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Stochastic Stability Analysis of Control Systems with Uncertain Communication ⋆

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Abstract: This paper presents conditions for determining the stability of a networked control system. We assume that a given system is designed to be stochastically stable, when disregarding the implementation of the controller on a network. Based on the system description and an associated Lyapunov function, we provide conditions for the quality of the network under which the networked system is stable. In particular, we provide a valid inter-sampling interval, mean communication delay, and a set to which the system converges in the mean.

Keywords: Control over networks; Communication constraints; Stochastic stability

1. INTRODUCTION

Advanced control is being implemented in vast spatially distributed systems such as the electricity grids. Such systems might use wireless communication or some shared communication link between subsystems to reduce the cost of the system. It is a challenge to design control systems that exploit unreliable and limited communication, since the information exchange including delays, packet loss, and sampling effects must be taken into account in the controller design. On the other hand, when the need for wireless information exchange arises in an already existing system design, it is relevant to determine the quality of service requirements for the communication such that the system maintains its functionality.

There are several works on the analysis and design of networked control systems. An overview of methods for stability analysis for networked control systems is provided in Zhang et al. (2001), where network-induced delays and packet losses are considered in the model of the transmission path. In Walsh et al. (2002), a detailed analysis is accomplished for maximum-error-first scheduling with the try-once-discard protocol, and bounds on the maximum allowable transfer interval are given for maintaining stability.

Also work on networked control systems with stochastic elements has been conducted. In Antunes et al. (2012), necessary and sufficient conditions are provided for the stability of impulsive renewal systems, and it is shown that the model formalism can be used for stability analysis of linear networked systems with stochastic inter-sampling times. Similar sufficient conditions are given for systems with nonlinear dynamics in Hespanha and Teel (2006).

In addition, there is much work on modeling and analysis of different classes of hybrid systems with stochastic elements and different notions of stability. A survey is provided in Teel et al. (2014), where different notions of stability are considered for different classes of hybrid systems with stochastic elements. Finally, literature also exists for controlling particular stochastic processes Hanson (2007).

In this work, we consider jump diffusion processes described by Øksendal and Sulem (2007) for modeling networked control systems similar to Hespanha (2006). We do not go to the full generality of this formalism - as the jumps have finite intensity, we model the jumps by a compound Poisson process.

Along the lines of Walsh et al. (2002), we assume that the closed loop system is asymptotically stable (in probability), when network-induced effects are neglected, and provide a specification of a communication network under which the system remains stable. We provide two different conditions, where the first condition gives results on stability in the mean and the second condition provides a condition for mean square stability based on Hespanha and Teel (2006).

The paper is organized as follows. Section 2 provides a problem formulation and introduces networked control systems; subsequently, jump diffusion processes are recalled for consistency in Section 3, and used for modeling networked control systems in Section 4. The main stability results of the paper are presented in Section 5, and finally conclusions are provided in Section 6.

2. PROBLEM FORMULATION

This section presents the considered problem of analyzing the stability of a networked control system subject to communication with stochastic delays.

We consider a scenario, where the sensors and the controller are physically separated, as shown in Fig. 1.

Due to the spatial distribution of the system, some communication network is necessary for information exchange. The communication network introduces a delay that might

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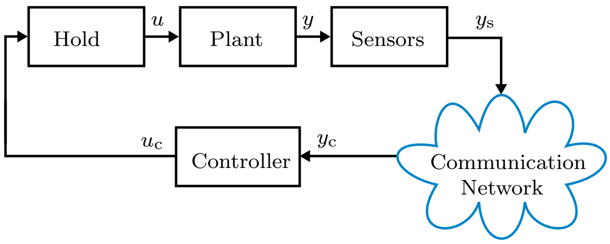

Fig. 1. Diagram of a system where measurements are exchanged between sensors and a controller via a communication network.

be stochastic; in particular, if the network is wireless the communication delay might vary significantly and packets will be lost. A timing diagram of the communication is shown in Fig. 2. We consider a setup where the sensor is clock-driven, and the controller and actuators are event-driven, i.e., the controller and actuator computes new outputs immediately after receiving an input.

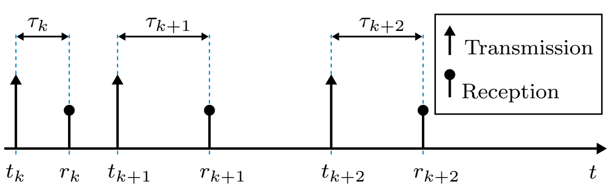

Fig. 2. Timing diagram of communication between the sensors and the controller. Only sensor information is sent over the network.

