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Electrodynamics with magnetic monopoles: Photon wave mechanical theory

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The microscopic Maxwell-Lorentz equations show an intrinsic symmetry if magnetic charge and current densities are included. Starting from the symmetrized set of field equations a propagator approach is used to describe the transverse (photon) electrodynamics in space-time. Two dyadic propagators are needed: (i) a transverse electric propagator with a near-field part and (ii) a magnetic propagator with a midfield part. The first quantized photon wave mechanical theory, based on the analytical parts of the two transverse Riemann-Silberstein vectors, is extended to include magnetic monopole dynamics. The dynamical equations for the photon helicity eigenvectors and the local energy conservation in the photon field are discussed. The Dirac string concept and the fiber bundle approach are avoided using the double-potential formalism of Cabibbo and Ferrari. The transverse (T) parts of the electric ($\mathbf{A}_T^e$) and magnetic ($\mathbf{A}_T^m$) vector potentials relate in a simple manner to our propagator theory if this is formulated in terms of the Huygens scalar propagator and the transverse electric and magnetic current densities. A transformation of the transverse vector potentials, which is nonlocal in space and time, allows one to express the transverse parts of the electric and magnetic fields as curls of combinations of the original ($\mathbf{A}_T^e$, $\mathbf{A}_T^m$) and the transformed ($\boldsymbol{\mathcal{A}}_T^e$, $\boldsymbol{\mathcal{A}}_T^m$) transverse vector potentials. The momentum of the particle-photon system is discussed, and it is shown that the electromagnetic parts of the canonical electric (charge e) and magnetic (charge g) particle momenta are given by $e(\boldsymbol{A}_T^e + \boldsymbol{\mathcal{A}}_T^m)$ and $g(\boldsymbol{A}_T^m - \boldsymbol{\mathcal{A}}_T^e)$, respectively. The angular momentum of the particle-photon system is analyzed and contact is made to the well-known nonretarded Saha-Wilson orbital angular momentum for an (e, g) pair. The near field of a magnetic monopole is studied based on the overlooked fact that the magnetic near field contains both longitudinal (L) (with $\nabla \cdot \mathbf{B}_L = 0$) and transverse *vector field* components. The sum of the two parts always is Einstein retarded, with proper account of the limitation caused by the lack of complete spatial photon localization. Finally, the relativistic electron-photon Hamiltonian in an external (prescribed) magnetic monopole field is discussed, paying particular attention to a determination of the transformed transverse magnetic vector potential, $\boldsymbol{\mathcal{A}}_T^m$, and its positive-frequency part.

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I. INTRODUCTION

In 1929 Weyl [1] showed that there exists a specific connection between the nonintegrability of a space- and time-dependent phase of a charged particle wave function and the vector and the scalar potential of the electromagnetic field. In modern notation, the gauge invariance of the potential (Lorenz) description of the Maxwell-Lorentz equations with a gauge function χ remains an invariance in quantum mechanics provided a unitary transformation $\exp[i(q/\hbar)\chi]$ is made on the particle (charge q) wave function In 1931, Dirac [2] introduced the hypothesis of a new particle, the magnetic monopole, and argued that if the new particle were to fit into the conventional quantum mechanics of an electron the product of the electron (e) and magnetic monopole (g) charges must satisfy a quantization condition. In 1948, Dirac [3] generalized his theory. The generalization was made possible by supposing the magnetic monopole to be at the end of a semi-infinite (unphysical) string of magnetic dipoles (a magnetic flux line). The string was needed in order to deal with the singular magnetic field of a point monopole.

In the wake of Dirac's renowned work the interest in magnetic monopole's possible structure [4,5] and their electrodynamics [6,7] has waxed and waned but never fallen to zero, despite the fact that there is no experimental evidence of the monopole's existence. In many studies the problem of an electrically charged Dirac particle in the field of a fixed magnetic monopole has been examined, often regarding the monopole vector potential as a connection on a fiber bundle [8–11]. The two different parametrizations of the vector potential imply that the wave function must be regarded as a section rather than an ordinary wave function. The fiber bundle approach replaces the Dirac string concept [12,13].

A semiclassical derivation of the eg-quantization condition was given by Saha [14,15] and Wilson [16]. The somewhat obscure relation of the Saha-Wilson argument to Dirac's was clarified by Goldhaber [8].

In an important article Cabibbo and Ferrari [17] showed that it was possible to avoid the pathological string concept (singular vector potential) by means of a double-potential formalism.

In previous studies of the electric-magnetic monopole interaction, the role of the photons, always intermediating the interaction, has not been addressed (the charge renormalization of the electric and magnetic charge *has* been examined

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by Schwinger using quantum-field theory [18,19]). Since the electron usually is assumed to be in the electromagnetic near field ($\sim r^{-2}$ field) of the fixed magnetic monopole, longitudinal and scalar photons are responsible for the dominating part of the interaction [20]. This implies that the need for a rigorous photon description often is less important in practice. However, the conceptual understanding of the near-field interaction is important, yet quite complicated in the photon picture [21].

The particular aspects of the present paper are the following: (i) a transverse propagator formalism of photon wave mechanics in symmetrized electrodynamics, (ii) an extension of the double-potential formalism which, via a special space-time nonlocal transformation of the vector potentials, allows one to uphold a minimal coupling principle free of singularities in quantum electrodynamics, (iii) a study of the structure of the momentum and angular momentum of the particle-photon system in symmetrized electrodynamics, emphasizing in particular the change in the electromagnetic part of the canonical electron momentum caused by the presence of a magnetic monopole in the electron's near-field zone, (iv) a calculation of the near field of a dynamic magnetic monopole, which shows that this field is the sum of an Einsteinian retarded part and a self-field part related to the lack of complete spatial photon confinement, and (v) an examination of the relativistic electron-photon Hamiltonian in a prescribed (external) magnetic monopole field.

The present paper is organized in the following manner. In Sec. II A, a few aspects of the well-known microscopic Maxwell-Lorentz equations [22,23] extended to allow for the presence of microscopic magnetic charges and current densities are summarized in both three-vector and covariant notation. Thereafter, the various vector fields are divided into their longitudinal (L) and transverse (T) parts [24]. Although this division is not relativistically invariant, it is of crucial importance for the subsequent part of this work. The L-T division is briefly discussed in Secs. II B and II C. In Sec. II D, an electromagnetic propagator formalism is established for the transverse electrodynamics and the spatial photon localization problem is discussed [25]. The propagator description is particularly convenient as the individual terms in the integral equations can be related separately to the transverse parts of the electric and magnetic vector potentials. In Sec. III the theory of photon wave mechanics in first quantization [25–28] is extended to include the presence of magnetic monopole dynamics. The first-quantized formalism can be extended to second quantization as described in Ref. [21]. In Sec. III A, the employed energy wave-function approach for transverse photons is presented and, in Secs. III B and III C, the dynamical equations for the photon helicity eigenvectors and the local law of energy flow in the photon field is discussed. In Sec. IV A, the double-potential formalism of Cabibbo and Ferrari [17] is briefly summarized and, in Sec. IV B, a space-time nonlocal transformation of the transverse electric and magnetic vector potentials is introduced. The transformation allows one to uphold the usual minimal coupling substitution without singularities. In Secs. V A–V D, we analyze the momentum of the coupled photon-particle ($E + M$) system, starting from the expression given for the total field momentum in the extended Maxwell-Lorentz theory. In particular, we identify the electromagnetic parts of the canonical particle momenta for electric and magnetic monopoles (Sec. V C), and discuss the photon-field momentum in the first-quantized theory. In Secs. VI A–VI D, we extend the Saha-Wilson approach for the angular momentum originating in the combined longitudinal parts of the electric and magnetic fields [14–16]. The field parts of the canonical particle angular momentum are identified (Sec. VI C), and an integral expression for the angular momentum of the photon field given in the terms of the photon energy wave function is derived (Sec. VI D). In Sec. VII we discuss the near field of a magnetic monopole, paying particular attention to the roles of longitudinal and scalar photons, the importance of the Einsteinian retarded transverse photon dynamics, and the limitation put on the spatial localization of transverse photons. Disregarding the spatial photon localization, it is shown that the total magnetic field always propagates in an Einstein-retarded manner. The conceptional problem of the Dirac approach is discussed. In Sec. VIII, we discuss the relativistic electron-photon Hamiltonian as this appears in an external (prescribed) magnetic monopole field. The main goal of the section is an explicit determination of the nonlocally transformed transverse magnetic vector potential ($\boldsymbol{\mathcal{A}}_T^m$), needed in the canonical magnetic monopole momentum. The established integral expression for $\boldsymbol{\mathcal{A}}_T^m$ over the $(\omega, \mathbf{q})$ domain allows a direct extraction of its analytical (positive-frequency) part. This part enters the radiation Hamiltonian. In Sec. IX, concluding remarks are given.

II. SYMMETRIZED ELECTRODYNAMICS AND TRANSVERSE-PROPAGATOR FORMALISM

A. Extended Maxwell-Lorentz equations

If magnetic monopoles exist, the Maxwell-Lorentz equations for the microscopic electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) field must be extended to the form [$c = (\varepsilon_0\mu_0)^{-1/2}$ being the vacuum speed of light]

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho^e, \tag{1}$$

$$\nabla \cdot \mathbf{B} = \frac{1}{c\varepsilon_0}\rho^m, \tag{2}$$

$$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + c\mu_0 \mathbf{J}^m, \tag{3}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}^e, \tag{4}$$

where ρ^e and ρ^m are the microscopic electric (e) and magnetic (m) charge densities and $\mathbf{J}^e$ and $\mathbf{J}^m$ are the related current densities. The prefactors to ρ^m and $\mathbf{J}^m$ have been chosen such that the electric and the magnetic charge and current densities have the same dimensionality. As a consequence of Eqs. (1) and (4), and Eqs. (2) and (3), the electric and magnetic charges satisfy *separate* equations of continuity, i.e.,

$$\nabla \cdot \mathbf{J}^e + \frac{\partial \rho^e}{\partial t} = 0, \tag{5}$$

$$\nabla \cdot \mathbf{J}^m + \frac{\partial \rho^m}{\partial t} = 0. \tag{6}$$

The set of extended field equations are *form invariant* under the duality transformation $[(\mathbf{E},\mathbf{B}) \leftrightarrow (\mathbf{E}_{\mathrm{NEW}},\mathbf{B}_{\mathrm{NEW}})]$ through the arbitrary (pseudorotation) angle Θ, viz.,

$$\mathbf{E}_{\mathrm{NEW}} = \mathbf{E}\cos\Theta + c\mathbf{B}\sin\Theta, \tag{7}$$

$$\mathbf{B}_{\mathrm{NEW}} = \mathbf{B}\cos\Theta - \frac{1}{c}\mathbf{E}\sin\Theta, \tag{8}$$

$$\rho^e_{\mathrm{NEW}} = \rho^e\cos\Theta + \rho^m\sin\Theta, \tag{9}$$

$$\rho^m_{\mathrm{NEW}} = \rho^m\cos\Theta - \rho^e\sin\Theta, \tag{10}$$

$$\mathbf{J}^e_{\mathrm{NEW}} = \mathbf{J}^e\cos\Theta + \mathbf{J}^m\sin\Theta, \tag{11}$$

$$\mathbf{J}^m_{\mathrm{NEW}} = \mathbf{J}^m\cos\Theta - \mathbf{J}^e\sin\Theta. \tag{12}$$

The form of Eqs. (9) and (10) show that in symmetrized electrodynamics it is a matter of convention to speak of a particle possessing an electric charge, but not a magnetic charge (or *vice versa*). However, *if* the magnetic and the electric charge densities coexist in a universal ratio K (synonymous with the demand that *all* particles have the same ratio of magnetic and electric charge), indicated by

$$\frac{\rho^m}{\rho^e} = K, \tag{13}$$

the choice $\Theta = \arctan K$ in the duality transformation leads to $\rho^m_{\mathrm{NEW}} = 0$. In consequence the set of extended Maxwell-Lorentz equations is reduced to the form usually known (accepted).

In passing it may be mentioned that the extended field equations have proven to be of value in classical vectorial diffraction theory even though the magnetic charge and current densities are fictitious quantities in this case. Thus, for the (ideal) special case of diffraction from an infinitely thin perfectly conducting plane metal screen, S, with an aperture (a hole), A, a duality transformation with $\Theta = \pi/2$ is of particular interest. The new (dual) electromagnetic field $(\mathbf{E}_{\mathrm{NEW}},\mathbf{B}_{\mathrm{NEW}}) = (c\mathbf{B}, -\mathbf{E}/c)$ is called the complementary field in diffraction theory. The name originates in the fact that a rigorous form of Babinet's principle may be established by comparing the original diffraction problem with incident field $(\mathbf{E}^0,\mathbf{B}^0)$ with the diffraction of the incident field $(\mathbf{E}^0_c,\mathbf{B}^0_c) = (c\mathbf{B}^0, -\mathbf{E}^0/c)$ from a "complimentary" screen, obtained by replacing the aperture by a screen ($A \to S$) and the screen by an aperture ($S \to A$). In the half-space behind the screens the original field $(\mathbf{E},\mathbf{B})$ and the complementary (subscript c) field $(\mathbf{E}_c,\mathbf{B}_c)$ are related according to $\mathbf{E} - c\mathbf{B}_c = \mathbf{E}^0$ and $\mathbf{B} + c\mathbf{E}_c = \mathbf{B}^0$ [29].

The central problem in classical diffraction from an aperture in an infinitely thin, perfectly conducting, plane screen is the determination of the field in the aperture region. Once this has been calculated (approximately, in general) the diffracted field everywhere in space is easily obtained. For the aperture field calculation it is useful to introduce a fictitious magnetic surface current density in the aperture domain [30,31].

