Besov space $B_{\vec{p}q}^s$, is the collection of all $f \in \mathcal{S}'$ such that

$$(3.7) \qquad \|f\|_{B_{\vec{p}q}^s} := \Big(\sum_{\nu=0}^\infty (2^{\nu s}\|\phi_\nu * f\|_{\vec{p}})^q \Big)^{1/q} < \infty,$$

with the ℓ_q-norm replaced by the $\sup_\nu$ if $q = \infty$.

(ii) The inhomogeneous mixed-norm Triebel-Lizorkin space $F_{\vec{p}q}^s$, is the collection of all $f \in \mathcal{S}'$ such that

$$(3.8) \qquad \|f\|_{F_{\vec{p}q}^s} := \Big|\Big| \Big(\sum_{\nu=0}^\infty (2^{\nu s}|\phi_\nu * f(\cdot)|)^q \Big)^{1/q} \Big|\Big|_{\vec{p}} < \infty,$$

with the ℓ_q-norm replaced by the $\sup_\nu$ if $q = \infty$.

4. Homogeneous mixed-norm Besov spaces

In this section we present the extension of the classical homogeneous Besov spaces (see Triebel [30], Peetre [27] and Frazier-Jawerth [10]), which we developed in [7] using mixed-norms.

We will say that a test function $\varphi \in \mathcal{S}$ is admissible when it satisfies (3.5) and (3.6). Furthermore, we set $\varphi_\nu(x) := 2^{\nu n}\varphi(2^\nu x), \ \forall \nu \in \mathbb{Z}$. We present the following:

Definition 4.1. *[7] For $s \in \mathbb{R}$, $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n$, $q \in (0, \infty]$ and φ admissible, we define the homogeneous mixed-norm Besov space $\dot{B}_{\vec{p}q}^s$, as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that*

$$(4.9) \qquad \|f\|_{\dot{B}_{\vec{p}q}^s} := \Big(\sum_{\nu \in \mathbb{Z}} (2^{\nu s}\|\varphi_\nu * f\|_{\vec{p}})^q \Big)^{1/q} < \infty,$$

4   G. CLEANTHOUS, A. G. GEORGIADIS, AND M. NIELSEN

with the ℓ_q-norm replaced by the $\sup_\nu$, if $q = \infty$.

Remark 4.2. *Several remarks regarding the homogeneous mixed-norm Besov spaces defined above and some results proven in [7] are in order.*

(α) By (3.6) we have that $\|f\|_{\dot{B}^s_{\vec{p}q}} = 0 \Leftrightarrow f \in \mathcal{P}$, which is why we work over the quotient $\mathcal{S}'/\mathcal{P}$.

(β) When $\vec{p} = (p, \ldots, p)$, then $\dot{B}^s_{\vec{p}q}$, coincides with $\dot{B}^s_{pq}$, the standard homogeneous Besov space.

(γ) Homogeneous mixed-norm Besov space $\dot{B}^s_{\vec{p}q}$ is quasi-Banach for all $s \in \mathbb{R}$, $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n$ and $q \in (0, \infty]$. The triangle inequality does not hold in general in $\dot{B}^s_{\vec{p}q}$. Instead we have the sub-additivity

$$\|f + g\|^r_{\dot{B}^s_{\vec{p}q}} \leq \|f\|^r_{\dot{B}^s_{\vec{p}q}} + \|g\|^r_{\dot{B}^s_{\vec{p}q}}, \quad \text{where } r := \min(1, p_1, \ldots, p_n, q).$$

Furthermore $\dot{B}^s_{\vec{p}q}$ is a Banach space when $\vec{p} \in [1, \infty)^n$, $q \in [1, \infty]$.

(δ) The quasi-norm in the definition of $\dot{B}^s_{\vec{p}q}$ depends on the choice of the admissible function φ, but for different admissible functions, we get equivalent quasi-norms. Therefore $\dot{B}^s_{\vec{p}q}$ space is independent of the admissible function φ.

