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Adaptive estimating function inference for non-stationary determinantal point processes

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Abstract

Estimating function inference is indispensable for many common point process models where the joint intensities are tractable while the likelihood function is not. In this paper we establish asymptotic normality of estimating function estimators in a very general setting of non-stationary point processes. We then adapt this result to the case of non-stationary determinantal point processes which are an important class of models for repulsive point patterns. In practice often first and second order estimating functions are used. For the latter it is common practice to omit contributions for pairs of points separated by a distance larger than some truncation distance which is usually specified in an ad hoc manner. We suggest instead a data-driven approach where the truncation distance is adapted automatically to the point process being fitted and where the approach integrates seamlessly with our asymptotic framework. The good performance of the adaptive approach is illustrated via simulation studies for non-stationary determinantal point processes and by an application to a real dataset.

Keywords: asymptotic normality, determinantal point processes, estimating functions, joint intensities, non-stationary, repulsive.

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1 Introduction

A common feature of spatial point process models (except for the Poisson process case) is that the likelihood function is not available in a simple form. Numerical approximations of the likelihood function are available (see e.g. Møller and Waagepetersen, 2004, 2007, for reviews) but the approaches are often computationally demanding and the distributional properties of the approximate maximum likelihood estimates may be difficult to assess. Therefore much work has focused on establishing computationally simple estimation methods that do not require knowledge of the likelihood function.

In this paper we focus on estimation methods for point processes which have known joint intensity functions. This includes many cases of Cox and cluster point process models (Møller and Waagepetersen, 2004; Illian *et al.*, 2008; Baddeley *et al.*, 2015) as well as determinantal point processes (Macchi, 1975; Soshnikov, 2000; Shirai and Takahashi, 2003; Lavancier *et al.*, 2015). These classes of models are quite different since realizations of Cox and cluster point processes are aggregated while determinantal point processes produce regular point pattern realizations.

Knowledge of an nth order joint intensity enables the use of the so-called Campbell formulae for computing expectations of statistics given by random sums indexed by n-tuples of distinct points in a point process. Unbiased estimating functions can then be constructed from such statistics by subtracting their expectations. So far mainly the cases of first and second order joint intensities have been considered where the first order joint intensity is simply the intensity function.

Theoretical results have been established in a variety of special cases of first and second order estimating functions for Cox and cluster processes (Schoenberg, 2005; Guan, 2006; Waagepetersen, 2007; Guan and Loh, 2007; Waagepetersen and Guan, 2009) and for the closely related Palm likelihood estimators (Tanaka *et al.*, 2008; Prokešová and Jensen, 2013; Prokešová *et al.*, 2016). The common general structure of the estimating functions on the other hand calls for a general theoretical set-up which is the first contribution of this paper. Our set-up also covers third or higher order estimating functions and combinations of such estimating functions, providing a general unifying framework.

The literature on statistical inference for continuous determinantal point processes is quite limited. A Bayesian approach is considered in Affandi *et al.* (2014), while likelihood and minimum contrast estimation methods

are discussed in Lavancier *et al.* (2015). In fact, maximum likelihood is not feasible in general and only an approximated version of the likelihood is proposed in the stationary case in Lavancier *et al.* (2015), without theoretical guarantees. On the other hand consistency and asymptotic normality of minimum contrast estimators based on the pair correlation function or the Ripley's K function have been established for determinantal point processes in Biscio and Lavancier (2017), but only in the stationary case. Based on the general set-up our second main contribution is to provide a detailed theoretical study of estimating function estimators for general non-stationary determinantal point processes.

Specializing to second-order estimating functions, a common approach (Guan, 2006; Tanaka *et al.*, 2008) is to restrict the random sum to pairs of R-close points for some user-specified $R > 0$. This may lead to faster computation and improved statistical efficiency. The properties of the resulting estimators depend strongly on R but only ad hoc guidance is available for the choice of R. Moreover, it is difficult to account for ad hoc choices of R when establishing theoretical results. Our third contribution is a simple intuitively appealing adaptive choice of R which leads to a theoretically tractable estimation procedure. We demonstrate its usefulness in simulation studies for determinantal point processes as well as an example of a cluster process. The practical advantage of the adaptive choice is further illustrated by an application to a dataset of locations of Japanese pines.