In Eqs. (1)–(4) the set of Maxwell-Lorentz equations was expressed in three-vector notation. In covariant notation the symmetrized field equations are given in terms of the covariant antisymmetric field tensor

$$\{F_{\mu\nu}\} \equiv \frac{1}{c}\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{pmatrix}, \tag{14}$$

its dual partner

$$\{G_{\mu\nu}\} \equiv \{F_{\mu\nu}(\mathbf{E}\to c\mathbf{B}, \mathbf{B}\to -\mathbf{E}/c)\}$$
$$= \frac{1}{c}\begin{pmatrix} 0 & -cB_1 & -cB_2 & -cB_3 \\ cB_1 & 0 & -E_3 & E_2 \\ cB_2 & E_3 & 0 & -E_1 \\ cB_3 & -E_2 & E_1 & 0 \end{pmatrix}, \tag{15}$$

and the electric and magnetic covariant four-current densities $\{J^e_\nu\} = (-c\rho^e, \mathbf{J}^e)$, $\{J^m_\nu\} = (-c\rho^m, \mathbf{J}^m)$. Thus, with $\{\partial^\mu\} \equiv (-c^{-1}\partial/\partial t, \nabla)$ one obtains

$$\partial^\mu F_{\mu\nu} = -\mu_0 J^e_\nu, \quad \nu = 0\text{–}3, \tag{16}$$

$$\partial^\mu G_{\mu\nu} = -\mu_0 J^m_\nu, \quad \nu = 0\text{–}3. \tag{17}$$

Above the usual summation convention for repeated super- and subscripts, and a matrix tensor signature $(-1,1,1,1)$, have been employed. In the subsequent analysis the three-vector notation will appear particularly useful. In Sec. IV A a double-potential formalism with electric ($\{A^{e,\mu}\}$) and magnetic ($\{A^{m,\mu}\}$) four potentials is introduced. In terms of these potentials $F_{\mu\nu} = \partial_\nu A^e_\mu - \partial_\nu A^e_\mu$ and $G_{\mu\nu} = \partial_\nu A^m_\mu - \partial_\nu A^m_\mu$.

B. Longitudinal parts of the field equations: Dynamical particle position variables

For what follows it is convenient to divide the extended Maxwell-Lorentz equations into sets describing respectively the curl-free [called longitudinal (L)] and divergence-free [transverse (T)] electrodynamics. With the unique (up to a space-independent constant of no physical importance) separation $\mathbf{E} = \mathbf{E}_L + \mathbf{E}_T$, $\mathbf{B} = \mathbf{B}_L + \mathbf{B}_T$ of the fields and $\mathbf{J}^e = \mathbf{J}^e_L + \mathbf{J}^e_T$, $\mathbf{J}^m = \mathbf{J}^m_L + \mathbf{J}^m_T$ of the current densities, one obtains from Eqs. (1)–(4) the following set of field equations for the longitudinal dynamics:

$$\nabla\cdot\mathbf{E}_L = \frac{1}{\varepsilon_0}\rho^e, \tag{18}$$

$$c\nabla\cdot\mathbf{B}_L = \frac{1}{\varepsilon_0}\rho^m, \tag{19}$$

$$c\frac{\partial}{\partial t}\mathbf{B}_L = -\frac{1}{\varepsilon_0}\mathbf{J}^m_L, \tag{20}$$

$$\frac{\partial}{\partial t}\mathbf{E}_L = -\frac{1}{\varepsilon_0}\mathbf{J}^e_L. \tag{21}$$

It appears from these equations that $\mathbf{E}_L$ and $\mathbf{B}_L$ depend respectively on ρ^e and ρ^m alone. By combining Eqs. (18) and (21) [Eqs. (19) and (20)] one obtains the equation of continuity in Eq. (5) [Eq. (6)] for electric (magnetic) charge. In the absence of magnetic monopoles $\nabla\cdot\mathbf{B}_L (= \nabla\cdot\mathbf{B}) = 0$. In the framework of classical (extended) electrodynamics the electric and magnetic monopoles are assumed to be point particles. If one denotes the various (particle label: α) electric and magnetic monopole charges by e_α and g_α, respectively, and the positions vectors of these by $\mathbf{r}_\alpha^e(t)$ and $\mathbf{r}_\alpha^m(t)$, the four-current densities become

$$\begin{aligned}\{J^{e,\mu}\} &\equiv [c\rho^e(\mathbf{r},t),\mathbf{J}^e(\mathbf{r},t)]\\ &= \sum_\alpha e_\alpha\big(c,\dot{\mathbf{r}}_\alpha^e(t)\big)\delta\big(\mathbf{r}-\mathbf{r}_\alpha^e(t)\big)\end{aligned} \tag{22}$$

and

$$\begin{aligned}\{J^{m,\mu}\} &\equiv [c\rho^m(\mathbf{r},t),\mathbf{J}^m(\mathbf{r},t)]\\ &= \sum_\alpha g_\alpha\big(c,\dot{\mathbf{r}}_\alpha^m(t)\big)\delta\big(\mathbf{r}-\mathbf{r}_\alpha^m(t)\big),\end{aligned} \tag{23}$$

where δ is the Dirac delta function. From Eqs. (18) and (19) and the first component of Eqs. (22) and (23) one obtains the following expressions for the longitudinal fields:

$$\mathbf{E}_L(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\sum_\alpha e_\alpha\frac{\mathbf{r}-\mathbf{r}_\alpha^e(t)}{\left|\mathbf{r}-\mathbf{r}_\alpha^e(t)\right|^3}, \tag{24}$$

$$\mathbf{B}_L(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0 c}\sum_\alpha g_\alpha\frac{\mathbf{r}-\mathbf{r}_\alpha^m(t)}{\left|\mathbf{r}-\mathbf{r}_\alpha^m(t)\right|^3}. \tag{25}$$

From these well-known results it appears that the longitudinal fields are just the instantaneous Coulomb (electric and magnetic) fields of the charge-density distributions. The elimination of the longitudinal field variables in favor of the particle position coordinates implies that $\mathbf{E}_L$ and $\mathbf{B}_L$ in a quantum-mechanical context are hidden in the (extended) Dirac (or Schrödinger) equation: the particles do not "see" their own longitudinal fields. The fact that

$$\nabla\cdot\mathbf{B}_L(\mathbf{r},t) = 0, \quad \mathbf{r}\neq\mathbf{r}_\alpha^m(t), \tag{26}$$

except at the positions of the magnetic charges (as indicated) where $\mathbf{B}_L$ is infinite, made it possible for Dirac to establish a certain quantum-mechanical description of the electrodynamics of an electron in the presence of a magnetic monopole. In order to keep the interaction Hamiltonian in its standard (minimal coupling) form the *physical* singularity in the vector potential $\mathbf{A}$ ($\mathbf{B}=\nabla\times\mathbf{A}$) at the position of the monopole Dirac handled by his now famous (Dirac) string concept [3].

The singularity in the Dirac point-particle model of the magnetic monopole is avoided in the 't Hooft–Polyakov theory from 1974 [4,5]. In this theory so-called hedgehog solutions were obtained. Such solutions are "lumps" of a quantum field with finite size everywhere. If a lump is small enough, it appears as a pointlike magnetic monopole [32]. In the 't Hooft–Polyakov description the Maxwell electrodynamics is extended to contain a gauge field and an isovector Higgs field [12,33–38]. The 't Hooft–Polyakov theory carries electric charge only, but viewed asymptotically the hedgehog solution has a radial magnetic field, corresponding to the presence of a magnetic monopole. The usual definition of the electromagnetic-field tensor (components) is replaced by

$$F_{\mu\nu}^{\text{ext}} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e|\phi|^3}\epsilon_{abc}\phi^a(\partial_\mu\phi^b)(\partial_\nu\phi^c), \tag{27}$$

where $\{\phi^a\}$ is the isovector Higgs field ($a=1-3$) and ϵ_{abc} is the completely antisymmetric Levi-Civita tensor. The gauge potential $\{A_\mu\}=\phi^a A_\mu^a/|\phi|$ represents a generalization of the electric vector potential appearing in the potential formulation of the standard Maxwell-Lorentz theory. When $\{\phi\}$ becomes fixed in isospace, $\{F_{\mu\nu}^{\text{ext}}\}$ reduces the $\{F_{\mu\nu}\}$ of Eq. (14), with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $A_\mu\equiv A_\mu^F$, F being the index for the fixed (F) isovector Higgs field. In the t' Hooft–Polyakov theory the responsibility for the monopole is transferred from the first (Dirac) part to the second (Higgs) one [Eq. (27)]. In a forthcoming paper, I shall show that a photon wave mechanical theory based on potentials [21] may be extended to include the isovector Higgs field.

A particle carrying both electric and magnetic charge (dual-charged particle) was named a dyon by Schwinger [6] and first studied in 1968 [39]. Work on a photon wave mechanical theory for dyons (with extension to the Higgs field) is in progress.

C. Transverse parts of the field equations

Although the division of the vector fields into T and L parts is not relativistically invariant the disadvantage of not retaining the manifest covariance of the field most often in photon and nonrelativistic electrodynamics is compensated for by simplicity, e.g., in the canonical quantization procedure raising photon wave mechanics to the second-quantized level.

The transverse electrodynamics is governed by the transverse part of the Maxwell-Lorentz equations in Eqs. (3) and (4). Thus

$$-\nabla\times\mathbf{E}_T = \frac{\partial}{\partial t}\mathbf{B}_T + c\mu_0\mathbf{J}_T^m, \tag{28}$$

$$\nabla\times\mathbf{B}_T = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}_T + \mu_0\mathbf{J}_T^e. \tag{29}$$

For the subsequent theoretical development it is important to understand that the transverse (longitudinal) part of a given current density is related in a spatially nonlocal but timely local manner to the current density itself, viz.,

$$\mathbf{J}_{T,L}(\mathbf{r},t) = \int_{-\infty}^{\infty}\boldsymbol{\delta}_{T,L}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}',t)d^3r', \tag{30}$$

where $\boldsymbol{\delta}_T$ and $\boldsymbol{\delta}_T$ are the transverse and longitudinal delta functions, tensorial quantities (in dyadic representation) satisfying the relation

$$\boldsymbol{\delta}_T(\mathbf{R}) + \boldsymbol{\delta}_L(\mathbf{R}) = \mathbf{U}\delta(\mathbf{R}), \tag{31}$$

$\mathbf{U}$ being the 3×3 unit tensor. In Eq. (30) the integration is over all space.

D. Propagator approach: Spatial field localization

A combination of the transverse set of extended Maxwell-Lorentz equations [Eqs. (28) and (29)] leads to the following wave equations for $\mathbf{E}_T$ and $c\mathbf{B}_T$:

$$\Box\mathbf{E}_T = \mu_0\frac{\partial}{\partial t}\mathbf{J}_T^e + c\mu_0\nabla\times\mathbf{J}_T^m, \tag{32}$$

$$\Box(c\mathbf{B}_T) = \mu_0\frac{\partial}{\partial t}\mathbf{J}_T^m - c\mu_0\nabla\times\mathbf{J}_T^e, \tag{33}$$

where $\Box=\nabla^2-c^{-2}\partial^2/\partial t^2$ is the d'Lambertian operator. The physically acceptable general solution of these equations can be obtained using the retarded (outgoing) Huygens scalar propagator

$$g(R,\tau)=\frac{1}{4\pi R}\delta\left(\frac{R}{c}-\tau\right) \tag{34}$$

in a scattering theory approach. Thus

$$\mathbf{E}_T(\mathbf{r},t)=\mathbf{E}_T^0(\mathbf{r},t)-\mu_0\int_{-\infty}^{\infty}g(R,\tau)\times\left[\frac{\partial}{\partial t'}\mathbf{J}_T^e(\mathbf{r}',t')+c\nabla'\times\mathbf{J}^m(\mathbf{r}',t')\right]d^3r'dt', \tag{35}$$

$$c\mathbf{B}_T(\mathbf{r},t)=c\mathbf{B}_T^0(\mathbf{r},t)-\mu_0\int_{-\infty}^{\infty}g(R,\tau)\times\left[\frac{\partial}{\partial t'}\mathbf{J}_T^m(\mathbf{r}',t')-c\nabla'\times\mathbf{J}^e(\mathbf{r}',t')\right]d^3r'dt', \tag{36}$$

where $\mathbf{E}_T^0$ and $\mathbf{B}_T^0$ are the electric and magnetic components of the incoming field, and $\mathbf{R}=\mathbf{r}-\mathbf{r}'$ ($R=|\mathbf{R}|$) and $\tau=t-t'$. In writing down the solutions we have used that $\nabla\times\mathbf{J}_T=\nabla\times\mathbf{J}$ ($\mathbf{J}=\mathbf{J}^e$ or $\mathbf{J}^m$). The integrations in Eqs. (35) and (36) are over all space and time.

