Vibro-acoustics of infinite and finite elastic fluid-filled cylindrical shells

Ledet, L. S.; Sorokin, S. V.

Published in:
Procedia Engineering

DOI (link to publication from Publisher):
10.1016/j.proeng.2017.09.356

Publication date:
2017

Document Version
Publisher's PDF, also known as Version of record

Available online at www.sciencedirect.com

ScienceDirect

Procedia Engineering 199 (2017) 1362–1367

Procedia Engineering

www.elsevier.com/locate/procedia

X International Conference on Structural Dynamics, EURODYN 2017

Vibro-acoustics of infinite and finite elastic fluid-filled cylindrical shells

L. S. Ledet[a], S. V. Sorokin[a]

[a]*Department of Mechanical and Manufacturing Engineering, Aalborg University, Fibigerstraede 16, 9220 Aalborg, Denmark*

Abstract

The classical model of an elastic fluid-filled cylindrical shell is used for analysis of its vibrations. The model is based on thin shell theory, standard linear acoustics and the heavy fluid-loading coupling concept. First, several important features of performance of a fluid-filled shell not yet fully explored in literature e.g. the difference between kinematic/forcing excitations, acoustic source type identification (monopole/dipole) and energy transfer between fluid and shell are studied. Then the discussion is extended to finite fluid-filled shells by application of the Boundary Integral Equations Method (BIEM). Two techniques for solving the equations deduced from the BIEM are discussed and investigated with respect to convergence, respectively, Boundary Elements (BE) and modal expansion. Successively, the implementation of the BIEM is validated against numerical and experimental results for the simplified case of an empty shell. Finally, impedance boundary conditions for a fluid-filled shell in an assembled piping system and computations of its resonances and forced response using the BIEM are discussed.

Keywords: Vibro-acoustics; Infinite and finite cylindrical shells; Wave propagation; Heavy fluid loading; Boundary Integral Equations Method (BIEM); Impedance boundary conditions

1. Introduction

In industrial pump systems vibrations are a common cause of leakage due to reduced fatigue durability. For such pump systems the receiving pipeline can, to some extent, be represented as a straight pipe with a proper choice of impedance boundary conditions (BC). However, analysis of structures subjected to such conditions increase substantially in complexity and thereby in computational time. Thus, the motivation of this paper is the study of convergence rate of different solution techniques to improve computational efficiency such as to allow performance assessment and parameter studies of such systems.

The model used in this paper is based on thin shell theory, standard linear acoustics and the heavy fluid loading coupling concept, as described in e.g. [1–6]. The waveguide properties of an infinite shell in the framework of the latter is a classical subject of vibro-acoustics, see also [7–9]. In this paper we therefore focus on infinite, finite and compound shell structures exposed to structural and acoustic excitations/sources. For this purpose the Boundary

* Corresponding author. Tel.: +45 26 84 25 12.
E-mail address: L.S. Ledet: *lsl@m-tech.aau.dk*, S.V. Sorokin: *svs@m-tech.aau.dk*

1877-7058 © 2017 The Authors. Published by Elsevier Ltd.
Peer-review under responsibility of the organizing committee of EURODYN 2017.
10.1016/j.proeng.2017.09.356

Integral Equations Method (BIEM) is, due to its versatility, the obvious choice, provided that Green's matrix can be derived, see e.g. [3–5,9,10]. Green's matrix is for this model found using an advanced new method based on bi-orthogonality conditions, see [6,11,12]. The points of particular interest are the action of monopole vs. dipole sources in the fluid and kinematic excitations of the structure. For 1D multi-modal finite structures such as an empty cylindrical shell, the BIEM yields the exact solution of problems in forced and free vibrations. For 2- and 3D problems such as vibration of a fluid-filled shell, the BIEM is converted to the Boundary Element Method (BEM), see e.g. [9] or thousands of other references. The alternative method to consider a shell under heavy fluid loading is to use modal Green's matrix for the acoustic loadings, see [5]. To the best of our knowledge, the comparison of convergence rate and computational efficiency of these techniques in analysis of forced vibration of a fluid-filled shell has not yet been fully explored in literature. The subject of our ongoing work is in particular the analysis of vibrations of compound shells with impedance boundary conditions.