2 Estimating functions based on joint intensities

A point process X on $\mathbb{R}^d$, $d \geqslant 1$, is a locally finite random subset of $\mathbb{R}^d$. For $B \subseteq \mathbb{R}^d$, we let $N(B)$ denote the random number of points in $X \cap B$ and $|B|$ the Lebesgue measure of B. That X is locally finite means that $N(B)$ is finite almost surely whenever B is bounded. The so-called joint intensities of a point process are described in Section 2.1. In this paper we mainly focus on determinantal point processes, detailed in Section 3. A prominent feature of determinantal point processes is that they have known joint intensity functions of any order.

2.1 Joint intensity functions and Campbell formulae

For integer $n \geqslant 1$, the joint intensity $\rho^{(n)}$ of nth order is defined by

$$\mathbb{E} \sum_{u_1,\ldots,u_n \in X}^{\neq} \mathbb{1}_{u_1 \in B_1,\ldots,u_n \in B_n} = \int_{\times_{i=1}^n B_i} \rho^{(n)}(u_1,\ldots,u_n)\mathrm{d}u_1 \cdots \mathrm{d}u_n \qquad (1)$$

for Borel sets $B_i \subseteq \mathbb{R}^d$, $i = 1,\ldots,n$, assuming that the left hand side is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. The $\neq$ over the summation sign means that the sum is over pairwise distinct points in X. Of special interest are the cases $n = 1$ and $n = 2$ where the intensity function $\rho = \rho^{(1)}$ and the second order joint intensity $\rho^{(2)}$ determine the first and second order moments of the count variables $N(B)$, $B \subseteq \mathbb{R}^d$. The pair correlation function $g(u,v)$ is defined as

$$g(u,v) = \frac{\rho^{(2)}(u,v)}{\rho(u)\rho(v)}$$

whenever $\rho(u)\rho(v) > 0$ (otherwise we define $g(u,v) = 0$). The product $\rho(u)g(u,v)$ can be interpreted as the intensity of X at u given that $v \in X$. Hence $g(u,v) > 1$ (< 1) means that presence of a point at v increases (decreases) the likeliness of observing yet another point at u. The Campbell formula

$$\mathbb{E} \sum_{u_1,\ldots,u_n \in X}^{\neq} f(u_1,\ldots,u_n) = \int f(u_1,\ldots,u_n)\rho^{(n)}(u_1,\ldots,u_n)\mathrm{d}u_1 \cdots \mathrm{d}u_n$$

follows immediately from the definition of $\rho^{(n)}$ for any non-negative function $f : (\mathbb{R}^d)^n \to [0,\infty[$.

2.2 A general asymptotic result for estimating functions

Consider a parametric family of distributions $\{\mathbb{P}_\theta : \theta \in \Theta\}$ of point processes on $\mathbb{R}^d$, where Θ is a subset of $\mathbb{R}^p$. We assume a realization of the point process X with distribution $\mathbb{P}_{\theta^*}$, $\theta^* \in \mathrm{Int}(\Theta)$, is observed on a bounded window $W_n \subset \mathbb{R}^d$. We estimate the unknown parameter θ^* by the solution

$\hat{\theta}_n$ of $e_n(\theta) = 0$ (or one of the solutions if there are many) where

$$e_n(\theta) = \begin{pmatrix} \sum^{\neq}_{u_1,\cdots,u_{q_1}\in X\cap W_n} f_1(u_1,\cdots,u_{q_1};\theta) - \int_{W_n^{q_1}} f_1(u;\theta)\rho^{(q_1)}(u;\theta)\mathrm{d}u \\ \vdots \\ \sum^{\neq}_{u_1,\cdots,u_{q_l}\in X\cap W_n} f_l(u_1,\cdots,u_{q_l};\theta) - \int_{W_n^{q_l}} f_l(u;\theta)\rho^{(q_l)}(u;\theta)\mathrm{d}u \end{pmatrix}$$

for l given functions $f_i : (\mathbb{R}^d)^{q_i} \times \Theta \to \mathbb{R}^{k_i}$ such that $\sum_i k_i = p$.