(ε) All the construction has been based on the dyadic decomposition of the frequency space. We can use instead, any other number $\beta > 1$ in all the procedure of Subsection 2.2, as well as in the Definition 4.1 of Besov spaces (replace $2^{\nu s}$ by $\beta^{\nu s}$) and get the same spaces with equivalent norms.

($\sigma\tau$) Some embeddings between homogeneous mixed-norm Besov spaces, provided in [7] are presented below:

($\sigma\tau 1$) Let $s \in \mathbb{R}$, $\vec{p} \in (0, \infty)^n$ and $0 < q < r \leq \infty$. Then we have the embedding

$$\dot{B}^s_{\vec{p}q} \hookrightarrow \dot{B}^s_{\vec{p}r},$$

coming from the well known embedding between the sequence spaces; $\ell_q \hookrightarrow \ell_r$.

($\sigma\tau 2$) Homogeneous mixed-norm Besov spaces and the classes $\mathcal{S}_\infty, \mathcal{S}'_\infty$ are connected in the following way:

Proposition 4.3. *Let $s \in \mathbb{R}$, $\vec{p} = (p_1, \cdots, p_n) \in (0, \infty)^n$ and $q \in (0, \infty]$. Then*

$$\mathcal{S}_\infty \hookrightarrow \dot{B}^s_{\vec{p}q} \quad \text{and} \quad \dot{B}^s_{\vec{p}q} \hookrightarrow \mathcal{S}'_\infty.$$

($\sigma\tau 3$) Spaces of different smoothness levels are connected as below:

Proposition 4.4. *Let $s, t \in \mathbb{R}$, $\vec{p} = (p_1, \cdots, p_n)$, $\vec{r} = (r_1, \ldots, r_n) \in (0, \infty)^n$ and $q \in (0, \infty]$ be such that*

$$t < s, \quad p_1 \leq r_1, \ldots, p_n \leq r_n, \quad \text{and} \quad s - \frac{1}{p_1} - \cdots - \frac{1}{p_n} = t - \frac{1}{r_1} - \cdots - \frac{1}{r_n},$$

then

$$\dot{B}^s_{\vec{p}q} \hookrightarrow \dot{B}^t_{\vec{r}q}.$$

Specifically we have the following relation between mixed and unmixed spaces:

Let $s \in \mathbb{R}$, $\vec{p} = (p_1, \cdots, p_n) \in (0, \infty)^n$ and $q \in (0, \infty]$. We set $p_m := \min(p_1, \ldots, p_n)$ and $p_M := \max(p_1, \ldots, p_n)$, then

$$\dot{B}^t_{p_m q} \hookrightarrow \dot{B}^s_{\vec{p}q} \hookrightarrow \dot{B}^\tau_{p_M q},$$

where

$$t = s - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right) + \frac{n}{p_m} \quad \text{and} \quad \tau = s - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right) + \frac{n}{p_M}.$$

4.1. **Homogeneous mixed-norm Triebel-Lizorkin spaces.** The development of homogeneous mixed-norm Triebel-Lizorkin spaces has been obtained in [14]. Let us present here only the definition of these spaces.

Definition 4.5. *[14] For $s \in \mathbb{R}$, $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n$, $q \in (0, \infty]$ and φ admissible, we define the homogeneous mixed-norm Triebel-Lizorkin space $\dot{F}^s_{\vec{p}q}$, as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that*

$$(4.10) \qquad \|f\|_{\dot{F}^s_{\vec{p}q}} := \left\| \left(\sum_{\nu \in \mathbb{Z}} (2^{\nu s} |\varphi_\nu * f(\cdot)|)^q \right)^{1/q} \right\|_{\vec{p}} < \infty,$$

with the ℓ_q-norm replaced by the $\sup_\nu$ if $q = \infty$.

Note that the remarks we presented for the case of homogeneous mixed-norm Besov spaces, apply for $\dot{F}^s_{\vec{p}q}$ spaces too.

5. Comparison of inhomogeneous and homogeneous spaces

In this section we give some new results, inspired by the unmixed case presented in [13]. We give the relation connecting the inhomogeneous and homogeneous mixed-norm Besov and Triebel-Lizorkin spaces, but let us first justify the title "homogeneous" which we use for some of our spaces.