The timing diagram shows that sensor information is send at times t_k for $k = 1, 2, \ldots$ and that the information is received by the controller at times r_k for $k = 1, 2, \ldots$ after being delayed τ_k by the communication network. Subsequently, a new control input is applied to the plant, i.e., the input is piecewise constant. In lines with Persis et al. (2010), it is decided that the sampling rate of the sensor should depend on the state of the system, such that the sampling rate is higher when the state is close to the desired setpoint; for details see Section 4. The case of fixed sampling intervals is covered by Corollary 1.

An example of a system output being measured and send to the controller is illustrated in Figure 3. It is seen in the figure that the sampling time of the sensor is shorter when y is small, and that the information available to the control is updated at irregular time intervals.

There are several works describing the distribution of different components of communication delays for different types of networks. In Walsh et al. (2002), the packet arrival events are modeled with a Poisson process, and in Wang et al. (2012) the delay of a particular wireless network is shown to have an exponential distribution.

The purpose of this paper is to verify the stability (in a stochastic sense) of a networked control system with the presented properties. Thus, we aim at solving the following problem, which is restated formally later.

Problem 1. Derive an algorithm for verifying the stability of a networked control system with stochastic communication delays.

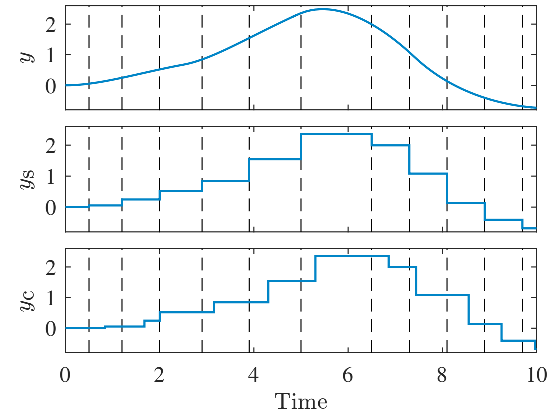

Fig. 3. Graph of an output y that is measured at output-dependent time intervals (dashed lines) giving the piecewise constant sensor output y_s. The sensor output is send to the controller, which updates its information about the plant output y_c. The variable y_c is updated at varying time intervals due to varying communication delays.

The next section presents jump diffusion processes that are subsequently used for modeling the networked control system.

3. JUMP DIFFUSION PROCESSES

The considered system is modelled by a jump diffusion process. We define a Poisson random measure $\mathcal{P}(dt, dq) \equiv \mathcal{P}(dt, dq; P, Q)$ on the Borel sets of $\mathbb{R} \times \mathbb{R}^d$: For a compound Poisson process $\sum_{j=1}^{P(t)} Q_j$, where $P(t)$ is a Poisson process and $Q \equiv (Q_j)$ is a sequence of i.i.d. $\mathbb{R}^d$ vectors, we count the number of jumps occurring in the time-interval A that have value in the set B

$$\mathcal{P}(A, B) = \sum_{j=1}^{\infty} \delta_{(T_j, Q_j)}(A \times B),$$

where T_j are the time-instances of jumps. The studied networked system will be described by the stochastic differential equation

$$\begin{aligned} dX(t) &= f(X(t), t)dt + g(X(t), t)dW(t) \\ &\quad + \int_{\mathbb{R}^d} h(X(t), t, q)\mathcal{P}(dt, dq), \end{aligned} \tag{1}$$

where $X(t) \in \mathbb{R}^n$ is the jump diffusion process, $W(t) \in \mathbb{R}^n$ is a standard Wiener process. The first term of (1) is the deterministic drift, the second term is the stochastic diffusion, and the third term introduces jumps. The functions f, g, h are assumed to satisfy the linear growth and the Lipschitz continuity conditions as in Theorem 1.19 of existence and uniqueness solutions of (Lévy) SDEs Øksendal and Sulem (2007). The jump times are given by the jump rate of the Poisson process $P(t)$. We consider a space-dependent Poisson process, i.e., the intensity (jump rate) $\lambda(t, x)$ is state dependent; hence, the survivor function of the jump is $\exp^{-\int_0^t \lambda(s, X(s))ds}$. The jump destinations are given the random variables (Q_j), each of which is assumed to have probability density function ϕ_Q.

A realization of a jump diffusion process is illustrated in Fig. 4, where the process is initialized at $X(0)$ and jumps at times t_1 and t_2.

