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# Investigations of the Switching Dynamics in Sliding Mode Control

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**Abstract:** In this paper, a (switching) sliding mode controller, applied to a mechanical system with additive white process noise, is investigated. The practical relevance of this study is a statistical characterization of system performance in terms of the stationary variance of the control error.

The system is modeled with a two-dimensional stochastic differential equation, whose coordinate functions to an extent are analyzed separately. In order to determine the stationary variance of one of the coordinate functions, the auto-correlation function for the other coordinate function is approximated with a Fourier series. Finally, analytical results are compared to results from Euler-Maruyama simulation over a wide range of model parameter settings.

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**Keywords:** Stochastic differential equation, Sliding mode, Euler-Maruyama simulation, Switching dynamics, Fokker Planck equation.

## 1. INTRODUCTION

Sliding mode control is a nonlinear control method which typically applies a discontinuous control signal to force a system to behave according to prescribed closed loop dynamics. Essentially, the control procedure consists of two parts: Firstly, the controller forces the system state to approach a so called sliding surface and, secondly, to slide along the surface towards the operating point. Near the operating point, the main purpose of the sliding mode controller is to keep this position and respond accordingly to any external disturbance (noise) which affects the system. The sliding surface is found as the sub-manifold of the desired closed loop dynamics. Robustness to external disturbances is achieved by the design of a feedback control, which is discontinuous across the sliding surface. The discontinuity creates in practice rapid switching and in theory additional challenges w.r.t. e.g. existence and uniqueness of solutions to model equations.

The discontinuity induced by the controller brings the main challenges in the analysis of the system and is a main motivation behind the investigations of the switching dynamics. A solid amount of literature exists on the application and analysis of sliding mode control. Among others, see Utkin et al. (1999); Liu and Wang (2012). Recent papers on application of sliding mode control are Herrera et al. (2015); Sakamoto et al. (2016).

In this paper, the system is modeled with stochastic differential equations (SDEs). Solutions to the SDE's are then considered by using appropriate approximations of practical implementations of switching, where the latter may include various imprecisions such as delay, hysteresis and continuous approximation of switching discontinuity. As a result, the system is represented with a two-dimensional

SDE with discontinuous drift coefficient and constant diffusion coefficient.

Solutions to SDEs have over a long period been a subject of great interest, both in the form of existence, uniqueness, explicit closed form solutions and construction of numerical approximations, see Kloeden and Platen (1992); Øksendal (2003); Mao and Yuan (2006). Whereas existence and uniqueness are established for the discontinuous bounded drift case (see Zvonkin (1974); Veretennikov (1981)), it is not yet proven for SDEs with unbounded discontinuous drift coefficients. Neither specific characteristics such as transient and stationary distributions, nor auto- and cross-covariance characteristics have been established for SDEs with discontinuous drift coefficient.

In order to apply some of the known results on SDEs with discontinuous bounded drift, a coordinate transformation of the system is introduced. This transformation implies that the discontinuous dynamics is isolated to only one of the coordinate functions of the two-dimensional SDE, which, additionally, has bounded drift coefficient. By this approach we are able to initiate the study of the two-dimensional SDE through well-known existence results of one of its coordinate functions. We apply the theory of Fourier series to the Fokker-Planck equation for the SDE with discontinuous bounded drift coefficient (Sundararajan (2001); Risken (1989)). This gives, by approximation methods, a time-dependent conditional density function from which an estimate on the auto-correlation is deduced. Finally, by including the auto-correlation function into the analysis of the variance, an estimate of the stationary variance is determined.

**Notation:** Let  $x : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{S}$  denote a (stochastic) process on a probability space  $(\Omega, \Sigma, P)$  with values in the state space  $\mathbb{S}$ . Throughout this paper we suppress the processes

dependency on the variable from the measure space  $(\Omega, \Sigma)$ , that is, we write  $x(t)$  in place of  $x(\cdot, t)$ .

The outline of the paper is as follows: In section 2 we provide the definition of the system, which is subject to subsequent analysis. In section 3 we initiate the analysis of system (3) through a coordinate transformation producing new coordinate functions  $\eta_1$  and  $\eta_2$ . In section 4 the  $\eta_1$  coordinate function is analyzed in order to give a bound on its variance. To estimate the variance of  $\eta_1$ , an estimate of the autocorrelation function of  $\eta_2$  is needed, which is presented in section 5 and 6. Finally, the main result is presented in section 7. Numerical results are presented in section 8 and compared with results generated by Euler-Maruyama simulations.

## 2. SYSTEM DEFINITION

Consider the idealized model of a mechanical system (with mass 1) given by the (control) system of first order differential equations

$$\dot{x}_1 = x_2 \quad (1a)$$

$$\dot{x}_2 = F(x, u), \quad (1b)$$

where  $x_1 : [0, T] \rightarrow \mathbb{R}$  is the position at time  $t \in [0, T]$ ,  $x = (x_1, x_2) : [0, T] \rightarrow \mathbb{R}^2$ ,  $u$  is the control variable and  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  comprises control forces, conservative forces, friction forces and other deterministic external forces.