2. Green's matrix and excitation conditions

In this model the system of governing equations is denoted by the differential operator, $\mathcal{D}$, and consist of three linear Partial Differential Equations (PDE) of dynamics of a thin cylindrical shell and the equation of linear acoustics. These equations are, by virtue of axi-symmetry, decoupled into circumferential modes, $m \in \mathbb{Z}$, indicated by the subscript, m. To construct a time-harmonic Green's matrix the problem to be solved is

$$\mathcal{D}_m \boldsymbol{G}_m^{0j} = \boldsymbol{q}_m^{0j} \tag{1}$$

where ^{0j} indicates a loading condition, $\boldsymbol{q}_m$ are external forces; $\lfloor q_{1m}, q_{2m}, q_{3m}, T_m \rfloor^T$ and $\boldsymbol{G}_m$ is Green's function (vector) containing the field variables (displacements); $\lfloor u_m, v_m, w_m, \phi_m \rfloor^T$ (axial, circumferential, radial and velocity potential).

Green's function is, by definition, found as the solution to the problem in Eq.(1) with a delta-function applied alternately on the right-hand-side of the equation in the direction of the waveguide, x, at the excitation point, ξ, see e.g. [3–5], such that for $j = 1$, $\boldsymbol{q}_m = \lfloor \delta(x-\xi), 0, 0, 0 \rfloor^T$ etc. Thus the complete Green's matrix is assembled using all Green's functions. In this particular case we employ tailored Green's functions following [13] which already satisfy the interfacial conditions at the fluid-structure interface.

Upon solving Eq.(1) the vibration of the fluid-filled shell is uniquely characterised by a set of generalised forces/-acoustic variables (axial, circumferential, radial, bending and acoustic velocity) and displacements (field variables) given as $\{Q_{1m}, Q_{2m}, Q_{3m}, Q_{4m}, \vartheta_m\}$ and $\{u_m, v_m, w_m, w'_m, p_m\}$ where w'_m and p_m is the rotation and acoustic pressure. Details and definitions can be found in [3,4,6].

The infinite cylindrical shell can, by virtue of symmetry, be presented as two semi-infinite shells separated at the excitation point with continuity and unit-jump conditions satisfied at $x \to \xi$. These conditions are formulated in Eq.(2) for $j = 1$.

$$Q_{1m}(x,\xi) = -\frac{1}{2}\mathrm{sgn}(x-\xi), \quad Q_{4m}(x,\xi) = 0, \quad p_m(x,\xi,r) = 0, \quad v_m(x,\xi) = 0, \quad w_m(x,\xi) = 0, \quad x \to \xi \tag{2}$$

Formulation of remaining loading conditions to construct Green's matrix can be found in [3,4,6]. In what follows in this section, we compare the response of a fluid-filled shell to a monopole source with its response to a dipole one. The former is the solution of Eq. (1) to the source; $T_m = \frac{1}{r}\delta(r-r_0)\delta(x-\xi)$ applied at the excitation point (ξ, r_0).

$$Q_{3m}(x,\xi) = 0, \quad Q_{2m}(x,\xi) = 0, \quad \vartheta_m(x,\xi,r) = -\frac{1}{2r}\mathrm{sgn}(x-\xi)\delta(r-r_0), \quad u_m(x,\xi) = 0, \quad w'_m(x,\xi) = 0, \quad x \to \xi \tag{3}$$

where the choice of continuity conditions depend on the loaded component, see [6,14]. This solution also constitutes a part of Green's matrix.