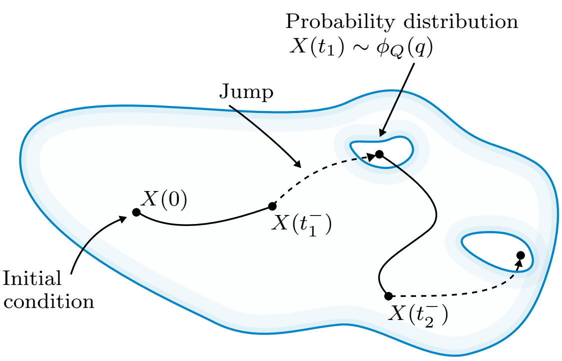

Fig. 4. Realization of jump diffusion process $X(t)$ with jump rate $\lambda(X(t))$. The realization has jumps at times t_1 and t_2.

We use the standard stochastic integration. This allows a reformulation of the integral solution as

$$X(t) = X(0) + \int_0^t f(x(s), s)ds + \int_0^t g(X(s), s)dW(s) + \sum_{k=1}^{P(t)} h(X(t_k^-), t_k, Q_k),$$

where $h(X(t_k^-), t_k, Q_k)$ is the jump amplitude of the k^{th} jump at time t_k, $X(t_k^-)$ is the prejump value of the state, and $P(t)(\omega)$ is the state-dependent number of jumps up to time t.

To analyze jump diffusion processes in Section 5, we rely on the following result on the backward generator given by Theorem 7.1 in Hanson (2007).

Theorem 1. Let $X(t)$ be a jump diffusion process satisfying (1), with continuous differentiable coefficients f, g, h. Let $v(x, t)$ be twice continuously differentiable in x and once in t. Then the conditional expectation of the composite process $v(X(t), t)$ satisfies

$$\mathbb{E}[v(X(t), t)|\ X(t_0) = x_0] = v(x_0, t_0) + \mathbb{E}\left[\int_0^t \left(\frac{\partial v}{\partial t}(X(s), s) + \mathcal{L}(v)(X(s), s)\right) ds|\ X(t_0) = x_0\right]$$

where

$$\mathcal{L}(v)(x,t) := f(x,t)\frac{\partial v}{\partial x}(x,t) + \frac{1}{2}\text{Tr}\left(g(x,t)g^T(x,t)\frac{\partial^2 v}{\partial x^2}(x,t)\right) + \lambda(t,x)\int_{\mathbb{R}^d} (v(x + h(x,t,q),t) - v(x,t))\phi_Q(q)dq.$$

In this paper, we will make no explicit use of probability spaces. We will neither address any issues related to the existence and uniqueness of stochastic differential equations. However, it is assumed that such a unique solution exists for the system at hand. Therefore, we will not make a distinction between weak and strong solutions and simply write a solution.

4. SYSTEM DESCRIPTION

The purpose of this section is to describe the networked control system presented in Section 2 as a jump diffusion process on the form presented in Section 3. To simplify notation, we assume that the stochastic differential equation is time homogeneous.

The dynamics of the system are given by

$$dX(t) = f(X(t))dt + g(X(t))dW(t) + h(X(t))u(t)dt \quad (2)$$
$$Y(t_k) = r(X(t_k)) + \xi(t_k), \quad (3)$$

where the process $X(t) \in \mathbb{R}^n$ represents the system state, $u(t) \in \mathbb{R}^m$ is the control, $W(t) \in \mathbb{R}^n$ is the process noise, $Y(t) \in \mathbb{R}^p$ is the measurement, and $\xi(t) \in \mathbb{R}^p$ is the measurement noise. It is assumed that a control $k : \mathbb{R}^p \to \mathbb{R}^m$ is given, and as described in Section 2, the control is only updated when new measurements are acquired. Thus, the control is given by

$$u(t) = k(Y(t_k)) \qquad \forall t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1})$$

where t_k is the k^{th} sample time of the sensors, and τ_k is a random variable - the stopping time - associated to the communication delay.

