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Asymptotic set-point regulation for a large class of non-linear hydraulic networks

Tom Nørgaard Jensen * **Carsten Skovmose Kallesøe** *,**
Rafał Wisniewski *

* *Aalborg University, Fredrik Bajers Vej 7c, DK-9220 Aalborg, Denmark (e-mail: {raf,tnj}@es.aau.dk).*
** *Grundfos, Poul Due Jensens Vej 7, DK-8850 Bjerringbro, Denmark (e-mail: ckallesoe@grundfos.com).*

Abstract: This work considers an industrial case study which is centred in a large-scale hydraulic network underlying a district heating system. We propose a set of decentralised proportional-integral control actions which attain asymptotic set-point regulation. The result is an extension of previous results on branched district heating networks to also include a class of looped district heating networks. The decentralised control architecture along with global asymptotic stability of the closed loop equilibrium point means that infrequent changes in network topology such as addition/removal of end-users can easily be commissioned without jeopardising the stability properties of the resulting closed loop system. Thus the decentralised control architecture is capable of plug-and-play commissioning.

1. INTRODUCTION

In this work, we examine the output regulation properties of a class of large-scale non-linear hydraulic networks in closed loop with a set of decentralised proportional-integral control actions. In particular, the class we consider represent district heating networks, which are used for transporting heated water in a closed network from a central heating source to a number of end-users and return the cooled water from the end-users back to the heating source to be reheated.

Our motivation in the following exposition is an industrial case study which considers a a district heating system where multiple pumps are distributed across the network as suggested in Bruus et al. (2004). In particular, we consider the problem of controlling a set of outputs to a set of desired reference values. The system is arbitrary but finite in size, meaning that the number of end-users in the system is arbitrary. The proposed control architecture is decentralised which means control signals can be calculated and applied locally at the end-users in the system. This fits well with the philosophy behind Plug-and-Play Process Control as expressed in Stoustrup (2009). Models and networks similar to those presented here also occur in mine ventilation networks and cardiovascular systems, which is the motivation behind the works Hu et al. (2003); Koroleva and Krstić (2005); Koroleva et al. (2006), where non-linear and adaptive controllers were proposed.

The work presented here, represents an extension of our previous works De Persis and Kallesøe (2011); De Persis et al. (2013), which considered a branched network topology; whereas in this work, we extend to a looped network topology. To understand the difference between the two types of networks, we refer to Figs. 1 and 2. From Fig. 1, it can be seen why the two topologies are denoted branched and looped respectively. Referring to the conceptual diagram in Fig. 2, we see that the difference between the two topologies is that the looped topology includes additional pipe connections both in the forward network (hot water) and the return network (cooled water) and in which no end-users are present. These additional connections add some robustness to the system in the sense that heating can still be provided in situations where pipes are disconnected, e.g., due to breakage or maintenance.
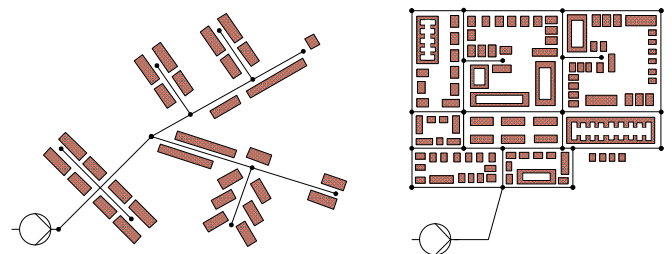
Fig. 1. Illustration of the two network topologies: branched (left) and looped (right), (Val, 2016).

The original contribution in the current exposition is a formal proof of the fact that asymptotic set-point regulation is still attained in the looped network topology in output feedback with the decentralised proportional-integral control actions which we introduced in previous works. In fact, the closed loop system has a global asymptotically stable equilibrium point in which the (output) set-point is attained.

