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A Condition of Equivalence Between Bus Injection and Branch Flow Models in Radial Networks

Tao Ding, Member, IEEE, Runzhao Lu, Student Member, IEEE, Yongheng Yang, Senior Member, IEEE, and Frede Blaabjerg, Fellow, IEEE

Abstract—This paper presents an exact bijection between the branch flow model (BFM) and bus injection model (BIM) in radial systems. Moreover, the equivalence and the corresponding condition are investigated and rigorously proved. The exploration reveals that the bijection exists if and only if the network is connected and there is no zero-impedance branch.

Index Terms—Branch flow model, bus injection model, bijection function, convexification

I. INTRODUCTION

The optimal power flow problem is fundamental in power system operations, which is practical in economic dispatch, unit commitment, transmission system expansion planning and reactive power operation, etc. Extensive research efforts have been made since the first formulation of the economic dispatch problem by Carpenter in 1962 [1].

The AC Optimal Power Flow (ACOPF) problem can accurately describe the power flow, but it is nonconvex due to the nonconvex AC power flow equation, challenging the global optimality.

Recently, many papers have discussed the convex relaxation of the AC power flow equation to attain the global optimality of ACOPF [2-25]. The majority of convex relaxation models are based on the second-order cone programming (SOCP) and semidefinite programming (SDP). These convex formulations have gained much attractiveness, as they can: 1) guarantee the global optimality, 2) certify the infeasibility, and 3) offer a lower bound for the optimization [2]. Particularly, for radial networks, SOCP and SDP relaxation techniques are proved equivalent, while the SOCP is more efficient in terms of convergence speed and it is thus suitable for radial networks [3-6].

In general, there are two types of SOCP formulations to model the power flow equation. One is the bus injection model (BIM) and the other is the branch flow model (BFM). In the BIM, the optimization variables are nodal voltages including voltages, currents and power injections [7-12]. In the BFM, the corresponding optimization variables are currents and powers of the branches [3-6, 13-15].

It has been presented that the BIM and BFM are equivalent in [4, 16], especially for the equivalence between the relaxed BIM and BFM. However, certain questions remain unsolved:

1) Are the BIM and BFM always equivalent?
2) What is the exact relationship between the BIM and BFM?
3) Are the two models still equivalent after relaxation?

In this paper, it has been found that the BIM and BFM are equivalent under the normal and relaxation if and only if the network is connected and there is no zero-impedance branch. Notably, the zero-impedance problem has widely been studied in power systems [26-29] and the islanding problem may exist in faulty or emergent conditions. Moreover, an exact bijection function is given and proved. It is interesting to find that only one injection \( \Gamma_{BIM} \rightarrow \Gamma_{BFM} \) exists for the case where the network is disconnected; and only one injection \( \Gamma_{BFM} \rightarrow \Gamma_{BIM} \) exists for the case where there is zero impedance. For the above, the surjection does not exist.

The rest of this paper is organized as follows. Section II presents the explicit derivation of the BIM and BFM. Then, a bijection and the corresponding condition from the BIM to BFM is given in Section III. Numerical results are provided in Section IV and conclusions are drawn in Section V.

II. BUS INJECTION AND BRANCH FLOW MODELS

A. General AC power flow model

For a power grid, the general power flow is described as

\[
U_0 = U_0^{ref}, \quad \theta_0 = \theta_0^{ref}
\]

\[
p_i^j = U_i^{\sum} U_j \left( G_{ij} \cos \theta_{ij} + B_{ij} \sin \theta_{ij} \right), \quad i \in \mathbb{B}, ij \in \mathcal{L} \tag{1}
\]

\[
q_i^j = U_i^{\sum} U_j \left( G_{ij} \sin \theta_{ij} - B_{ij} \cos \theta_{ij} \right)
\]

where \( U_i \) and \( \theta_i \) are the voltage magnitude and angle of bus \( i \); \( G_{ij} \) and \( B_{ij} \) are real and imaginary parts of the admittance matrix; \( p_i^j \) and \( q_i^j \) are controllable active and reactive power; \( \mathbb{B} \) and \( \mathcal{L} \) are the sets of branches and buses, respectively; \( n \) is the number of buses; \( \theta_i = \theta_i - \theta_j \) is the angle difference of the branch with the from bus \( i \) to bus \( j \); \( U_i^{ref} \) and \( \theta_i^{ref} \) are voltage magnitude and angle of the reference bus.