However, any realistic model describing the behavior of a mechanical system should include the influence of non-deterministic (random) forces. Such forces are often modeled by means of the Wiener process (appearing as white noise in some expositions). Therefore, let  $W : [0, T] \rightarrow \mathbb{R}$  (with  $W(t) = W_t$ ) denote a real valued standard Wiener process with initial value  $W_0 = 0$  and consider, instead of (1), the (control) system of SDEs

$$dx_1 = x_2 dt \quad (2a)$$

$$dx_2 = F(x, u)dt + dW_t. \quad (2b)$$

In the sequel, we study the stochastic behavior of the SDE given in (2) when a sliding mode controller is applied. The analysis is limited to the case where the force  $F$  comprises only friction force modeled as  $-\alpha x_2$  where  $\alpha > 0$  is a viscous friction coefficient as well as feedback control forces. In summary, we consider the system

$$dx_1 = x_2 dt \quad (3a)$$

$$dx_2 = -\alpha x_2 dt + udt + dW_t. \quad (3b)$$

The control  $u$  is designed as a sliding mode controller with switching across a sliding surface

$$S = \{(x_1, x_2) : ax_1 + x_2 = 0\},$$

where the design constant  $a$  is chosen to ensure  $S$  to be a stable manifold. To ensure reaching the surface and maintain sliding, the control  $u$  is designed as

$$u = -k \operatorname{sgn}(ax_1 + x_2),$$

with  $k$  being a constant gain. Next, we fix  $a = \alpha$ . This brings useful properties to the dynamics of the system which is advantageous in the following analysis of the system. However, there is a tradeoff as this implies that the ability to influence the control variable  $u$  is restricted to the constant gain  $k$ .

## 3. PRELIMINARY ANALYSIS OF THE SYSTEM

In system 3, the control function  $u$  contributes with discontinuous dynamics driven by both coordinate functions. In order to simplify this challenge, we introduce a coordinate transformation of the system.

Let a (linear) coordinate transformation from  $(x_1, x_2)$  coordinates to  $(\eta_1, \eta_2)$  coordinates be defined by

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (4)$$

The coordinates are defined such that the  $\eta_1$ -axis is parallel with the sliding surface and the  $\eta_2$ -axis is perpendicular to the sliding surface.

The new coordinates system 3 then reads

$$\begin{aligned} d\eta_1 &= dx_1 - \alpha dx_2 \\ &= (-\alpha\eta_1 + \eta_2)dt - \alpha udt - \alpha dW_t \end{aligned} \quad (5a)$$

$$d\eta_2 = \alpha dx_1 + dx_2 = udt + dW_t. \quad (5b)$$

where  $u : [0, T] \rightarrow \mathbb{R}$  is the control signal given by

$$u = -k \operatorname{sgn}(\eta_2),$$

with  $k$  being a constant gain.

### 3.1 The $\eta_2$ coordinate function

The SDE given in (5b) depends only on  $\eta_2$  and the Wiener process  $W_t$  and, therefore, it can be treated as an independent one-dimensional SDE. The drift coefficient is equal to the identity. For such an SDE, it is proven by A. J. Veretennikov that a strongly unique solution exists (Veretennikov (1981)). Furthermore, for the particular discontinuous SDE in (5b) some additional information has previously been obtained.

Firstly, the stationary density function is known (see Simonsen et al. (2013)), and is given by

$$\Phi_{\eta_2}(\eta_2) = \mathbb{I}_{(\eta_2 < 0)} k e^{2k\eta_2} + \mathbb{I}_{(\eta_2 > 0)} k e^{-2k\eta_2}, \quad (6)$$

where  $\mathbb{I}_{(\cdot)}$  denotes the indicator function. From the stationary density function, the stationary mean and variance are determined to be

$$\mathbb{E}[\eta_2] = 0 \quad \text{and} \quad \operatorname{Var}[\eta_2] = \frac{1}{2k^2}. \quad (7)$$

Secondly, it is proven that numerical solutions produced with the Euler-Maruyama method converge to the strong solution of (5b) (see Simonsen et al. (2014)).

However, there are still open questions regarding the auto- and cross-correlation functions for the  $\eta_2$  coordinate function. The auto-correlation analysis is addressed in Section 6 of this paper.

### 3.2 The $\eta_1$ coordinate function

Investigations of the dynamics in the  $\eta_1$  coordinate function is the main contribution of this paper. More specifically, we will derive an estimate of the system behavior near the system's operating point based on an analysis of the variance.

The following section initiates this analysis by application of Ito calculus to the  $\eta_1$  coordinate function.

#### 4. ITO CALCULUS ON THE $\eta_1$ COORDINATION FUNCTION

In the following, Ito's lemma (Øksendal, 2003, Thm.4.1.2) is applied to the stochastic process  $Y(\eta_1, t) = \eta_1 e^{\alpha t}$ , where  $Y : \mathbb{R}^2 \rightarrow \mathbb{R}; (z, t) \mapsto z e^{\alpha t}$  and  $\eta_1$  is given as in (5a). We remark that the SDE in (5a) can be treated independently of (5b) in Ito's lemma, as long as the local drift and diffusion coefficient are non-anticipating. For further information see (Arnold, 1974, p.102).

Application of Ito's lemma gives

$$dY = e^{\alpha t} (\eta_2 - \alpha u) dt - \alpha e^{\alpha t} dW_t .$$

which, when multiplied with  $e^{-\alpha t}$ , yields the following integral expression for the  $\eta_1$  coordinate function

$$\begin{aligned} \eta_1(t) &= \eta_1(0)e^{-\alpha t} + \int_0^t e^{\alpha(s-t)} (\eta_2 - \alpha u) ds \\ &\quad - \int_0^t \alpha e^{\alpha(s-t)} dW_s . \end{aligned} \quad (8)$$

By a similar procedure, Ito's lemma is applied to  $Z(\eta_1, t) = \eta_1^2$  to obtain

$$\begin{aligned} dZ &= (-2\alpha\eta_1^2 + 2\eta_1(\eta_2 + \alpha k \operatorname{sgn}(\eta_2)) + \alpha^2) dt \\ &\quad - 2\eta_1 \alpha dW_t . \end{aligned} \quad (9)$$