For what follows it is useful to write Eqs. (35) and (36) in a form involving in the integrals only time derivatives of the relevant current densities. The new form also makes the limitation in the possibility for precise photon localization in space manifest. We reach our goal by making use of the following well-known results [40]:

$$\int_{-\infty}^{\infty}g(R,\tau)\frac{\partial}{\partial t'}\mathbf{J}_T(\mathbf{r}',t')d^3r'=\int_{-\infty}^{\infty}\mathbf{G}_T(\mathbf{R},\tau)\cdot\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')d^3r'+\mathbf{W}_T^{SF}(\mathbf{r},t) \tag{37}$$

and

$$\int_{-\infty}^{\infty}g(R,\tau)\nabla'\times\mathbf{J}(\mathbf{r}',t')d^3r'=-\frac{1}{c}\int_{-\infty}^{\infty}\mathbf{G}_M(\mathbf{R},\tau)\cdot\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')d^3r', \tag{38}$$

where $\mathbf{J}=\mathbf{J}^e$ or $\mathbf{J}^m$. On the right sides of these equations appear the transverse (subscript T) (electric) field propagator, $\mathbf{G}_T(\mathbf{R},\tau)$, and the magnetic (subscript M) field propagator, $\mathbf{G}_M(\mathbf{R},\tau)$, both tensorial quantities. The vector $\mathbf{W}_T^{SF}(\mathbf{r},t)$ represents the transverse self-field (superscript SF) contribution to the scattered (electric or magnetic) field. By utilizing the two relations above, Eqs. (35) and (36) can be rewritten as follows:

$$\mathbf{E}_T(\mathbf{r},t)=\mathbf{E}_T^0(\mathbf{r},t)+\mathbf{E}_T^{SF}(\mathbf{r},t)-\mu_0\int_{-\infty}^{\infty}\left[\mathbf{G}_T(\mathbf{R},\tau)\times\frac{\partial}{\partial t'}\mathbf{J}^e(\mathbf{r}',t')-\mathbf{G}_M(\mathbf{R},\tau)\cdot\frac{\partial}{\partial t'}\mathbf{J}^m(\mathbf{r}',t')\right]d^3r'dt' \tag{39}$$

and

$$c\mathbf{B}_T(\mathbf{r},t)=c\mathbf{B}_T^0(\mathbf{r},t)+c\mathbf{B}_T^{SF}(\mathbf{r},t)-\mu_0\int_{-\infty}^{\infty}\left[\mathbf{G}_T(\mathbf{R},\tau)\cdot\frac{\partial}{\partial t'}\mathbf{J}^m(\mathbf{r}',t')+\mathbf{G}_M(\mathbf{R},\tau)\cdot\frac{\partial}{\partial t'}\mathbf{J}^e(\mathbf{r}',t')\right]d^3r'dt'. \tag{40}$$

In spherical coordinates centered on the singular point $\mathbf{R}=\mathbf{0}$ the explicit expression for $\mathbf{G}_T(\mathbf{R},\tau)$ and $\mathbf{G}_M(\mathbf{R},\tau)$ are particularly simple, viz. [40],

$$\mathbf{G}_T(\mathbf{R},\tau)=\frac{1}{4\pi R}\delta\left(\frac{R}{c}-\tau\right)(\mathbf{U}-\mathbf{e_R}\mathbf{e_R})-\frac{c^2\tau}{4\pi R^3}\Theta(\tau)\Theta\left(\frac{R}{c}-\tau\right)(\mathbf{U}-3\mathbf{e_R}\mathbf{e_R}) \tag{41}$$

and

$$\mathbf{G}_M(\mathbf{R},\tau)=\left[\frac{1}{4\pi R}\delta\left(\frac{R}{c}-\tau\right)+\frac{c}{4\pi R^2}\Theta\left(\tau-\frac{R}{C}\right)\right]\mathbf{U}\times\mathbf{e_R}, \tag{42}$$

where Θ is the Heaviside unit step function and $\mathbf{e_R}=\mathbf{R}/R$ is a unit vector in the radial direction. The $\mathbf{G}_M$ tensor only behaves as a genuine magnetic *propagator* in the absence of persistent electric and magnetic currents [25]. Such currents are of no interest in this work, and we thus assume they are absent. The physics hidden in the near (R^{-3}) and mid (R^{-2}) -field parts of the electric and magnetic propagators are discussed in detail elsewhere [40] and not to be repeated here. The terms proportional to $\delta(R/c-\tau)$ relate to the far-field (R^{-1}) parts of the electric and magnetic fields.

The transverse self-field contribution, $\mathbf{W}_T^{SF}(\mathbf{r},t)$, depends on the contraction geometry used in the propagator formalism around the singular point [40]. In the spherical contraction scheme one has

$$\mathbf{E}_T^{SF}(\mathbf{r},t)=-\frac{\mathbf{P}_T^e(\mathbf{r},t)}{3\varepsilon_0}, \tag{43}$$

$$c\mathbf{B}_T^{SF}(\mathbf{r},t)=-\frac{\mathbf{P}_T^m(\mathbf{r},t)}{3\varepsilon_0}, \tag{44}$$

where $\mathbf{P}_T$ (with superscript e or m) is the transverse part of a generalized polarizability, $\mathbf{P}$, related to the current density via the definition

$$\mathbf{J}(\mathbf{r},t)\equiv\frac{\partial\mathbf{P}(\mathbf{r},\mathbf{t})}{\partial t}. \tag{45}$$

The self-fields are nonvanishing only in the so-called rim (near-field) zones of the electric and magnetic current density distributions. For a point source (located at $\mathbf{r}'$) the rim zone has an extension given by the $|\mathbf{r}-\mathbf{r}'|^{-3}$ tail; cf. the explicit expression for $\boldsymbol{\delta}_T$ (or $\boldsymbol{\delta}_L$) in spherical contraction [40]. The extension of the rim zone represents the initial (best possible) spatial confinement of a photon emitted from a given source. The rim zone is of utmost importance for understanding the field-matter momentum problem in the presence of magnetic monopoles, and for our suggested manner of introducing generalized canonical particle momenta for electric and magnetic monopoles in near-field contact; see Sec. V C.

III. PHOTON WAVE MECHANICS IN FIRST QUANTIZATION

A. Energy wave-function approach

The expressions given for $\mathbf{E}_T(\mathbf{r},t)$ and $c\mathbf{B}_T(\mathbf{r},t)$ in Eqs. (35) and (36) form the basis for a propagator description of photon energy wave mechanics [21,25,27] in the so-called photon perspective [28]. In this perspective the photon source is identified with (the time derivatives of) the transverse vector-field parts, $\mathbf{J}_T^e(\mathbf{r},t)$ and $\mathbf{J}_T^m(\mathbf{r},t)$, of the electric and magnetic current densities distributions. Starting from the two transverse Riemann-Silberstein vectors

$$\mathbf{F}_\pm(\mathbf{r},t)=\sqrt{\frac{\varepsilon_0}{2}}[\mathbf{E}_T(\mathbf{r},t)\pm ic\mathbf{B}_T(\mathbf{r},t)], \tag{46}$$

which relate to electromagnetic fields composed of positive ($\mathbf{F}_+$) and negative ($\mathbf{F}_-$) helicity species, one projects out their analytical (positive-frequency) parts [indicated by superscript (+) below], i.e.,

$$\mathbf{F}_\pm^{(+)}(\mathbf{r},t)=\int_0^\infty \mathbf{F}_\pm(\mathbf{r};\omega)e^{-i\omega t}\frac{d\omega}{2\pi}, \tag{47}$$

where the $\mathbf{F}_\pm(\mathbf{r};\omega)$'s are the Fourier integral transforms of the $\mathbf{F}_\pm(\mathbf{r},t)$'s. In the electromagnetic interaction of an incoming photon field with electric and magnetic charge distributions, we subsequently characterize (in spinorial notation) the state of the photon field by the six-component object

$$\mathbf{\Psi}(\mathbf{r},t)=\begin{pmatrix}\mathbf{F}_+^{(+)}(\mathbf{r},t)\\ \mathbf{F}_-^{(+)}(\mathbf{r},t)\end{pmatrix}. \tag{48}$$

The name Riemann-Silberstein vector has its roots in Riemann and Silberstein's cast of the Maxwell equations into complex form; see Bateman [41] (and also Refs. [25] and [27]).

To underline the close relationship between the wave mechanical theories of massive and massless elementary particles photon wave functions are connected to the positive-frequency part of the electromagnetic field. The negative-frequency part of the spectrum then relates to antiphotons. Since the photon and its antiparticle are identical (because the information carried by the positive and negative frequencies in the Maxwell-Lorentz equations are identical) there is of course no absolute need for connecting a photon wave mechanical formalism alone to the positive-frequency part of the electromagnetic spectrum.

In our scattering theory approach a prescribed incident free-photon wave function,

$$\mathbf{\Phi}^0(\mathbf{r},t)=\begin{pmatrix}\mathbf{F}_+^{0(+)}(\mathbf{r},t)\\ \mathbf{F}_-^{0(+)}(\mathbf{r},t)\end{pmatrix}=\sqrt{\frac{\varepsilon_0}{2}}\begin{pmatrix}\mathbf{E}_T^{0(+)}(\mathbf{r},t)+ic\mathbf{B}_T^{0(+)}(\mathbf{r},t)\\ \mathbf{E}_T^{0(+)}(\mathbf{r},t)-ic\mathbf{B}_T^{0(+)}(\mathbf{r},t)\end{pmatrix}, \tag{49}$$

gives rise to a scattered (superscript s) photon state

$$\mathbf{\Psi}^s(\mathbf{r},t)=\mathbf{\Psi}(\mathbf{r},t)-\mathbf{\Phi}^0(\mathbf{r},t) \tag{50}$$

$$=\begin{pmatrix}\mathbf{F}_+^{s(+)}(\mathbf{r},t)\\ \mathbf{F}_-^{s(+)}(\mathbf{r},t)\end{pmatrix}. \tag{51}$$

The explicit expressions for the analytic Riemann-Silberstein vectors entering $\mathbf{\Psi}^s(\mathbf{r},t)$ are readily obtained from the analytical parts of Eqs. (35) and (36). Hence

$$\mathbf{F}_\pm^{s(+)}(\mathbf{r},t)=-\mu_0\sqrt{\frac{\varepsilon_0}{2}}\int_{-\infty}^{\infty} g(R,\tau)\times\left(\frac{\partial}{\partial t'}\mp i\nabla'\times\right)\mathcal{J}_\pm^{(+)}(\mathbf{r}',t')d^3r'dt', \tag{52}$$

where

$$\mathcal{J}_\pm^{(+)}(\mathbf{r},t)=\int_0^\infty[\mathbf{J}^e(\mathbf{r};\omega)\pm i\mathbf{J}^m(\mathbf{r};\omega)]e^{-i\omega t}\frac{d\omega}{2\pi}, \tag{53}$$

$\mathbf{J}^e(\mathbf{r};\omega)$ and $\mathbf{J}^m(\mathbf{r};\omega)$ being the Fourier integral transforms of $\mathbf{J}^e(\mathbf{r},t)$ and $\mathbf{J}^m(\mathbf{r},t)$. With the help of Eq. (38) it is possible to express Eq. (52) in the following compact form:

$$\mathbf{F}_\pm^{s(+)}(\mathbf{r},t)=-\mu_0\sqrt{\frac{\varepsilon_0}{2}}\int_{-\infty}^{\infty}\mathcal{G}_\pm(\mathbf{R},\tau)\cdot\frac{\partial}{\partial t'}\mathcal{J}_\pm^{(+)}(\mathbf{r}',t')d^3r'dt', \tag{54}$$

with generalized propagators

$$\mathcal{G}_\pm(\mathbf{R},\tau)=g(R,\tau)\mathbf{U}\pm i\mathbf{G}_M(\mathbf{R},\tau). \tag{55}$$

The result in Eq. (54) forms a good starting point for the second quantization of the photon energy wave function formalism in the Coulomb gauge, in analogy with the procedure employed in the absence of magnetic monopoles [28,42]. Perhaps, it is possible to establish a second-quantized theory starting from Eqs. (39) and (40). In such an approach it is the difference fields $\mathbf{E}_T-\mathbf{E}_T^{SF}$ and $\mathbf{B}_T-\mathbf{B}_T^{SF}$ which upon quantization relate to the photons. In the absence of magnetic monopoles ($\mathbf{B}_T^{SF}=0$) it *is* possible to establish such a theory. However, a change from the canonical field momentum operator $-\varepsilon_0\hat{\mathbf{E}}_T$ to a new $-\varepsilon_0(\hat{\mathbf{E}}_T-\hat{\mathbf{E}}_T^{SF})$ requires that the Columb gauge representation is replaced by the so-called G representation [42].

In the present scattering theory it is implicitly assumed that the induced current densities $\mathbf{J}^e(\mathbf{r},t)$ and $\mathbf{J}^m(\mathbf{r},t)$ both vanish outside the finite-time interval $(0,T)$.

Upon completion of the scattering process ($t>T$) the scattered photon state takes some (asymptotic) form, say

$$\mathbf{\Phi}^s(\mathbf{r},t)=\mathbf{\Psi}^s(\mathbf{r},t(>T)). \tag{56}$$

The final photon state

$$\mathbf{\Phi}(\mathbf{r},t)=\mathbf{\Phi}^0(\mathbf{r},t)+\mathbf{\Phi}^s(\mathbf{r},t) \tag{57}$$

as well as the incident photon state are free-photon states, which implies that they have constant energies (E and E_0), given in the photon energy wave function approach by

$$\int_{-\infty}^{\infty}[\mathbf{\Phi}^0(\mathbf{r},t)]^\dagger\cdot\mathbf{\Phi}^0(\mathbf{r},t)d^3r=E_0 \tag{58}$$

and

$$\int_{-\infty}^{\infty} \mathbf{\Phi}^{\dagger}(\mathbf{r},t)\cdot\mathbf{\Phi}(\mathbf{r},t)d^3r = E, \tag{59}$$

where † stands for Hermitian conjugation. In the first-quantized description the free-photon energy equals the energy in the classical electromagnetic field.

In the low-frequency quantum optical regime it is often possible in elastic scattering processes to assume (approximately) that the initial and the final photon state have the same energy. In such cases $E = E_0$, and a combination of Eqs. (58) and (59) then shows that

$$\int_{-\infty}^{\infty}[\mathbf{\Phi}^s(\mathbf{r},t)]^{\dagger}\cdot\mathbf{\Phi}^s(\mathbf{r},t)d^3r + \int_{-\infty}^{\infty}\{[\mathbf{\Phi}^0(\mathbf{r},t)]^{\dagger}\cdot\mathbf{\Phi}^s(\mathbf{r},t) + [\mathbf{\Phi}^s(\mathbf{r},t)]^{\dagger}\cdot\mathbf{\Phi}^0(\mathbf{r},t)\}d^3r = 0. \tag{60}$$

In principle the condition in Eq. (60) allows one to determine a common (up to this point unknown) overall amplitude of the prevailing generalized current density $\mathcal{J}_{\pm} = \mathbf{J}^e \pm i\mathbf{J}^m$. If the incoming part of the outgoing state can be neglected, the scattered state satisfies the condition $\int_{-\infty}^{\infty}(\mathbf{\Phi}^s)^{\dagger}\cdot\mathbf{\Phi}^s d^3r = E^0$. In Ref. [28] a model calculation of the amplitude of $\mathcal{J} \equiv \mathbf{J}^e(\mathbf{J}^m = \mathbf{0})$ for a single atom emitting a sinusoidal wave train of finite time length has been given.

