Fast Harmonic Chirp Summation

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ABSTRACT

The harmonic chirp signal model has only very recently been introduced for modelling approximately periodic signals with a time-varying fundamental frequency. A number of estimators for the parameters of this model have already been proposed, but they are either inaccurate, non-robust to noise, or very computationally intensive. In this paper, we propose a fast algorithm for the harmonic chirp summation method which has been demonstrated in the literature to be accurate and robust to noise. The proposed algorithm is orders of magnitude faster than previous algorithms which is also demonstrated via timing studies.

Index Terms— Harmonic chirp model, harmonic chirp summation, fast algorithm

1. INTRODUCTION

Many real-world signals are approximately periodic on a short timescale. These signals are encountered in many applications such as music processing [1, 2], speech processing [3, 4], sonar [5], order analysis [6], and electrocardiography [7]. Since the underlying parameters of most real-world signals are time-varying, the periodic assumption is no longer accurate when the signal is viewed on a larger timescale. An example of this is illustrated in Fig. 1 where the spectrogram of an accelerating car engine is shown. When viewed on a short time scale, the frequency components in the signal are approximately constant, but the spectrogram clearly reveals that they are increasing. Thus, we get a more accurate model of the signal by replacing the constant frequency components in the harmonic model with time-varying ones. The harmonic chirp model (HCM) is such a generalisation of the harmonic model where the frequency components are modelled as varying linearly in time, and we here discuss how the harmonic chirp summation (HCS) method can be implemented efficiently for the estimation of the intercept, i.e., the fundamental frequency, and the slope, i.e., the fundamental chirp rate, of such linearly varying frequency components.

The HCM has only very recently been used in [8] as an alternative to modelling non-stationary speech using amplitude modulation models [9]. The HCM was also used in the context of speech processing in [10, 11], but was considered in a more general framework in [12, 13] in which animal sound signals were analysed. In all of these papers, the complex-valued HCM was used although we know of no application where such signals naturally occur. Since the

![Fig. 1. A spectrogram of an accelerating car engine. The recording is downloaded from http://tinyurl.com/prosigorder.](image)
2. The harmonic chirp model

The real-valued harmonic chirp model (HCM) for a data set \( \{ x(n) \}_{n=0}^{N-1} \) is the real part of the complex-valued HCM introduced in [8] and given by

\[
x(n) = \sum_{i=1}^{l} A_i \cos(\theta_i(n)) + e(n)
\]

where \( n_0 \) is a start index, \( A_i \geq 0 \) is an amplitude, \( l \in \{1, \cdots, L\} \) is the model order, \( e(n) \in \mathbb{R} \) is additive noise, and \( \theta_i(n) \in [0, 2\pi) \) is the instantaneous phase. In the HCM, this phase is modelled with a second order polynomial as

\[
\theta_i(n) = \phi_i + \omega_0 n + \beta_0 n^2/2
\]

where \( \phi_i, \omega_0 \) and \( \beta_0 \) are the phase at \( n = 0 \), the fundamental frequency, and the fundamental chirp rate, respectively. The signal model in (1) can be rewritten into a vector form given by

\[
x = Z_l(\omega_0, \beta_0) \alpha_l + e
\]

with the following definitions

\[
x = [x(n_0) \cdots x(n_0 + N - 1)]^T
\]

\[
e = [e(n_0) \cdots e(n_0 + N - 1)]^T
\]

\[
Z_l(\omega, \beta) = \begin{bmatrix} C_l(\omega, \beta) & S_l(\omega, \beta) \end{bmatrix}
\]

\[
C_l(\omega, \beta) = \begin{bmatrix} c(\omega, \beta) & \cdots & c(\omega, l\beta) \end{bmatrix}
\]

\[
S_l(\omega, \beta) = \begin{bmatrix} s(\omega, \beta) & \cdots & s(\omega, l\beta) \end{bmatrix}
\]

\[
c(\omega, \beta) = \cos(\omega_0 n_0 + \frac{1}{2} \beta n_0^2)
\]

\[
s(\omega, \beta) = \begin{bmatrix} \sin(\omega_0 n_0 + \frac{1}{2} \beta n_0^2) & \cdots & \sin(\omega_0 n_0 + \frac{1}{2} \beta (n_0 + N - 1)^2) \end{bmatrix}^T
\]

\[
\alpha_l = \begin{bmatrix} a_1 & \cdots & a_l & -b_l & \cdots & -b_l \end{bmatrix}^T
\]

where \( a_i = A_i \cos(\phi_i) \) and \( b_l = A_l \sin(\phi_l) \). We assume that the signal has been Nyquist sampled so that the instantaneous frequency \( \omega_i(n) = \theta_i(n) + i(\omega_0 + \beta_0 n) \) satisfies

\[
0 < \omega_i(n) < \pi, \quad \text{for } i \in \{1, 2, \ldots, l\}.
\]

