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Phasor Alternatives to Friis' Transmission Equation

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Abstract—Two alternatives to Friis' transmission equation in terms of phasor voltage waves are presented. In one formulation antennas are characterized by the complex effective length vectors. An additional form introducing field gain, that serves effectively as a phasor counterpart to the power gain, is proposed. Both forms show the same degree of symmetry and modularity as the original Friis' equation, but thanks to using phasors instead of power quantities they allow for superposition of fields or voltages. Although the new transmission equations are formulated in frequency domain, they also constitute a simple and intuitive way to calculate time domain responses of antennas, such as in ultrawideband (UWB) systems. Application of the equations is demonstrated on a simple example of coupling between two non-aligned dipoles in the vicinity of a conductive reflector.

Index Terms—Antenna theory, Radiowave propagation, Ultra-wideband antennas, Receiving antennas.

I. INTRODUCTION

Since it was introduced in 1946, the Friis' transmission formula [1] has become extremely popular among wireless communication engineers. The main reason, apart from its clear simplicity, might be that the structure of the formula logically follows the flow of electromagnetic energy along the wireless channel. The expression is modular, allowing the user to identify particular blocks with their specific functions such as mismatch terms, gain, path loss, etc., and possibly plug and unplug parts of the chain according to their impact on the analyzed wireless link. Depending on the particular formulation, sections of the expression may correspond to many useful quantities such as accepted power, total radiated power, equivalent isotropic radiated power or power density of the field.

However, the original Friis' formula cannot be used in scenarios with multiple transmitters or receivers, or with multipath propagation, where contributions from many directions need to be added with appropriate phase to yield the correct result. Another situation where the formula cannot be applied is when the link needs to be analyzed in time domain, having signals of pulse waveforms such as in ranging systems or ultrawideband (UWB) communication systems.

In the latter case the antenna community started employing various time domain versions of the transmission equation [2]–[4]. Still, most of publications on this topic deals with frequency domain phasor quantities, where an antenna is characterized either by complex effective length [3]–[9] or some form of transfer function [9]–[13], each of them either as scalar or vector, and not always the same for transmitting and

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receiving antennas owing to Lorentz reciprocity theorem [14]. Some authors have proposed formulations of the wireless link in which the antennas and the wireless channel are treated as separate blocks [4], [9], [13], but none of them features the same degree of simplicity and intuitiveness as Friis' original.

In this letter, we propose a new form of transmission equation in frequency domain, which uses phasor quantities, therefore giving information about the phase shift rendered by the wireless link and allowing for superposition of fields or voltages thus obtained. The proposed formulation uses the complex effective length vector and a new definition of field gain to characterize the antennas. In contrast to all previous formulations, the proposed one is designed to be symmetrical and modular in the same spirit as the widely used Friis' equation. By symmetry we mean that the equation has its terms arranged in the same order as they appear in the wireless link, whereas modularity means that each term characterizes one functional block in the link, such as input port, antenna, or propagation path. For example, it is possible to easily express some intermediate quantities, such as the E-field of the incident wave, by multiplying only some terms ("modules") of the equation. The specific cascaded arrangement of the terms in the equation attempts to follow intuitively the functions of the particular blocks in the wireless chain from left to right and should therefore be easy to remember.

II. TRANSMISSION EQUATIONS

A. Friis' Transmission Equation

As published in [1], the transmission equation relates the power available at the receiver (RX) to the power supplied by the transmitter (TX). The equation was originally formulated using effective areas A, here we repeat this form extended by mismatch terms [15]

$$\frac{P_2}{P_1} = (1 - |\Gamma_1|^2) \cdot A_1 \cdot \left(\frac{1}{\lambda r}\right)^2 \cdot A_2 \cdot (1 - |\Gamma_2|^2) \quad (1)$$

where P_1 and P_2 are the TX and RX powers, respectively, Γ_1 and Γ_2 are the antenna reflection coefficients, λ is the free space wavelength and r is the distance between the TX and RX antennas. An alternative form uses antenna gains G instead of effective areas:

$$\frac{P_2}{P_1} = (1 - |\Gamma_1|^2) \cdot G_1 \cdot \left(\frac{\lambda}{4\pi r}\right)^2 \cdot G_2 \cdot (1 - |\Gamma_2|^2) \quad (2)$$