Next, we substitute the expression of  $\eta_1$  given in (8) into (9) to obtain an integral form for  $r \in [0, T]$ . Then, taking the expectation of  $\eta_1(t)$  and  $Z(\eta_1, r)$  and thereafter applying Fubini's theorem gives

$$\begin{aligned} \mathbb{E}[\eta_1] &= \eta_1(0)e^{-\alpha t} + \int_0^t e^{\alpha(s-t)} \mathbb{E}[\eta_2 - \alpha u] ds \\ &\quad - \mathbb{E} \left[ \int_0^t \alpha e^{\alpha(s-t)} dW_s \right] , \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[Z(\eta_1, r)] &= \mathbb{E}[Z(\eta_1, 0)] + \int_0^r -2\alpha \mathbb{E}[Z(\eta_1, t)] dt \\ &\quad + 2 \int_0^r \int_0^t e^{\alpha(s-t)} \mathbb{E} \left[ (\eta_2(t) + \alpha k \operatorname{sgn}(\eta_2(t))) \right. \\ &\quad \left. \cdot (\eta_2(s) + \alpha k \operatorname{sgn}(\eta_2(s))) \right] ds dt \\ &\quad - 2 \int_0^r \mathbb{E} \left[ \int_0^t \alpha e^{\alpha(s-t)} (\eta_2(t) + \alpha k \operatorname{sgn}(\eta_2(t))) dW_s \right] dt \\ &\quad + \int_0^r \mathbb{E}[\alpha^2] dt - 2\mathbb{E} \left[ \int_0^r \eta_1(t) \alpha dW_t \right] . \end{aligned}$$

Recall that the stochastic Ito integral  $I(f) = \int_0^t f(\cdot, s) dW_s$  is a martingale if  $\mathbb{E}[\int_0^t f^2(\cdot, s) ds] < \infty$  and  $f(\cdot, s)$  is non-anticipating. Hence, in this case the expectation of  $I(f)$  is zero and the last terms in the expression of  $\mathbb{E}[\eta_1]$  and  $\mathbb{E}[Z(\eta_1, r)]$  vanish. Therefore, the expectation  $\mathbb{E}[\eta_1]$  is reduced to

$$\begin{aligned} \mathbb{E}[\eta_1] &= \eta_1(0)e^{-\alpha t} + \int_0^t e^{\alpha(s-t)} (\mathbb{E}[\eta_2] + \alpha k \mathbb{E}[\operatorname{sgn}(\eta_2)]) ds \\ &= \eta_1(0)e^{-\alpha t} , \end{aligned}$$

under stationary assumption on the  $\eta_2$  coordinate function (see (6) and (7)).

Furthermore, using the result of Appendix A (see (A.8)) we have

$$\begin{aligned} \mathbb{E}[Z(\eta_1, r)] &\approx Z(\eta_1, 0) + \int_0^r -2\alpha \mathbb{E}[Z(\eta_1, t)] dt \\ &\quad + 2 \int_0^r \int_0^t e^{\alpha(s-t)} \left( \mathbb{E}[\eta_2(t)\eta_2(s)] \right. \\ &\quad + \alpha k \mathbb{E}[\operatorname{sgn}(\eta_2(t))\eta_2(s)] + \alpha k \mathbb{E}[\eta_2(t) \operatorname{sgn}(\eta_2(s))] \\ &\quad \left. + \alpha^2 k^2 \mathbb{E}[\operatorname{sgn}(\eta_2(t)) \operatorname{sgn}(\eta_2(s))] \right) ds dt \\ &\quad - 2 \int_0^r \frac{\alpha(\alpha k^2 + 1)}{\alpha + k^2} dt + \int_0^r \alpha^2 dt , \end{aligned}$$

By differentiating observe that

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[Z(\eta_1, t)] &\approx -2\alpha \mathbb{E}[Z(\eta_1, t)] + 2 \int_0^t e^{\alpha(s-t)} \left( \mathbb{E}[\eta_2(t)\eta_2(s)] \right. \\ &\quad + \alpha k \mathbb{E}[\operatorname{sgn}(\eta_2(t))\eta_2(s)] + \alpha k \mathbb{E}[\eta_2(t) \operatorname{sgn}(\eta_2(s))] \\ &\quad \left. + \alpha^2 k^2 \mathbb{E}[\operatorname{sgn}(\eta_2(t)) \operatorname{sgn}(\eta_2(s))] \right) ds \\ &\quad - 2 \frac{\alpha(\alpha k^2 + 1)}{\alpha + k^2} + \alpha^2 , \end{aligned} \quad (10)$$

and, therefore, an approximated stationary variance of  $\eta_1$  can be determined from  $0 = \frac{d}{dt} \mathbb{E}[Z(\eta_1, t)] = \frac{d}{dt} \mathbb{E}[\eta_1^2]$  whenever the four auto- and cross-covariance terms in the integrand are known.

Since all auto- and cross-covariances only depend on the  $\eta_2$  coordinate function, the density of  $\eta_2$  is analyzed in the following section. This analysis will be based on a Fourier series expansion of the density obtained via the Fokker-Planck equation.