To model (2) as a jump diffusion process, we extend the state vector of the system to preserve the Markov property of the system, when taking into account delays. Thus, we define a new stochastic differential equation, with state $Z = (X, Y, Z)$ as follows

$$\begin{bmatrix} dX(t) \\ dY(t) \\ dU(t) \end{bmatrix} = \begin{bmatrix} f(X(t)) + h(X(t))U(t) \\ 0 \\ 0 \end{bmatrix} dt + \begin{bmatrix} g(X(t)) \\ 0 \\ 0 \end{bmatrix} dW(t) + \int_{\mathbb{R}^p} \begin{bmatrix} 0 & 0 \\ h_y(Z(t), q) & 0 \\ 0 & h_u(Z(t)) \end{bmatrix} \begin{bmatrix} \mathcal{P}_y(dt, dq) \\ \mathcal{P}_u(dt) \end{bmatrix},$$

where the jump maps are given by

$$h_y : ((x, y, u), q) \mapsto r(x) + q - y$$
$$h_u : (x, y, u) \mapsto k(y) - u.$$

and $\mathcal{P}_y$ is the Poisson random measure defined by a Poisson process $P_y(t)$ and the family (Q_k) of i.i.d random variables, which represents measurement noise with distribution ϕ_Q. The family (Q_k) may also be used to model packet loss as in Hespanha and Teel (2006). The random measure $\mathcal{P}_u$ is defined by $\mathcal{P}_u(A) = \sum_{j=1}^{\infty} \delta_{T_j}(A)$, where (T_j) is the sequence of jumps in a Poisson process $P_u(t)$.

As initially explained, the jump rate of the Poisson process P_y depends on $X(t)$; however, the jump rate of the Poisson process P_u is assumed to be constant. Thus, the expected number of jumps of P_u in a time interval only dependents on the length of the interval. If P_u makes several jumps between two jumps of P_y, then $U(t)$ will not change its value, since $Y(t)$ is constant in the considered time interval. The Poisson process $P_y(t)$ making several jumps without $P_u(t)$ making a jump corresponds to discarding old measurements, which is reasonable in this application.

5. STABILITY ANALYSIS

This section provides two stability results. First, we provide a condition for stability in the mean, and subsequently a condition for mean square stability.

To lighten the notation, we simplify the system described in the previous section by assuming that there is only one Poisson process $P(t) \equiv P_y(t)$. In addition, the jump rate λ is independent of the state.

We strive to examine asymptotic behaviour of $X(t)$. To this end, we will study both $\mathbb{E}[X(t)]$ and $\mathbb{E}[X^2(t)]$. We will show that there is time T such that for $t > T$, $\mathbb{E}[X(t)] \in B_{r_1}(\lambda)$ and $\mathbb{E}[X^2(t)] \in B_{r_2}(\lambda)$, where $B_{r_i}(\lambda)$

is a disk of radius r_i, and λ is the jump rate of the Poisson process $P(t)$. The radii r_i ($i = 1, 2$) can be made arbitrarily small by making λ sufficiently large. In particular, apart from having information about the expectation of the process $X(t)$ by the application of the Markov inequality, we are able to evaluate the probability of $X(t)$ getting values greater than some ϵ

$$\mathbb{P}[||X(t)|| > \epsilon] \leq \frac{1}{\epsilon^2}\mathbb{E}[X^2(t)].$$

We consider two systems Σ_1:

$$dZ(t) = (f(Z(t)) + k(Z(t)))dt + g(Z(t))dW(t) \tag{4}$$

and its Poisson-sampled counterpart Σ_2 :

$$\begin{bmatrix} dX(t) \\ dY(t) \end{bmatrix} = \begin{bmatrix} (f(X(t)) + k(Y(t)))dt + g(X(t))dW(t) \\ (X(t) - Y(t))\mathcal{P}(dt) \end{bmatrix} \tag{5}$$

The stochastic differential equation Σ_2 can be seen as a realisation of Σ_1 with piecewise-constant control $k(Y(t))$ as $Y(t)$ is a compound Poisson process taking on constant values between sample-times t_k, where the time intervals $t_{k+1} - t_k$ are exponentially distributed with some jump rate λ.

5.1 *Stability in the Mean*

We want to formally verify that if the system Σ_1 is stochastically asymptotically stable, then the realisation of Σ_2 will almost surely converge to a disk D containing the equilibrium 0. The radius of D will depend on the jump rate λ - the bigger jump rate, the smaller radius of D.

We suppose that Σ_1 is stochastically asymptotically stable (asymptotically stable in probability) in the large Khasminskii (2012), i.e., the next two conditions are satisfied

(1) for every $\epsilon > 0$, $\lim_{c\to 0} \mathbb{P}\left[\sup_{0\leq t<\infty} ||Z(t)|| \leq \epsilon |\ Z(0) = c\right] = 1$,
(2) $\mathbb{P}\left[\lim_{t\to\infty} Z(t) = 0\right] = 1$,

where $||Z(t)||$ denotes the Euclidian norm. In other words, we assume that to prove stochastic asymptotic stability of Σ_1, the designer has applied a positive definite function v with the following property $\mathcal{L}(v)(z) \leq -w(z)$ for some positive definite function w, where

$$\mathcal{L} := \sum_i (f_i + k_i)\frac{\partial}{\partial z_i} + \frac{1}{2}\sum_{i,j}[gg^T]_{ij}\frac{\partial^2}{\partial z_i \partial z_j}. \tag{6}$$

The following assumptions about the Lyapunov function v, the drift f, and the diffusion g will be instrumental.