Alternative, dipole excitation is specified as

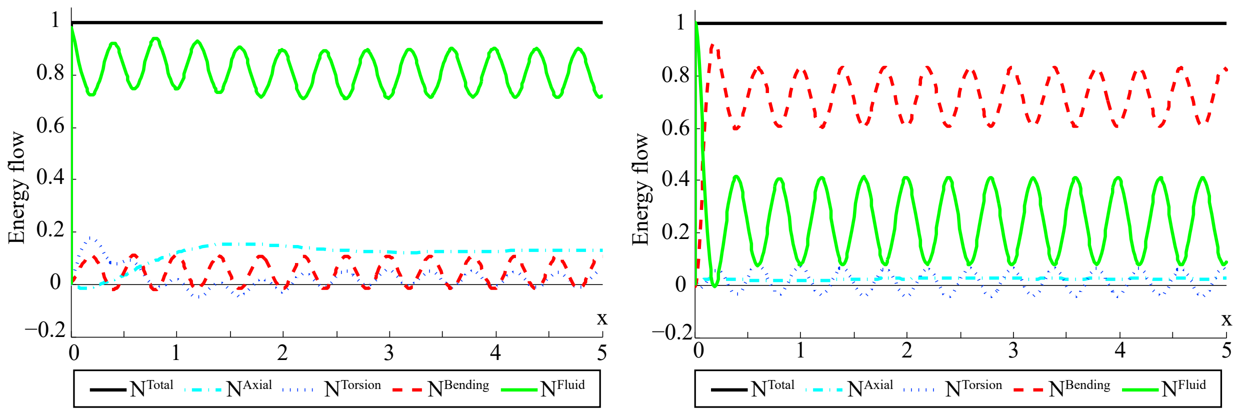
$$Q_{1m}(x,\xi) = 0, \quad Q_{4m}(x,\xi) = 0, \quad p_m(x,\xi,r) = -\frac{1}{2r}\mathrm{sgn}(x-\xi)\delta(r-r_0), \quad v_m(x,\xi) = 0, \quad w_m(x,\xi) = 0, \quad x \to \xi \tag{4}$$

For this particular problem we can derive an equivalent external source to the initial problem in Eq.(1) as well as the equivalent source for a load applied in Q_{4m} is $q_{3m} = \frac{\mathrm{d}\delta(x-\xi)}{\mathrm{d}x}$, see [3,4,6]. The equivalent acoustic source is derived from the relationship between pressure and velocity in linear acoustics and becomes; $T_m = \frac{1}{r}\delta(r - r_0)\frac{\mathrm{d}\delta(x-\xi)}{\mathrm{d}x}$, which can immediately be identified as an acoustic dipole source in x. From this it follows directly that the study of acoustic monopoles and dipoles is equivalent to the study of acoustic velocity vs. pressure sources.

Although, in acoustics monopoles and dipoles (force/kinematic) are accepted models of sources; in structural dynamics jumps in kinematic variables are meaningful only for semi-infinite structures. Thus, for structural kinematic excitations there are no straightforward equivalent excitation since the choice of 'continuity' conditions is not unique. The study of the choice of these conditions is a subject of our ongoing work.

2.1. Comparison – velocity vs. pressure source

To demonstrate the difference between a velocity and a pressure source (monopole/dipole) a particularly interesting example of near- and far-field energy flow (normalised) is shown in Fig.1 for $f = 68$kHz, vibrating at $m = 3$ with the sources located at $r_0 = 0.95$. This example is chosen based on [3] against which the model is also validated. In the figure the energy flow is divided into the physical transmission paths (structure and fluid) following [3] and all quantities are transformed into non-dimensional form using: $\rho = \frac{\rho_{fl}}{\rho_{str}}$, $\gamma = \frac{c_{str}}{c_{fl}}$, $\mu = \frac{h}{R}$, $\Omega = \frac{\omega^2 R^2}{c_{str}^2}$, $r = \frac{\tilde{r}}{R}$ and $x = \frac{\tilde{x}}{R}$.



(a) (b)

Fig. 1. Normalised energy flow for (a) velocity source and (b) pressure source at $m = 3$, $f = 68$kHz and $r_0 = 0.95$.