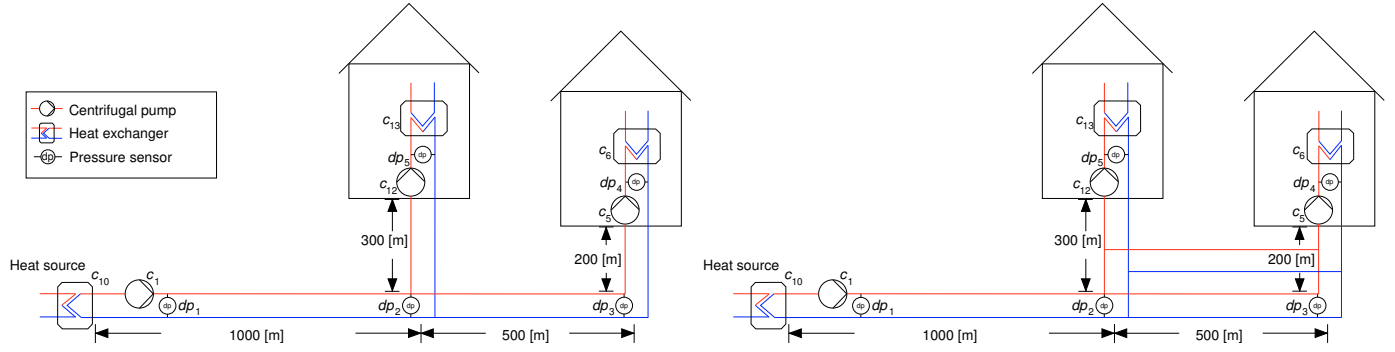Fig. 2. Conceptual diagram of the two network topologies: branched (left) and looped (right) where additional connections are present both in the forward (red) part of the network and the return (blue) part of the network.

The structure of the paper is as follows. In Section 2, we introduce a generalised component model along with assumptions on the network structure. Section 3 presents the structure of the state-space model of the hydraulic network. In Section 4, we show that the steady-state condition of the system has a unique solution. The decentralised control actions are introduced in Section 5 along with the proof that the closed loop system has a global asymptotically stable equilibrium point in which the desired output reference is attained. In Section 6, we present the results of a laboratory experiment which supports the findings. Lastly, conclusions and suggestions to future research directions will be drawn in Section 7.

Nomenclature:

x_i:	the ith component of the vector $x \in \mathbb{R}^n$.
$\langle x, y \rangle$:	the standard inner product of $x, y \in \mathbb{R}^n$.
$vec(x_i)$:	the column vector with entries x_i.
$diag(x_i)$:	the diagonal matrix with entries x_i.
$Df(x)$:	the Jacobian matrix of $f(\cdot)$ at x.
$\mathcal{S}_d(x)$:	the sphere with centre in x and radius d.
$M(n, m; X)$:	the set of n-by-m matrices with entries from the set X.

A map $f : X \to Y$ is said to be *proper* if the pre-image of f of every compact set in Y is compact in X.
The map $f : X \to Y$ with $X, Y \subseteq \mathbb{R}^n$ is said to be (strict) monotonically increasing if it fulfils $\langle x - y, f(x) - f(y) \rangle (>) \geq 0$ for every $x, y \in X$ such that $x \neq y$.

2. PRELIMINARIES

As in De Persis and Kallesøe (2011), which we refer to for additional details on the network modelling, we will here model the network by considering it as a number of two-terminal components and interconnections between them. In this way, the network can be viewed as a connected graph $\mathcal{G}$ where the edges correspond to the components and the vertices correspond to the interconnections. Fig. 3 shows the hydraulic network diagram of the looped network illustrated in Fig. 2. In Fig. 3, it is seen that three types of network components are present: pipes, pumps and valves. The pressure losses across heat exchangers are here modelled by valves.

In (1), we recall the general component model from De Persis et al. (2013) which covers all three types of components

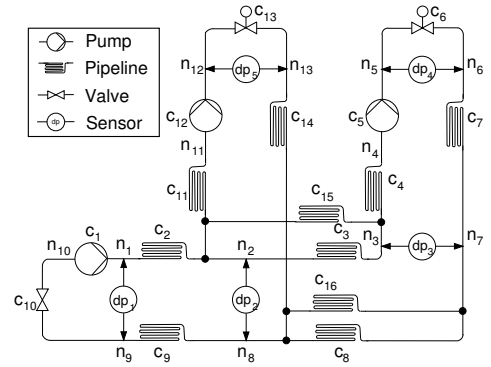$$\Delta p_i = \mathcal{J}_i \dot{q}_i + f_i(q_i) - w_i, \qquad (1)$$

Fig. 3. Hydraulic network diagram corresponding to the conceptual network diagram of the looped topology illustrated on the right in Fig. 2.

where Δp_i is the pressure drop across the ith component; q_i is the fluid flow through the component; the water mass inertia $\mathcal{J}_i > 0$ for pipe components and zero for other components; the hydraulic resistance $f_i(\cdot)$ is strict monotonically increasing and $f_i(0) = 0$ for pipe and valve components and identically zero everywhere for pump components; w_i is a control signal which is non-zero for pump components only, i.e., the pressure drop across the pump can be set to a desired value independently of the fluid flow.