Assumption 1.

$$||f(x) - f(y)|| \leq K||x - y||, \tag{7a}$$
$$||f(x)|| \leq M \tag{7b}$$
$$||k(x)|| \leq H \tag{7c}$$
$$v(x) \geq W||x|| \tag{7d}$$
$$||v(x) - v(y)|| \leq F||x - y|| \tag{7e}$$
$$||\frac{\partial v}{\partial x}(x) - \frac{\partial v}{\partial x}(y)|| \leq N||x - y|| \tag{7f}$$
$$\left|\left|\left[\frac{\partial^2 v(x)}{\partial x_i \partial x_j}\right]_{ij} - \left[\frac{\partial^2 v(y)}{\partial x_i \partial x_j}\right]_{ij}\right|\right| \leq R||x - y||; \tag{7g}$$
$$||g(x)g^T(x) - g(y)g^T(y)|| \leq S||x - y|| \tag{7h}$$
$$||g(x)|| \leq T. \tag{7i}$$

When stability analysis is carried out on a compact subset, Assumption 1 is not considered conservative. The first stability result is formulated in the following proposition.

Proposition 1. Consider the systems Σ_2 in (5). Suppose that on a subset $U \subseteq \mathbb{R}^n$, there is a positive definite function $v : U \to \mathbb{R}$ of Class C^2 such that $\mathcal{L}(v)(z) \leq -w(z)$ on U, where $\mathcal{L}$ is defined in (6). Suppose also that Assumption 1 holds on U, and the Poisson process $P(t)$ has a jump rate λ. Then there is a disk $D(\lambda)$ containing 0 such that for each $x_0 \in U$, there is $\bar{T} > 0$ such that $\mathbb{E}[||X(t)|| \mid X(t_0) = x_0] \leq d(\lambda)$ for $t > \bar{T}$. Furthermore, $d(\lambda)$ converges to 0 as $\lambda \to \infty$.

Proof 1. We shall evaluate the function v on the realisations of Σ_2 between the samples, which correspond to the Poisson jumps at $\{t_i\}_{i\in\mathbb{N}_0}$. Subsequently, at each sampling instant t_i, $X(t_i) = Y(t_i)$. To this end, we use that $X(t)$, $t < t_k$ is independent of t_k (jump rate λ of $P(t)$ is independent of $X(t)$). The expectation of $v(X(t_{k+1}(\omega))(\omega))$ is by the calculus of conditional expectations, Eq. 6.8.14 in Section 6.8 in Hoffmann-Jørgensen (1994)

$$\mathbb{E}\left[v(X(t_{k+1}))|\ t_k = t_0, X(t_k) = x_0\right] = \mathbb{E}[\phi(t_{k+1})], \text{ where}$$
$$\phi(t) = \mathbb{E}\left[v(X(t))|\ t_k = t_0, X(t_k) = x_0\right].$$

By Itô formula and the mean value theorem, we have

$$\begin{aligned}\phi(t) - v(x_0) &= \mathbb{E}_{x_0}\left[\int_{t_0}^{t}\mathcal{L}(v)(X(s))ds\right] \\ &= (t - t_0)\mathbb{E}_{x_0}\left[\mathcal{L}(v)(X(t^*))\right],\end{aligned} \tag{8}$$

where t^* is a random variable with values in (t_0, t), and $\mathbb{E}_{x_0}$ denotes the conditional expectation, conditioned on $X(t_0) = x_0$. We use the definition of $\mathcal{L}$ on the right hand side of (8)

$$\mathbb{E}_{x_0}\left[\sum_i (f_i(X(t^*) + k_i(x_0))\frac{\partial v(X(t^*))}{\partial x_i} + \frac{1}{2}\sum_{i,j}[g(X(t^*))g^T(X(t^*))]_{ij}\frac{\partial^2 v(X(t^*))}{\partial x_i \partial x_j}\right].$$