From the figure it is clear that there is a profound difference between applying a velocity and a pressure source near the shell wall. In (a) the energy remains in the fluid and in (b) the energy escapes rapidly from the fluid to the structure where it is conveyed as bending in the shell wall. This example, in particular, illustrates that the choice of source (velocity or pressure) is not trivial. This discussion is in particular relevant when acoustic sources are deduced from Computational Fluid Dynamics (CFD) simulations where the impedance at the radiating boundaries is rarely coherent with the impedance of the acoustic sources in an elastic fluid-filled shell. Presumably this is caused by the rigid body or incompressible fluid assumption in CFD. Thus, errors in impedance of sources deduced from CFD simulations may produce behaviours counter to the actual physical behaviour.

3. Boundary Integral Equations Method

To advance from infinite to finite and compound shells the Green's matrix is employed in the Boundary Integral Equations Method. Following [5] (or [3] for *in vacuo* shells) Somigliana's identity can be expressed in its non-dimensional form as shown in Eq.(5) when using Green's matrix as kernel.

$$\frac{1}{\mu}\left[\delta_{1j}u_m(\xi)+\delta_{2j}v_m(\xi)+\delta_{2j}w_m(\xi)+\delta_{4j}\mu w'_m(\xi)+\delta_{5j}i\frac{\rho}{\gamma^3\Omega}p_m(\xi,r_0)\right]= \tag{5}$$
$$\left[Q_{1m}(x)u_m^{0j}(x,\xi)+Q_{2m}(x)v_m^{0j}(x,\xi)+Q_{3m}(x)w_m^{0j}(x,\xi)+\mu Q_{4m}(x)w_m^{'0j}(x,\xi)+i\frac{\rho}{\gamma^3\mu\Omega}\int_0^1 p_m^{(0j)}(x,\xi,r)\vartheta_m(x,r)r\mathrm{d}r\right]\Bigg|_{x=a}^{x=b}$$
$$-\left[Q_{1m}^{0j}(x,\xi)u_m(x)+Q_{2m}^{0j}(x,\xi)v_m(x)+Q_{3m}^{0j}(x,\xi)w_m(x)+\mu Q_{4m}^{0j}(x,\xi)w'_m(x)+i\frac{\rho}{\gamma^3\mu\Omega}\int_0^1 p_m(x,r)\vartheta_m^{(0j)}(x,\xi,r)r\mathrm{d}r\right]\Bigg|_{x=a}^{x=b}$$
$$+\int_a^b\frac{1}{\mu}\left[q_{1m}(x)u_m^{0j}(x,\xi)+q_{2m}(x)v_m^{0j}(x,\xi)+q_{3m}(x)w_m^{0j}(x,\xi)+i\frac{\rho}{\gamma^3\Omega}\int_0^1 T_m(x,r)p_m^{0j}(x,\xi,r)r\mathrm{d}r\right]\mathrm{d}x$$

where δ_{ij} is Kronecker's delta, i the complex operator, j refer to the loading cases of Green's matrix and ξ has becomes the observation point as oppose to an excitation point previously. Note that for a monopole the solution become a function of the ring source location, r_0, see e.g. [3,4,6]. The remaining forces/displacements ($u, v, w, w', p, \vartheta, Q_1, Q_2, Q_3, Q_4$) constitute a total of 20 unknown boundary coefficients/functions; 10 at $x = a$ and 10 at $x = b$ – assuming, for now, that the acoustic integral can be evaluated straight-away.

Now, moving the observation point alternately towards the boundaries, a and b, from inside the domain we arrive at 10 Boundary (Integral) Equations (BIE) and by introducing the 10 additional BC's in Eq.(6) we arrive at a complete equation system, which constitutes the Boundary Integral Equations Method.

$$\begin{aligned}\chi_{11}^{\alpha}Q_{1m}+\chi_{12}^{\alpha}u_m&=0 & \chi_{21}^{\alpha}Q_{2m}+\chi_{22}^{\alpha}v_m&=0 & \\ \chi_{31}^{\alpha}Q_{3m}+\chi_{32}^{\alpha}w_m&=0 & \chi_{41}^{\alpha}Q_{4m}+\chi_{42}^{\alpha}w'_m&=0 & \chi_{51}^{\alpha}\vartheta_m+\chi_{52}^{\alpha}p_m=0\end{aligned} \tag{6}$$

where a set of arbitrary boundary conditions can be obtained by appropriate choice of the coefficients, χ_{ij}^{α}, at $\alpha = a, b$.