Let $f \in \mathcal{S}'$. We set $f_\mu(x) := 2^{\mu n} f(2^\mu x)$ for every $\mu \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. We will show that

$$(5.11) \qquad \|f_\mu\|_{\dot{B}^s_{\vec{p}q}} = 2^{\mu N} \|f\|_{\dot{B}^s_{\vec{p}q}}, \ \forall s \in \mathbb{R}, \ \vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n, \ q \in (0, \infty],$$

where N is an exponent depending only on the parameters s, $\vec{p}$, q.

Indeed, let $\nu, \mu \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. By changing variables we obtain that

$$(5.12) \qquad \varphi_\nu * f_\mu(x) = 2^{\mu n} \big(\varphi_{\nu-\mu} * f\big)(2^\mu x).$$

Now the mixed Lebesgue norm of $\big(\varphi_{\nu-\mu} * f\big)(2^\mu x)$, by changing the variables $2^\mu x_j =: y_j$, for every direction $j = 1, \ldots, n$, equals to

$$(5.13) \qquad \|(\varphi_{\nu-\mu} * f)(2^\mu \cdot)\|_{\vec{p}} = 2^{-\mu\left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)} \|\varphi_{\nu-\mu} * f\|_{\vec{p}}.$$

From (5.12) and (5.13), it follows that

$$
\begin{aligned}
\|f_\mu\|_{\dot{B}^s_{\vec{p}q}} &= \left(\sum_{\nu \in \mathbb{Z}} (2^{\nu s} \|\varphi_\nu * f_\mu\|_{\vec{p}})^q \right)^{1/q} \\
&= \left(\sum_{\nu \in \mathbb{Z}} \left(2^{\nu s} 2^{\mu n} 2^{-\mu\left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)} \|\varphi_{\nu-\mu} * f\|_{\vec{p}} \right)^q \right)^{1/q} \\
&= 2^{\mu\left(s + n - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)\right)} \left(\sum_{\nu \in \mathbb{Z}} \left(2^{(\nu-\mu)s} \|\varphi_{\nu-\mu} * f\|_{\vec{p}} \right)^q \right)^{1/q} \\
&= 2^{\mu\left(s + n - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)\right)} \|f\|_{\dot{B}^s_{\vec{p}q}}.
\end{aligned}
$$

So (5.11) holds true for $N := s + n - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)$. Note that (5.11) remains true for the homogeneous mixed-norm Triebel-Lizorkin spaces as well (with the same N) and does not hold for the inhomogeneous spaces.

The exponent N is called the *homogeneous dimension* of $\dot{B}^s_{\vec{p}q}$ (or $\dot{F}^s_{\vec{p}q}$) space. Note that for the unmixed case the homogeneous dimension we derived turns to $N = s + n\left(1 - \frac{1}{p}\right)$ as in [13].

6   G. CLEANTHOUS, A. G. GEORGIADIS, AND M. NIELSEN

Now let us present the relation connecting the inhomogeneous and homogeneous spaces with mixed-norms, inspired by the classical, unmixed, situation, see [13].

Theorem 5.1. *Let $s > 0$, $\vec{p} = (p_1, \ldots, p_n)$ with $\min(p_1, \ldots, p_n) \geq 1$ and $0 < q \leq \infty$. Then*

$$\text{(i) } B^s_{\vec{p}q} = L_{\vec{p}} \cap \dot{B}^s_{\vec{p}q} \text{ and (ii) } F^s_{\vec{p}q} = L_{\vec{p}} \cap \dot{F}^s_{\vec{p}q}.$$

Proof. (i) Let $f \in B^s_{\vec{p}q}$. Let also $\phi_0, \phi \in \mathcal{S}$ satisfying (3.3)-(3.6) be such that

$$\sum_{\nu \geq 0} \widehat{\phi_\nu}(\xi) = 1, \text{ for every } \xi \in \mathbb{R}^n.$$

Then

$$f = \sum_{\nu \geq 0} \phi_\nu * f \quad (\text{convergence in } \mathcal{S}').$$