B. Dynamical equations for the helicity eigenvectors

Combining Eqs. (28) and (29) it appears that the transverse Riemann-Silberstein vectors [Eq. (46)] satisfy the dynamical equations

$$i\hbar\frac{\partial}{\partial t}\mathbf{F}_{\pm}(\mathbf{r},t) = \pm c\hbar\nabla\times\mathbf{F}_{\pm}(\mathbf{r},t) - \frac{i\hbar}{\sqrt{2\varepsilon_0}}\left[\mathbf{J}_T^e(\mathbf{r},t) \pm i\mathbf{J}_T^m(\mathbf{r},t)\right], \tag{61}$$

where $\hbar$ is Planck's constant divided by 2π. In the framework of the first-qunatized theory, $\hbar$ is "just" a multiplicative constant. In the absence of magnetic monopoles, Eq. (61) is reduced to a well-known form, used previously as a starting point for studies of the interaction of photons with atoms [28], mesoscopic particles [40], and continuous (condensed-matter) media [43].

The vectorial character of $\mathbf{F}_{\pm}$ incorporates the spin ($\mathbf{s}$) of the photon (and antiphoton): a spin one can be represented by a vector in a three-dimensional complex vector space. The appearance of the photon spin in the dynamical equations is manifest when the analytical part of Eq. (61) is written in the form

$$i\hbar\frac{\partial}{\partial t}\mathbf{F}_{\pm}^{(+)}(\mathbf{r},t) = \pm c(\hat{\mathbf{\Sigma}}\cdot\hat{\mathbf{p}})\mathbf{F}_{\pm}^{(+)}(\mathbf{r},t) - \frac{i\hbar}{\sqrt{2\varepsilon_0}}\mathcal{J}_{\pm}^{(+)}(\mathbf{r},t), \tag{62}$$

where $\hat{\mathbf{\Sigma}} = \hat{\mathbf{s}}/\hbar$ is the dimensional-less spin-one operator of the photon [in the (3×3)-matrix representation the Cartesian components of $\hat{\mathbf{\Sigma}}$ are given by $(\hat{\Sigma}_k)_{ij} = \varepsilon_{ijk}/i$, where ε_{ijk} is the completely antisymmetric Levi-Civita tensor], and $\hat{\mathbf{p}} = (\hbar/i)\nabla$ is the momentum operator in the $\mathbf{r}$ representation. The scalar product $\hat{\mathbf{\Sigma}}\cdot\hat{\mathbf{p}}$ connects via

$$\hat{\mathbf{\Sigma}}\cdot\hat{\mathbf{p}} = \hat{p}\hat{h} \tag{63}$$

to the photon helicity operator $\hat{h}$ (the plane-wave components of $\mathbf{F}_{+}^{(+)}$ and $\mathbf{F}_{-}^{(+)}$ are eigenstates of $\hat{h}$ with eigenvalues +1 and −1, respectively). In $\mathbf{r}$ space

$$\hat{p} = (\hat{\mathbf{p}}\cdot\hat{\mathbf{p}})^{1/2} = \frac{\hbar}{i}\sqrt{\nabla^2} \tag{64}$$

in a symbolic notation dating back to the Landau-Peierls quantum theory of the photon [44]. The operator is defined via its action in $\mathbf{p}$ space and it can be shown that

$$\hat{h}\mathbf{F}_{\pm}^{(+)}(\mathbf{r},t) = \pm\mathbf{F}_{\pm}^{(+)}(\mathbf{r},t), \tag{65}$$

so that the two helicity species are eigenvectors of the helicity operator.

C. Local energy conservation in the photon field

From the dynamical equations for the two helicity species one can derive a continuity equation for the photon energy density

$$\mathbf{\Psi}^{\dagger}\cdot\mathbf{\Psi} = (\mathbf{F}_{+}^{(+)})^*\cdot\mathbf{F}_{+}^{(+)} + (\mathbf{F}_{-}^{(+)})^*\cdot\mathbf{F}_{-}^{(+)} = \sum_{s=+,-}(\mathbf{F}_s^{(+)})^*\cdot\mathbf{F}_s^{(+)}. \tag{66}$$

Thus an expression for the time derivative of the modulus squared of the wave function, viz.,

$$\frac{\partial}{\partial t}(\mathbf{\Psi}^{\dagger}\cdot\mathbf{\Psi}) = \sum_{s=+,-}\left\{\left[\frac{\partial}{\partial t}(\mathbf{F}_s^{(+)})^*\right]\cdot\mathbf{F}_s^{(+)} + (\mathbf{F}_s^{(+)})^*\cdot\frac{\partial}{\partial t}\mathbf{F}_s^{(+)}\right\}, \tag{67}$$

is obtained eliminating $\partial(\mathbf{F}_s^{(+)})^*/\partial t$ and $\partial\mathbf{F}_s^{(+)}/\partial t$ by means of the analytical part of Eqs. (61) and their complex conjugates. As the reader may verify, one arrives at the following results for the two helicity species:

$$\frac{\partial}{\partial t}[(\mathbf{F}_{\pm}^{(+)})^*\cdot\mathbf{F}_{\pm}^{(+)}] = \pm ic\nabla\cdot[(\mathbf{F}_{\pm}^{(+)})^*\times\mathbf{F}_{\pm}^{(+)}] - \frac{1}{\sqrt{2\varepsilon_0}}[(\mathbf{F}_{\pm}^{(+)})^*\cdot\mathcal{J}_{\pm}^{(+)} + \mathbf{F}_{\pm}^{(+)}\cdot(\mathcal{J}_{\pm}^{(+)})^*], \tag{68}$$

with $\mathcal{J}_{\pm}^{(+)} = \mathbf{J}^{e(+)} \pm i\mathbf{J}^{m(+)}$ [see Eq. (53)]. By combining Eqs. (66)–(68) one gets

$$\frac{\partial}{\partial t}(\mathbf{\Psi}^{\dagger}\cdot\mathbf{\Psi}) = ic\nabla\cdot[(\mathbf{F}_{+}^{(+)})^*\times\mathbf{F}_{+}^{(+)} - (\mathbf{F}_{-}^{(+)})^*\times\mathbf{F}_{-}^{(+)}] - \frac{1}{\sqrt{2\varepsilon_0}}\sum_{s=+,-}[(\mathbf{F}_s^{(+)})^*\cdot\mathcal{J}_s^{(+)} + \mathbf{F}_s^{(+)}\cdot(\mathcal{J}_s^{(+)})^*]. \tag{69}$$

The right side of Eq. (69) can be rewritten in compact spinorial notation. Since

$$\hat{h}\mathbf{\Psi} = \hat{h}\begin{pmatrix}\mathbf{F}_{+}^{(+)}\\ \mathbf{F}_{-}^{(+)}\end{pmatrix} = \begin{pmatrix}\mathbf{F}_{+}^{(+)}\\ -\mathbf{F}_{-}^{(+)}\end{pmatrix}, \tag{70}$$

one has

$$\boldsymbol{\Psi}^{\dagger}\times(\hat{h}\boldsymbol{\Psi})=[(\mathbf{F}_{+}^{(+)})^{*},(\mathbf{F}_{-}^{(+)})^{*}]\times\begin{pmatrix}\mathbf{F}_{+}^{(+)}\\-\mathbf{F}_{-}^{(+)}\end{pmatrix}$$
$$\equiv(\mathbf{F}_{+}^{(+)})^{*}\times\mathbf{F}_{+}^{(+)}-(\mathbf{F}_{-}^{(+)})^{*}\times\mathbf{F}_{-}^{(+)}. \quad (71)$$

Furthermore, by introduction of the six-component current density

$$\mathcal{J}=\begin{pmatrix}\mathcal{J}_{+}^{(+)}\\\mathcal{J}_{-}^{(+)}\end{pmatrix}, \quad (72)$$

one realizes that

$$\boldsymbol{\Psi}^{\dagger}\cdot\mathcal{J}+\mathcal{J}^{\dagger}\cdot\boldsymbol{\Psi}=\sum_{s=+,-}[(\mathbf{F}_{s}^{(+)})^{*}\cdot\mathcal{J}_{s}^{(+)}+\mathbf{F}_{s}^{(+)}\cdot(\mathcal{J}_{s}^{(+)})^{*}]. \quad (73)$$

By inserting Eqs. (71) and (73) into Eq. (69) one finally obtains

$$\frac{\partial}{\partial t}(\boldsymbol{\Psi}^{\dagger}\cdot\boldsymbol{\Psi})=-\nabla\cdot\left[\frac{c}{i}\boldsymbol{\Psi}^{\dagger}\times(\hat{h}\boldsymbol{\Psi})\right]$$
$$-\frac{1}{\sqrt{2\varepsilon_0}}(\boldsymbol{\Psi}^{\dagger}\cdot\mathcal{J}+\mathcal{J}^{\dagger}\cdot\boldsymbol{\Psi}). \quad (74)$$

Equation (74) is the sought for continuity equation for the photon energy density, $\boldsymbol{\Psi}^{\dagger}\cdot\boldsymbol{\Psi}$. Thus

$$\boldsymbol{\mathcal{S}}_{\Psi}=\frac{c}{i}\boldsymbol{\Psi}^{\dagger}\times(\hat{h}\boldsymbol{\Psi}) \quad (75)$$

is the energy current density of the photon energy wave function and the term $(\boldsymbol{\Psi}^{\dagger}\cdot\mathcal{J}+\mathcal{J}^{\dagger}\cdot\boldsymbol{\Psi})/\sqrt{2\varepsilon_0}$ represents the density of work done by the photon field on the electric and magnetic monopoles per unit time.

IV. DOUBLE-POTENTIAL FORMALISM

In Dirac's electromagnetic theory describing (among other things) the quantum problem of an electrically charged elementary particle (≈ electron) placed in the field of a (fixed) magnetic monopole a *singular* vector potential ($\mathbf{A}$) enters. In order to uphold the usual minimal coupling substitution $\nabla\rightarrow\nabla-(ie/\hbar)\mathbf{A}$, e being the particle charge, it is necessary to suppose that the pole is attached to (placed at the end of) a physically unobservable so-called Dirac string. On the string, stretching off to infinity, the vector potential is singular. It was shown by Cabibbo and Ferrari [17] that the pathological string description can be avoided in an elegant manner by introducing a second four potential. In Secs. V–VIII, we shall realize that this double-potential formalism is very useful in our photon wave mechanical description of photon-coupled electric and magnetic monopoles. In particular, an extra transformation of the double four-potentials, which is nonlocal in space and time, in a formal sense, leads to expressions for the conjugate particle momentum and angular momentum which take standard form.

A. Electric and magnetic four potentials

In the double-potential formalism the electric and magnetic fields are given by the dual forms

$$\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{A}^{e}-\nabla\phi^{e}-c\nabla\times\mathbf{A}^{m}, \quad (76)$$

$$\mathbf{B}=-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}^{m}-\frac{1}{c}\nabla\phi^{m}+\nabla\times\mathbf{A}^{e}, \quad (77)$$

where

$$\{A^{e,\mu}\}=\left(\frac{\phi^{e}}{c},\mathbf{A}^{e}\right) \quad (78)$$

and

$$\{A^{m,\mu}\}=\left(\frac{\phi^{m}}{c},\mathbf{A}^{m}\right) \quad (79)$$

are the electric (e) and magnetic (m) four potentials. In the absence of magnetic monopoles, Eqs. (76) and (77) express $\mathbf{E}$ and $\mathbf{B}$ in terms of the well-known electric vector and scalar potentials. By combining Eqs. (76) and (77) it appears that the total (transverse plus longitudinal) Riemann-Silberstein vectors are functions only of the complex four potentials

$$\{A^{e,\mu}\pm iA^{m,\mu}\}=\left[\frac{1}{c}(\phi^{e}\pm i\phi^{m}),\mathbf{A}^{e}\pm i\mathbf{A}^{m}\right]. \quad (80)$$

A division of Eqs. (76) and (77) into their longitudinal and transverse parts gives, respectively,

$$\mathbf{E}_{L}=-\frac{\partial}{\partial t}\mathbf{A}_{L}^{e}-\nabla\phi^{e}, \quad (81)$$

$$c\mathbf{B}_{L}=-\frac{\partial}{\partial t}\mathbf{A}_{L}^{m}-\nabla\phi^{m}, \quad (82)$$

and

$$\mathbf{E}_{T}=-\frac{\partial}{\partial t}\mathbf{A}_{T}^{e}-c\nabla\times\mathbf{A}_{T}^{m}, \quad (83)$$

$$c\mathbf{B}_{T}=-\frac{\partial}{\partial t}\mathbf{A}_{T}^{m}+c\nabla\times\mathbf{A}_{T}^{e}. \quad (84)$$

The Riemann-Silberstein vectors are given by

$$\sqrt{\frac{2}{\varepsilon_0}}\mathbf{F}_{\pm}=-\frac{\partial}{\partial t}\left(\mathbf{A}_{T}^{e}\pm i\mathbf{A}_{T}^{m}\right)\pm ic\nabla\times\left(\mathbf{A}_{T}^{e}\pm i\mathbf{A}_{T}^{m}\right). \quad (85)$$

It appears from Eqs. (83) and (84) that a mixing of the electric and magnetic vector potentials is needed in the potential form. Individual gauge transformations on the electric and magnetic four-potentials do not change the transverse part of the potentials: $\mathbf{A}_{T}^{e}$ and $\mathbf{A}_{T}^{m}$ are gauge-invariant quantities. No mixing of e and m potentials occurs in the longitudinal part of the fields; see Eqs. (81) and (82). In the Coulomb (subscript C) gauge, where $\nabla\cdot\mathbf{A}^{e}(=\nabla\cdot\mathbf{A}_{L}^{e})=\nabla\cdot\mathbf{A}^{m}(=\nabla\cdot\mathbf{A}_{L}^{m})=0$, one has

$$\mathbf{E}_{L}=-\nabla\phi_{C}^{e}, \quad (86)$$

$$c\mathbf{B}_{L}=-\nabla\phi_{C}^{m}, \quad (87)$$

since $\mathbf{A}_{L}^{e}=\mathbf{A}_{L}^{m}=\mathbf{0}$.