In both cases the power ratio (transducer gain) is related to the $\hat{s}_{21}$ generalized scattering parameter [16] by $P_2/P_1 = |\hat{s}_{21}|^2$. The effective areas and the power gains are dependent on both frequency f and direction of propagation, $A = A(f, \theta, \phi)$, $G = G(f, \theta, \phi)$, where θ and ϕ are the local spherical coordinates of the propagation vector.
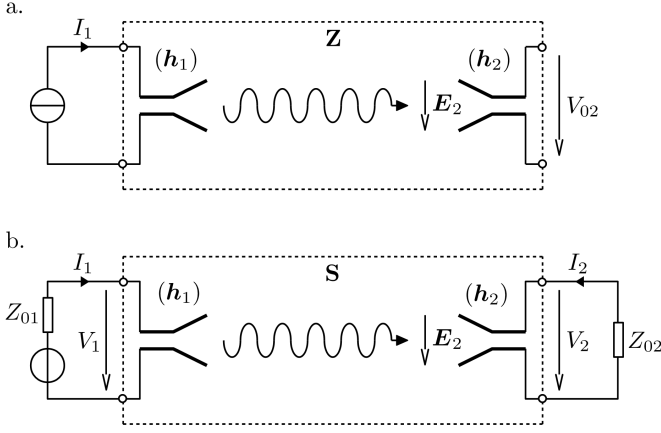
Fig. 1. Wireless link as a two-port network with a. impedance matrix, b. scattering matrix.

The symmetry and modularity (here emphasized by using dots for multiplications) of (1) and (2) allow us to follow the flow of power along the entire wireless link and clearly see the contribution of antenna mismatch, antenna gain, path loss, and, in other versions of the equation, also antenna efficiency, polarization mismatch, etc. In the following two subsections, we introduce phasor counterparts to (1) and (2) with the same level of symmetry and modularity, but this time allowing us to follow the voltage/field magnitudes and phase shifts.

B. Phasor Transmission Equation with Effective Lengths

In the following, we assume all field and circuit quantities as rms phasors with the $e^{j\omega t}$ convention, ω being the angular frequency of the sine wave and j the imaginary unit.

Let us assume a wireless link with transmitting and receiving antennas as depicted in Fig. 1. The open-circuit voltage induced at the RX antenna is

$$V_{02} = \boldsymbol{E}_2 \cdot \boldsymbol{h}_2 \qquad (3)$$

where $\boldsymbol{E}_2$ is the electric field vector of the uniform plane wave incident on the RX antenna due to radiation from the TX antenna [17]

$$\boldsymbol{E}_2 = \frac{j\eta_0 I_1 \boldsymbol{h}_1}{2\lambda r} e^{-jkr} \qquad (4)$$

The antennas are characterized by complex effective length vectors $\boldsymbol{h} = \boldsymbol{h}(f, \theta, \phi)$, which, together with $\boldsymbol{E}_2$, are orthogonal to the direction of propagation $\boldsymbol{r}$. The TX antenna is fed by current I_1, η_0 is the free-space wave impedance and k is the propagation constant in free space. Substituting (4) into (3) gives us the expression for coupling between the antennas in terms of mutual impedance

$$V_{02} = I_1 \cdot z_{21} = I_1 \cdot \boldsymbol{h}_1 \cdot \frac{j\eta_0 e^{-jkr}}{2\lambda r} \cdot \boldsymbol{h}_2 \qquad (5)$$

In the above equation and those to follow, there are three distinct kinds of multiplications occurring, denoted by dots ($\cdot$) or implicitly without notation: scalar-scalar, scalar-vector, and vector-vector dot product. The order by which the multiplications are taken is irrelevant thanks to the associative

and commutative rules. The purpose of splitting the vector multiplications apart and of the prolific use of the dots is to emphasize the actual order in which the logical modules of the wireless link appear in the real world (see Fig. 1).