#### 5. A DISCRETE FOURIER SERIES REPRESENTATION OF THE MARGINAL DENSITY FUNCTION FOR $\eta_2$

The Fokker-Planck equation related to  $\eta_2(t)$  is

$$\begin{aligned} \frac{\partial}{\partial t} p(\eta_2, t) &= \frac{\partial}{\partial \eta_2} [k \operatorname{sgn}(\eta_2) p(\eta_2, t)] + \frac{1}{2} \frac{\partial^2}{\partial \eta_2^2} p(\eta_2, t) \\ &= k \left[ \frac{\partial}{\partial \eta_2} \operatorname{sgn}(\eta_2) \right] p(\eta_2, t) \\ &\quad + k \operatorname{sgn}(\eta_2) \frac{\partial}{\partial \eta_2} p(\eta_2, t) + \frac{1}{2} \frac{\partial^2}{\partial \eta_2^2} p(\eta_2, t) , \end{aligned} \quad (11)$$

where  $p(\eta_2, t)$  is the marginal density function for the  $\eta_2$  coordinate function. A truncated version of the density function can be represented as a Fourier series over the interval  $[-L, L]$ ,

$$p(\eta_2, t) = \frac{a(t)}{2} + \sum_{n=1}^{\infty} b_n(t) \cos\left(\frac{\pi n \eta_2}{L}\right) + \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{\pi n \eta_2}{L}\right), \quad (12)$$

where

$$a(t) = \frac{1}{L} \int_{-L}^L p(\eta_2, t) d\eta_2, \quad (13a)$$

$$b_n(t) = \frac{1}{L} \int_{-L}^L p(\eta_2, t) \cos\left(\frac{\pi n \eta_2}{L}\right) d\eta_2, \quad (13b)$$

$$c_n(t) = \frac{1}{L} \int_{-L}^L p(\eta_2, t) \sin\left(\frac{\pi n \eta_2}{L}\right) d\eta_2. \quad (13c)$$

Furthermore, over the interval  $[-L, L]$ , the sign-function  $\text{sgn}(\eta_2)$  and its (distributional) derivative  $\frac{\partial}{\partial \eta_2} \text{sgn}(\eta_2) = 2\delta$ , with  $\delta$  denoting the Dirac delta distribution, can be represented as the Fourier series

$$\text{sgn}(\eta_2) = \sum_{n=1}^{\infty} \frac{2(1 - \cos(\pi n))}{\pi n} \sin\left(\frac{\pi n \eta_2}{L}\right), \quad (14)$$

and

$$\frac{\partial}{\partial \eta_2} \text{sgn}(\eta_2) = 2\delta(\eta_2) = \frac{1}{L} + \frac{2}{L} \sum_{n=1}^{\infty} \cos\left(\frac{\pi n \eta_2}{L}\right). \quad (15)$$

We now outline how the coefficient function

$$b_1(t), c_1(t), b_2(t), c_2(t), \dots$$

from (13) and, therefore, also the density in (12), can be approximated as solutions to a system of linear ordinary differential equations. Details are omitted due to limited space.

We write  $p(\eta_2, t)$  in terms of its Fourier series representation including the series for  $\text{sgn}(\eta_2)$  and  $\frac{\partial}{\partial \eta_2} \text{sgn}(\eta_2) = 2\delta(\eta_2)$  in the Fokker-Planck equation (11). Then for the series representation we expand the derivatives of the Fokker Planck equation and finally collect a truncated (finite) subset of trigonometric terms to obtain the following linear coefficient dynamics

$$\frac{d}{dt} \begin{pmatrix} z \\ a \end{pmatrix} = \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} \begin{pmatrix} z \\ a \end{pmatrix}, \quad (16)$$

where  $A \in \mathbb{R}^{2\ell \times 2\ell}$  (with  $\ell$  representing the number of terms),

$$z = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_\ell(t) \\ c_1(t) \\ c_2(t) \\ \vdots \\ c_\ell(t) \end{pmatrix}, \quad b = \begin{pmatrix} k \\ k \\ \vdots \\ k \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad z(t_0) = \begin{pmatrix} \cos\left(\frac{\pi \eta_2(t_0)}{L}\right) \\ \cos\left(\frac{2\pi \eta_2(t_0)}{L}\right) \\ \vdots \\ \cos\left(\frac{\ell \pi \eta_2(t_0)}{L}\right) \\ \sin\left(\frac{\pi \eta_2(t_0)}{L}\right) \\ \sin\left(\frac{2\pi \eta_2(t_0)}{L}\right) \\ \vdots \\ \sin\left(\frac{\ell \pi \eta_2(t_0)}{L}\right) \end{pmatrix}$$

and  $a(t_0) = \frac{1}{L}$ . The solution to (16) is given explicitly by

$$\begin{pmatrix} z(t) \\ a(t) \end{pmatrix} = \exp\left(\begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} t\right) \begin{pmatrix} z(t_0) \\ a(t_0) \end{pmatrix},$$

thus determines an approximated conditional density function  $\tilde{p}$  given by

$$\tilde{p}(\eta_2(t)|\eta_2(t_0)) = \frac{a}{2} + \sum_{n=1}^{\ell} b_n(t, \eta_2(t_0)) \cos\left(\frac{\pi n \eta_2}{L}\right) + \sum_{n=1}^{\ell} c_n(t, \eta_2(t_0)) \sin\left(\frac{\pi n \eta_2}{L}\right),$$

where  $\{b_n(t, \eta_2(t_0)), c_n(t, \eta_2(t_0)) : n \in \{1, \dots, \ell\}\}$  are the approximated coefficients.

In the following section, we return to the main problem to determine the auto- and cross-covariance terms appearing in the expression of  $\frac{d}{dt} \mathbb{E}[Z(\eta_1, t)] = \frac{d}{dt} \mathbb{E}[\eta_1^2]$ . More precisely, the approximated conditional density function will be applied to determine approximations of  $\mathbb{E}[\eta_2(t)\eta_2(s)]$ ,  $\mathbb{E}[\eta_2(t) \text{sgn}(\eta_2(s))]$  and  $\mathbb{E}[\text{sgn}(\eta_2(t)) \text{sgn}(\eta_2(s))]$  for  $t, s \in [0, T]$ .