A basic assumption for the following theorem is that a central limit theorem is available for $e_n(\theta^*)$ (assumption (X3)). In addition to this, a number of technical assumptions (F1) through (F3) (or (F3')), (X1) and (X2) regarding existence and differentiability of joint intensities as well as differentiability of the f_i are needed. All these conditions are detailed in Appendix A while the proof of the theorem is given in the supplementary material.

Theorem 2.1. *Under Assumptions* (F1) *through* (F3) *(or* (F3')*),* (X1) *and* (X2)*, with a probability tending to one as $n \to \infty$, there exists a $|W_n|^{1/2}$-consistent sequence of roots $\hat{\theta}_n$ of the estimating equations $e_n(\theta) = 0$. Precisely, for all $\varepsilon > 0$, there exists $A > 0$ such that*

$$\mathbb{P}(\exists \hat{\theta}_n : e_n(\hat{\theta}_n) = 0 \text{ and } |W_n|^{1/2} \|\hat{\theta}_n - \theta^*\| < A) > 1 - \varepsilon$$

for a sufficiently large n.

Moreover, if (X3) *holds true, then*

$$|W_n|\Sigma_n^{-1/2} H_n(\theta^*)(\hat{\theta}_n - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I_p),$$

where $\Sigma_n = Var(e_n(\theta^*))$, $H_n(\theta^*)$ is defined in (F3)*, and I_p is the $p\times p$ identity matrix.*

Remark 2.2. *While the parameter θ^* is generally uniquely defined (in the sense that $\theta \mapsto \mathbb{P}_\theta$ is injective) and verifies $\mathbb{E}(e_n(\theta^*)) = 0$, the solution to $e_n(\theta) = 0$ may not be unique. The above theorem states that there exists a consistent and asymptotically Gaussian sequence of solutions, but unicity is not guaranteed. This drawback is unfortunately common in most asymptotic results for estimating functions inference, see the references in introduction, Sørensen (1999), the handbook by Heyde (1997) or the discussion in (Van der Vaart, 2000, Section 5.6). Nonetheless it can be proved that the solution is unique for n sufficiently large whenever $\lim_{n\to\infty} e_n(\theta)/|W_n|$ admits a unique zero, see Jacod and Sørensen (2017). But simple sufficient conditions to ensure the latter condition elude us.*

2.3 Second order estimating functions

Referring to the previous section, much attention has been devoted to instances of the case $l = 1$, $q_1 = 2$ and $k_1 = p$. In this case we obtain a second-order estimating function of the form

$$e_n(\theta) = \sum_{u,v \in X \cap W_n}^{\neq} f(u, v; \theta) - \int_{W_n^2} f(u, v; \theta) \rho^{(2)}(u, v; \theta) \mathrm{d}u\mathrm{d}v. \tag{2}$$

In Guan (2006), the author noted that for computational and statistical efficiency it may be advantageous to use only close pairs of points rather than all pairs of points. Thus in (2) it is common practice to introduce an indicator $\mathbb{1}_{\|u-v\| \leqslant R}$ for some constant $0 < R$ or choose f so that $f(u, v) = 0$ whenever $\|u - v\| > R$. We discuss a method for choosing R in Section 2.4.