Throughout this exposition we assume that there is an inner loop controller on the actuator (centrifugal pump), such that the actuator provides a desired pressure drop regardless of the flow through it, such that (1) is fulfilled. Furthermore, a distinction will be made between so-called end-user pumps which are located at the end-users and are used for feedback control and booster pumps which are distributed throughout the network and have the purpose to cancel out large pressure gradients.

As in Jensen and Wisniewski (2011), we have the following assumption for resistive components which is motivated by turbulent flow in the system.

Assumption 1. The function $f_i(\cdot)$ in (1) is given by

$$f_i(x) = v_i |x| x, \qquad (2)$$

where $v_i > 0$ is a parameter of the resistive component.

Recall that for non-resistive components $f_i(\cdot)$ is zero on its entire domain.

Furthermore, we assume that the network topology fulfil the following.

Assumption 2. There exists a spanning tree [1] $\mathcal{T}$ of $\mathcal{G}$ such that

1) The chords of $\mathcal{T}$ corresponds to pipe components.
2) k of the fundamental cycles of $\mathcal{G}$ with respect to $\mathcal{T}$ contains a (end-user) pump and a (end-user) valve, both of which belongs only to one such cycle.

Assumption 2-2) defines the structure of the k end-user connections which are illustrated in Fig. 4. The pipe in Fig. 4 corresponds to the chord defining the fundamental cycle containing the end-user pump and the end-user valve.
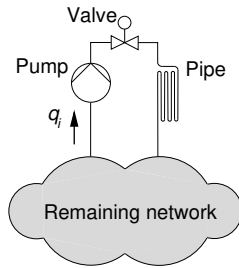

Fig. 4. Illustration of the structure related to end-users.

3. MODEL FORMULATION

Our starting point is a formulation of the network model. Let m be the number of edges in $\mathcal{G}$, n be the number of fundamental cycles defined by the chosen spanning tree $\mathcal{T}$ (recall that k of these cycles contain an end-user) and $B \in M(n, m; \{-1, 0, 1\})$ be the fundamental cycle matrix of $\mathcal{G}$ with respect to $\mathcal{T}$, that is, B indicates whether or not an edge belongs to a fundamental cycle. Then, the vector $q \in \mathbb{R}^n$ of flows through the chords ($\mathcal{G} \setminus \mathcal{T}$) of $\mathcal{T}$ constitutes a vector of free flow variables. By stacking the individual component models (1) into a vector $\Delta p = vec(\Delta p_i)$, Kirchhoffs voltage law can be expressed as $B\Delta p = 0.$, or in other words: the sum of pressure drops in a cycle is zero. From this constraint, the following network model can be derived

$$J\dot{q} = -F(q) + BGu \, , \, y(q) = h(CB^T q), \qquad (3)$$

where $q \in \mathbb{R}^n$ is the vector of independent flow variables; $u \in \mathbb{R}^l$ is the vector of actuator inputs; $y \in \mathbb{R}^k$ is the vector of outputs; $J = Bdiag(\mathcal{J}_i)B^T > 0$ is the mass inertia matrix; $-F(q) = -Bvec(f_i(B_i^T q))$ is the system damping due to hydraulic resistance (here, we recall that $B^T q \in \mathbb{R}^m$ is the vector of component flow variables, and we have used B_i to denote the ith column of B); $G \in M(m, l; \{0, 1\})$ maps actuators to the component vector (that is $vec(w_i) = Gu$); $C \in M(k, m; \{0, 1\})$ selects which (end-user) flows are mapped to the output; $h(z) = vec(h_i(z_i))$ is a map $h : \mathbb{R}^k \to \mathbb{R}^k$ where each $h_i(\cdot)$ is proper [2] and strict monotonically increasing with

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[1] We recall that a spanning tree of a connected graph $\mathcal{G}$ is a connected sub-graph of $\mathcal{G}$ which contains all vertices but has no cycles, see e.g. Deo (1974). The cycles obtained by adding edges in $\mathcal{G} \setminus \mathcal{T}$ to $\mathcal{T}$ are denoted the fundamental cycles of $\mathcal{G}$ with respect to $\mathcal{T}$.

[2] We can drop this requirement if we instead require that y_i^* is in the image of $h_i(\cdot)$.