## 6. THE AUTO-CORRELATION FUNCTION AND RELATED EXPRESSIONS

The auto-correlation function is given by

$$\begin{aligned} \mathbb{E}[\eta_2(t)\eta_2(s)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta_2 y f_{\eta_2(t), \eta_2(s)}(\eta_2, y) d\eta_2 dy \\ &= \int_{-\infty}^{\infty} y f_{\eta_2(s)}(y) \int_{-\infty}^{\infty} \eta_2 f_{\eta_2(t)|\eta_2(s)}(\eta_2|y) d\eta_2 dy \\ &= \int_{-\infty}^{\infty} y f_{\eta_2(s)}(y) \mathbb{E}[\eta_2(t)|y] dy, \end{aligned}$$

where  $f_{\eta_2(s)}(y)$  is the density function for the distribution of  $\eta_2(s)$ , which by assumption is equal to  $\Phi_{\eta_2}$  given in (6). The density  $f_{\eta_2(t)}(\eta_2|y)$  is substituted with  $\tilde{p}(\eta_2(t)|y)$ , which is the approximated conditional density function given an initial condition  $y = \eta_2(s)$ . Therefore, the conditional expectation,  $\mathbb{E}[\eta_2(t)|y]$  can be approximated by

$$\begin{aligned} \mathbb{E}[\eta_2(t)|y] &\approx \int_{-L}^L \eta_2 \tilde{p}(\eta_2(t)|y) d\eta_2 = \int_{-L}^L \eta_2 \frac{a(t)}{2} d\eta_2 \\ &\quad + \sum_{m=1}^{\ell} \int_{-L}^L \eta_2 b_m(t, y) \cos\left(\frac{\pi m \eta_2}{L}\right) d\eta_2 \\ &\quad + \sum_{m=1}^{\ell} \int_{-L}^L \eta_2 c_m(t, y) \sin\left(\frac{\pi m \eta_2}{L}\right) d\eta_2 \\ &= \sum_{m=1}^{\ell} c_m(t, y) \frac{-2L^2}{\pi m} \cos(\pi m). \end{aligned}$$

From this

$$\begin{aligned} \mathbb{E}[\eta_2(t)\eta_2(s)] &\approx \int_{-\infty}^{\infty} y \Phi_{\eta_2}(y) \sum_{m=1}^{\ell} c_m(t, y) \frac{-2L^2}{\pi m} \cos(\pi m) dy, \quad (17) \end{aligned}$$

where  $\Phi_{\eta_2}$  is the stationary density function for  $\eta_2$  given in (6). Since coefficients  $c_m(t)$  appear as solutions to (16), the

integrand appears as a weighted sum of products of linear functions and complex exponentials and in that way gives a closed form solution.

A similar procedure can be applied to determine

$$\mathbb{E}[\text{sgn}(\eta_2(t))\eta_2(s)]$$

for  $s \leq t$ . Observe that

$$\mathbb{E}[\text{sgn}(\eta_2(t))\eta_2(s)] = \int_{-\infty}^{\infty} y \mathbb{E}[\text{sgn}(\eta_2(t))|y] \Phi_{\eta_2}(y) dy.$$

The expectation  $\mathbb{E}[\text{sgn}(\eta_2(t))|y]$  is given by

$$\mathbb{E}[\text{sgn}(\eta_2(t))|y] = \int_{-\infty}^{\infty} \text{sgn}(\eta_2) f_{\eta_2(t)|y}(\eta_2) d\eta_2,$$

and can be approximated by

$$\begin{aligned} \mathbb{E}[\text{sgn}(\eta_2(t))|y] &\approx \int_{-L}^L \text{sgn}(\eta_2) \tilde{p}(\eta_2(t)|y) d\eta_2 = \int_{-L}^L \text{sgn}(\eta_2) \frac{a}{2} d\eta_2 \\ &+ \sum_{m=1}^{\ell} \int_{-L}^L \text{sgn}(\eta_2) b_m(t, y) \cos\left(\frac{\pi m \eta_2}{L}\right) d\eta_2 \\ &+ \sum_{m=1}^{\ell} \int_{-L}^L \text{sgn}(\eta_2) c_m(t, y) \sin\left(\frac{\pi m \eta_2}{L}\right) d\eta_2 \\ &= \sum_{m=1}^{\ell} c_m(t, y) \frac{2L}{\pi m} (1 - \cos(\pi m)). \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}[\text{sgn}(\eta_2(t))\eta_2(s)] &\approx \int_{-\infty}^{\infty} y \Phi_{\eta_2}(y) \sum_{m=1}^{\ell} c_m(t, y) \frac{2L}{\pi m} (1 - \cos(\pi m)) dy. \end{aligned} \quad (18)$$

By the same procedure we obtain

$$\begin{aligned} \mathbb{E}[\eta_2(t) \text{sgn}(\eta_2(s))] &\approx \int_{-\infty}^{\infty} \text{sgn}(y) \Phi_{\eta_2}(y) \sum_{m=1}^{\ell} c_m(t, y) \frac{-2L^2}{\pi m} \cos(\pi m) dy, \end{aligned} \quad (19)$$

and

$$\begin{aligned} \mathbb{E}[\text{sgn}(\eta_2(t)) \text{sgn}(\eta_2(s))] &\approx \int_{-\infty}^{\infty} \text{sgn}(y) \Phi_{\eta_2}(y) \sum_{m=1}^{\ell} c_m(t, y) \frac{2L}{\pi m} (1 - \cos(\pi m)) dy. \end{aligned} \quad (20)$$

In the sequel, the approximations of the expectations will be applied to determine an upper bound on the stationary variance of  $\eta_1$ .