B. Nonlocal transformation of transverse vector potentials

For the subsequent analysis of the momentum (Sec. V) and angular momentum (Sec. VI) of photon-coupled electric and magnetic monopoles it will be useful to replace the gauge-invariant transverse vector potentials $\mathbf{A}_T^e$ and $\mathbf{A}_T^m$ by new transverse ones, denoted by $\boldsymbol{\mathcal{A}}_T^e$ and $\boldsymbol{\mathcal{A}}_T^m$. The two sets are related by a certain transformation which is nonlocal in space-time. The usefulness of the transformation becomes manifest when applied for instance to the (relativistic) quantum theory of an electron coupled to a time-dependent magnetic monopole source (see Sec. VIII).

Let the old and the new vector potentials be given by the Fourier integral transformations

$$\mathbf{A}_T^\alpha(\mathbf{r},t) = (2\pi)^{-4}\int_{-\infty}^{\infty} \mathbf{A}_T^\alpha(\mathbf{q},\omega)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}d^3q\,d\omega, \quad (88)$$

$$\boldsymbol{\mathcal{A}}_T^\alpha(\mathbf{r},t) = (2\pi)^{-4}\int_{-\infty}^{\infty} \boldsymbol{\mathcal{A}}_T^\alpha(\mathbf{q},\omega)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}d^3q\,d\omega, \quad (89)$$

with $\alpha = e$ or m, and let us then connect the potentials via the following relation between the Fourier amplitudes:

$$\mathbf{A}_T^\alpha(\mathbf{q},\omega) = \frac{cq}{\omega}\hat{\mathbf{q}} \times \boldsymbol{\mathcal{A}}_T^\alpha(\mathbf{q},\omega), \quad \alpha = e \text{ or } m, \quad (90)$$

where $\hat{\mathbf{q}} = \mathbf{q}/q$ is a unit vector in the direction of the wave vector ($\mathbf{q}$). Since we have demanded that the new potentials must be transverse [implying that $\hat{\mathbf{q}}\cdot\boldsymbol{\mathcal{A}}_T^\alpha(\mathbf{q},\omega) = 0$], we obtain from Eq. (90) the inverse connection

$$\boldsymbol{\mathcal{A}}_T^\alpha(\mathbf{q},\omega) = -\frac{\omega}{cq}\hat{\mathbf{q}} \times \mathbf{A}_T^\alpha(\mathbf{q},\omega). \quad (91)$$

By writing Eq. (90) in the form

$$i\frac{\omega}{c}\mathbf{A}_T^\alpha(\mathbf{q},\omega) = i\mathbf{q} \times \boldsymbol{\mathcal{A}}_T^\alpha(\mathbf{q},\omega), \quad (92)$$

and making use of the fact that the multiplication by $-i\omega$ and $i\mathbf{q}$ in the four-dimensional Fourier space corresponds respectively to the operations $\partial/\partial t$ and ∇ in the space-time domain, it is obvious that the old $[\mathbf{A}_T^\alpha(\mathbf{r},t)]$ and new $[\boldsymbol{\mathcal{A}}_T^\alpha(\mathbf{r},t)]$ transverse vector potentials are connected as follows:

$$-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_T^\alpha(\mathbf{r},t) = \nabla \times \boldsymbol{\mathcal{A}}_T^\alpha(\mathbf{r},t). \quad (93)$$

By utilizing Eq. (93) in Eqs. (83) and (84) the transverse parts of the electric and magnetic fields can be expressed in the mixed potential forms

$$c^{-1}\mathbf{E}_T(\mathbf{r},t) = \nabla \times \left[\boldsymbol{\mathcal{A}}_T^e(\mathbf{r},t) - \mathbf{A}_T^m(\mathbf{r},t)\right], \quad (94)$$

$$\mathbf{B}_T(\mathbf{r},t) = \nabla \times \left[\mathbf{A}_T^e(\mathbf{r},t) + \boldsymbol{\mathcal{A}}_T^m(\mathbf{r},t)\right]. \quad (95)$$

The space-time nonlocal transformation of the potentials thus has enabled us to express $\mathbf{E}_T/c$ and $\mathbf{B}_T$ as curls of simple combinations of the old and new transverse potentials.

V. MOMENTUM OF PARTICLE ($E + M$)-PHOTON SYSTEM

A. Total field momentum

Within the framework of the extended microscopic first-quantized Maxwell-Lorentz theory the total field momentum is given by

$$\mathbf{P}(t) = \varepsilon_0\int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)d^3r. \quad (96)$$

To analyze the structure of $\mathbf{P}$ we divide the $\mathbf{E}$ and $\mathbf{B}$ fields in Eq. (96) into their longitudinal and transverse vector field parts, Hence

$$\mathbf{P}(t) = \mathbf{P}_{LL}(t) + \mathbf{P}_{LT}(t) + \mathbf{P}_{TL}(t) + \mathbf{P}_{TT}(t), \quad (97)$$

where

$$\mathbf{P}_{IJ}(t) = \varepsilon_0\int_{-\infty}^{\infty} \mathbf{E}_I \times \mathbf{B}_J d^3r. \quad (98)$$

The (IJ) subscript in Eq. (98) refers to one of the combinations (LL), (LT), (TL), and (TT); cf. Eq. (96). The division in Eq. (97) is analogous to one often used in the absence of magnetic monopoles [24,40]. In such studies the total momentum only consists of two parts, viz.,

$$\mathbf{P}(t|\rho^m = 0) = \mathbf{P}_{LT}(t,\rho^m = 0) + \mathbf{P}_{TT}(t,\rho^m = 0), \quad (99)$$

the reason stemming from the fact that the magnetic field has no longitudinal part when $\rho^m = 0$; see Eq. (19).

B. Proof that the LL part is zero

To prove the assertion above we transfer the relevant integral over direct space to an integral over wave-vector ($\mathbf{q}$) space using the Parseval-Plancherel identity. Thus

$$\mathbf{P}_{LL}(t) = \varepsilon_0\int_{-\infty}^{\infty} \mathbf{E}_L(\mathbf{r},t)\times\mathbf{B}_L(\mathbf{r},t)d^3r$$
$$= \frac{\varepsilon_0}{(2\pi)^3}\int_{-\infty}^{\infty} \mathbf{E}_L^*(\mathbf{q};t)\times\mathbf{B}_L(\mathbf{q};t)d^3q. \quad (100)$$

Since $\mathbf{E}_L^*(\mathbf{q};t) = \hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\mathbf{E}_L^*(\mathbf{q};t)$ and $\mathbf{B}_L(\mathbf{q};t) = \hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\mathbf{B}_L(\mathbf{q};t)$, one has $\mathbf{E}_L^*(\mathbf{q};t)\times\mathbf{B}_L(\mathbf{q};t) \propto \hat{\mathbf{q}}\times\hat{\mathbf{q}} = \mathbf{0}$. In consequence,

$$\mathbf{P}_{LL}(t) = \mathbf{0}, \quad (101)$$

as claimed. Although there is no field momentum associated to the product of the longitudinal fields, we shall realize in Sec. VI that the vector product of these fields do give rise to a net angular momentum of the (e,m)-monopole system, even in the absence of transverse (real) photons. As shown for a static system of an electric charge and a magnetic monopole by Saha [14,15] and Wilson [16], one may upon quantization of the semiclassical (first-quantized) results of Wilson in units of $\hbar/2$ recover the famous Dirac quantization for the product of the electric and magnetic monopole charges [2,3].

C. Electromagnetic parts of canonical particle momenta

It is known that $\mathbf{P}_{LT}(t)$ in the absence of magnetic monopoles can be identified as the electromagnetic momentum associated to the individual electric monopoles [24,40]. In order to analyze the situation in the presence of magnetic monopoles, it is useful to transfer the relevant integral over $\mathbf{r}$ space [Eq. (98)] to an integral over $\mathbf{q}$ space. With the help of the Parseval-Plancherel identity one obtains

$$\mathbf{P}_{LT}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \mathbf{E}_L(\mathbf{r},t) \times \mathbf{B}_T(\mathbf{r},t) d^3r$$
$$= \varepsilon_0 \int_{-\infty}^{\infty} \mathbf{E}_L^*(\mathbf{q},t) \times \mathbf{B}_T(\mathbf{q},t) \frac{d^3q}{(2\pi)^3}. \tag{102}$$

Since the Maxwell equation in (18) takes the form $i\mathbf{q} \cdot \mathbf{E}_L(\mathbf{q};t) = \rho^e(\mathbf{q};t)/\varepsilon_0$ in the wave-vector domain one gets

$$\mathbf{E}_L(\mathbf{q};t) = \hat{\mathbf{q}}\hat{\mathbf{q}} \cdot \mathbf{E}_L(\mathbf{q};t) = \frac{\hat{\mathbf{q}}}{i\varepsilon_0 q} \rho^e(\mathbf{q};t), \tag{103}$$

where $\hat{\mathbf{q}} = \mathbf{q}/q$ as before is a unit vector in the direction of $\mathbf{E}_L(\mathbf{q};t)$. An expression for the electric charge density in $\mathbf{q}$ space is readily obtained from the first component of Eq. (22). Hence

$$\rho^e(\mathbf{q};t) = \sum_\alpha e_\alpha \int_{-\infty}^{\infty} \delta\big(\mathbf{r} - \mathbf{r}_\alpha^e(t)\big) e^{-i\mathbf{q}\cdot\mathbf{r}} d^3r$$
$$= \sum_\alpha e_\alpha \exp[-i\mathbf{q} \cdot \mathbf{r}_\alpha^e(t)]. \tag{104}$$

The usefulness of the nonlocal transformation of the transverse vector potentials introduced in Sec. IV now shows up after a little algebra, Thus, if one inserts a combination of Eqs. (103) and (104) and the $\mathbf{q}$-space form of Eq. (95), i.e.,

$$\mathbf{B}_T(\mathbf{q};t) = iq\hat{\mathbf{q}} \times \big[\mathbf{A}_T^e(\mathbf{q};t) + \boldsymbol{\mathcal{A}}_T^m(\mathbf{q};t)\big], \tag{105}$$

into Eq. (102) one obtains

$$\mathbf{P}_{LT}(t) = -\sum_\alpha e_\alpha \int_{-\infty}^{\infty} \hat{\mathbf{q}}$$
$$\times \left\{\hat{\mathbf{q}} \times \big[\mathbf{A}_T^e(\mathbf{q};t) + \boldsymbol{\mathcal{A}}_T^m(\mathbf{q};t)\big]\right\} e^{i\mathbf{q}\cdot\mathbf{r}_\alpha^e(t)} \frac{d^3q}{(2\pi)^3}$$
$$= \sum_\alpha e_\alpha \int_{-\infty}^{\infty} \big[\mathbf{A}_T^e(\mathbf{q};t) + \boldsymbol{\mathcal{A}}_T^m(\mathbf{q};t)\big] e^{i\mathbf{q}\cdot\mathbf{r}_\alpha^e(t)} \frac{d^3q}{(2\pi)^3}. \tag{106}$$

The last expression in Eq. (106) is just the Fourier integral representative of

$$\mathbf{P}_{LT}(t) = \sum_\alpha e_\alpha \big[\mathbf{A}_T^e\big(\mathbf{r}_\alpha^e(t),t\big) + \boldsymbol{\mathcal{A}}_T^m\big(\mathbf{r}_\alpha^e,t\big)\big]. \tag{107}$$

The result in Eq. (107) shows that $\mathbf{P}_{LT}(t)$ may be expressed as a function of the various (instantaneous) position coordinates of the electric monopoles, $\mathbf{r}_\alpha^e(t)$, including also an explicit time dependence of the transverse vector potentials at these positions. The fact that only the set of position coordinates $[\{\mathbf{r}_\alpha^e(t)\}]$ enters Eq. (107) implies that one may characterize $\mathbf{P}_{LT}(t)$ as the *electromagnetic momentum* associated to the electric monopole particles. If $\boldsymbol{\Pi}_\alpha^e$ denotes the *mechanical momentum* of particle number α, the quantity

$$\mathbf{p}_\alpha^e = \Pi_\alpha^e + e_\alpha\big[\mathbf{A}_T^e\big(\mathbf{r}_\alpha^e(t),t\big) + \boldsymbol{\mathcal{A}}_T^m\big(\mathbf{r}_\alpha^e(t),t\big)\big] \tag{108}$$

represents the *total momentum* of this particle, also called the *canonical momentum*. A graphical representation of the physics of Eq. (108) is given in Fig. 1.

In retrospect, it appears that it is our nonlocal transformation of the transverse (and gauge-invariant) vector potential $[\mathbf{A}_T^m(\mathbf{r},t) \rightarrow \boldsymbol{\mathcal{A}}_T^m(\mathbf{r},t)]$ which has enabled us to bring the contribution from the magnetic monopoles to $\mathbf{P}_{LT}(t)$ into a form analogous to the one known in the absence of magnetic monopoles [24,40]. In a quantum physical setting the momentum $\mathbf{p}_\alpha^e$ translates into a momentum operator $(\hbar/i)\boldsymbol{\nabla}_\alpha^e$ in the configurational space representation, and thereby one is led to a Hamiltonian formalism free of singularities; see the remarks in Sec. VIII.