$h_i(0) = 0$. Lastly, we have that $m > n > k > 0$ and $l \geq k$ with equality only if no boosting pumps are present in the system. Due to 1) in Assumption 2, the inertia matrix J fulfils $J > 0$ (see Tahavori et al. (2013)) for every choice of minimal representation. The output y_i corresponds to the pressure drop across the ith end-user valve and thus $h_i(\cdot)$ is the hydraulic resistance of the end-user valve.

We will partition the vector $u \in \mathbb{R}^l$ of actuator inputs into two components $u_c \in \mathbb{R}^k$ and $u_b \in \mathbb{R}^{l-k}$ with $u = (u_c^T, u_b^T)^T$, where the former component corresponds to the k actuators in 2) of Assumption 2 and the latter component corresponds to boosting pumps. Likewise, we partition the matrix G into sub-matrices G_c and G_b such that $G = [G_c \ G_b]$. Lastly, we will assume that the boosting pumps in the system are providing a constant input pressure u_b^*. This gives the following structure of (3)

$$J\dot{q} = -F(q) + BG_c u_c + c \, , \, y(q) = h(CB^T q), \qquad (4)$$

where $c = BG_b u_b^* \in \mathbb{R}^n$ is a constant.

The point 2) in Assumption 2 corresponds to the fact that there exists k pairs of input/output associated with the same cycle and *only* this cycle. The latter means that for an appropriate indexing we have

$$B = \left[*_{n \times m-2k} \begin{bmatrix} I_k \\ 0_{n-k \times k} \end{bmatrix} \begin{bmatrix} I_k \\ 0_{n-k \times k} \end{bmatrix} \right] \qquad (5a)$$

$$G_c = \begin{bmatrix} 0_{m-2k \times k} \\ I_k \\ 0_{k \times k} \end{bmatrix} \qquad (5b)$$

$$C = [\, 0_{k \times m-2k} \ 0_{k \times k} \ I_k \,]. \qquad (5c)$$

The structure of B in (5a) is due to the fact that we have indexed the actuators in 2) in Assumption 2 from $m-2k+1$ to $m - k$ in the component vector. Likewise, the end-user valves in 2) are indexed from $m - k + 1$ to m in the component vector. Furthermore, since these components are included in the same k fundamental cycles and *only* these k cycles this means that we can define the direction of the components to align with the direction of the cycles such that we obtain the stated structure. The structure of G_c in (5b) is due to the fact that G_c maps the end-user pumps to the component vector. Lastly, the structure of C in (5c) is due to the fact that C selects the component flows corresponding to the end-user valves.

Equation (5) means that

$$BG_c = BC^T = \begin{bmatrix} I_k \\ 0_{n-k \times k} \end{bmatrix} \equiv T. \qquad (6)$$

Note that $T : \mathbb{R}^k \to \mathbb{R}^n$ is a linear injection which means that $z = Tx \neq 0$ for every $x \in \mathbb{R}^k \setminus 0$.

4. STEADY STATE UNIQUENESS

In this section, we will show that the steady-state condition of the system (4) has a unique solution (u_c^*, q^*) for every constant vector y^* of outputs.

The steady-state condition of the system (4) is given by

$$0 = -F(q^*) + Tu_c^* + c \, , \, y^* = h(T^T q^*). \qquad (7)$$

For the hydraulic resistance function given in (2), it was shown in Jensen and Wisniewski (2011) that $F(\cdot)$ is a homeomorphism which in particular means that it has a

continuous inverse $F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. This means that we can write

$$q^* = F^{-1}(Tu_c^* + c). \tag{8}$$

Furthermore, since $F(\cdot)$ is strict monotonically increasing (Jensen and Wisniewski, 2011), we also have that $F^{-1}(\cdot)$ is strict monotonically increasing (Chua and Lam, 1972). Since T is injective we have that the composition $F^{-1} \circ T : \mathbb{R}^k \to \mathbb{R}^n$ is injective as well, so q^* is unique for every unique choice of u_c^*.

Now, let $H : \mathbb{R}^k \to \mathbb{R}^k$ be the map

$$H(x) = T^T F^{-1}(Tx + c). \tag{9}$$

Then we can prove the following proposition

Proposition 1. The map $H : \mathbb{R}^k \to \mathbb{R}^k$ defined in (9) is a homeomorphism onto $\mathbb{R}^k$.