Firstly, we focus on the drift part

$$\begin{aligned}&\mathbb{E}_{x_0}\left[\sum_i (f_i(X(t^*)) + k_i(x_0))\frac{\partial v((X(t^*))}{\partial x_i}\right] \\ =&\sum_i (f_i(X(t_0)) + k_i(X(t_0)))\frac{\partial v(X(t_0))}{\partial x_i} \\ &+\mathbb{E}_{x_0}\left[\sum_i (f_i(X(t^*)) - f_i(X(t_0)))\frac{\partial v(X(t_0))}{\partial x_i}\right] \\ &+\mathbb{E}_{x_0}\left[\sum_i (f_i(X(t^*)) + k_i(x_0))\left(\frac{\partial v(X(t^*))}{\partial x_i} - \frac{\partial v(X(t_0))}{\partial x_i}\right)\right] \\ :=&A_1 + A_2 + A_3.\end{aligned}$$

We will evaluate the upper bounds of the terms A_2 and A_3 using Assumption 1, and observe

$$A_2 \leq KF\mathbb{E}_{x_0}[||X(t^*) - X(t_0)||].$$

We use Itô formula to evaluate

$$\begin{aligned}E_{x_0}[||X(t^*) - X(t_0)|| &\leq \sum_{i=1}^{n} E_{x_0}[||X_i(t^*) - X_i(t_0)||] \\ &\leq nM(t - t_0).\end{aligned}$$

We conclude that

$$A_2 \leq nKFM(t-t_0).$$

Similarly,

$$A_3 \leq (M+H)N\mathbb{E}_{x_0}[||X(t^*)-X(t_0)||] \\ \leq n(M+H)NM(t-t_0).$$

In the next step, our focus is on the diffusion part

$$\begin{aligned}&\mathbb{E}_{x_0}\left[\sum_{i,j}[g(X(t^*))g^T(X(t^*))]_{ij}\frac{\partial^2 v(X(t^*))}{\partial x_i \partial x_j}\right]\\ =&\sum_{i,j}[g(X(t_0))g^T(X(t_0))]_{ij}\frac{\partial^2 v(X(t_0))}{\partial x_i \partial x_j}\\ &+\mathbb{E}_{x_0}\left[\sum_{i,j}([g(X(t^*))g^T(X(t^*))]_{ij}\right.\\ &\left.-[g(X(t_0))g^T(X(t_0))]_{ij})\frac{\partial^2 v(X(t_0))}{\partial x_i \partial x_j}\right]\\ &+\mathbb{E}_{x_0}\left[[g(X(t^*))g^T(x(t^*))]_{ij}\left(\frac{\partial^2 v(X(t^*))}{\partial x_i \partial x_j}\right.\right.\\ &\left.\left.-\frac{\partial^2 v(X(t_0))}{\partial x_i \partial x_j}\right)\right]\\ =&B_1+B_2+B_3.\end{aligned}$$

We evaluate upper bounds of the terms B_2, and B_3

$$B_2 \leq SN\mathbb{E}[||X(t^*)-X(t_0)||] \leq nSNM(t-t_0)$$
$$B_3 \leq RT^2\mathbb{E}[||X(t^*)-X(t_0)||] \leq nRT^2M(t-t_0).$$

From the above evaluations and the initial hypothesis that

$$\mathcal{L}(v)(X(t_0)) \leq -w(X(t_0)),$$

we have the following evaluation of the expected value of $\phi(t_{k+1})$

$$\begin{aligned}&\mathbb{E}[v(X(t_{k+1}))|\ X(t_k)=x_0] \\ &\leq v(x_0)-w(x_0)\mathbb{E}[t_{k+1}-t_k]+M'\mathbb{E}[(t_{k+1}-t_k)^2]\\ &= v(x_0)-\frac{1}{\lambda}w(x_0)+\frac{2}{\lambda^2}M',\end{aligned} \tag{9}$$

where $M' \equiv n(KF+(M+H)N+\frac{1}{2}SN+\frac{1}{2}RT^2)M$.

The argument above shows that the expected value of the process $v(X(t))$ governed by the system Σ_2 decays to $\frac{2}{\lambda^2}M'$ or below. On the other hand, $W||x|| \leq v(x)$ on U. Define $d(\lambda) \equiv \frac{2}{\lambda^2 W}M'$. Now, it follows that

$$\mathbb{E}[||X(t)||\ |\ X(t_0)=x_0] \leq \frac{1}{W}E[v(X(t))|\ X(t_0)=x_0] \leq d(\lambda)$$

for sufficiently big t.