Next, let us connect a voltage source with internal impedance Z_{01} at the input of the TX antenna and load the RX antenna with impedance Z_{02}, as shown in Fig. 1b. Let us also introduce incoming (incident), $V^+ = I^+ Z_0$, and outgoing (reflected), $V^- = I^- Z_0$, voltage and current waves [16] at either port of the transmission network, with the total voltages and total currents defined as $V = V^+ + V^-$, $I = I^+ - I^-$ and the wave reflection coefficient defined as $\Gamma = (Z - Z_0)/(Z + Z_0)$ where Z is the input impedance of TX or RX antenna. After some manipulations we obtain the transmission formula using scattering parameter s_{21} relating the incident wave at the TX antenna V_1^+ and the outgoing wave at the RX antenna V_2^-:

$$V_2^- = V_1^+ \cdot s_{21} = V_1^+ \cdot (1 - \Gamma_1) \cdot \frac{\boldsymbol{h}_1}{2} \cdot \frac{\eta_0}{Z_{01}} \cdot \frac{j e^{-jkr}}{\lambda r} \cdot \frac{\boldsymbol{h}_2}{2} \cdot (1 - \Gamma_2) \qquad (6)$$

Analogously to Friis' formula (1) we have two terms containing the matching properties, $(1 - \Gamma)$, that reduce to 1 when the corresponding antenna is matched, one path loss term $j e^{-jkr}/\lambda r$, and phasor effective lengths $\boldsymbol{h}$ take the role of effective areas A characterizing the directional properties of the antennas. Conveniently, the part of the expression to the left of $\boldsymbol{h}_2/2$ equals the electric field vector incident on the RX antenna $\boldsymbol{E}_2$, whereas the same part including $\boldsymbol{h}_2/2$ gives the outgoing voltage wave V_2^- in case the load is matched to the antenna. The $\boldsymbol{h}/2$ terms in (6) are thus equivalent to the wave-type effective lengths.

The remaining extra term in (6), η_0/Z_{01}, arises from the transformation between the source and free space, and can be mitigated by normalizing the field and circuit quantities and effective lengths to their respective real and positive reference impedances. In particular, the normalized voltage waves [16] become $\hat{V}_1^+ = V_1^+/\sqrt{Z_{01}}$ and $\hat{V}_2^- = V_2^-/\sqrt{Z_{02}}$, i.e. normalized to the characteristic impedances of the transmission lines at either end of the link (hat symbol ˆ denotes normalization), whereas the normalized effective length vectors become

$$\hat{\boldsymbol{h}} = \frac{\boldsymbol{h}}{2} \sqrt{\frac{\eta_0}{Z_0}} \qquad (7)$$

because they constitute an interface between the transmission lines with characteristic impedances Z_0 and free space with wave impedance η_0. The resulting transmission formula, now in terms of generalized scattering parameter $\hat{s}_{21}$ [16], is:

$$\hat{V}_2^- = \hat{V}_1^+ \cdot \hat{s}_{21} = \hat{V}_1^+ \cdot (1 - \Gamma_1) \cdot \hat{\boldsymbol{h}}_1 \cdot \frac{j e^{-jkr}}{\lambda r} \cdot \hat{\boldsymbol{h}}_2 \cdot (1 - \Gamma_2) \qquad (8)$$