Proof: Our strategy for proving Proposition 1 will be as follows: we will show that 1) $H(\cdot)$ is a local homeomorphism and 2) $H(\cdot)$ is proper. Then it follows from (Wu and Desoer, 1972, Theorem 1) that $H(\cdot)$ is a homeomorphism onto $\mathbb{R}^k$. In the proof of 2) we will use the following result from Wu and Desoer (1972): For a map $f : \mathbb{R}^n \to \mathbb{R}^k$, properness is equivalent to the condition

$$\lim_{||x|| \to \infty} ||f(x)|| = \infty. \tag{10}$$

We will start by proving 2) that $H(\cdot)$ is proper. To this end, we first note that $F(\cdot)$ fulfils the following homogeneity property (Jensen and Wisniewski, 2011)

$$F(\lambda x) = \lambda|\lambda|F(x) \tag{11}$$

for every $\lambda \in \mathbb{R}$. Defining $y = F(x)$ (and consequently $x = F^{-1}(y)$) and taking $F^{-1}(\cdot)$ on both sides in (11) we obtain

$$\lambda F^{-1}(y) = F^{-1}(\lambda|\lambda|y). \tag{12}$$

Let $\varepsilon = \lambda|\lambda|$ then $\lambda = \text{sign}(\varepsilon)\sqrt{|\varepsilon|}$, where we have defined $\text{sign}(0) = 0$. If we replace into (12), we obtain the following homogeneity property of $F^{-1}(\cdot)$

$$F^{-1}(\varepsilon y) = \text{sign}(\varepsilon)\sqrt{|\varepsilon|}F^{-1}(y) \tag{13}$$

for every $\varepsilon \in \mathbb{R}$.

Now, let $\bar{H} : \mathbb{R}^k \to \mathbb{R}^k$ be the map given by

$$\bar{H}(z) = T^T F^{-1}(Tz), \tag{14}$$

consequently, since T is a linear map, we have that $\bar{H}(\cdot)$ fulfils the same homogeneity property as $F^{-1}(\cdot)$. Now, note that $\bar{H}(\cdot)$ is a local homeomorphism since

$$\langle x - y, \bar{H}(x) - \bar{H}(y) \rangle > 0 \tag{15}$$

for every $x, y \in \mathbb{R}^k$ such that $x \neq y$. This can be realised since T is injective and $F^{-1}(\cdot)$ is strict monotonically increasing. Furthermore, since $\bar{H}(0) = 0$ it follows that there exists $\gamma > 0$ such that

$$\alpha = \min\{||\bar{H}(z)|| \mid z \in \mathcal{S}_\gamma(0)\} > 0. \tag{16}$$

Let $\{z_n\}$ with $n \in \mathbb{N}$ be a sequence such that $||z_n|| \to \infty$ for $n \to \infty$. This sequence gives rise to another sequence $\{(y_n, \varepsilon_n)\}$ with $y_n \in \mathcal{S}_\gamma(0)$, from which it follows that $|\varepsilon_n| \to \infty$ for $n \to \infty$.

We now have that

$$H(z_n) = T^T F^{-1}(Tz_n + c) \tag{17}$$

$$= \text{sign}(\varepsilon_n)\sqrt{|\varepsilon_n|}F^{-1}\left(Ty_n + \frac{c}{\varepsilon_n}\right). \tag{18}$$

From (18) and continuity of $F^{-1}(\cdot)$ it follows that for every $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$||H(z_n)|| \geq |\text{sign}(\varepsilon_n)\sqrt{|\varepsilon_n|}(\alpha - \delta)| \tag{19}$$

for every $n > N$. Therefore we have that $||H(z_n)|| \to \infty$ for $n \to \infty$ and we conclude that 2) $H(\cdot)$ is a proper map.

Now, we need to show that 1) $H(\cdot)$ is a local homeomorphism. To this end, let $x, y \in \mathbb{R}^k$ with $x \neq y$ and $p, q \in \mathbb{R}^n$ be given by $p = Tx + c$ and $q = Ty + c$. Since T is an injection it follows that $p \neq q$.

Let $g(x, y) \equiv \langle x - y, H(x) - H(y) \rangle$, then by the definition of $H(\cdot)$ in (9) we have

$$
\begin{aligned}
g(x, y) &= \langle Tx - Ty, F^{-1}(Tx + c) - F^{-1}(Ty + c) \rangle \\
&= \langle Tx + c - Ty - c, F^{-1}(Tx + c) - F^{-1}(Tx + c) \rangle \\
&= \langle p - q, F^{-1}(p) - F^{-1}(q) \rangle > 0.
\end{aligned}
\tag{20}
$$

That is, $H(\cdot)$ is strict monotonically increasing. From this it follows that $H(\cdot)$ is a local homeomorphism (Chua and Lam, 1972, Lemma 3.1).