Using the fact that $\min(p_1, \ldots, p_n) \geq 1$ and hence $\|\cdot\|_{\vec{p}}$ turns to a norm, it follows that

$$
\begin{aligned}
\|f\|_{L_{\vec{p}}} &= \left\| \sum_{\nu \geq 0} \phi_\nu * f \right\|_{\vec{p}} \leq \sum_{\nu \geq 0} \|\phi_\nu * f\|_{\vec{p}} \\
&\leq \sum_{\nu \geq 0} 2^{-\nu s} \sup_{\mu \geq 0} 2^{\mu s} \|\phi_\mu * f\|_{\vec{p}} \\
&= c_s \sup_{\mu \geq 0} 2^{\mu s} \|\phi_\mu * f\|_{\vec{p}} \leq c\|f\|_{B^s_{\vec{p}q}},
\end{aligned}
$$

(5.14)

where for the last equality, we used the assumption $s > 0$.

Let now $\varphi \in \mathcal{S}$ satisfying (3.5) and (3.6). By Penedek-Panzone [3, Theorem 1.b, p. 319] and by the fact that $\min(p_1, \ldots, p_n) \geq 1$, we have the following behaviour for the mixed-norms of convolution operators:

(5.15) $\qquad \|\varphi_\nu * f\|_{\vec{p}} \leq \|\varphi_\nu\|_1 \|f\|_{\vec{p}} = c\|f\|_{\vec{p}}, \text{ for every } \nu \in \mathbb{Z},$

since we can easily observe that $\|\varphi_\nu\|_1 = \|\varphi\|_1$, for every $\nu \in \mathbb{Z}$ and hence we get immediately

$$
\begin{aligned}
\|f\|_{\dot{B}^s_{\vec{p}q}} &= \left(\sum_{\nu \in \mathbb{Z}} (2^{\nu s} \|\varphi_\nu * f\|_{\vec{p}})^q \right)^{1/q} \\
&\leq c \left(\sum_{\nu \leq 0} (2^{\nu s} \|\varphi_\nu * f\|_{\vec{p}})^q \right)^{1/q} + c \left(\sum_{\nu > 0} (2^{\nu s} \|\varphi_\nu * f\|_{\vec{p}})^q \right)^{1/q} \\
&\leq \left(\sum_{\nu \leq 0} 2^{\nu s q} \right)^{1/q} \|f\|_{\vec{p}} + c\|f\|_{B^s_{\vec{p}q}} \\
&\leq c(\|f\|_{\vec{p}} + \|f\|_{B^s_{\vec{p}q}}),
\end{aligned}
$$

(5.16)

where we used again the fact that $s > 0$. Combining (5.14) and (5.16) we have the embedding

$$B^s_{\vec{p}q} \hookrightarrow L_{\vec{p}} \cap \dot{B}^s_{\vec{p}q}.$$

For the other direction, note that (5.15) holds true for the functions ϕ_ν, $\nu \geq 0$ as well. Then,

$$
\begin{aligned}
\|f\|_{B^s_{\vec{p}q}} &= \Big(\sum_{\nu \geq 0} (2^{\nu s} \|\phi_\nu * f\|_{\vec{p}})^q \Big)^{1/q} \\
&\leq c \|\phi_0 * f\|_{\vec{p}} + c \Big(\sum_{\nu > 0} (2^{\nu s} \|\phi_\nu * f\|_{\vec{p}})^q \Big)^{1/q} \\
&\leq c (\|f\|_{\vec{p}} + \|f\|_{\dot{B}^s_{\vec{p}q}}),
\end{aligned}
$$

which guarantees the embedding

$$
L_{\vec{p}} \cap \dot{B}^s_{\vec{p}q} \hookrightarrow B^s_{\vec{p}q}.
$$

(ii) We will follow [13]. Let $f \in \mathcal{S}'$ and $\phi_0, \phi \in \mathcal{S}$ satisfying (3.3)-(3.6) be such that $\{\widehat{\phi_\nu}\}_{\nu \geq 0}$ to be a partition of unity. Then

$$
(5.17) \qquad f = \sum_{\nu \geq 0} \phi_\nu * f \quad \text{(convergence in } \mathcal{S}').
$$

We turn to estimate

$$
\sum_{\nu \geq 1} |\phi_\nu * f(x)|.
$$

We distinguish the cases $q \geq 1$ and $q < 1$.