This form can be directly compared with Friis' formula (1) with square of magnitude ($|\cdot|^2$) as the key. The input and output powers, the path loss term, and the realized effective areas can be expressed in terms of their phasor counterparts as $P_1 = |\hat{V}_1^+|^2$, $P_2 = |\hat{V}_2^-|^2$, and

$$\left(\frac{1}{\lambda r}\right)^2 = \left|\frac{j e^{-jkr}}{\lambda r}\right|^2$$

$$(1 - |\Gamma|^2) A = \left|(1 - \Gamma)\hat{h}\right|^2$$

where $\hat{h}$ stands for single polarization element of $\hat{\boldsymbol{h}}$ corresponding to A. From the last equation it is possible to express the relation between the effective area and the effective length of an antenna [18]

$$A = \frac{|1-\Gamma|^2}{1-|\Gamma|^2}|\hat{h}|^2 = \frac{Z_0}{R}|\hat{h}|^2 = \frac{\eta_0}{R}\left|\frac{h}{2}\right|^2 \qquad (9)$$

where R is the real part of the antenna input impedance Z. From (9) it follows that the effective length and its adjacent matching term cannot be treated separately, hence in practice it may sometimes be more convenient to characterize each antenna with the realized effective length $\hat{\boldsymbol{h}}_{\mathrm{R}} = \hat{\boldsymbol{h}}(1-\Gamma)$, as in [9]. This definition of realized effective length is also equivalent to the transfer function in [4].

Just like the plain form (6), also the normalized form (8) offers the plane wave quantity to be extracted: the part to the left of the $\hat{\boldsymbol{h}}_2$ term is the normalized electric field vector incident on the RX antenna, $\hat{\boldsymbol{E}}_2 = \boldsymbol{E}_2/\sqrt{\eta_0}$. This term, after applying again the $|\cdot|^2$ norm, corresponds to the plane wave power density, $S_2 = |\hat{E}_2|^2$, that is given by the Friis' formula (1) by multiplying everything to the left of A_2.

C. Phasor Transmission Equation with Field Gains

Since the way how an antenna radiates is proportional to its electrical size, it seems natural to normalize $\boldsymbol{h}$ also to the wavelength. To that end, we introduce a dimensionless antenna measure, a *field gain*, which constitutes a phasor counterpart to the more widely used power gain:

$$\hat{\boldsymbol{g}} = \hat{\boldsymbol{h}}\frac{\sqrt{4\pi}}{\lambda}\frac{1-\Gamma}{\sqrt{1-|\Gamma|^2}} = \frac{\hat{\boldsymbol{h}}}{\lambda}\sqrt{\frac{4\pi Z_0}{R}} = \frac{\boldsymbol{h}}{\lambda}\sqrt{\frac{\pi\eta_0}{R}} \qquad (10)$$

The field gain vector $\hat{\boldsymbol{g}} = \hat{\boldsymbol{g}}(f,\theta,\phi)$ is defined as the *complex electric field vector radiated in a given direction divided by the total complex electric field of a fictious lossless isotropic antenna with the same accepted power and zero phase contribution*. Thus, the absolute values of the vector components are equal to the square roots of corresponding partial gains of the antenna and the phases represent the complete phase shifts introduced by the antenna to the corresponding polarization components, excluding the propagation delay and the $\pi/2$ phase shift already contained in the free space term. Note that this definition of field gain is slightly different from that in [19]. The transmission formula using the field gain is then

$$\hat{V}_2^- = \hat{V}_1^+\cdot\hat{s}_{21} = \hat{V}_1^+\cdot\sqrt{1-|\Gamma_1|^2}\cdot\hat{\boldsymbol{g}}_1\cdot\frac{\mathrm{j}\lambda\mathrm{e}^{-\mathrm{j}kr}}{4\pi r}\cdot\hat{\boldsymbol{g}}_2\cdot\sqrt{1-|\Gamma_2|^2} \qquad (11)$$

and we can again easily establish connection of the above equation with Friis' formula (2) applying the $|\cdot|^2$ operator to each of the terms in (11), leading to, in particular:

$$\left(\frac{\lambda}{4\pi r}\right)^2 = \left|\frac{\mathrm{j}\lambda\mathrm{e}^{-\mathrm{j}kr}}{4\pi r}\right|^2$$
$$G = |\hat{g}|^2$$

where $\hat{g}$ represents the polarization component of $\hat{\boldsymbol{g}}$ corresponding to partial gain G. The phasor path loss term in (11) is identical to the one in [11], which indicates that the realized

field gain, ie. the field gain extended by the mismatch term, $\hat{\boldsymbol{g}}_{\mathrm{R}} = \hat{\boldsymbol{g}}\sqrt{1-|\Gamma|^2}$, is equivalent to the vector transfer function in [11].