Let \( r_{ij} \) and \( x_{ij} \) be the resistance and reactance of branch \( ij \).

According to the definition of admittance matrix, \( G_{ij}, B_{ij}, r_{ij}, \) and \( x_{ij} \) can be correlated as

\[
G_{ij} = -\frac{r_{ij}}{r_{ij}^2 + x_{ij}^2}, \quad B_{ij} = \frac{x_{ij}}{r_{ij}^2 + x_{ij}^2}, \quad G_{ij} = G_{ji}, \quad B_{ij} = B_{ji} \tag{2}
\]

which helps to derive the following:

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\[ x_i B_{ij} - r_i G_{ij} = \frac{x_i^2}{x_i^2 + r_i^2} - \frac{r_i^2}{x_i^2 + r_i^2} = 1 \]
\[ r_i B_{ij} + x_i G_{ij} = \frac{r_i x_i}{x_i^2 + r_i^2} + \frac{-x_i r_i}{x_i^2 + r_i^2} = 0 \]  
\( \text{(3)} \)

For convenience, the self-admittance can be expressed as 
\[ G_{ii} - \sum_{j \neq i} G_{ij} = G_i \quad \text{and} \quad B_{ii} - \sum_{j \neq i} B_{ij} = B_i, \]  
where \( G_{ii} \) and \( B_{ii} \) are shunt admittance. Then, (1) can be reformulated as 
\[ U_i = U_i^{ref}, \quad \theta_i = \theta_i^{ref} \]
\[ -G_i u_i^2 + p_i^* = U_i \sum_{j \neq i} U_j (G_j \cos \theta_j + B_j \sin \theta_j), \quad i \in \mathbb{B}, \quad ij \in \mathbb{L} \] 
\[ B_j u_j^2 + q_i^* = U_i \sum_{j \neq i} U_j (G_j \cos \theta_j - B_j \sin \theta_j) \]  
\( \text{(4)} \)

For a radial network, the voltage angle of the BIM and BFM can be eliminated since there is no circle and the angles can be uniquely determined by the voltage magnitudes and the branch flows. The transformations can be derived as follows.

**B. Bus Injection Model**

A transformation is given as 
\[ u_i = U_i^2, \]
\[ T_j = U_i U_j \sin \theta_j, \quad ij \in \mathbb{L}, \quad i, j \in \mathbb{B} \]
\[ R_j = U_i U_j \cos \theta_j \]  
\( \text{(5)} \)

which utilizes the squared voltage magnitudes, voltage inner and cross products of the connected buses for each line. Taking the square for \( T_j \) and \( R_j \) will eliminate the voltage angle. This gives the BIM formulation as 
\[ -G_i u_i + p_i^* = -u_i \sum_{j \neq i} G_j + \sum_{j \neq i} (G_j R_j + B_j T_j), \quad i \in \mathbb{B} \] 
\( \text{(6a)} \)
\[ B_i u_i + q_i^* = u_i \sum_{j \neq i} B_j + \sum_{j \neq i} (G_j T_j - B_j R_j), \quad i \in \mathbb{B} \] 
\( \text{(6b)} \)
\[ R_j^2 + T_j^2 = u_i u_j, \quad ij \in \mathbb{L} \] 
\( \text{(6c)} \)
\[ R_j = R_j^2, \quad T_j = -T_j^2, \quad ij \in \mathbb{L} \] 
\( \text{(6d)} \)
\[ u_o = (U_0^{ref})^2 \]  
\( \text{(6e)} \)

which is defined as the feasible region of the BIM being \( \Gamma_{BIM} \). In addition, the variables in the BIM should include \( (R_j, T_j) \) for each line \( ij \), and \( (u_i, p_i^*, q_i^*) \) for each bus \( i \).