To find the resonances we set q_{nm} and T_m to zero and seek non-trivial solutions of Eq.(5). Note that this is not equivalent to the classical eigenvalue problem as Green's matrix is an implicit functions of frequency by virtue of the dispersion of free-waves. Thus, resonances are found by sweeping across the spectrum. Likewise, Eq.(5) can be solved for any arbitrary external load and through Somigliana's identity obtain the forced response.

For waveguides, such as the empty shell, supporting only a finite number of waves the solution to Eq.(5) is easily obtained and is exact in the framework of the theory used. However, when continuous or mixed waveguides are considered such as the fluid-filled shell, the boundary equations become boundary integral equations due to the acoustic integral; causing the complexity of the system to increase considerably and the solutions become approximate. In general, there are two obvious ways of solving this transcendental system. Unfortunately, both methods involve an expansion of the equation system to, in general, the number of boundary coefficients times the number of modes/elements, M, retained in the analysis. For the fluid-filled shell only the acoustic domain is a continuous waveguide and therefore the equation system expands only to; $16 + 4M$. The two methods are, respectively:

1. **The Boundary Element Method**, [9]. We introduce M boundary elements with any local shape function across the radius such that the integral in Eq.(5) can be approximated by a summation. Depending on the order of the shape functions and number of elements the equation system expands accordingly. Thus, for elements with constant shape functions and equal size the equation system for the fluid-filled shell expands proportional to the number of elements i.e. $4M$ and the acoustic integral can be approximated by the summation in Eq.(7).

$$\int_0^1 p_m^{(0j)}(x,\xi,r)\vartheta_m(x,r)r\mathrm{d}r\Bigg|_{x=a}^{x=b} \quad\Rightarrow\quad \frac{1}{M}\sum_{n=0}^{M-1}p_m^{(0j)}(x,\xi,r_n)\vartheta_{nm}(x,r_n)r_n\Bigg|_{x=a}^{x=b} \quad \text{where} \quad r_n=\frac{(2n+1)}{2M} \tag{7}$$

To get additional equations to the additional $4M$ unknowns we formulate Somigliana's identity and boundary conditions for each Boundary Element such that $r_0 = r_n$ for $n = 0, \ldots, M-1$ in Eq.(5). This is done only for the acoustic source.

2. **The method of modal expansion**, [5]. We assume a certain profile for the radial distribution of the pressure and velocity at each boundary. The obvious choice of profile is the modal expansion on M free waves from the

dispersion equation such that;

$$\vartheta_m(x,r) = \sum_{n=1}^{M} \overline{\vartheta}_{nm}(x) J_m\left(r\sqrt{k_n^2+\gamma^2\Omega^2}\right) \qquad p_m(x,r) = \sum_{n=1}^{M} \overline{p}_{nm}(x) J_m\left(r\sqrt{k_n^2+\gamma^2\Omega^2}\right)$$
$$\Rightarrow \quad \int_0^1 p_m^{(0j)}(x,\xi,r)\vartheta_m(x,r) r\mathrm{d}r\Bigg|_{x=a}^{x=b} = \int_0^1 p_m^{(0j)}(x,\xi,r) \sum_{n=1}^{M} \overline{\vartheta}_{nm}(x) J_m\left(r\sqrt{k_n^2+\gamma^2\Omega^2}\right) r\mathrm{d}r\Bigg|_{x=a}^{x=b} \tag{8}$$

where k is the wave-number in the direction of the waveguide. In this case the additional equations to the $4M$ additional unknowns are retrieved using tailored modal Green's matrices as kernels for the acoustic loadings, see [5]. Thus, the tailored modal Green's matrices solves the PDE's for a set of modal loads; $T_m = J_m(\kappa_n r)$ (Bessel-function) for $n = 1, \dots, M$ and we set-up Somigliana's identity for each modal Green's matrix with boundary conditions formulated for each kernel.