This proves the thesis. $\square$

Using Proposition 1, we can now prove the following result:

Proposition 2. The composition $h \circ H : \mathbb{R}^k \to \mathbb{R}^k$ is a homeomorphism onto $\mathbb{R}^k$.

Proof: The map $h : \mathbb{R}^k \to \mathbb{R}^k$ is diagonal meaning

$$h(z) = (h_1(z_1), h_2(z_2), \ldots, h_k(z_k))^T. \tag{21}$$

Since each $h_i : \mathbb{R} \to \mathbb{R}$ is strict monotonically increasing it is a local homeomorphism (Chua and Lam, 1972). Furthermore, since $h_i(\cdot)$ is proper it follows that $h_i(\cdot)$ is a homeomorphism onto $\mathbb{R}$ (Wu and Desoer, 1972). Consequently, the diagonal map $h : \mathbb{R}^k \to \mathbb{R}^k$ is a homeomorphism onto $\mathbb{R}^k$.

Since both $h(\cdot)$ and $H(\cdot)$ are homeomorphisms onto $\mathbb{R}^k$ it follows that the composition $h \circ H(\cdot)$ is a homeomorphism onto $\mathbb{R}^k$. $\square$

As a consequence of Proposition 2, the composition $h \circ H(\cdot)$ is invertible, which means that for every unique y^* there exists a unique vector u_c^* given by

$$u_c^* = (h \circ H)^{-1}(y^*). \tag{22}$$

Furthermore, since $F^{-1} \circ T(\cdot)$ is an injection, we have that the steady-state flow vector $q^* = F^{-1}(Tu_c^* + c)$ is unique as well.

Now that we have established existence and uniqueness of the steady-state solution we will next propose a set of control actions to attain the desired steady-state.

5. SET-POINT REGULATION

In this section, we introduce a set of proportional-integral (PI) control actions with the purpose of controlling the output to a desired set-point. Subsequently, we use Lyapunov theory to prove that the output of the closed loop system converges to the desired reference.

Let, $r \in \mathbb{R}^k$ be a vector of output reference values. We wish to use feedback control on the system (4) to obtain the steady-state output $y^* = r$ of the closed loop system. To this end, we define the following set of (PI) controllers

$$\dot{\xi} = -K(y - y^*) \,, \quad u_c = \xi - N(y - y^*), \qquad (23)$$

with $K = diag(K_i)$ and $N = diag(N_i)$ where $K_i, N_i > 0$.

This gives us the following closed loop system

$$J\dot{q} = -F(q) + T\xi - TN(h(T^T q) - y^*) + c \qquad (24)$$

$$\dot{\xi} = -K(h(T^T q) - y^*). \qquad (25)$$

We can derive the following second order dynamic expression

$$J\ddot{q} = -\left(DF(q) + TNDh(T^T q)T^T\right)\dot{q} + T\dot{\xi} \qquad (26)$$

Recall from De Persis et al. (2013) that $DF(q)$ is a positive semi-definite matrix and that it can be written as

$$DF(q) = \Lambda_1(q) + F\Lambda_2(q)F^T, \qquad (27)$$

where $\Lambda_1(q) = diag(\frac{\partial}{\partial q_1} f_1(q_1), \ldots, \frac{\partial}{\partial q_n} f_n(q_n)) \geq 0$, $\Lambda_2(q) = diag(\frac{\partial}{\partial q_{n+1}} f_{n+1}(q_{n+1}), \ldots, \frac{\partial}{\partial q_m} f_m(q_m)) \geq 0$ and $\frac{\partial}{\partial q_i} f_i(q_i) \neq 0$ when $q_i \neq 0$ for every $i = 1, 2, \ldots, n$.

Furthermore, since $N_i > 0$ and each $h_i(\cdot)$ is strict monotonically increasing we have that the matrix

$$TNDh(T^T q)T^T = Tdiag(N_i)diag\left(\frac{\partial}{\partial T_i^T q} h_i(T_i^T q)\right)T^T \qquad (28)$$

is positive semi-definite [3].