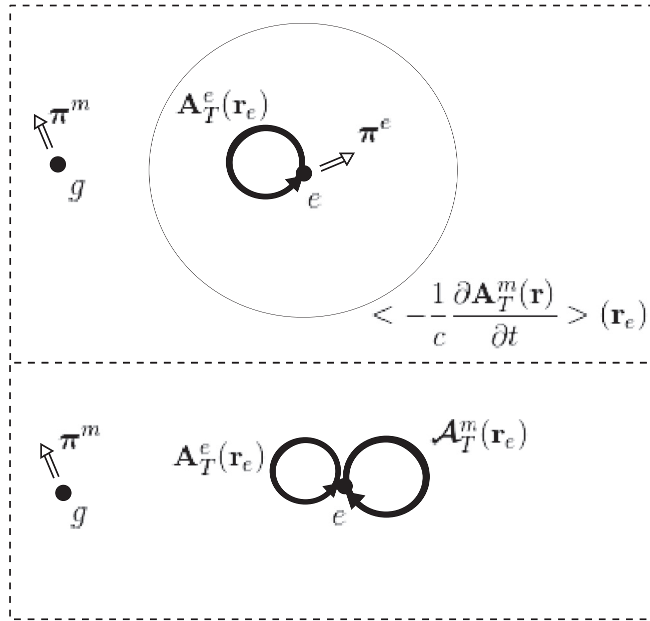

FIG. 1. Electromagnetic momentum of an electron (e) in the field of a nonuniformly moving magnetic monopole (g). The mechanical momentum of the particles are denoted $\boldsymbol{\pi}^e$ and $\boldsymbol{\pi}^m$. In the upper part of the figure is shown the e-self-field contribution $e\mathbf{A}_T^e(\mathbf{r}_e)$ and the spatially nonlocal g contribution obtained as a weighted spatial average $e\langle -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_T^m(\mathbf{r})\rangle(\mathbf{r}_e)$ over the near-field zone of the electron. The effective range of this zone is indicated by the big circle. Upon a space-time nonlocal transformation, the effect of the magnetic monopole appears as a magnetic self-field contribution $e\boldsymbol{\mathcal{A}}_T^m(\mathbf{r}_e)$, as shown in the lower part of the figure. The e and g self-field contributions are proportional to e and eg, respectively. The form of the canonical electron momentum, viz., $\mathbf{p}^e = \boldsymbol{\pi}^e + e[\mathbf{A}_T^e(\mathbf{r}_e) + \boldsymbol{\mathcal{A}}_T^m(\mathbf{r}_e)]$ allows a minimal coupling procedure in quantum electrodynamics. For notional simplicity the reference to time has been omitted from the various arguments.

By now it is obvious that the quantity

$$\mathbf{P}_{TL}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \mathbf{E}_T(\mathbf{r},t) \times \mathbf{B}_L(\mathbf{r},t) d^3r \tag{109}$$

may be identified as the electromagnetic momentum associated to the system of magnetic monopoles. If one makes use of the expression give for $\mathbf{E}_T(\mathbf{r},t)$ in Eq. (94), and notes that $\mathbf{E}_T \times \mathbf{B}_L = -\mathbf{B}_L \times \mathbf{E}_T$, a moment of reflection shows that $\mathbf{P}_{TL}(t)$ can be written in the following form:

$$\mathbf{P}_{TL}(t) = \sum_\alpha g_\alpha \big[\mathbf{A}_T^m\big(\mathbf{r}_\alpha^m(t),t\big) - \boldsymbol{\mathcal{A}}_T^e\big(\mathbf{r}_\alpha^m(t),t\big)\big]. \tag{110}$$

The quantity

$$\mathbf{p}_\alpha^m = \pi_\alpha^m + g_\alpha\big[\mathbf{A}_T^m\big(\mathbf{r}_\alpha^m(t),t\big) - \boldsymbol{\mathcal{A}}_T^e\big(\mathbf{r}_\alpha^m(t),t\big)\big], \tag{111}$$

hence is the total (canonical) momentum of magnetic monopole particle number α, π_α^m denoting the mechanical momentum of this particle. The electric monopole adds a momentum contribution $-g_\alpha \boldsymbol{\mathcal{A}}_T^e(\mathbf{r}_\alpha^m, t)$ to the one $[g_\alpha \mathbf{A}_T^m(\mathbf{r}_\alpha^m(t), t)]$ stemming from a system consisting of magnetic monopoles only.

Since the contribution to $\boldsymbol{\mathcal{A}}_T^m(\mathbf{r}_\alpha^e, t)$ from magnetic monopole particle number β is proportional to g_β it appears that the "new" term to the canonical momentum $\mathbf{p}_\alpha^e$ originating in particle β is proportional to the charge product $e_\alpha g_\beta$. The same of course is the case for $\mathbf{p}_\alpha^m$, $[g_\alpha e_\beta]$.

D. Photon-field momentum

For the massless photon the relation between the momentum density, $\mathbf{P}_{TT}(t)$, and the energy current density, $\boldsymbol{\mathcal{S}}_\Phi(\mathbf{r}, t)$, necessarily is given by

$$\boldsymbol{\mathcal{S}}_\Psi(\mathbf{r}, t) = c^2 \boldsymbol{\mathcal{P}}_{TT}(\mathbf{r}, t). \tag{112}$$

The momentum of the photon field

$$\mathbf{P}_{TT}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \mathbf{E}_T(\mathbf{r}, t) \times \mathbf{B}_T(\mathbf{r}, t) d^3 r = \int_{-\infty}^{\infty} \boldsymbol{\mathcal{P}}_{TT}(\mathbf{r}, t) d^3 r \tag{113}$$

now can be related to the photon energy wave function using the expression for $\boldsymbol{\mathcal{S}}_\Psi(\mathbf{r}, t)$ given in Eq. (75). Hence

$$\mathbf{P}_{TT}(t) = \frac{1}{ic} \int_{-\infty}^{\infty} \boldsymbol{\Psi}^\dagger(\mathbf{r}, t) \times [\hat{h} \boldsymbol{\Psi}(\mathbf{r}, t)] d^3 r. \tag{114}$$

The result in Eq. (114) also can be obtained in a more tedious manner starting from the first member of Eq. (113), expressing $\mathbf{E}_T$ and $\mathbf{B}_T$ in terms of $\mathbf{F}_+$ and $\mathbf{F}_-$ [Eqs. (46)], and utilizing afterwards Eqs. (47) and (48). Although we have made use of various density concepts ($\boldsymbol{\Psi}^\dagger \cdot \boldsymbol{\Psi}$, $\boldsymbol{\mathcal{P}}_{TT}$, $\boldsymbol{\mathcal{S}}_\Psi$) it must be remembered that these quantities have no strict physical meaning because of our inability to localize a photon precisely in space-time [25].

The momentum of the final state $[\boldsymbol{\Phi}(\mathbf{r}, t)]$, viz.,

$$\mathbf{P}_{TT}^{\text{out}}(t) = \frac{1}{ic} \int_{-\infty}^{\infty} \boldsymbol{\Phi}^\dagger(\mathbf{r}, t) \times [\hat{h} \boldsymbol{\Phi}(\mathbf{r}, t)] d^3 r, \tag{115}$$

is time independent due to momentum conservation. The incoming free-photon state $[\boldsymbol{\Phi}^0(\mathbf{r}, t)]$ likewise possesses a time-independent momentum, $\mathbf{P}_{TT}^{\text{in}}$. The difference $\mathbf{P}_{TT}^{\text{in}} - \mathbf{P}_{TT}^{\text{out}}$ accounts for the final transfer of momentum to the system of electric and magnetic monopoles. It is instructive to prove by an explicit calculation that $d\mathbf{P}_{TT}^{\text{out}}/dt = 0$. Using the Parseval-Plancherel relation $\mathbf{P}_{TT}^{\text{out}}$ can be expressed as an integral over $\mathbf{q}$ space, viz.,

$$\mathbf{P}_{TT}^{\text{out}} = \frac{1}{ic} \int_{-\infty}^{\infty} \boldsymbol{\Phi}^\dagger(\mathbf{q}; t) \times [\hat{h} \boldsymbol{\Phi}(\mathbf{q}, t)] \frac{d^3 q}{(2\pi)^3}, \tag{116}$$

where the upper $[\mathbf{F}_+^{(+)}(\mathbf{q}; t)]$ and lower $[\mathbf{F}_-^{(+)}(\mathbf{q}; t)]$ spinorial components of $\boldsymbol{\Phi}(\mathbf{q}; t)$ satisfy the dynamical free-space equations

$$i\hbar \frac{\partial}{\partial t} \mathbf{F}_\pm^{(+)}(\mathbf{q}; t) = \pm i c \hbar \mathbf{q} \times \mathbf{F}_\pm^{(+)}(\mathbf{q}; t), \tag{117}$$

cf. Eq. (61). The general solutions to Eqs. (117) are well known [25,27], and from these it follows that

$$\boldsymbol{\Phi}(\mathbf{q}; t) = \begin{pmatrix} \mathbf{F}_+^{(+)}(\mathbf{q}) \hat{\mathbf{e}}_+(\hat{\mathbf{q}}) \\ \mathbf{F}_-^{(+)}(\mathbf{q}) \hat{\mathbf{e}}_-(\hat{\mathbf{q}}) \end{pmatrix} \exp(-icqt), \tag{118}$$

where $\hat{\mathbf{e}}_\pm(\hat{\mathbf{q}})$ are helicity unit vectors (the unit vectors $\hat{\mathbf{q}} = \mathbf{q}/q$ and $\mathbf{e}_+$ and $\mathbf{e}_-$ form a right-handed triad). The time dependence of $\boldsymbol{\Phi}(\mathbf{q}; t)$ immediately shows that the integrand of Eq. (116) is time independent and therefore $\mathbf{P}_{TT}^{\text{out}}$ (and $\mathbf{P}_{TT}^{\text{in}}$) is constant in time.

VI. ANGULAR MOMENTUM OF PARTICLE-PHOTON SYSTEM

A. Total field angular momentum

It appears from the symmetrized set of Maxwell-Lorentz equations that the total angular momentum $\mathbf{I}$ of the electromagnetic field with respect to a reference point $\mathbf{r}_0$ is given by the integrated moment of the momentum density, $\varepsilon_0 \mathbf{E} \times \mathbf{B}$, about $\mathbf{r}_0$, i.e.,

$$\mathbf{I}(t|\mathbf{r}_0) = \varepsilon_0 \int_{-\infty}^{\infty} (\mathbf{r} - \mathbf{r}_0) \times [\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)] d^3 r. \tag{119}$$

In analogy with the structural division made for the field momentum in Sec. V, $\mathbf{I}(t)$ is decomposed as follows:

$$\mathbf{I}(t|\mathbf{r}_0) = \sum_{I,J} \mathbf{I}_{IJ}(t|\mathbf{r}_0), \tag{120}$$

where

$$\mathbf{I}_{IJ}(t|\mathbf{r}_0) = \varepsilon_0 \int_{-\infty}^{\infty} (\mathbf{r} - \mathbf{r}_0) \times [\mathbf{E}_I(\mathbf{r}, t) \times \mathbf{B}_J(\mathbf{r}, t)] d^3 r. \tag{121}$$

As before, the (IJ) subscript runs over the four combinations (LL), (LT), (TL), and (TT). Since

$$\mathbf{I}_{IJ}(t|\mathbf{r}_0) = I_{IJ}(t|\mathbf{0}) - \mathbf{r}_0 \times \mathbf{P}_{IJ}(t), \tag{122}$$

it is sufficient to analyze the various contributions to the angular momentum about the origin of our coordinate system, viz.,

$$\mathbf{I}_{IJ}(t|\mathbf{0})[\equiv \mathbf{I}_{IJ}(t)] = \varepsilon_0 \int_{-\infty}^{\infty} \mathbf{r} \times [\mathbf{E}_I(\mathbf{r}, t) \times \mathbf{B}_J(\mathbf{r}, t)](d)^3 r. \tag{123}$$

In the absence of magnetic monopoles only the $\mathbf{I}_{LT}$ and the $\mathbf{I}_{TT}$ parts survive.

B. Dynamic Saha-Wilson part

Let us consider first the case of a single pair of monopoles with charge e and g. Since $\mathbf{P}_{LL}(t) = \mathbf{0}$ the angular momentum $\mathbf{I}_{LL}(t)$ is independent of the choice of reference point $(\mathbf{r}_0)$. Conveniently, we calculate $\mathbf{I}_{LL}(t)$ about the *instantaneous* position of the magnetic monopole; $\mathbf{r}_0 = \mathbf{r}^m(t)$. By inserting the relevant Coulomb expressions for $\mathbf{E}_L$ and $\mathbf{B}_L$ [Eqs. (24) and (25)], the LL part of the angular momentum, i.e.,

$$\mathbf{I}_{LL}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \mathbf{r} \times [\mathbf{E}_L(\mathbf{r}, t) \times \mathbf{B}_L(\mathbf{r}, t)] d^3 r, \tag{124}$$

becomes

$$\mathbf{I}_{LL}(t) = \frac{eg}{(4\pi)^2\varepsilon_0 c}\int_{-\infty}^{\infty}\mathbf{r}\times\left[\frac{[\mathbf{r}-\mathbf{R}_{ge}(t)]\times\mathbf{r}}{|\mathbf{r}-\mathbf{R}_{ge}|^3 r^3}\right]d^3r$$
$$= \frac{eg}{(4\pi)^2\varepsilon_0 c}\int_{-\infty}^{\infty}\left[\frac{\mathbf{r}-\mathbf{R}_{ge}(t)}{|\mathbf{r}-\mathbf{R}_{ge}|^3}\right]\cdot\left[\frac{\mathbf{U}-\mathbf{e_r e_r}}{r}\right]d^3r, \tag{125}$$

where $\mathbf{R}_{ge}(t) = \mathbf{r}^e(t) - \mathbf{r}^m(t)$ [$= \mathbf{r}^e(t)$, here] and $\mathbf{e_r} = \mathbf{r}/r$. Using the relation

$$(\mathbf{a}\cdot\nabla)\mathbf{e_r} = \mathbf{a}\cdot(\mathbf{U}-\mathbf{e_r e_r})\frac{1}{r}, \tag{126}$$

with $\mathbf{a} = \mathbf{r} - \mathbf{R}_{ge}(t)/|\mathbf{r}-\mathbf{R}_{ge}(t)|^3$ in Eq. (125), a subsequent partial integration gives

$$\mathbf{I}_{LL}(t) = -\frac{eg}{(4\pi)^2\varepsilon_0 c}\int_{-\infty}^{\infty}\mathbf{e_r}\nabla\cdot\left[\frac{\mathbf{r}-\mathbf{R}_{ge}(t)}{|\mathbf{r}-\mathbf{R}_{ge}|^3}\right]d^3r. \tag{127}$$

Since

$$\nabla\cdot\left[\frac{\mathbf{r}-\mathbf{R}_{ge}(t)}{|\mathbf{r}-\mathbf{R}_{ge}|^3}\right] = 4\pi\delta(\mathbf{r}-\mathbf{R}_{ge}(t)), \tag{128}$$

one finally obtains

$$\mathbf{I}_{LL}(t) = \frac{eg}{4\pi\varepsilon_0 c}\hat{\mathbf{R}}_{eg}(t), \tag{129}$$

where $\hat{\mathbf{R}}_{eg}(t)[= -\hat{\mathbf{R}}_{ge}(t)] = \mathbf{R}_{eg}(t)/R_{eg}(t)$ is a unit vector pointing from the electric monopole towards the magnetic monopole. The result in Eq. (129) is a generalization of the result obtained originally by Saha [14,15] and Wilson [16] for a static system (time independent $\mathbf{R}_{eg}$). The expression in Eq. (129) is not new, and has been used in classical studies of the scattering of a charge e in the field of a fixed monopole of charge g [8]. The angular momentum $\mathbf{I}_{LL}(t)$ may not be attached to either of the monopole particles. It depends on both and is nonvanishing even in the static case.