Finally, as we are by no means restricted to radial profiles formulated as an expansion on free waves there are many other possible solution techniques for this problem. Thus, the motivation of this paper is to study also other tailored Green's matrices and their solution with respect to convergence rate and computational efficiency. This is the subject of our ongoing work.

3.1. Validation of implementation of the BIEM

As correct implementation the BIE system can be a challenging task results from the shell model are compared to both experimental results and numerical results from commercial software (ANSYS). To ensure correct implementation of the BIEM it is expedient, from an experimental viewpoint, to reduce the problem to an empty shell since experimental sources of error associated with a fluid-filled shell are substantial compared to the empty shell. In addition, the BIEM for the empty shell provide exact solutions in the framework of the theory used, as discussed in section 1, and is therefore less sensitive to experimental and numerical inconsistencies. Thus, letting $\rho \to 0$ and $\gamma \to \infty$ in Eq.(5) the BIE's reduce to simple algebraic boundary equations with exact solutions.

The comparison between resonances from the shell model, numerical model and experiments are presented in table 1 for free-free boundary conditions, a frequency spectrum from 0 to 1600Hz and numerical results calculated using the Finite Element Method with 3D quadratic solid elements (SOLID186, see ANSYS documentation). The associated fluid and mechanical parameters are; $R = 68.3$mm, $h = 1.6$mm, $L = 743.5$mm, $E = 205$GPa, $\nu = 0.3$, $\rho_{str} = 7967$kg/m^{-3}, $\rho_{fl} = 1000$kg/m^{-3}, $c_{fl} = 1440$m/s^{-1} and $c_{str} = \sqrt{\frac{E}{\rho_{str}(1-\nu^2)}}$.

Mode m	Exp. [Hz]	Shell model [Hz]	ANSYS [Hz]	Rel. err. Shell	Rel. err. ANSYS
1	1315.25	1325.50	1325.60	0.77%	0.78%
2	224.50	224.50	224.66	0%	0.07%
2	227.25	227.50	227.67	0.11%	0.18%
2	536.00	537.50	537.64	0.28%	0.31%
2	1247.00	1255.50	1255.50	0.68%	0.68%
3	635.50	635.25	635.13	-0.04%	-0.06%
3	638.25	639.00	638.84	0.12%	0.09%
3	693.50	695.25	694.88	0.25%	0.20%
3	920.75	921.75	921.45	0.11%	0.08%
3	1358.25	1361.00	1360.90	0.20%	0.19%
4	1216.00	1218.00	1217.00	0.16%	0.08%
4	1221.75	1221.75	1220.70	0%	-0.09%
4	1245.50	1247.25	1246.00	0.14%	0.04%
4	1321.50	1322.00	1320.80	0.04%	-0.05%
4	1482.00	1485.75	1484.50	0.25%	0.17%

Table 1. Comparison between resonances from experiments, shell and numerical model. Relative errors are calculated with respect to experiments.

As seen from table 1 we may conclude that the implementation of the BIEM is successful since the shell model agrees well with both experimental and numerical results – below 1%. With the BIEM formulated correctly expansion to larger systems is straightforward and the BIEM may therefore be used for studies on fluid-filled shells subjected to impedance conditions.

3.2. Discussion of impedance boundary conditions

For structures with complex-valued impedance boundary conditions the resonances are complex-valued as well. Therefore the frequency sweep is extended from the real line to the complex domain. Inevitably, the computational effort of finding these resonances increase significantly and so, fast converging solutions of the BIE's will obviously improve the overall computational efficiency considerably − especially for fluid-filled shells.