**C. Branch Flow Model**

The BFM formulation employs the squared voltage and current magnitudes to eliminate the voltage angles, following 
\[ v_i = U_i^2, \quad ij \in \mathbb{L}, \quad i, j \in \mathbb{B} \]  
\( \text{(7)} \)

Subsequently, the BFM formulation can be derived from (1) as 
\[ -G_{ij} v_i + P_i^* = \sum_{k \neq i} P_{ik} - \sum_{j \neq i} (P_{ij} - r_j I_{ij}), \quad i \in \mathbb{B} \] 
\( \text{(8a)} \)
\[ B_{ij} v_i + Q_i^* = \sum_{k \neq i} Q_{ik} - \sum_{j \neq i} (Q_{ij} - x_j I_{ij}), \quad i \in \mathbb{B} \] 
\( \text{(8b)} \)
\[ v_j = v_i - 2 (r_j P_j + x_j Q_j) + (r_i^2 + x_i^2) I_{ij}, \quad ij \in \mathbb{L} \] 
\( \text{(8c)} \)

\[ l_q v_i = P_i^2 + Q_i^2, \quad ij \in \mathbb{L} \]  
\( \text{(8d)} \)
\[ v_0 = (U_0^{ref})^2 \]  
\( \text{(8e)} \)

where \( P_i^* \) and \( Q_i^* \) are the controllable active and reactive powers in the BFM. Here, the feasible region of the BFM in (8) is defined as \( \Gamma_{BFM} \). Moreover, the variables in the BFM should include \((P_{ij}, Q_{ij}, I_{ij})\) for the line \( ij \), and \((v_i, P_i^*, Q_i^*)\) for each bus \( i \).

It should be noted that \( I_{ij} \) for each line is just as an intermediate variable that can be uniquely determined by \( P_{ij}, Q_{ij}, v_i \). Thus, \( I_{ij} \) is not considered in the computing variables.

Consequently, it can be found that the number of the computing variables for the BIM and BFM is equal. Moreover, by the transformations in (5) and (7), only the voltage magnitudes are preserved, while the voltage angles are eliminated. For radial systems, since the number of angle differences equals to the number of unknown voltage angles, the voltage angles can be uniquely determined and therefore this angle elimination can be exactly recovered once the branch flows and voltage magnitudes are determined [3-6].

**III. EQUIVALENCE OF BIM AND BFM BY A LINEAR BIJECTION**

**A. Proof on the Equivalence of the BIM and BFM**

As shown in Fig. 1, proving the equivalence of the BIM and BFM requires finding a bijective function between them, such that for any feasible point in one set is uniquely paired with one feasible point of the other set [13, 16].

\[ \mathcal{F}: \mathcal{V} \rightarrow \mathcal{W} \text{ is an injection and } \mathcal{F} \text{ is a bijection if and only if } \mathcal{F} \text{ is invertible.} \]

According to Lemma 1, we can define an injection \( \mathcal{F} \) between the BIM and BFM as (9). In the following, we will prove that \( \mathcal{F} \) is a bijection between \( \Gamma_{BFM} \) and \( \Gamma_{BIM} \), i.e., \( \mathcal{F} \) is an invertible injection from \( \Gamma_{BFM} \) to \( \Gamma_{BIM} \).

\[ \mathcal{F}: \begin{cases} v_i = u_i & i \in \mathbb{B} \\ P_j = -u_j G_j + G_i R_j + B_i T_j & ij \in \mathbb{L} \\ Q_j = u_j B_j + G_j T_j - B_j R_j & ij \in \mathbb{L} \\ P_i^* = p_i^* & i \in \mathbb{B} \\ Q_i^* = q_i^* & i \in \mathbb{B} \end{cases} \]  
\( \text{(9)} \)

(i) Prove \( \mathcal{F} \) is a \( \Gamma_{BIM} \rightarrow \Gamma_{BFM} \) linear injection

At first, it can be found that \( \mathcal{F} \) is a linear injection. Now, we want to show that for any given feasible solution \((u_i, u_j, R_j, T_j, P_i^*, Q_i^*)\) obtained by (9) belongs to \( \Gamma_{BIM} \).
\[ \left( v_i, v_j, P_{ij}, Q_{ij}, P_{ij}', Q_{ij}' \right) \]  

\[ P_{ij}' = -u_i G_j + G_j R_j + B_j T_j \]  

\[ Q_{ij}' = u_i B_j + G_j T_j - B_j R_j \]  

Proof: As \( \left( u_i u_j, R_{ij}, T_{ij}, P_{ij}, Q_{ij} \right) \) s.t. \( u_i u_j \in \Gamma_{MM} \), it satisfies

\[ -G_{ij} u_i + P_{ij}' = -u_i \sum_{(p, q) \in \Gamma} G_{pq} + \sum_{(p, q) \in \Gamma} \left( G_{pq} R_{pq} + B_{pq} T_{pq} \right), \quad i \in I \]  

\[ B_{ij} u_i + Q_{ij}' = u_i \sum_{(p, q) \in \Gamma} B_{pq} + \sum_{(p, q) \in \Gamma} \left( G_{pq} T_{pq} - B_{pq} R_{pq} \right), \quad i \in I \]  

\[ R_{ij} = R_j, \quad T_{ij} = -T_j, \quad i, j \in I \]  

\[ \overline{u_i} = (U_i'')^2 \]  