Case α : $1 \leq q \leq \infty$. By Hölder's inequality, denoting by q' the conjugate index of q, we obtain

$$
\begin{aligned}
\sum_{\nu \geq 1} |\phi_\nu * f(x)| &\leq \Big(\sum_{\nu \geq 1} 2^{-\nu s q'} \Big)^{1/q'} \Big(\sum_{\nu \geq 1} \big(2^{\nu s} |\phi_\nu * f(x)| \big)^q \Big)^{1/q} \\
&\leq c_{s,q} \Big(\sum_{\nu \geq 1} \big(2^{\nu s} |\phi_\nu * f(x)| \big)^q \Big)^{1/q},
\end{aligned}
$$

thanks to the assumption $s > 0$.

Case β : $0 < q < 1$. Using the q-triangle inequality and the fact that $s > 0$, we derive

$$
\sum_{\nu \geq 1} |\phi_\nu * f(x)| \leq \sum_{\nu \geq 1} 2^{\nu s} |\phi_\nu * f(x)| \leq \Big(\sum_{\nu \geq 1} \big(2^{\nu s} |\phi_\nu * f(x)| \big)^q \Big)^{1/q}.
$$

Since now $\min(p_1, \ldots, p_n) \geq 1$, by assumption, relation (5.17) and the bounds above lead us to

$$
\begin{aligned}
\|f\|_{\vec{p}} &\leq c \|\phi_0 * f\|_{\vec{p}} + c \Big\| \Big(\sum_{\nu \geq 1} (2^{\nu s} |\phi_\nu * f(\cdot)|)^q \Big)^{1/q} \Big\|_{\vec{p}} \\
(5.18) \qquad &\leq c \Big\| \Big(\sum_{\nu \geq 0} (2^{\nu s} |\phi_\nu * f(\cdot)|)^q \Big)^{1/q} \Big\|_{\vec{p}} = c \|f\|_{F^s_{\vec{p}q}}.
\end{aligned}
$$

Let now $\varphi \in \mathcal{S}$ satisfying (3.5) and (3.6). Then,

$$
\begin{aligned}
\|f\|_{\dot{F}^s_{\vec{p}q}} &\leq c \Big\| \Big(\sum_{\nu \leq 0} (2^{\nu s} |\varphi_\nu * f(\cdot)|)^q \Big)^{1/q} \Big\|_{\vec{p}} + c \Big\| \Big(\sum_{\nu > 0} (2^{\nu s} |\varphi_\nu * f(\cdot)|)^q \Big)^{1/q} \Big\|_{\vec{p}} \\
(5.19) \qquad &=: c (\Sigma_1 + \Sigma_2).
\end{aligned}
$$

Of course it holds that

$$\Sigma_2 \leq \|f\|_{F_{\vec{p}q}^s} \tag{5.20}$$

and so we restrict our interest to Σ_1. We consider separately the cases when q is smaller than 1 or not.

Case α : $0 < q \leq 1$. By Hölder's inequality, denoting by $(1/q)'$ the conjugate index of $1/q$, we obtain

$$
\begin{aligned}
\sum_{\nu \leq 0} \left(2^{\nu s} |\varphi_\nu * f(\cdot)| \right)^q &\leq \left(\sum_{\nu \leq 0} 2^{(\nu s q/2)(1/q)'} \right)^{1/(1/q)'} \left(\sum_{\nu \leq 0} 2^{\nu s/2} |\varphi_\nu * f(\cdot)| \right)^q \\
&\leq c_{s,q} \left(\sum_{\nu \leq 0} 2^{\nu s/2} |\varphi_\nu * f(\cdot)| \right)^q.
\end{aligned}
$$

The last inequality gives us

$$
\begin{aligned}
\Sigma_1 &\leq c \left\| \sum_{\nu \leq 0} 2^{\nu s/2} |\varphi_\nu * f(\cdot)| \right\|_{\vec{p}} \leq c \sum_{\nu \leq 0} 2^{\nu s/2} \|\varphi_\nu * f\|_{\vec{p}} \\
&\leq c \left(\sum_{\nu \leq 0} 2^{\nu s/2} \right) \|f\|_{\vec{p}} \leq c \|f\|_{\vec{p}},
\end{aligned}
\tag{5.21}
$$

where for the second inequality we used the fact that $\| \cdot \|_{\vec{p}}$ is a norm under our assumptions, for the third the inequality (5.15) and for the last the assumption $s > 0$.