Now, let $q^* = F^{-1} \circ T \circ (h \circ H)^{-1}(r)$ which is unique by Proposition 2 and injectivity of T. Furthermore, define the change of coordinates $\tilde{q} = q - q^*$ and $\tilde{\rho} = T^T \tilde{q}$ which consequently are well defined. Lastly, we define the function

$$\tilde{h}(\tilde{\rho}) \equiv h(T^T \tilde{q} + T^T q^*) - h(T^T q^*) \qquad (29)$$

which is zero if and only if $\tilde{\rho} = T^T \tilde{q} = 0$.

Now, we are ready to state the main result of the exposition.

Proposition 3. The point $(\tilde{\rho}^T, \dot{q}^T) = 0$ is the global asymptotically stable equilibrium point of the closed loop system (4), (23).

Proof: Let the candidate Lyapunov function $V(\tilde{\rho}, \dot{q})$ be given by

$$V(\tilde{\rho}, \dot{q}) = \frac{1}{2}\dot{q}^T J\dot{q} + \sum_{i=1}^{k} K_i \int_0^{\tilde{\rho}_i} \tilde{h}_i(s)ds, \qquad (30)$$

which is positive definite and radially unbounded.

The time derivative of $V(\cdot, \cdot)$ is given by (recall that $\tilde{\rho} = T\tilde{q}$)

$$\frac{d}{dt}V(\tilde{\rho}, \dot{q}) = \dot{q}^T J\ddot{q} + \dot{\tilde{\rho}}^T K\tilde{h}(\tilde{\rho})$$

$$= -\dot{q}^T\left(DF(q) + TNDh(T^T q)T^T\right)\dot{q} \qquad (31)$$

$$\leq -\sum_{i=1}^{n} \frac{\partial}{\partial q_i} f_i(q_i)\dot{q}_i^2.$$

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[3] T_i denotes the ith column of T.

Recall that $\frac{\partial}{\partial q_i} f_i(q_i) > 0$ for every $q_i \neq 0$ and that $\frac{\partial}{\partial q_i} f_i(q_i) = 0$ if and only if $q_i = 0$. Thus we conclude that the system converges to the set $\mathcal{Q}$ given by

$$\mathcal{Q} = \{(q^T, \dot{q}^T) \in \mathbb{R}^{2n} \mid q_i = 0 \vee \dot{q}_i = 0\}. \qquad (32)$$

The condition $q_i = 0$ implies that $\dot{q}_i = 0$ which in turn implies that $\ddot{q}_i = 0$. Then from (26) it follows that $T\dot{\xi} = 0$ is the largest invariant set of the closed loop system fulfilling $\frac{d}{dt}V(\tilde{\rho}, \dot{q}) = 0$. From injectivity of T it follows that $\dot{\xi} \to 0$ as $t \to \infty$. Furthermore, since K is non-singular it follows from (23) that $y \to y^*$ for $t \to \infty$. By definition of of $\tilde{h}(\cdot)$ in (29) it follows that $\tilde{\rho} \to 0$ as $t \to \infty$. This concludes the proof. $\square$

The proof of Proposition 3 shows that the closed loop system attains the desired output $y^* = r$ as $t \to \infty$. Furthermore, since the equilibrium of the closed loop system is global asymptotically stable, i.e., the result holds for any initial condition, then we conclude that infrequent changes in network topology such as addition/removal of end-users can be introduced without jeopardising this stability property. This is because the initial condition of the newly attained system after the introduced structural change is guaranteed to be in the basin of attraction of the equilibrium point of the system (whose existence and uniqueness is guaranteed by Proposition 2).

6. EXPERIMENTAL RESULTS

To support the main result of the current paper, we have conducted an experiment on a laboratory setup. The hydraulic network diagram of the setup is illustrated in Fig. 5. As it can be seen in Fig. 5, the setup is able to emulate up to four end-users of the type in Assumption 2-2). However, in the experiment we will assume that only end-users 1 and 2 are present in the system and the loops containing end-users 3 and 4 will in this experiment be considered as simple connections between the forward and return parts of the network, i.e., extraneous loops added for robustness purposes (so $k = 2$ and $n = 4$). Even though these extraneous loops are different in nature than what was illustrated in Figs. 2 and 3, since they represent short circuits between the forward and return part of the network, note that Assumption 2 does not impose constraints on these extraneous loops.
Consequently, pumps C_5, C_{19} and C_{23} are not used in the experiment, while pumps C_9 and C_{27} will be used to control the differential pressure across valves C_{10} and C_{28} respectively using the control actions (23). Lastly, the boosting pump C_1 will be operating with a constant differential pressure of 1 bar. The results of the experiment are given in Fig. 6. It can be seen that the output of the closed loop system follows the reference while all flows in the system converge to some steady-state value.