Additionally, the transmission losses can be formulated as

\[ P_{loss} = r_i L_i = \overline{P}_i + \overline{P}_i' \]  

\[ \sum_{k, j \neq i} P_{k, j} = -u_i G_j + G_j R_j + B_j T_j + u_i B_j + G_j T_j - B_j R_j \]  

\[ Q_{loss} = x_i L_i = \overline{Q}_i + \overline{Q}_i' \]  

Then, it derives.

\[ (r_i^2 + x_i^2) L_i = r_i P_{loss} + x_i Q_{loss} = \overline{u}_i + u_i - 2 \overline{R}_i \]  

(10)

(11)

(12)

(13)

(14)

(15)

(16)

(17)

For any \( i \in I \), substitute \( P_{ij} \) by the linear injection (10) into (8a) and its right-hand expression can be obtained by

\[ \sum_{k, j \neq i} P_{k, j} = -u_i G_j + G_j R_j + B_j T_j + u_i B_j + G_j T_j - B_j R_j \]  

(10a)

\[ \sum_{k, j \neq i} \left( -u_i G_j + G_j R_j + B_j T_j \right) - \left( -u_i G_j + G_j R_j + B_j T_j - r_i L_i \right) \]  

(12a)

\[ \sum_{k, j \neq i} \left( -u_i G_j + G_j R_j + B_j T_j \right) - \sum_{k, j \neq i} \left( -u_i G_j + G_j R_j + B_j T_j - r_i L_i \right) \]  

(11b)

For any \( i \in I \), substitute \( Q_{ij} \) by the linear injection (10) into (8b) and its right-hand expression can be obtained by

\[ \sum_{k, j \neq i} \left( -u_i G_j + G_j R_j + B_j T_j \right) - \sum_{k, j \neq i} \left( -u_i G_j + G_j R_j + B_j T_j - r_i L_i \right) \]  

(11a)

(12b)

(12c)

(13a)

(13b)
With the above proof (1-5), it is shown that for any given feasible solution of \( \Gamma_{BIM} \), the corresponding image solution \( \left( v_i, v_j, P_{ij}^c, Q_{ij}^c, P_{ij}^r, Q_{ij}^r \right) \) \( \forall i \in \mathbb{N} \), \( \forall j \in \mathbb{N} \), by the linear injection \( \mathcal{F} \), \( \mathcal{F} \) belongs to \( \Gamma_{BFM} \). Therefore, \( \mathcal{F} \) is a linear injection from \( \Gamma_{BIM} \) to \( \Gamma_{BFM} \).

(ii) Prove \( \mathcal{F} \) is an invertible injection

The linear injection in (9), \( \mathcal{F} : \Gamma_{BIM} \rightarrow \Gamma_{BFM} \), can be written as

\[
\begin{pmatrix}
P_{ij}^c \\
Q_{ij}^c \\
v_i \\
P_{ij} \\
Q_{ij}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -G_{ij} & G_{ij} & B_{ij} \\
0 & 0 & B_{ij} & -B_{ij} & G_{ij}
\end{pmatrix}
\begin{pmatrix}
P_{ij}^r \\
Q_{ij}^r \\
v_i \\
P_{ij} \\
Q_{ij}
\end{pmatrix}
\]  
\tag{18}

The determinant of the coefficient matrix can be calculated by

\[
\begin{vmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -G_{ij} & G_{ij} & B_{ij} \\
0 & 0 & B_{ij} & -B_{ij} & G_{ij}
\end{vmatrix} = \det(G_{ij}G_{ij} + B_{ij}B_{ij})
\]  
\tag{19}

**Theorem 1:** If the network is connected and there is no zero-impedance, the injection (9) is invertible and the linear bijection \( \Gamma_{BIM} \rightarrow \Gamma_{BFM} \) exists.