If this constraint is satisfied by \( \omega_i(n) \), then it is also satisfied for all lower model orders, so we here focus on \( \omega_i(n) \). As we argue in Sec. 3, the best choice of the start index is \( n_0 = -(N-1)/2 \). For this start index, the feasible region \( \mathbb{Q}_l \) for the fundamental frequency \( \omega_0 \) and chirp rate \( \beta_0 \) can be found by inserting the minimum and maximum values of \( \omega_i(n) \) into (12). Thus,

\[
\mathbb{Q}_l = \left\{ (\omega_0, \beta_0) \mid 0 < \omega_0 - |\beta_0| (N-1)/2 \right. \\
\left. \hfill \wedge \omega_0 + |\beta_0| (N-1)/2 < \pi/l \right\}.
\]

Often, the feasible region is not only determined by Nyquist sampling, but also by prior knowledge on the parameters. Here, we assume that such prior knowledge is represented by upper and lower bounds \( \omega_{\min}, \omega_{\max}, \beta_{\min}, \beta_{\max} \) and \( \mathbb{F}_0 \) on the fundamental frequency and chirp rate. This, the total feasible region is the intersection between \( \mathbb{Q}_l \) and these bounds, i.e.,

\[
\mathbb{P}_l = \left\{ (\omega_0, \beta_0) \in \mathbb{Q}_l \mid \omega_{\min} \leq \omega_0 \leq \omega_{\max} \wedge \beta_{\min} \leq \beta_0 \leq \beta_{\max} \right\}.
\]

The feasible sets for a number of different model orders are illustrated in Fig. 2.

2.1. Harmonic Chirp Summation

As shown in [8, 13], the nonlinear least squares (NLS) estimator of the fundamental frequency and chirp rate for a given \( l \) is

\[
(\hat{\omega}_0, \hat{\beta}_0) = \underset{(\omega_0, \beta_0) \in \mathbb{P}_l}{\text{argmax}} C_l(\omega_0, \beta_0)
\]

where the NLS objective is given by

\[
C_l(\omega_0, \beta_0) = x^T Z_l^T(\omega_0, \beta_0) Z_l(\omega_0, \beta_0)^{-1} Z_l^T(\omega_0, \beta_0) x.
\]

As alluded to in the introduction, the NLS objective has a very oscillatory behaviour, but the maximiser can be found reliably using a grid search over \( \mathbb{P}_l \) as in [13]. However, this is very costly since we must invert a \( 2l \times 2l \) matrix for all points on the grid (or solve a linear system of the same size) for all candidate model orders \( l \in \{1, \ldots, L\} \). In the harmonic chirp summation (HCS) method, this inversion is avoided by replacing the matrix with its asymptotic expression (in \( N \)). Specifically, we have from the results in [14] that

\[
\lim_{N \to \infty} 2N^{-1} Z_l^T(\omega_0, \beta_0) Z_l(\omega_0, \beta_0) = I_{2l}
\]

where \( I_{2l} \) is the \( 2l \times 2l \) identity matrix. Using this as an approximation for a finite \( N \), we obtain the approximate NLS objective

\[
J_l(\omega_0, \beta_0) = x^T Z_l^T(\omega_0, \beta_0) Z_l(\omega_0, \beta_0) z
\]

which is the objective of the HCS method. The cost of making the approximation based on (17) is that the HCS method breaks down when the fundamental frequency is low [13]. Low here refers to the number of periods in a segment of data. Thus, we can get a better time-frequency resolution by using the NLS estimator since we can work with shorter segment of data before the estimator breaks down.