## 7. THE MAIN RESULT

For  $s < t$  let  $C(t-s)$  be the sum of the auto- and cross-covariance terms appearing in (10), that is

$$\begin{aligned} C(t-s) &= \mathbb{E}[\eta_2(t)\eta_2(s)] + \alpha k \mathbb{E}[\text{sgn}(\eta_2(t))\eta_2(s)] \\ &+ \alpha k \mathbb{E}[\eta_2(t) \text{sgn}(\eta_2(s))] \\ &+ \alpha^2 k^2 \mathbb{E}[\text{sgn}(\eta_2(t)) \text{sgn}(\eta_2(s))]. \end{aligned}$$

Notice that  $C(t-s)$  can be approximated by the sum of (17)-(20) and by this approximation  $C(t-s)$  is bounded.

Substituting  $C(t-s)$  into (10) yields

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[\eta_1^2] &\approx -2\alpha \mathbb{E}[\eta_1^2] - 2 \frac{\alpha(\alpha k^2 + 1)}{\alpha + k^2} + \alpha^2 + 2 \int_0^t e^{\alpha(s-t)} C(t-s) ds \\ &\approx -2\alpha \mathbb{E}[\eta_1^2] - 2 \frac{\alpha(\alpha k^2 + 1)}{\alpha + k^2} + \alpha^2 + g(t), \end{aligned}$$

where  $g(t)$  denotes the integral term. Since boundedness of  $C$  implies a well defined limit,  $g_{\infty}$ , of  $g(t)$  for  $t \rightarrow \infty$ , the approximated stationary variance of  $\eta_1$  can be determined as

$$\mathbb{E}[\eta_1^2] \approx -\frac{\alpha k^2 + 1}{\alpha + k^2} + \frac{\alpha}{2} + \frac{g_{\infty}}{2\alpha}. \quad (21)$$

### 7.1 Original coordinates

The analysis above also gives approximated stationary variances of the original coordinates. From (4) we get

$$\begin{aligned} x_1 &= \frac{1}{1 + \alpha^2} \eta_1 + \frac{\alpha}{1 + \alpha^2} \eta_2 \\ x_2 &= -\frac{\alpha}{1 + \alpha^2} \eta_1 + \frac{1}{1 + \alpha^2} \eta_2, \end{aligned}$$

implying

$$\begin{aligned} \text{Var}[x_1] &= \frac{1}{(1 + \alpha^2)^2} \text{Var}[\eta_1] + \frac{\alpha^2}{(1 + \alpha^2)^2} \text{Var}[\eta_2] \\ &+ \frac{2\alpha}{(1 + \alpha^2)^2} \mathbb{E}[\eta_1 \eta_2], \end{aligned}$$

with a similar expression for  $\text{Var}[x_2]$ . The first two terms are given in (21) and (7). The last term can be approximated by identifying the term  $\eta_1 \eta_2$  from (9), and then using (17) and (19). More precisely

$$\mathbb{E}[\eta_1 \eta_2] \approx \lim_{t \rightarrow \infty} \int_0^t e^{\alpha(s-t)} \tilde{C}(t-s) ds.$$

with  $\tilde{C}(t-s)$  approximated by the sum of (17) and (19), that is

$$\tilde{C}(t-s) = \mathbb{E}[\eta_2(t)\eta_2(s)] + \alpha k \mathbb{E}[\eta_2(t) \text{sgn}(\eta_2(s))].$$

In the following section, system (5) will be simulated with the Euler-Maruyama method to illustrate the validity of the method.

## 8. SIMULATION

This section presents a comparison between the stationary variance (21) and an estimated variance determined from simulations via the Euler-Maruyama method.

The  $(\eta_1, \eta_2)$  coordinate functions are simulated with the Euler-Maruyama method with step size  $h = 2^{-10}$  for  $t \in (0, 50)$ . For fixed value of  $k$  and  $\alpha$ , the simulations are repeated 4000 times and the variance of the end-points of the realizations is determined. The estimated variances are presented in Table 1. The stationary variances given by (21) and (7) are presented in Table 2 for comparison.

Similar, from section 7.1 and the values in Table 1 and 2 we also get Table 3 and 4 comprising estimated and stationary variances of the original  $x_1$  coordinate.

Finally, figures 1 and 2 show simulated trajectories for  $\eta_1, \eta_2$  and  $x_1, x_2$  respectively and a variation of values of

$k \backslash \alpha$	0.1	0.3	0.5	0.7
0.1	(4438, 49.82)	(538.43, 50.27)	(209.09, 53.31)	(99.76, 49.82)
0.3	(322.30, 5.68)	(50.14, 5.84)	(20.00, 5.72)	(10.77, 5.68)
0.5	(68.95, 2.08)	(13.02, 2.08)	(5.72, 2.03)	(3.47, 2.08)
0.7	(20.19, 1.01)	(4.90, 1.01)	(2.33, 1.04)	(1.53, 1.08)

Table 1. The (Euler-Maruyama) estimated variance of  $(\eta_1(50), \eta_2(50))$  determined for different values of  $k$  and  $\alpha$ .