7. CONCLUSION

We extended our previous results on set-point regulation in large-scale hydraulic networks to a larger class of networks which represents the looped topology of district heating systems. The results show that the network in closed loop with the proposed decentralised proportional-integral controllers attains a global asymptotically stable equilibrium
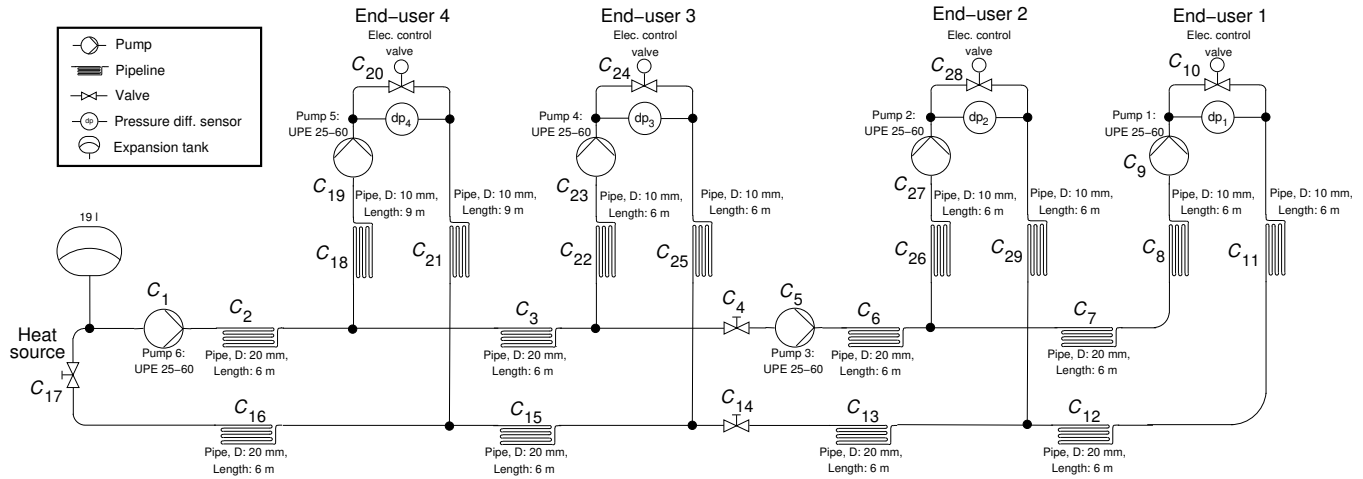
Fig. 5. Hydraulic network diagram of the laboratory setup used in the experiment.
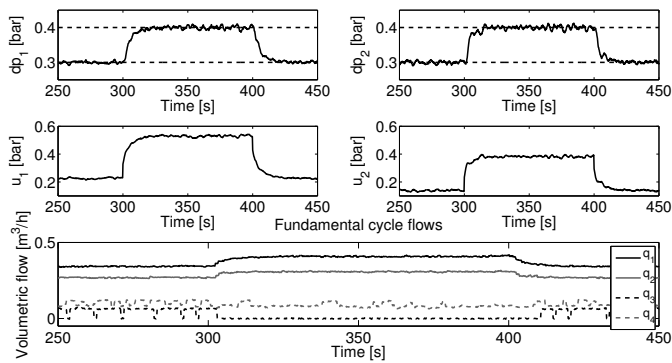

Fig. 6. Result of the conducted experiment where a step response from 0.3 to 0.4 bar is obtained. The top plots show the outputs (dp_1 and dp_2) of the end-users 1 and 2 in Fig. 5. The centre plots show the inputs to the pumps C_9 and C_{27} resp. Lastly, the bottom plot shows the flows through the valves C_{10}, C_{28}, C_{24} and C_{20} resp., where the two flows mapping to outputs dp_1 and dp_2 are given by solid lines.

point. Furthermore, in this equilibrium point the desired reference output is attained. Due to the decentralised control architecture and the fact that the result is global, the architecture lends itself well to plug-and-play commissioning. This is due to the fact that infrequent changes in network structure such as addition and removal of end-users can be introduced without jeopardising the stability properties of the system, since the initial conditions of the newly attained system is guaranteed to belong to the basin of attraction of the equilibrium point. In future research it will be desirable to allow the boosting pumps to be dynamically controlled as well, since this will allow minimisation of energy consumption such as we described in (Jensen et al., 2014) for the branched network topology.

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