Case β : $1 < q \leq \infty$. By the identity $|a + b|^{1/q} \leq |a|^{1/q} + |b|^{1/q}$, we derive

$$\sum_{\nu \leq 0} \left(2^{\nu s} |\varphi_\nu * f(\cdot)| \right)^q \leq \left(\sum_{\nu \leq 0} 2^{\nu s} |\varphi_\nu * f(\cdot)| \right)^q.$$

So with the same steps as before we get for this case too

$$\Sigma_1 \leq c \|f\|_{\vec{p}}. \tag{5.22}$$

Combining (5.18)-(5.22) we have that

$$\|f\|_{\dot{F}_{\vec{p}q}^s} \leq c \|f\|_{F_{\vec{p}q}^s}$$

which together with (5.18) offers the inclusion

$$F_{\vec{p}q}^s \hookrightarrow L_{\vec{p}} \cap \dot{F}_{\vec{p}q}^s.$$

The converse embedding comes straight from the expression (5.17) and the estimation (5.15), indeed

$$
\begin{aligned}
\|f\|_{F_{\vec{p}q}^s} &\leq c \|\phi_0 * f\|_{\vec{p}} + c \left\| \left(\sum_{\nu > 0} (2^{\nu s} |\phi_\nu * f(\cdot)|)^q \right)^{1/q} \right\|_{\vec{p}} \\
&\leq c \|f\|_{\vec{p}} + c \|f\|_{\dot{F}_{\vec{p}q}^s}
\end{aligned}
$$

and the proof is complete.

$\square$

SPACES OF DISTRIBUTIONS WITH MIXED LEBESGUE NORMS   9

REFERENCES

[1] N. Antonić and I. Ivec, On the Hörmander-Mihlin theorem for mixed-norm Lebesgue spaces, J. Math. Anal. Appl. 433 (2016), no. 1, 176-199.

[2] R.J. Bagby, An extended inequality for the maximal function, Proc. Amer. Math. Soc. 48 (1975), 419-122.

[3] A.I. Benedek and R. Panzone, The spaces L^P with mixed norm, Duke Math. J. 28 (1961), 301-324.

[4] L. Borup and M. Nielsen, On anisotropic Triebel-Lizorkin type spaces, with applications to the study of pseudo-differential operators, J. Funct. Spaces Appl., 6 (2008), no. 2, 107-154.

[5] M. Bownik, Atomic and molecular decompositions of anisotropic Besov spaces, Math. Z. 250 (2005), no. 3, 539-571.

[6] M. Bownik, K.P. Ho, Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces, Trans. Amer. Math. Soc. 358 (2006), no. 4, 1469-1510.

[7] G. Cleanthous, A.G. Georgiadis, and M. Nielsen, Discrete decomposition for homogeneous mixed-norm Besov spaces, Contemp. Math., to appear.

[8] I. Daubechies, Ten lectures on wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[9] D.L. Fernandez, Vector-valued singular integral operators on L^p-spaces with mixed norms and applications, Pac. J. Math. 129 (1987), no. 2, 257-275.

[10] M. Frazier and B. Jawerth, Decomposition of Besov Spaces, Indiana Univ. Math. J. 34 (1985), 777-799.

[11] M. Frazier and B. Jawerth, The φ-transform and applications to distribution spaces, Function spaces and applications (Lund, 1986), 223-246, Lecture Notes in Math., 1302, Springer, Berlin, 1988.

[12] M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution, J. of Funct. Anal. 93 (1990), 34-170.