If Theorem 1 holds, the impedance of each branch is infinite and non-zero, so the coefficient matrix of (19) is invertible, whose inverse matrix can be formulated as

\[
\begin{pmatrix}
P_{ij}^r \\
Q_{ij}^r \\
v_i \\
P_{ij} \\
Q_{ij}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -r_{ij} & -x_{ij} & P_{ij} \\
0 & 0 & x_{ij} & -r_{ij} & Q_{ij}
\end{pmatrix}
\begin{pmatrix}
P_{ij}^c \\
Q_{ij}^c \\
v_i \\
P_{ij} \\
Q_{ij}
\end{pmatrix}
\]  
\tag{20}

**Case 1:** If the branch \( ij \) is opened and the network is not connected, the values of \( r_{ij} \) and \( x_{ij} \) are infinite (e.g., \( G_{ij} \) and \( B_{ij} \) are zero). At this time, \( P_{ij} \), \( Q_{ij} \), \( u_i \), and \( u_j \) should be strictly zero, no matter what \( \left( u_i, u_j, R_{ij}, P_{ij}^r, Q_{ij}^r, P_{ij}^c, Q_{ij}^c \right) \) is. As a result, we could only find a non-invertible injection \( \Gamma_{BIM} \rightarrow \Gamma_{BFM} \) by

\[
\begin{pmatrix}
P_{ij}^r \\
Q_{ij}^r \\
v_i \\
P_{ij} \\
Q_{ij}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
P_{ij}^c \\
Q_{ij}^c \\
v_i \\
P_{ij} \\
Q_{ij}
\end{pmatrix}
\]  
\tag{21a}

**Case 2:** If the impedance of the branch \( ij \) is zero, the values of \( r_{ij} \) and \( x_{ij} \) are zero (e.g., \( G_{ij} \) and \( B_{ij} \) are infinite). At this time, \( u_i \) and \( u_j \) are strictly equal, no matter what \( \left( v_i, v_j, P_{ij}^c, Q_{ij}^c, P_{ij}^r, Q_{ij}^r \right) \) is. This gives \( R_{ij} = v_i \) and \( T_j = 0 \). As a result, we could only find a non-invertible injection \( \Gamma_{BFM} \rightarrow \Gamma_{BIM} \) by

\[
\begin{pmatrix}
P_{ij}^r \\
Q_{ij}^r \\
v_i \\
P_{ij} \\
Q_{ij}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
P_{ij}^c \\
Q_{ij}^c \\
v_i \\
P_{ij} \\
Q_{ij}
\end{pmatrix}
\]  
\tag{21b}

For the two cases, only the injection exists but the bijection does not exist. Take Case 1 for illustration. If the network is not connected, the injection \( \Gamma_{BIM} \rightarrow \Gamma_{BFM} \) exists in (21a) and it is unique, but there are many surjections \( \Gamma_{BFM} \rightarrow \Gamma_{BIM} \). This suggests that the two formulations are not equivalent anymore.

**B. Proof on Equivalence of Relaxed BIM and BFM by SOCP**

The feasible region of the BIM and BFM is non-convex due to the quadratic equalities. To convexify the feasible regions, conic relaxation techniques were utilized for the BFM [3-6, 13-15] and BIM [7-12] to relax the quadratic equalities into inequalities, such that

\[
\text{BIM: } R_{ij}^2 + T_{ij}^2 = u_i u_j \rightarrow R_{ij}^2 + T_{ij}^2 \leq u_i u_j, \quad ij \in \mathbb{N}
\]  
\tag{22a}

\[
\text{BFM: } l_v v_i = P_{ij}^c + Q_{ij}^c \rightarrow l_v v_i \geq P_{ij}^c + Q_{ij}^c, \quad ij \in \mathbb{N}
\]  
\tag{22b}

Replacing (6c) by (22a) and (8d) by (22b), the relaxed feasible region of the BIM and BFM is denoted as \( \bar{\Gamma}_{BIM} \) and \( \bar{\Gamma}_{BFM} \), respectively. If Theorem 1 holds, we will prove that the function \( \mathcal{F} \) in (7) is also a bijection from \( \bar{\Gamma}_{BIM} \) to \( \bar{\Gamma}_{BFM} \).