Since the usual orbital angular momentum, $\mathbf{L} = \mathbf{R}_{eg}\times\mu\dot{\mathbf{R}}_{eg}$ (μ being the reduced mass of the two-particle system) has no component along $\hat{\mathbf{R}}_{eg}$, quantization of the total orbital angular momentum along the connection line of the particles leads to the *charge quantization condition*

$$\frac{1}{4\pi\varepsilon_0}\frac{eg}{\hbar c} = n, \quad n = 0, \pm1, \pm2, \ldots. \tag{130}$$

With the admission of half-integer $(n + 1/2)$ values, the result in Eq. (130) is identical to Dirac's ingenious finding. In the purely orbital case it seems plausible to limit n to integer values.

The result in Eq. (130) readily may be generalized to an assembly of monopoles. Hence

$$\mathbf{I}_{LL}(t) = \frac{1}{4\pi\varepsilon_0 c}\sum_{\alpha,\beta}e_\alpha g_\beta\hat{\mathbf{R}}_{e_\alpha g_\beta}(t) \tag{131}$$

in an obvious notation.

C. Field parts of the canonical particle angular momentum

The angular momentum

$$\mathbf{I}_{LT}(t) = \varepsilon_0\int_{-\infty}^{\infty}\mathbf{r}\times[\mathbf{E}_L(\mathbf{r},t)\times\mathbf{B}_T(\mathbf{r},t)]d^3r$$
$$= \varepsilon_0\int_{-\infty}^{\infty}\mathbf{r}\times\big(\mathbf{E}_L(\mathbf{r},t)$$
$$\times\left\{\nabla\times\left[\mathbf{A}_T^e(\mathbf{r},t)+\boldsymbol{\mathcal{A}}_T^m(\mathbf{r},t)\right]\right\}\big)d^3r \tag{132}$$

is calculated in a manner completely identical to the one used previously in the absence of magnetic monopoles [$\boldsymbol{\mathcal{A}}_T^m(\mathbf{r},t) = \mathbf{0}$]; see Refs. [24,40]. As expected, the result

$$\mathbf{I}_{LT} = \sum_\alpha e_\alpha\mathbf{r}_\alpha^e(t)\times\left[\mathbf{A}_T^e\big(\mathbf{r}_\alpha^e(t),t\big)+\boldsymbol{\mathcal{A}}_T^m\big(\mathbf{r}_\alpha^e(t),t\big)\right] \tag{133}$$

can be written as a function of the instantaneous position coordinates of the electric monopoles and the transverse vector potential prevailing at these positions. The result for $\mathbf{I}_{LT}(t)$ is gauge invariant. The sum of the mechanical angular momentum of the electrons and the electromagnetic angular momentum is given by

$$\sum_\alpha\mathbf{r}_\alpha^e(t)\times\boldsymbol{\pi}_\alpha^e(t)+\mathbf{I}_{LT}(t) = \sum_\alpha\mathbf{r}_\alpha^e(t)\times\mathbf{p}_\alpha^e(t), \tag{134}$$

where $\mathbf{p}_\alpha^e(t)$ is the canonical momentum of particle number α; see Eq. (108).

The field angular momentum

$$\mathbf{I}_{TL}(t) = \varepsilon_0\int_{-\infty}^{\infty}\mathbf{r}\times[\mathbf{E}_T(\mathbf{r},t)\times\mathbf{B}_L(\mathbf{r},t)]d^3r \tag{135}$$

is readily calculated if one rewrites it in the form

$$\mathbf{I}_{TL}(t) = \varepsilon_0\int_{-\infty}^{\infty}\mathbf{r}\times\big(c\mathbf{B}_L(\mathbf{r},t)$$
$$\times\left\{\nabla\times\left[\mathbf{A}_T^m(\mathbf{r},t)-\boldsymbol{\mathcal{A}}_T^e(\mathbf{r},t)\right]\right\}\big)d^3r. \tag{136}$$

To obtain Eq. (136) we have used $\mathbf{E}_T\times\mathbf{B}_L = -\mathbf{B}_L\times\mathbf{E}_T$ and Eq. (93). Since $c\mathbf{B}_L(\mathbf{r},t)$ and $\mathbf{E}_L(\mathbf{r},t)$ are given by form-identical magnetic and electric Coulomb field expressions [see Eqs. (24) and (25)], it is obvious that the same type of calculation which led us from Eq. (132) to Eq. (133) now gives

$$\mathbf{I}_{TL} = \sum_\alpha g_\alpha\mathbf{r}_\alpha^m(t)\times\left[\mathbf{A}_T^m\big(\mathbf{r}_\alpha^m(t),t\big)+\boldsymbol{\mathcal{A}}_T^e\big(\mathbf{r}_\alpha^m(t),t\big)\right]. \tag{137}$$

The gauge-invariant result in Eq. (137) is just the field angular momentum of the system of magnetic monopoles. Grouped together with the total mechanical angular momenta of the magnetic monopoles one obtains immediately

$$\sum_\alpha\mathbf{r}_\alpha^m(t)\times\boldsymbol{\pi}_\alpha^m(t)+\mathbf{I}_{TL}(t) = \sum_\alpha\mathbf{r}_\alpha^m(t)\times\mathbf{p}_\alpha^m(t), \tag{138}$$

where $\mathbf{p}_\alpha^m(t)$ is the canonical momentum of magnetic monopole number α; see Eq. (111).

D. Photon-field angular momentum

The angular momentum of the photon field, i.e.,

$$\mathbf{I}_{TT}(t) = \varepsilon_0\int_{-\infty}^{\infty}\mathbf{r}\times[\mathbf{E}_T(\mathbf{r},t)\times\mathbf{B}_T(\mathbf{r},t)]d^3r, \tag{139}$$

can be expressed in terms of the energy current density of the photon energy wave function. Hence

$$\mathbf{I}_{TT}(t) = \frac{1}{ic}\int_{-\infty}^{\infty} \mathbf{r}\times[\mathbf{\Psi}^{\dagger}(\mathbf{r},t)\times[\hat{h}\mathbf{\Psi}(\mathbf{r},t)]]. \tag{140}$$

The reader may prove this to herself, taking a glance at Eqs. (112)–(114). The angular momentum of the outgoing free-photon state $[\mathbf{\Phi}(\mathbf{r},t)]$ is time independent due to angular momentum conservation. An explicit proof can be given calculating $d\mathbf{I}_{TT}(t)/dt$ from Eq. (139); see, e.g., Refs. [24,40].

VII. NEAR FIELD OF A MAGNETIC MONOPOLE

In the perspective of photon wave mechanics we now discuss the magnetic field in the near-field zone of a single magnetic monopole of charge g. The longitudinal and transverse parts of the magnetic field are given by Eqs. (25) and (40) reducing the number of monopoles to one.

Let us first consider the field of a monopole fixed at the origin of our coordinate system. In this static (ST) case the magnetic field has a longitudinal component only ($\mathbf{B}=\mathbf{B}_L^{ST}$). Thus

$$c\mathbf{B}_L^{ST}(\mathbf{r}) = \frac{g}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3} = -\frac{g}{4\pi\varepsilon_0}\nabla\frac{1}{r}, \tag{141}$$

with $\nabla\times\mathbf{B}_L^{ST}(\mathbf{r})=\mathbf{0}$. Except at the monopole position, the longitudinal field satisfies the condition

$$\nabla\cdot\mathbf{B}_L^{ST}(\mathbf{r}) = 0, \quad \mathbf{r}\neq\mathbf{0}. \tag{142}$$

The static field is *not* a genuine transverse vector field as $c\nabla\cdot\mathbf{B}_L^{ST}(\mathbf{r}) = (g/\varepsilon_0)\delta(\mathbf{r})$: the transversality condition is satisfied in all space except at $\mathbf{r}=\mathbf{0}$. In Dirac's discussion of the quantum mechanics of an electron, Eq. (142) was used as a basis for the introduction of a singular vector potential $\mathbf{A}^{\mathrm{Dirac}}$ to represent the field of a fixed monopole:

$$\mathbf{B}_L^{ST}(\mathbf{r}) = \nabla\times\mathbf{A}^{\mathrm{Dirac}}(\mathbf{r}), \quad \mathbf{r}\neq\mathbf{0}. \tag{143}$$

Thus $\mathbf{A}^{\mathrm{Dirac}}(\mathbf{r})$ corresponds not to an isolated monopole charge, but rather to a semi-infinite line of magnetic dipoles ending at the monopole (Dirac string).

In studies of the Dirac equation for an electrically charged particle in a fixed magnetic monopole field one usually parametrizes the vector potential in two different ways, corresponding to two overlapping but not identical regions. In such a fiber bundle formulation the Dirac string is avoided, but the wave function becomes a section rather than an ordinary function.

In a quantum physical setting the monopole is never completely at rest, and the electric-magnetic monopole interaction is due to photon exchange with canonical electric and magnetic monopole momenta given by Eqs. (108) and (111), respectively. Let us therefore consider the L and T parts of the magnetic monopole field as time-dependent (dynamic) quantities.

The longitudinal component $[\mathbf{B}_L(\mathbf{r},t)]$ of the dynamic magnetic field satisfies Eqs. (19) and (20). A replacement of $\mathbf{B}_L$ by the relevant combination of magnetic potentials [Eq. (81)] in these equations results in the following coupled equations between the magnetic vector ($\mathbf{A}_L^m$) and scalar (ϕ^m) potentials:

$$-\frac{\partial^2\mathbf{A}_L^m}{\partial t^2} - \nabla\left(\frac{\partial\phi^m}{\partial t}\right) = -\frac{1}{\varepsilon_0}\mathbf{J}_L^m, \tag{144}$$

$$-\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}_L^m) - \nabla^2\phi^m = \frac{1}{\varepsilon_0}\rho^m. \tag{145}$$

The Lorenz gauge condition,

$$\nabla\cdot\mathbf{A}_L^m + \frac{1}{c^2}\frac{\partial\phi^m}{\partial t} = 0, \tag{146}$$

transfers Eqs. (144) and (145) to the uncoupled wave equations

$$\Box\mathbf{A}_L^m = -\frac{1}{\varepsilon_0}\mathbf{J}_L^m, \tag{147}$$

$$\Box\phi^m = -\frac{1}{\varepsilon_0}\rho^m. \tag{148}$$

In obtaining Eq. (147) we have used the vector identity $\mathbf{0}=\nabla\times(\nabla\times\mathbf{A}_L^m)=\nabla\nabla\cdot\mathbf{A}_L^m-\nabla^2\mathbf{A}_L^m$. On the basis of Eqs. (147) and (148) longitudinal (L) and scalar (S) magnetic (m) photons can be introduced in analogy to electric (e) L and S photons in both first and second quantization [20,21,24,45]. The m-photon types play a particularly important role in the interaction process between magnetic monopoles in near-field contact. An electric and a magnetic monopole cannot exchange m photons nor e photons.