III. EXAMPLE

Application of the above presented equations is demonstrated on a simple example of far-field coupling between two dipoles in the presence of an infinite conductive reflector. This example is also intended to serve as a validation of the proposed approach, as the results are directly compared to output from numerical code using the well-established finite-difference time-domain (FDTD) method [20].

Let us assume two dipoles of length $l = 32$ mm with their centers placed at positions given by position vectors $\boldsymbol{p}_1 = [0,0,1]$ mm and $\boldsymbol{p}_2 = [321,200,120]$ mm, and oriented along the z and x axes, respectively (Fig. 2). One of the dipoles is excited with Gaussian modulated sine pulse covering frequency band 3.1–5.3 GHz. The reflector is an infinitely large perfectly conducting surface at position $z = -44$ mm.

If we now assume a local coordinate system where the z axis coincides with the symmetry axis of the dipole, then the effective length of a thin dipole shorter than λ can be expressed as [17]

$$\boldsymbol{h} = \hat{\boldsymbol{\theta}}\frac{\lambda}{\pi}\frac{\cos\left(\frac{\pi l}{\lambda}\cos\theta\right)-\cos\frac{\pi l}{\lambda}}{\sin\theta\sin\frac{\pi l}{\lambda}} \qquad (12)$$

where $\hat{\boldsymbol{\theta}}$ is the spherical coordinate unit vector given by coordinates θ and ϕ. To evaluate (12) for each antenna, we define the link vector pointing from TX to RX as $\boldsymbol{r} = \boldsymbol{p}_2 - \boldsymbol{p}_1$ and its length $r = \|\boldsymbol{r}\| \doteq 396$ mm is the link distance between the two antennas. The local cosine and sine terms in (12) are calculated as

$$\cos\theta = \hat{\boldsymbol{d}}\cdot\hat{\boldsymbol{r}} \qquad \sin\theta = \|\hat{\boldsymbol{d}}\times\hat{\boldsymbol{r}}\|$$

where $\hat{\boldsymbol{r}}$ is the radius unit vector for each antenna, namely $\hat{\boldsymbol{r}}_1 = \boldsymbol{r}r^{-1}$, $\hat{\boldsymbol{r}}_2 = -\boldsymbol{r}r^{-1}$, and $\hat{\boldsymbol{d}}$ is the directional unit vector, in our case $\hat{\boldsymbol{d}}_1 = \hat{\boldsymbol{z}}$, $\hat{\boldsymbol{d}}_2 = \hat{\boldsymbol{x}}$. The $\hat{\boldsymbol{\theta}}$ vector is then

$$\hat{\boldsymbol{\theta}} = \frac{\hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{d}}}{\sin\theta}$$

The respective effective lengths (12) are inserted into (6) or after normalization using (7) into (8). In order to determine the matching terms, we use the following approximate formula for impedance of a thin wire dipole

$$Z = \frac{\eta_0}{2\pi}\frac{(1+\mathrm{e}^{\mathrm{j}kl})\mathrm{Ein}\,\mathrm{j}kl - \frac{\mathrm{e}^{\mathrm{j}kl}}{2}\mathrm{Ein}\,2\mathrm{j}kl + \mathrm{j}\sin kl\ln\frac{2a}{l}}{\sin^2\frac{kl}{2}} \qquad (13)$$

where a is the wire radius and Ein is the entire exponential integral [21]