**Proof:** Suppose \( \forall \left( u_i, u_j, R_{ij}, T_j, P_{ij}^r, Q_{ij}^r \right) \), \( \forall i \in \mathbb{N} \), \( \forall j \in \mathbb{N} \), and its image solution is \( \left( v_i, v_j, P_{ij}^c, Q_{ij}^c, P_{ij}^r, Q_{ij}^r \right) \), obtained by (9). According to (1-5), it suggests that (6a), (6b), (6c) and (6e) are still satisfied. As for the relaxed inequalities (22), if (22a) holds, we have \( R_{ij}^2 + T_{ij}^2 \leq u_i u_j \). According to (4), it gives

\[
\bar{P}_{ij}^2 - \bar{Q}_{ij}^2 - l_v v_i \bar{R}_{ij} - \bar{Q}_{ij} \bar{T}_{ij} \leq u_i u_j
\]  
\tag{23}

The constraint \( R_{ij}^2 + T_{ij}^2 \leq u_i u_j \) leads to \( \bar{P}_{ij}^2 + \bar{Q}_{ij}^2 - l_v v_i \leq 0 \) and the image solution satisfies (22b). Therefore, the image solution belongs to \( \bar{\Gamma}_{BFM} \). \( \mathcal{F} \) is a \( \bar{\Gamma}_{BIM} \rightarrow \bar{\Gamma}_{BFM} \) linear bijection and the two relaxed models are equivalent.

**IV. NUMERICAL SIMULATIONS**

An example two-bus system with three sets of branch impedances is depicted in Fig. 2, where Bus 1 is the reference bus with the voltage being 1∠0° and Bus 2 is connected to a microgrid that can be operated in an island. The controllable power \( \left( p_{ij}^c, q_{ij}^c \right) \) is equal to the branch flow \( \left( P_{ij}, Q_{ij} \right) \) since the shunt impedance is ignored. Three cases are considered (C1, C2, and C3). C1 is a normal case; C2 contains the zero impedance; and C3 is disconnected since the impedance is infinite. The power flow of the three cases is shown in Table I.
Fig. 2. A simple two-bus example.

For C1, $P_1=0.0205, Q_1=0.0111, R_1=0.9573, T_1=0.0300$ and $P_2=0.0205, Q_2=0.0111, 0.0205, 0.0111$. The linear bijection between $\Gamma_{BIM}$ and $\Gamma_{BFM}$ in (18) and (20) is satisfied.

For C2, the impedance of the branch is zero and the shunt is neglected, so the voltage of Bus 2 is equal to Bus 1, i.e., $R_2=1$, $T_2=0$ and the branch flow $(P_2, Q_2)$ should be equal to the power injection of Bus 2, yielding $(P_2=p_1^i, Q_2=q_1^i)$. At this time, the injection $\Gamma_{BFM} \rightarrow \Gamma_{BIM}$ is (21), but the surjection is not unique, which can be expressed as (24). Choosing different $\alpha$ and $\beta$ yields different image solutions.

$$P_1=\alpha R_1+\beta T_1, \quad Q_1=\alpha R_2+\beta T_1, \quad \forall \alpha, \beta $$ (24)

For C3, the branch is opened and there is no power flow on the branch, such that $(P_1=0, Q_1=0)$. Meanwhile, Bus 2 is in the microgrid and the voltage magnitude is determined by the island (i.e., any possible value). Thus, the injection $\Gamma_{BIM} \rightarrow \Gamma_{BFM}$ is (21a), but the surjection is not unique, which can be expressed as (25). Choosing different voltage values of Bus 2 yields different $(R_2, T_2)$ and therefore different image solutions.

$$R_2=\alpha v_2+\beta T_2, \quad T_2=\sqrt{1-\alpha^2 v_2+\beta^2 T_2}, \quad \forall \alpha, \beta, \gamma $$ (25)

Table I. Power flow solutions under different parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Power Flow Solutions (p.u.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{12}$</td>
<td>$x_{12}$</td>
</tr>
<tr>
<td>C1</td>
<td>1</td>
</tr>
<tr>
<td>C2</td>
<td>0</td>
</tr>
<tr>
<td>C3</td>
<td>inf</td>
</tr>
</tbody>
</table>

V. CONCLUSIONS

This paper aims to investigate the condition on the equivalence between the BIM and BFM in radial systems. It has been demonstrated that if the network is connected and there is no zero-impedance, the bijection exists and the two models are strictly equivalent under both normal and relaxation. Moreover, it has also been indicated that if the condition does not hold, only the injection exists and the two formulations are not equivalent anymore.

REFERENCES