Let us next turn our attention towards the sum of the longitudinal $[\mathbf{B}_L(\mathbf{r},t)]$ and propagating transverse near $[\mathbf{B}_T^{NF}(\mathbf{r},t)]$ fields. From Eq. (20) one obtains

$$c\mathbf{B}_L(\mathbf{r},t) = -\frac{1}{\varepsilon_0}\int_{-\infty}^{t}\mathbf{J}_L^m(\mathbf{r},t')dt' = -\frac{1}{\varepsilon_0}\mathbf{P}_L^m(\mathbf{r},t), \tag{149}$$

since $\mathbf{J}_L^m=\partial\mathbf{P}_L^m/\partial t$ [see the definition in Eq. (45)] and $\mathbf{P}_L^m(\mathbf{r},-\infty)=\mathbf{0}$ (assuming no permanent source polarization). The transverse part of the near field is given by

$$c\mathbf{B}_T^{NF}(\mathbf{r},t) = -\mu_0\int_{-\infty}^{\infty}\mathbf{G}_T^{NF}(\mathbf{R},\tau)\cdot\frac{\partial}{\partial t'}\mathbf{J}^m(\mathbf{r}',t')dt'd^3r', \tag{150}$$

with

$$\mathbf{G}_T^{NF} = -\frac{c^2\tau}{4\pi R^3}\Theta(\tau)\Theta\left(\frac{R}{c}-\tau\right)(\mathbf{U}-3\mathbf{e_R}\mathbf{e_R}), \tag{151}$$

as it appears from Eqs. (40) and (41), respectively. The integral expression for $c\mathbf{B}_T^{NF}$ can be simplified carrying out the integral over time. Thus

$$\begin{aligned}&\int_{-\infty}^{\infty}(t-t')\Theta(t-t')\Theta\left(\frac{R}{c}-t+t'\right)\frac{\partial}{\partial t'}\mathbf{J}^m(\mathbf{r}',t')dt'\\&=\int_{t-R/c}^{t}(t-t')\frac{\partial}{\partial t'}\mathbf{J}^m(\mathbf{r}',t')dt\\&=-\frac{R}{c}\mathbf{J}^m\left(t-\frac{R}{c}\right)+\int_{t-R/c}^{t}\mathbf{J}^m(\mathbf{r}',t')dt'.\end{aligned} \tag{152}$$

The last member of Eq. (152) was obtained by a partial integration. Using Eq. (45) one gets

$$\int_{-\infty}^{\infty}\tau\Theta(\tau)\Theta\left(\frac{R}{c}-\tau\right)\frac{\partial}{\partial t'}\mathbf{J}^m(\mathbf{r}',t')dt' = \mathbf{P}^m(\mathbf{r}',t)-\mathbf{P}^m\left(\mathbf{r}',t-\frac{R}{c}\right)-\frac{R}{c}\mathbf{J}^m\left(\mathbf{r}',t-\frac{R}{c}\right). \tag{153}$$

Gathering the information in Eqs. (150), (151), and (153) one reaches the following expression:

$$c\mathbf{B}_T^{NF}(\mathbf{r},t) = \frac{1}{\varepsilon_0}\int_{-\infty}^{\infty}\frac{1}{4\pi R^3}(\mathbf{U}-3\mathbf{e_R}\mathbf{e_R}) \cdot\left[\mathbf{P}^m(\mathbf{r}',t)-\mathbf{P}^m\left(\mathbf{r}',t-\frac{R}{c}\right) -\frac{R}{c}\mathbf{J}^m\left(\mathbf{r}',t-\frac{R}{c}\right)\right]d^3r'. \tag{154}$$

The term in Eq. (154) which involves $\mathbf{P}^m(\mathbf{r}',t)$ can be rewritten making use of the expression for the longitudinal delta function (a dyadic quantity), as this is given in spherical contraction viz. [24,25],

$$\boldsymbol{\delta}_L(\mathbf{R}) = \frac{1}{3}\delta(\mathbf{R})\mathbf{U}+\frac{1}{4\pi R^3}(\mathbf{U}-3\mathbf{e_R}\mathbf{e_R}). \tag{155}$$

Hence one obtains

$$\frac{1}{\varepsilon_0}\int_{-\infty}^{\infty}\frac{1}{4\pi R^3}(\mathbf{U}-3\mathbf{e_R}\mathbf{e_R})\cdot\mathbf{P}^m(\mathbf{r}',t)d^3r' = \frac{1}{\varepsilon_0}\int_{-\infty}^{\infty}\boldsymbol{\delta}_L(\mathbf{r}-\mathbf{r}')\cdot\mathbf{P}^m(\mathbf{r}',t)d^3r'-\frac{1}{3\varepsilon_0}\mathbf{P}^m(\mathbf{r},t) = \frac{1}{\varepsilon_0}\mathbf{P}_L^m(\mathbf{r},t)-\frac{1}{3\varepsilon_0}\mathbf{P}^m(\mathbf{r},t). \tag{156}$$

The first term in the last member of Eq. (156) is equal to $-c\mathbf{B}_L(\mathbf{r},t)$; see Eq. (149). Altogether, the following result is obtained for the sought after sum:

$$c\left[\mathbf{B}_L(\mathbf{r},t)+\mathbf{B}_T^{NF}(\mathbf{r},t)\right] = -\frac{1}{3\varepsilon_0}\mathbf{P}^m(\mathbf{r},t) - \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{1}{R^3}(\mathbf{U}-3\mathbf{e_R}\mathbf{e_R}) \cdot\left[\mathbf{P}^m\left(\mathbf{r}',t-\frac{R}{c}\right)+\frac{R}{c}\mathbf{J}^m\left(\mathbf{r}',t-\frac{R}{c}\right)\right]d^3r'. \tag{157}$$

It appears form Eq. (157) that the sum $\mathbf{B}_L(\mathbf{r},t)+\mathbf{B}_T^{NF}(\mathbf{r},t)$ satisfies the Einstein causality criterion outside the magnetic monopole domain, where $\mathbf{P}^m(\mathbf{r},t)=\mathbf{0}$. The retarded nature of

$$\mathbf{B}_L(\mathbf{r},t)+\mathbf{B}_T^{NF}(\mathbf{r},t) = \left[\mathbf{B}_L+\mathbf{B}_T^{NF}\right]\left(\mathbf{r},t-\frac{R}{c}\right) \tag{158}$$

in the rim zone is remarkable perhaps, since $\mathbf{B}_L$ is nonretarded and $\mathbf{B}_T^{NF}$ is nonvanishing in front of the light cone ($R>c\tau$); cf. Eq. (151).

In the near-field zone one also has a self-field contribution to the magnetic field, given by Eq. (44) in spherical contraction. The term $c\mathbf{B}_T^{SF}=-\mathbf{P}_T^m/(3\varepsilon_0)$ relates to the spatial confinement problem for a transverse photon emitted from a magnetic current density distribution $\mathbf{J}_T^m(\mathbf{r},t)$ [in complete analogy to the previously analyzed electric case, $\mathbf{J}_T^e(\mathbf{r},t)$ (Ref. [42])]. Since $\mathbf{P}_T^m=-\mathbf{P}_L^m$ outside the particle domain [$\boldsymbol{\delta}_T(\mathbf{R})=-\boldsymbol{\delta}_L(\mathbf{R})$ for $\mathbf{R}\neq\mathbf{0}$; see Eq. (31)], the confinement region for the photon has the same range as that of $\mathbf{B}_L+\mathbf{B}_T^{NF}$.

VIII. ELECTRON-PHOTON HAMILTONIAN IN AN EXTERNAL MAGNETIC MONOPOLE FIELD

Let us consider the photon mediated interaction between a single electron-magnetic monopole pair. It is assumed that the electromagnetic field of the magnetic monopole has a prescribed time dependence, so that only the electron and photon time evolutions are described by dynamical variables. For simplicity, it is further assumed that the monopoles are not in near-field contact. The interacting Dirac and photon fields hence are described by the relativistic Hamiltonian

$$H = H_P+H_R+H_I. \tag{159}$$

The particle Hamiltonian (H_P)

$$H_P = c\boldsymbol{\alpha}\cdot\left(\mathbf{p}^e-e\boldsymbol{\mathcal{A}}_T^m\right)+\beta mc^2 \tag{160}$$

includes the external transverse magnetic monopole field. In the standard choice, the four quantities $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$ and β are represented by the four-dimensional Dirac matrices [12,45]. The inclusion of the nonlocally transformed magnetic monopole vector potential in H_P originates in the fact that only the electron position coordinate is a dynamical variable in $\boldsymbol{\mathcal{A}}$. The radiation (R) (transverse photon) Hamiltonian (H_R) is given by

$$H_R = \int_{-\infty}^{\infty}\boldsymbol{\Psi}^\dagger\cdot\boldsymbol{\Psi}d^3r \tag{161}$$

in photon wave mechanics and the interaction (I) Hamiltonian (H_I) has the form

$$H_I = -ce\boldsymbol{\alpha}\cdot\mathbf{A}_T^e. \tag{162}$$

The main goal in this section is a determination of $\boldsymbol{\mathcal{A}}_T^m(\mathbf{r},t)$. It appears from Eq. (36) that the magnetic part ($\mathbf{B}_T^m$) of the transverse magnetic field ($\mathbf{B}_T$) is given by the propagator integral expression

$$c\mathbf{B}_T^m(\mathbf{r},t) = -\mu_0\int_{-\infty}^{\infty}g(R,\tau)\frac{\partial}{\partial t'}\mathbf{J}_T^m(\mathbf{r}',t')d^3r'dt'. \tag{163}$$

A partial integration in time (with vanishing contributions in the limits $t'=\pm\infty$) and use of the relation $\partial g/\partial t'=-\partial g/\partial t$ gives

$$c\mathbf{B}_T^m(\mathbf{r},t) = -\mu_0\frac{\partial}{\partial t}\int_{-\infty}^{\infty}g(R,\tau)\mathbf{J}_T^m(\mathbf{r}',t')d^3r'dt'. \tag{164}$$

Since $c\mathbf{B}_T^m=-\partial\mathbf{A}_T^m/\partial t$ [see Eq. (84)] one obtains

$$\mathbf{A}_T^m(\mathbf{r},t) = \mu_0\int_{-\infty}^{\infty}g(R,\tau)\mathbf{J}_T^m(\mathbf{r}',t')d^3r'dt'. \tag{165}$$

In order to relate $\boldsymbol{\mathcal{A}}_T^m$ to $\mathbf{A}_T^m$ we use the nonlocal transformation discussed in Sec. IV B. By means of the folding integral theorem, one has from Eq. (165)

$$\mathbf{A}_T^m(\mathbf{q},\omega) = \mu_0 g(\mathbf{q},\omega)\mathbf{J}_T^m(\mathbf{q},\omega), \tag{166}$$

where [40]

$$g(\mathbf{r},\omega) = \frac{c^2}{(cq)^2 - \omega^2} \tag{167}$$

is the Huygens scalar propagator in the $(\omega, \mathbf{q})$ domain. Inserting Eq. (166) in Eq. (92) gives

$$\boldsymbol{\mathcal{A}}_T^m(\mathbf{q},\omega) = -\frac{\mu_0\omega}{cq} g(\mathbf{q},\omega)\hat{\mathbf{q}} \times \mathbf{J}_T^m(\mathbf{q},\omega). \tag{168}$$

The Fourier integral expression in Eq. (90) finally takes the explicit form

$$\boldsymbol{\mathcal{A}}_T^m(\mathbf{r},t) = \mu_0 c \int_{-\infty}^{\infty} \frac{\omega}{q} \frac{\hat{\mathbf{q}} \times \mathbf{J}_T^m(\mathbf{q},\omega)}{\omega^2 - (cq)^2} e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \frac{d\omega\, d^3q}{(2\pi)^4}. \tag{169}$$

For specific $\mathbf{J}_T^m(\mathbf{q},\omega)$ functions it may be possible to carry out the ω integration by a residue calculation, with a retarded propagating form originating in the first-order pole located at $\omega = cq(> 0)$.

The analytical part of $\boldsymbol{\mathcal{A}}_T^m(\mathbf{r},t)$ appears in the radiation Hamiltonian; cf. Eqs. (46)–(48), (66), and (94). This part of $\boldsymbol{\mathcal{A}}_T^m(\mathbf{r},t)$ is obtained by a limitation of the ω integral in Eq. (169) to positive frequencies $[\int_{-\infty}^{\infty}(\ldots)d\omega \rightarrow \int_0^{\infty}(\ldots)d\omega]$. Also the analytical part of $\boldsymbol{\mathcal{A}}_T^e(\mathbf{r},t)$ enters H_R.

IX. CONCLUDING REMARKS

The scattering of a charged Dirac particle by an assumed fixed magnetic monopole involves a number of complicated problems related to relativistic electron dynamics, but the role of the photon in electric (e) and magnetic (g) monopole interactions seldom is discussed. In order to obtain a good physical understanding of fundamental aspects of the e-g interaction it is necessary to study the coupled photon-particle $(e+g)$ electrodynamics. Even in Coulomb-like (r^{-2}) interactions the photon plays a conceptually important role, despite the Einsteinian retardation being extremely small.

In the renowned quantum approach of Dirac it is attempted to treat the longitudinal (L) magnetic Coulomb field of the monopole ($\mathbf{B}_L$) as a transverse quantity, $\nabla \cdot \mathbf{B}_L = 0$. To qualify as a *genuine transverse (T) vector field*, the condition $\nabla \cdot (\mathbf{B}_T + \mathbf{B}_L) = \nabla \cdot \mathbf{B}_L = 0$ must be satisfied in the entire space. The condition fails in one point, viz., at the position of the magnetic point monopole. Once the inevitable present dynamical behavior of the monopole is taken into account the magnetic monopole near field has both L and T components and the total field is retarded. In the near-field zone an *apparently* nonretarded field component arises from the spatial photon localization problem. No propagation effects can be associated to this quantum localization phenomenon, however. For e and g poles outside each other's near-field zone the electromagnetic interaction solely is related to exchange of transverse photons.

In between classical and quantum electrodynamics stands photon wave mechanics, i.e., the first-quantized theory of the photon. This theory can be upgraded to second quantization and a number of phenomena difficult to implement in quantum electrodynamics are easily studied in photon wave mechanics. In the present work photon wave mechanics has been combined with a transverse electromagnetic propagator description. The combined theory is particularly convenient when the e-g interaction is treated on basis of a double-potential formalism, since compact and simple integral (propagator) expressions relate the transverse electric and magnetic vector potentials to their (e, g)-particle source current density distributions.

A space-time nonlocal transformation of the transverse electric and magnetic vector potentials constitute a key result of the present paper since it leads to expressions for the electromagnetic parts of the particle canonical momentum and angular momentum. These expressions, via the forms $\mathbf{p} \cdot (\mathbf{A}_T^e + \boldsymbol{\mathcal{A}}_T^m)$ and $\mathbf{p} \cdot (\mathbf{A}_T^m - \boldsymbol{\mathcal{A}}_T^e)$, allow one to uphold the minimum coupling principle in quantum electrodynamics with magnetic monopoles.

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