■

The sampling schedule indicated in Section 2, is slightly different than the one addressed in Proposition 1. To accommodate for the schedule that involves the sampling of the sensor data with a constant delay and subsequently sending the data to the actuator with the Poisson distribution, we define the following process. Let $T_s > 0$ be the inter-sampling time, and let (W_i) be a family of i.i.d. exponentially distributed random variables with jump rate λ. Suppose that $T_m \equiv \sum_{i=1}^m W_i$ and $P(t)$ is the following process

$$P(t) = \sharp\{n \in \mathbb{N}_0|\ nT_\mathrm{s}+T_n \leq t\} \tag{10}$$

where $\sharp A$ denotes the number of elements in A. In other words, $P(t)$ is a process kept constant in the time interval $[t_k, t_{k+1})$ with $t_{k+1} = T_s + W_{k+1}$. By (9)

$$\begin{aligned}\mathbb{E}[v(X(t_{k+1}))|\ X(t_k)=x_0] \leq v(x_0)-w(x_0)(T_s+\frac{1}{\lambda})\\ +M'(T_\mathrm{s}^2+\frac{2}{\lambda^2}+T_\mathrm{s}\frac{2}{\lambda}),\end{aligned}$$

where $M' = n(KF+(M+H)N+\frac{1}{2}SN+\frac{1}{2}RT^2)M$, and the coefficients H,K,F,M,S,R,T are given in Assumption 1. As a consequence, we have the corollary.

Corollary 1. Let $T_\mathrm{s} > 0$. With the same assumptions as in Proposition 1 and $P(t)$ defined by (10), there is a disk $D(\lambda, T_\mathrm{s})$ containing 0 such that for each $x_0 \in U$, there is $\bar{T} > 0$ such that $\mathbb{E}[||X(t)||\ |\ X(t_0)=x_0] \leq d(\lambda, T_\mathrm{s})$ for $t > \bar{T}$. Furthermore, $d(\lambda, T_\mathrm{s})$ converges to 0 as $\lambda \to \infty$ and $T_\mathrm{s} \to 0$.

Corollary 1 can be used for determining a ball to which the realizations of a networked control system converges to. The size of the ball depends both on the sampling time T_s and $\bar{T}$.

5.2 Mean Square Stability

This section provides a guideline for choosing the jump rate to ensure that a given dynamical system is stable in a mean square sense. The stability condition is derived based on Proposition 2 below. It leans upon Theorem 3 and Corollary 2 in Hespanha and Teel (2006), and its main component is Theorem 1 in Section 3.

Proposition 2. Suppose that that there exists a nonnegative function $w : \mathbb{R}^n \to \mathbb{R}$ and constants $L \in \mathbb{R}$, $c, l \geq 0$ such that for all $x \in \mathbb{R}^n$

$$\frac{\partial w}{\partial x}(x)f(x)+\frac{1}{2}\sum_{i,j}[gg^T]_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j}(x) \leq Lw(x)+c \tag{11a}$$

$$\int_{\mathbb{R}^d} w(h(x,q))\phi_Q(q)dq \leq lw(x) \tag{11b}$$

$$w(x) \geq \alpha||x||^2 \tag{11c}$$

and that $\frac{l}{1-LT} < 1$. Then every solution process $X(t)$ to (1) with $\lambda = \frac{1}{T}$ for which $\mathbb{E}[w(x_0)] < \infty$ is mean-square stable and satisfies

$$\mathbb{E}[||X(t)||^2] \leq \frac{e^{-\epsilon t}}{\alpha a}\mathbb{E}[w(x_0)]+\frac{cb}{\epsilon\alpha a}, \forall t \geq 0$$

for some constants $\epsilon > 0$, $0 < a \leq b < \infty$.

The Proposition 2 provides a very strong stability condition, which is desirable to satisfy. Thus, the next proposition provides easy computable conditions under which a given system can be implemented as a networked control system, with stochastic jumps. Again, we consider the systems Σ_1 and Σ_2 introduced in (4) and (5).