$k \backslash \alpha$	0.1	0.3	0.5	0.7
0.1	(4482, 50)	(531.5, 50)	(194.3, 50)	(100.0, 50)
0.3	(324.96, 5.56)	(49.95, 5.56)	(19.73, 5.56)	(10.68, 5.56)
0.5	(70.02, 2)	(13.20, 2)	(7.5, 2)	(3.44, 2)
0.7	(23.30, 1.02)	(5.18, 1.02)	(2.47, 1.02)	(1.57, 1.02)

Table 2. The stationary variance of  $(\eta_1, \eta_2)$  calculated for different values of  $k$  and  $\alpha$ .

$k \backslash \alpha$	0.1	0.3	0.5	0.7
0.1	4439	538.74	209.06	100.00
0.3	322.20	50.17	19.78	10.45
0.5	68.97	13.00	5.59	3.22
0.7	20.20	4.84	2.21	1.36

Table 3. The (Euler-Maruyama) estimated variance of  $x_1(50)$  determined for different values of  $k$  and  $\alpha$ .

$k \backslash \alpha$	0.1	0.3	0.5	0.7
0.1	4483	531.9	194.7	100.3
0.3	325.03	50.21	20.04	10.89
0.5	70.02	13.69	6.13	3.63
0.7	23.32	5.30	2.65	1.72

Table 4. The stationary variance of  $x_1$  calculated for different values of  $k$  and  $\alpha$ .

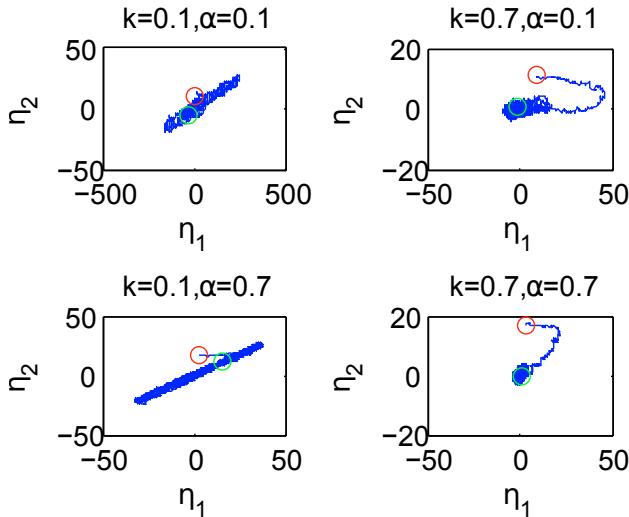


Fig. 1. Simulated  $\eta_1, \eta_2$ . Initial values shown by red circle and final values shown by green circle.

$k$  and  $\alpha$ . Initial values are in all cases  $(x_1(0), x_2(0)) = (10, 10)$ . Notice that for  $k = 0.1$  and  $\alpha = 0.1$  the reaching and sliding behavior are hidden in large fluctuations, whereas for the remaining cases reaching and sliding phases are observable.

## 9. PERSPECTIVES

In this paper, we give an estimate of the control error which is induced by the application of a sliding mode controller to a mechanical system. The significance of this result is rooted in the attempt to describe (or estimate) the collected behavior of a family of physical systems

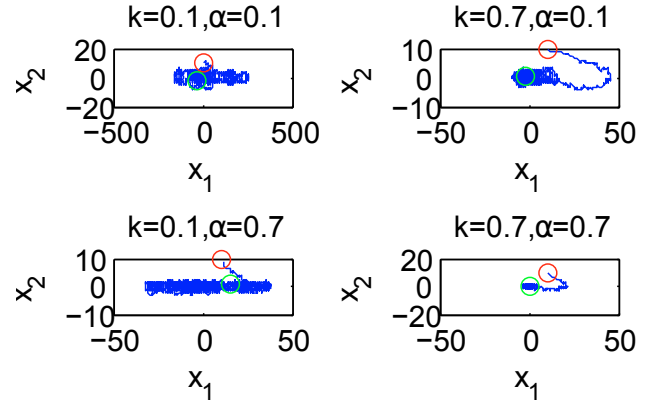


Fig. 2. Simulated  $x_1, x_2$ . Initial values shown by red circle and final values shown by green circle.

connected via certain switching laws, which are governed by some kind of sliding mode control.

The presented results are for a two-dimensional system with the control input isolated to one coordinate function. Thereby, it is possible to introduce a coordinate transformation which reduces the complexity of the discontinuous dynamics provided by the sliding mode controller. For systems of higher dimension with the same structure, i.e. system that can be formulated in the Phase Variable Form, a corresponding coordinate transformation will result in a similar simplification of the discontinuous challenges. Thus, by the same procedure presented in this paper, an estimate of the system behavior can be evaluated.

## Appendix A

Consider the integral term appearing in the expression of  $\mathbb{E}[Z(\eta_1, r)]$

$$\int_0^t \alpha e^{\alpha(s-t)} (\eta_2(t) + \alpha k \operatorname{sgn}(\eta_2(t))) dW_s \quad (\text{A.1})$$

$$= \alpha k \operatorname{sgn}(\eta_2(t)) x(t) + \eta_2(t) x(t),$$

where the process  $x$  is defined by

$$x(t) = \int_0^t \alpha e^{\alpha(s-t)} dW_s.$$

It is easily recognized that  $x$  is the solution of the following SDE

$$dx = -\alpha x dt + \alpha dW_t, \quad x(0) = 0. \quad (\text{A.2})$$

Let the functions  $F, Q : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$F(z) = z_1 z_2 \quad \text{and} \quad Q(z) = z_1 \operatorname{sgn}(z_2).$$

The gradients and Hessians are

$$F_z(z) = \begin{bmatrix} z_2 \\ z_1 \end{bmatrix} \quad \text{and} \quad Q_z(z) = \begin{bmatrix} \operatorname{sgn}(z_2) \\ z_1 \delta'(z_2) \end{bmatrix},$$

$$F_{zz}(z) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad Q_{zz}(z) = \begin{bmatrix} 0 & \delta'(z_2) \\ \delta(z_2) & z_1 \delta''(z_2) \end{bmatrix},$$

with  $\delta'$  denoting the distributional derivative of the Dirac delta distribution.