2.1.1. Efficient implementation

A closer inspection of (18) reveals that it can be rewritten as

\[
J_l(\omega_0, \beta_0) = \sum_{i=1}^{l} \left| f^H(i\omega_0) (g^T(i\beta_0) \odot x) \right|^2 = \sum_{i=1}^{l} J_i(i\omega_0, i\beta_0)
\]

where \( \odot \) denotes element-wise multiplication and

\[
f(\omega) = [1 \exp(j\omega) \cdots \exp(j\omega(N-1))]^T
\]

\[
g(\beta) = [\exp(j\beta n_0) \cdots \exp(j\beta (n_0 + N - 1)^2)]^T.
\]

Thus, for each candidate \( \beta_0 \), we can efficiently compute the objective on a uniform grid using an FFT algorithm. Moreover, the objective \( J_i(i\omega_0, i\beta_0) \) evaluated on a uniform grid covering \( \mathbb{Q}_l \) acts
as a mother objective for the objectives of higher model orders since each of these can be computed by recursively summing the appropriate elements of $J_l(\omega, \beta)$. Finally, $J_l(\omega, -\beta) = J_l(-\omega, -\beta)$ so if we use a symmetric rate grid around 0, we basically get one half of the objective evaluations for free since the FFT algorithm already computes the objective for negative candidate fundamental frequencies. In total, the time complexity is $O(KF \log F)$ for calculating $J_l(\omega, \beta)$ and $O(\sum_{i=1}^{L} (K/i)(F/i)) = O(KF)$ for calculating $\{J_l(\omega, \beta)\}_{l=2}^{L}$ where $K$ and $F$ are the number of candidate fundamental chirp rates for $l = 1$ and the number of FFT grid points, respectively. Thus, the total time complexity of our proposed algorithm for the HCS method is dominated by $O(KF \log F)$ which means that, in terms of the order of time complexity, we get the objectives $\{J_l(\omega, \beta)\}_{l=2}^{L}$ for free. In Sec. 4, we establish how $K$ and $F$ depend on $N$ and $L$.

3. THE START INDEX, THE HESSIAN, AND THE CRLB

Traditionally, the start index $n_0$ is set to either 0 or 1 without much thought. This was also the case in [13]. In many applications, however, even a non-integer choice for this index can be made since the important information lies in the distance between the sampling times, not an arbitrary starting time. This was also noted in [15] where it was shown that selecting the start index as $n_0 = -(N - 1)/2$ minimised the CRLB of the phase of a chirp signal (i.e., the HCM with $l = 1$). This indexing was also adopted in [8, 11], but the consequences of this choice for the general HCM with $l > 1$ were not analysed, and we, therefore, do this here. Specifically, we show (asymptotically) that a start index of $n_0 = -(N - 1)/2$ 1) diagonalises the Hessian of $J_l(\omega, \beta)$, and 2) minimises the CRLB of any unbiased estimator of the fundamental frequency.

3.1. The Hessian

Unfortunately, the derivation of the Hessian is lengthy, so we skip it here, but if we make use of the asymptotic result in (17) and assume that the SNR is high so that $x \approx Z(\omega, \beta)\alpha$, it can be shown that

$$H_l(\xi) = \frac{\partial^2}{\partial \xi \partial \xi} J_l(\xi) \approx -\frac{N^2(N^2 - 1)}{24} \sum_{i=1}^{L} A_i^2 \times \left[\frac{1}{n_0 + \frac{N-1}{2}} \frac{n_0 + \frac{N-1}{2}}{\Delta \omega} + \frac{n_0 + \frac{N-1}{2}}{\Delta \beta} \right]$$

(21)

where $\xi = [\omega, \beta]^T$. Clearly, setting $n_0 = -(N - 1)/2$ makes the Hessian diagonal, and this can make simple iterative procedures for refining a coarse grid-based estimate converge much faster to the maximum. In Fig. 3, an example of the objective for $n_0 = -(N - 1)/2$ is shown. For comparison, an example for $n_0 = 0$ can be found in [13]. Another advantage of having a diagonal Hessian is that the grid size can be designed for the fundamental frequency and chirp rate independently. We discuss this more thoroughly in Sec. 4.

3.2. The CRLB

The asymptotic CRLB for the parameter vector $\eta_l = [\alpha^T \omega \beta]^T$ can be derived under the same approximations as made in the derivation of the Hessian. Assuming that the noise is white and Gaussian with variance $\sigma^2$, we obtain the Fisher information matrix (FIM) in (22) on the top of page 4 where $n_l = [n_0 \cdots n_0 + N - 1]^T$ and $l = [1 \cdots l]^T$. As we are here only interested in the bound on the fundamental frequency and chirp rate, we only focus on the lower $2 \times 2$ matrix of the asymptotic inverse FIM which is given by

$$X_l^{-1}(\omega, \beta) = \frac{24n^2}{(N^2 - 1)(N^2 - 4)} \sum_{i=1}^{L} A_i^2 \times \left[\frac{(N^2 - 4) + 60 (n_0 + \frac{N-1}{2})^2}{-60 (n_0 + \frac{N-1}{2})^2} \right].$$

(23)

The CRLB for $\beta$ does not depend on the start index, but the CRLB for $\omega$ does and is minimised by $n_0 = -(N - 1)/2$. In fact, the ratio between the CRLB for $\omega$ for $n_0 = 0$ and $n_0 = -(N - 1)/2$ approaches 16 for $N \rightarrow \infty$. Thus, we get a much more accurate estimate of the fundamental frequency for $n_0 = -(N - 1)/2$.