Proposition 3. Let Σ_1 be given by (4). Suppose that a nonnegative function $w : \mathbb{R}^n \to \mathbb{R}$ and constants $L \in \mathbb{R}$, $\overline{\alpha}, \underline{\alpha}, c \geq 0$ are such that

$$\begin{aligned}&\frac{\partial w}{\partial z}(z)(f(z)+k(z))+\frac{1}{2}\sum_{i,j}[gg^T]_{ij}(z)\frac{\partial^2 w}{\partial z_i \partial z_j}(z)\\ &\leq Lw(z)+c\end{aligned} \tag{12a}$$

$$\overline{\alpha}||z||^2 \geq w(z) \geq \underline{\alpha}||z||^2 \tag{12b}$$

for all $z \in \mathbb{R}^n$ and that

$$\frac{\partial w}{\partial z}(z)(k(z) - k(y)) \leq K_2||(z, y)||^2 + K_1||(z, y)|| \quad (13)$$

for all $z, y \in \mathbb{R}^n$. Suppose that there exists $\gamma := \gamma(\bar{T}) > 0$ such that

$$0 < \frac{\gamma\overline{\alpha}}{\underline{\beta} - (K + L\overline{\alpha})\bar{T}} < 1, \quad (14)$$

and $\underline{\beta} := \min(1, \gamma)\underline{\alpha}$. Then every weak solution process $(X(t), Y(t))$ to the networked system Σ_2 in (5) is mean-square stable and satisfies

$$\mathbb{E}[||x||^2] \leq \frac{e^{-\epsilon t}}{\underline{\beta} a}\mathbb{E}[w(x_0)] + \frac{(c + c')b}{\epsilon\underline{\beta} a}, \forall t \geq 0$$

for mean inter-sampling interval $\bar{T} = \frac{1}{\lambda}$ and some constants $\epsilon > 0$, $0 < a \leq b < \infty$.

Proof 2. We prove the stability of Σ_2 by showing that $v : (x, y) \mapsto w(x) + \gamma w(y)$ for some $\gamma > 0$ is a nonnegative function that complies with Proposition 2 for Σ_2, i.e.

$$\frac{\partial w}{\partial x}(x)(f(x) + k(y)) + \frac{1}{2}\sum_{i,j}[gg^T]_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j}(x) \leq L_2 v(x, y) + c \quad (15a)$$

$$v(0, x - y) \leq lv(x, y) \quad (15b)$$

$$\overline{\beta}||(x, y)||^2 \geq v(x, y) \geq \underline{\beta}||(x, y)||^2. \quad (15c)$$

It is seen from the definition of v and (12b) that (15c) is satisfied for $\underline{\beta} := \min(1, \gamma)\underline{\alpha}$ and $\bar{\beta} := \max(1, \gamma)\bar{\alpha}$.

We use (12b) and (15c) to bound the terms in (15b) as

$$v(0, x - y) = \gamma w(x - y) \leq \gamma\overline{\alpha}||(x, y)||^2$$
$$lv(x, y) \geq l\underline{\beta}||(x, y)||^2.$$

This implies that (15b) is satisfied if $l \geq \frac{\gamma\overline{\alpha}}{\underline{\beta}}$.

From (12a) it is known that

$$\frac{\partial w}{\partial x}(x)(f(x) + k(y)) + \frac{1}{2}\sum_{i,j}[gg^T]_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j}(x) \leq \frac{\partial w}{\partial x}(x)(k(y) - k(x)) + Lw(x) + c$$

which by (13) and the existence of constants K and c' such that $K_2||(x, y)||^2 + K_1||(x, y)|| \leq K||(x, y)||^2 + c'$ implies that

$$\begin{aligned}
\frac{\partial w}{\partial x}(x)(f(x) + k(y)) + \frac{1}{2}\sum_{i,j}[gg^T]_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j}(x) &\leq K||(x, y)||^2 + Lw(x) + c + c' \\
&\leq K||(x, y)||^2 + L\overline{\alpha}||x||^2 + c + c' \\
&\leq (K + L\overline{\alpha})||(x, y)||^2 + c + c' \\
&\leq \frac{(K + L\overline{\alpha})}{\underline{\beta}}v(x, y) + c + c'.
\end{aligned}$$

Thus, the system is stable for mean $\bar{T}$ satisfying

$$\frac{\gamma\overline{\alpha}}{\underline{\beta} - (K + L\overline{\alpha})\bar{T}} < 1.$$

■

The presented condition is concerned with mean-square stability, which is a very strong notion of stability. Therefore, it might be relevant to only use it for local stability analysis in a sufficiently small neighborhood of 0. It is seen from Proposition 3 that a stabilizing controller can be implemented by means of a communication network with stochastic delay, if the mean inter-sampling interval $\bar{T}$ satisfies (14).

6. CONCLUSIONS

In this paper, two stability conditions for networked control systems modeled by jump diffusion processes were presented. The conditions provide easy checkable conditions under which a given system is stable when implemented in a networked control setting. Both conditions provide a compromise between the size of an expected value (or the variance) of the state and the mean sampling rate.

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