As a consequence of impedance conditions the corresponding complex-valued solutions are manifestations of damped resonances. The physical interpretation of these resonances is the radiation or leakage of energy at the boundaries. This dissipation of energy is analogue to conventional internal (material) damping. However, in assembled piping systems, it is related to structure- and fluid-borne sound propagation.

4. Conclusion

The classical model of a thin cylindrical shell, standard linear acoustic and the heavy fluid loading coupling concept is revisited with the aim to develop a computationally efficient and robust tool for assessment of vibro-acoustic properties of assembled piping systems. To accomplish this task, the Boundary Integral Equations Method is chosen using a time-harmonic tailored Green's matrix. This solution strategy allows for a consistent investigation of force/kinematic sources in the acoustic medium. For the fluid-filled shell an example illustrates the profound difference between pressure and velocity sources near the shell wall. A study of kinematic versus forcing structural excitations is a subject of our ongoing work. The BIEM, which permit analysis of forced and free vibrations of fluid-filled shells with arbitrary boundary conditions including the complex-valued impedance boundary conditions, has been validated for the empty shell with free-free conditions in comparison with both numerical and experimental results, while different solution techniques have been discussed for the transcendental boundary integral equations. The study of convergence rate and computational efficiency using alternative techniques to implement the BIEM is the subject of our ongoing work.

Acknowledgements

This work was supported by the Innovation Fund Denmark and is greatly appreciated by the authors.

References

[1] C. R. Fuller, F. J. Fahy, Characteristics of wave propagation and energy distributions in cylindrical elastic shells filled with fluid, J. Sound and Vibration 81 (1982) 501–518.
[2] A. S. Knudsen, S. V. Sorokin, Modelling of linear wave propagation in spatial fluid filled pipe systems consisting of elastic curved and straight elements, J. Sound and Vibration 329 (2010) 5116–5146.
[3] S. V. Sorokin, J. B. Nielsen, N. Olhoff, Green's matrix and the boundary integral equation method for the analysis of vibration and energy flow in cylindrical shells with and without internal fluid, J. Sound and Vibration 271 (2004) 815–847.
[4] S. V. Sorokin, O. A. Ershova, Analysis of the energy transmission on compound cylindrical shells with and without internal heavy fluid loading by boundary integral equations and by floquet theory, J. Sound and Vibration 291 (2006) 81–99.
[5] S. V. Sorokin, A. V. Terentiev, Flow-induced vibrations of an elastic cylindrical shell conveying a compressible fluid, J. Sound and Vibration 296 (2006) 777–796.
[6] L. S. Ledet, S. V. Sorokin, J. B. Larsen, M. M. Lauridsen, Bi-orthogonality conditions for power flow analysis in fluid-loaded elastic cylindrical shells: Theory and applications, Intl. Conf. on Noise and Vibration - Emerging Technologies (NOVEM) (2015) Dubrovnik, Croatia.
[7] G. Pavic, Vibrational energy flow in elastic cylindrical shells, J. Sound and Vibration 142 (1990) 293–310.
[8] C. R. Fuller, The input mobility of an infinite circular cylindrical elastic shell filled with fluid, J. Sound and Vibration 87 (1983) 409–427.
[9] F. Fahy, P. Gardonio, Sound and Structural Vibration, 2nd ed., Elsevier, 2007.
[10] G. C. Hsiao, W. L. Wendland, Boundary Integral Equations, 1st ed., Springer, 2008.
[11] J. D. Achenbach, Reciprocity in elastodynamics, 2nd ed., Cambridge University Press, 2003.
[12] R. D. Gregory, A note on bi-orthogonality relations for elastic cylinders of general cross section, J. Elasticity 13 (1983) 351–355.
[13] B. O. Olsen, Vibroacoustic power flow in infinite compliant pipes excited by mechanical forces and internal acoustic sources, Ph.D. thesis, University of Southampton, 2001.
[14] D. J. Mead, Wave propagation and natural modes in periodic systems: ii. multi-coupled systems, with and without damping, J. Sound and Vibration 40 (1975) 19–39.