[13] M. Frazier, B. Jawerth, and G. Weiss, Littlewood-Paley Theory and the Study of Function Spaces, CBMS 79 (1991), AMS.

[14] A.G. Georgiadis, J. Johnsen and M. Nielsen, Wavelet transforms for homogeneous mixed-norm Triebel-Lizorkin spaces, Arxiv e-prints: arXiv:1608.03782.

[15] A.G. Georgiadis and M. Nielsen, Pseudodifferential operators on mixed-norm Triebel-Lizorkin and Besov spaces. Math. Nachr., 289 (2016), no. 16, 2019-2036. .

[16] J. Johnsen, S. Munch Hansen, W. Sickel, Characterisation by local means of anisotropic Lizorkin-Triebel spaces with mixed norms, Z. Anal. Anwend. 32 (2013), no. 3, 257-277.

[17] J. Johnsen, S. Munch Hansen, W. Sickel, Anisotropic Lizorkin-Triebel spaces with mixed normstraces on smooth boundaries, Math. Nachr. 288 (2015), no. 11-12, 1327-1359.

[18] J. Johnsen and W. Sickel, A direct proof of Sobolev embeddings for quasi-homogeneous Lizorkin-Triebel spaces with mixed norms, J. Funct. Spaces Appl. 5 (2007), no. 2, 183-198.

[19] J. Johnsen and W. Sickel, On the trace problem for Lizorkin-Triebel spaces with mixed norms, Mathematische Nachrichten, 281 (2008), no. 5, 669-696.

[20] G. Kerkyacharian and P. Petrushev, Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces, Trans. Amer. Math. Soc. 367 (2015), 121-189.

[21] G. Kyriazis, P. Petrushev, Rational bases for spaces of holomorphic functions in the disc, J. Lond. Math. Soc. (2) 89 (2014), no. 2, 434-460.

[22] G. Kyriazis and P. Petrushev, On the construction of frames for spaces of distributions, J. Funct. Anal. 257 (2009), no. 7, 2159-2187.

[23] G. Kyriazis, P. Petrushev, and Y. Xu, Decomposition of weighted Triebel-Lizorkin and Besov spaces on the ball, Proc. Lond. Math. Soc. (3) 97 (2008), no. 2, 477-513.

[24] Y. Liang, Y. Sawano, T. Ullrich, D. Yang and W. Yuan, New characterizations of Besov-Triebel-Lizorkin-Hausdorff spaces including coorbits and wavelets, J. Fourier Anal. Appl. 18 (2012), no. 5, 1067-1111.

10   G. CLEANTHOUS, A. G. GEORGIADIS, AND M. NIELSEN

[25] P. I. Lizorkin, Multipliers of fourier integrals and bounds of convolutions in spaces with mixed norms, Applications, Izv. Akad. Nauk SSSR Ser. Mat. 34 (1970), 225-255, Engl. transl. Math. USSR Izv. 4 (1970).

[26] F. Narcowich, P. Petrushev and J. Ward, Decomposition of Besov and Triebel-Lizorkin spaces on the sphere, J. Funct. Anal. 238 (2006), no. 2, 530-564.

[27] J. Peetre, New thoughts on Besov spaces, Duke University Math. Series 1, Dept. Math., Duke Univ., Durham, N.C., 1976.

[28] P. Petrushev and Y. Xu, Localized polynomial frames on the interval with Jacobi weights, J. Fourier Anal. Appl. 11 (2005), no. 5, 557-575.

[29] R. Torres and E. Ward, Leibniz's rule, sampling and wavelets on mixed Lebesgue spaces, J. Fourier Anal. Appl. 21 (2015), 1053-1076.

[30] H. Triebel, Theory of function spaces, Monographs in Math. Vol. 78, Birkhäuser, Verlag, Basel, 1983.

DEPARTMENT OF MATHEMATICS, ARISTOTLE UNIVERSITY OF THESSALONIKI
E-mail address: gkleanth@math.auth.gr

DEPARTMENT OF MATHEMATICAL SCIENCES, AALBORG UNIVERSITY
E-mail address: nasos@math.aau.dk
E-mail address: mnielsen@math.aau.dk