4. GRID SIZE SELECTION

As we have alluded to earlier, we recommend that the maximiser of the HCS objective in (18) is found by evaluating the objective over a coarse grid followed by a refinement procedure such as the Nelder-Mead method [16]. This ensures that the number of required objective evaluations is low. In [13], the authors did not use a refinement step and recommended that the number of grid points was based on the CRLB. Specifically, they suggested that $F = O(N^{5/2})$ and $K = O(N^{3/2})$ grid points were used for the fundamental frequency and chirp rate, respectively. By following the procedure suggested in [17] instead, we show how a much lower value can be selected by combining a coarse grid search with a refinement step.

Suppose that we set the grid sizes in $\Delta \xi = [\Delta \omega \Delta \beta]^T$ so that the objective has decreased by a factor of $g > 1$ when we move $\pm \Delta \xi$ away from the maximiser $\hat{\xi}$. That is,

$$J_l(\hat{\xi} \pm \Delta \xi) = J_l(\hat{\xi})/g.$$ 

(24)

Unfortunately, this equation cannot be solved for $\Delta \xi$. However, we can obtain an approximate solution by replacing $J_l(\hat{\xi} \pm \Delta \xi)$ by its second order Taylor approximation around $\hat{\xi}$. Since $J_l(\xi) \approx (N^2/4) \sum_{i=1}^{L} A_i^2$ and the Hessian in (21) is diagonal for $n_0 = -(N - 1)/2$, we obtain that

$$\Delta \omega_0 \approx \sqrt{\frac{2 - 1}{g} \frac{J_l(\hat{\xi})}{H_l(\hat{\xi})_{11}}} > \sqrt{\frac{12g - 1}{g} \frac{1}{NL}}$$

(25)

and

$$\Delta \beta_0 \approx \sqrt{\frac{2 - 1}{g} \frac{J_l(\hat{\xi})}{H_l(\hat{\xi})_{22}}} > \sqrt{\frac{12g - 1}{g} \frac{\sqrt{60}}{N^2L}}$$

(26)

where the lower bounds are obtained by selecting the worst case value of the amplitudes, i.e., $A_i = 0$ for $i = 1, \ldots, L - 1$ and

![Fig. 3. Example of an objective function for $N = 50$, $l = 5$, $n_0 = -(N - 1)/2$, and no noise. The red dot indicates the maximum, and the red rectangle shows a scaled grid cell.](image367to549.png)
Due to the limited space and that we here propose a fast algorithm of an already known estimator, we neither evaluate the accuracy of the estimator nor apply it to real world data. For such evaluations, we refer the interested reader to [8, 11, 13]. Instead, we here only find a suitable value for the grid cells and evaluate the computation times suggesting that using a combination of a coarse grid followed by a refinement works as well as using a much finer grid with refinement.

The former approach, however, dramatically reduces the computation time as illustrated in Fig. 5 for various data length and model orders\(^2\). The fine-grid version of our HCS algorithm is roughly 50 times faster than the naïve HCS algorithm in [13]. Since the only difference between the two HCS algorithms is the coarseness of the grid we are confident that our HCS algorithm is much more than a factor of 50 times faster than the naïve HCS algorithm in [13].

6. CONCLUSION

In this paper, we have developed a fast algorithm for the harmonic chirp summation method which has been demonstrated in the literature to be an accurate and robust estimator of the parameters in the harmonic chirp model. The proposed estimator is orders of magnitudes faster than a recently suggested algorithm. The speed up was obtained by using an FFT algorithm in the evaluation of the cost function and by using a combination of a coarse grid search and a local refinement instead of a fine grid search. Numerical evaluations showed that using the former approach is as accurate as the latter, but leads to a significantly lower computation time. We believe that our fast algorithm for the HCS method now makes the method practically useful in a large number of applications.

\(^2\)For orders 3 and 4, the HCS (10 dB) method is costly compared to higher orders. This is due to MATLAB’s FFT implementation whose time complexity can be quite high when the FFT size is prime or has large prime factors.
7. REFERENCES