Using (5b) and (A.2) we see that the process  $z$  defined by  $z = \begin{bmatrix} x \\ \eta_2 \end{bmatrix}$ , satisfies the SDE

$$dz = \mu dt + GdW_t = \begin{bmatrix} -\alpha x \\ -k \operatorname{sgn}(\eta_2) \end{bmatrix} dt + \begin{bmatrix} \alpha \\ 1 \end{bmatrix} dW_t.$$

Hence, by Ito's lemma

$$\begin{aligned} dF &= (F_z(x, \eta_2))^T \mu + \frac{1}{2} \operatorname{tr}(G^T F_{zz}(x, \eta_2) G) dt \\ &\quad + F_z(x, \eta_2)^T G dW_t \\ &= (-\alpha F - kQ + \alpha) dt + F_z(x, \eta_2)^T G dW_t, \end{aligned}$$

and

$$\begin{aligned} dQ &= (Q_z(x, \eta_2))^T \mu + \frac{1}{2} \operatorname{tr}(G^T Q_{zz}(x, \eta_2) G) dt \\ &\quad + Q_z(x, \eta_2)^T G dW_t \\ &= (-\alpha + k\delta(\eta_2)) Q \\ &\quad + \frac{1}{2} (2\alpha\delta(\eta_2) + x\delta'(\eta_2)) dt + Q_z(x, \eta_2)^T G dW_t. \end{aligned}$$

Taking expectations, differentiating and solving for stationarity yields

$$\mathbb{E}[F] = \frac{\alpha - k\mathbb{E}[Q]}{\alpha}, \quad (\text{A.3})$$

and

$$\mathbb{E}[(\alpha + k\delta(\eta_2))Q] = \frac{1}{2} \mathbb{E}[(2\alpha\delta(\eta_2) + x\delta'(\eta_2))]. \quad (\text{A.4})$$

Let  $P_{x|\eta_2}$  and  $P_{\eta_2}$  denote the conditional distribution of  $x$  given  $\eta_2$  and the marginal distribution of  $\eta_2$  respectively. We then get

$$\begin{aligned} \mathbb{E}[\delta(\eta_2)Q] &= \int \int x \operatorname{sgn}(\eta_2) dP_{x|\eta_2} \delta(\eta_2) dP_{\eta_2} \\ &= \int \mathbb{E}[x|\eta_2] \operatorname{sgn}(\eta_2) \delta(\eta_2) dP_{\eta_2} = 0, \end{aligned} \quad (\text{A.5})$$

Moreover

$$\begin{aligned} \mathbb{E}[x\delta'(\eta_2)] &= \int \int x dP_{x|\eta_2} \delta'(\eta_2) dP_{\eta_2} \\ &= \int \mathbb{E}[x|\eta_2] \delta'(\eta_2) \Phi_{\eta_2} d\eta_2 \\ &= -\mathbb{E}'[x|\eta_2 = 0] \Phi_{\eta_2}(0) - \mathbb{E}[x|\eta_2 = 0] \Phi'_{\eta_2}(0) \\ &= -\mathbb{E}'[x|\eta_2 = 0] \Phi_{\eta_2}(0), \end{aligned} \quad (\text{A.6})$$

where  $\Phi_{\eta_2}$  is the (stationary) marginal density of  $\eta_2$  and ' indicates the derivative with respect to  $\eta_2$ . Using (A.5) and (A.6) in (A.4) we obtain

$$\begin{aligned} \alpha\mathbb{E}[Q] &= \alpha\mathbb{E}[\delta(\eta_2)] + \frac{1}{2} \mathbb{E}[x\delta'(\eta_2)] \\ &= \alpha\Phi_{\eta_2}(0) - \frac{1}{2} \mathbb{E}'[x|\eta_2 = 0] \Phi_{\eta_2}(0) \\ &= k(\alpha - \frac{1}{2} \mathbb{E}'[x|\eta_2 = 0]). \end{aligned}$$

And with the following approximation

$$\mathbb{E}'[x|\eta_2 = 0] \approx \frac{\mathbb{E}[x\eta_2]}{\operatorname{Var}[\eta_2]} = \frac{\mathbb{E}[F]}{\operatorname{Var}[\eta_2]},$$

we obtain

$$\alpha\mathbb{E}[Q] \approx k(\alpha - k^2\mathbb{E}[F]). \quad (\text{A.7})$$

Solving (A.3) and (A.7) yields

$$\mathbb{E}[F] \approx \frac{\alpha}{\alpha + k^2}, \quad \text{and} \quad \mathbb{E}[Q] \approx k\mathbb{E}[F].$$

The expectation of the integral appearing in the expression of  $\mathbb{E}[Z(\eta_1, r)]$  is therefore

$$\begin{aligned} &\mathbb{E}\left(\int_0^t \alpha e^{\alpha(s-t)} (\eta_2(t) + \alpha k \operatorname{sgn}(\eta_2(t))) dW_s\right) \\ &\approx \alpha k \mathbb{E}[Q] + \mathbb{E}[F] \approx (\alpha k^2 + 1) \mathbb{E}[F] \approx \frac{\alpha(\alpha k^2 + 1)}{\alpha + k^2} \end{aligned} \quad (\text{A.8})$$

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