$$\mathrm{Ein}\,z = \int_0^z \frac{1-\mathrm{e}^{-t}}{t}\mathrm{d}t$$

The reference impedance is $Z_0 = 50\ \Omega$ on both sides of the link and the free space impedance is $\eta_0 \doteq 377\ \Omega$. The simulations have been performed with our in-house FDTD code, with cell size $\Delta = 2$ mm. The dipoles have been modeled as thin wires [20] with radius $a = 0.208\Delta$ and the $50\ \Omega$ impedances as lumped elements at the dipole centers.
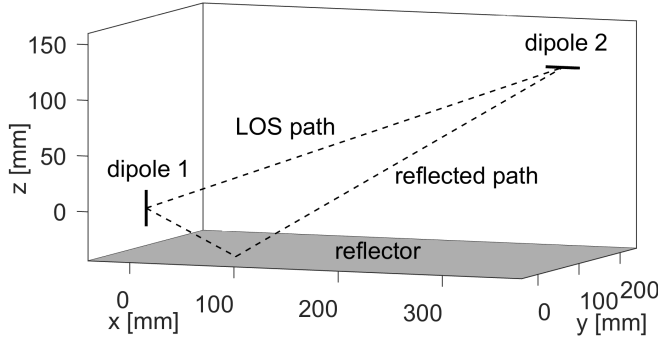
Fig. 2. Example of coupling between two arbitrary oriented dipoles in the presence of an infinite perfectly conducting reflector.
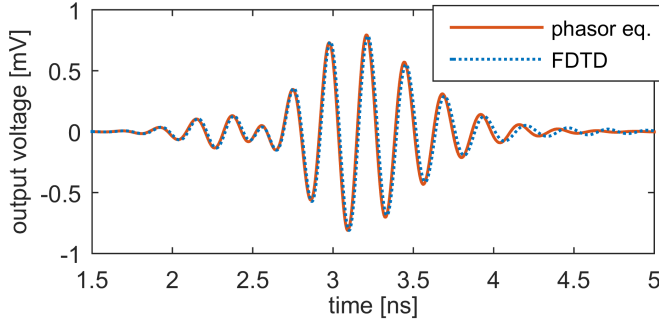

Fig. 3. Output voltage at the RX antenna calculated by the phasor transmission equation and obtained from FDTD simulation.

Propagation via the reflector is calculated the same way as above, by introducing a mirror image of dipole 1 with position vector $\boldsymbol{p}'_1 = [0, 0, -89]$ and unit directional vector $\hat{\boldsymbol{d}}'_1 = -\hat{\boldsymbol{z}}$. The waveform received by dipole 2 is then composed of the line-of-sight (LOS) waveform from dipole 1 and the waveform originating from its image, resulting in total coupling

$$s_{21}^{(\text{total})} = s_{21}^{(\text{LOS})} - s_{21}^{(\text{image})}$$

The output voltage waveforms calculated by the phasor transmission formula and obtained from FDTD are presented in Fig. 3. Both the phasor equation and the FDTD method agree on both the shape and magnitude of the transmitted signal, and correctly model fading due to the reflector. The remaining differences between the two methods can be attributed to finite accuracy of the FDTD method and the approximate nature of expressions (12) and (13).

IV. CONCLUSION

We have presented two versions of phasor transmission equation, one with effective length and the other with newly defined field gain characterizing each antenna in the wireless link. These formulations should serve as alternatives to Friis' transmission equation with effective areas and power gains, respectively, in situations in which phase of the signals cannot be neglected in the calculations, such as when the time domain waveform is required and/or when the received signal is composed of multipath components. Application of the proposed equations has been demonstrated on a canonical problem of coupling between two arbitrarily oriented dipoles

and the results have been verified by comparing with the FDTD method.

Even though the calculations including phase can certainly be done in many other possible ways already presented in the literature, the proposed equations have the advantage of having symmetric and cascaded structure, where the arrangement of the terms follows intuitively the particular functional blocks along the wireless link. We sincerely hope that this specific modular arrangement will help adoption of the equation in the professional community.

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