The general form (2) includes e.g. the score functions of second-order composite likelihood (Guan, 2006; Waagepetersen, 2007) and Palm likelihood functions (Tanaka *et al.*, 2008; Prokešová and Jensen, 2013; Prokešová *et al.*, 2016) as well as score functions of minimum contrast object functions based on non-parametric estimates of summary statistics as the Ripley's K or the pair correlation function. For the second-order composite likelihood defined in equation (4) in Guan (2006),

$$f(u, v; \theta) = \frac{\nabla_\theta \rho^{(2)}(u, v; \theta)}{\rho^{(2)}(u, v; \theta)} - \frac{\int_{W^2} \nabla_\theta \rho^{(2)}(u, v; \theta) \mathrm{d}u\mathrm{d}v}{\int_{W^2} \rho^{(2)}(u, v; \theta) \mathrm{d}u\mathrm{d}v}$$

while

$$f(u, v; \theta) = \frac{\nabla_\theta \rho^{(2)}(u, v; \theta)}{\rho^{(2)}(u, v; \theta)} \tag{3}$$

for the second-order composite likelihood proposed in Waagepetersen (2007). The score of the Palm likelihood as generalized to the inhomogeneous case in Prokešová *et al.* (2016) is obtained with

$$f(u, v; \theta) = \frac{\nabla_\theta \frac{\rho^{(2)}(u,v;\theta)}{\rho(u;\theta)}}{\rho^{(2)}(u, v; \theta)/\rho(u; \theta)} - \frac{1}{N(W) - 1} \int_W \nabla_\theta \left(\frac{\rho^{(2)}(u, w; \theta)}{\rho(u; \theta)} \right) \mathrm{d}w.$$

In Prokešová *et al.* (2016), the authors also regarded the second-order composite likelihood proposed in Waagepetersen (2007) as a generalization of the stationary case Palm likelihood but the interpretation as a second-order composite likelihood given in Waagepetersen (2007) is more straightforward.

Considering a class of estimating functions of the form (2) a natural question is what is the optimal choice of f? A solution to this problem is provided in Deng *et al.* (2017) where an approximation of the optimal f is obtained by solving numerically a certain integral equation. This yields a statistically optimal estimation procedure but is computationally demanding and requires specification of third and fourth order joint intensities. When computational speed and ease of use is an issue, there is still scope for simpler methods. Moreover, given several (simple) estimation methods, it is possible to combine them adaptively in order to build a final estimator that achieves better properties than each initial estimator, see Lavancier and Rochet (2016, 2017).

2.4 Adaptive version

Consider second-order composite likelihood using (3) but only R close pairs. The resulting weight function is then of the form

$$f_R(u, v; \theta) = \mathbb{1}_{\|u-v\| \leqslant R} \frac{\nabla_\theta \rho^{(2)}(u, v; \theta)}{\rho^{(2)}(u, v; \theta)}. \tag{4}$$

As mentioned in the previous section, using only R close pairs may be beneficial both for statistical efficiency and computational tractability. However, the possible improvement depends strongly on the chosen R. Simulation studies such as in Prokešová *et al.* (2016) and Deng *et al.* (2017) usually compare results for several values of R corresponding to different multiples of some parameter associated with 'range of correlation'. For a cluster process this parameter could e.g. be the standard deviation of the distribution for dispersal of offspring around parents. For a determinantal point process the parameter would typically be a correlation scale parameter in the kernel of the determinantal point process, see Section 3. In practice these parameters are not known and among the quantities that need to be estimated. In Guan (2006) it is suggested to choose an R that minimizes a goodness of fit criterion for the fitted point process model while the choice of R in Tanaka *et al.* (2008) and Waagepetersen and Guan (2009) is done by inspection of a non-parametric estimate of the pair correlation function (a similar appproach is suggested by Heagerty and Lele (1998) and Bai *et al.* (2014) in the context of pairwise composite likelihood for random fields). Both approaches imply extra work and ad hoc decisions by the user and it becomes very complex to determine the statistical properties of the resulting parameter estimates.

A typical behaviour of many pair correlation functions is that $g(u, v; \theta)$ converges to a limiting value of 1 when $\|u-v\|$ increases and $|g(u, v; \theta)-1| \leqslant M(u, v; \theta)$ where

$$M(u, v; \theta) = \max_{s \in \{u,v\}} |g(s, s; \theta) - 1|.$$

Note that for DPPs, $M(u, v; \theta) = 1$ (see the next section) and for stationary point processes, $M(u, v; \theta)$ does not depend on u and v. If $g(u, v; \theta) = 1$ for $\|u - v\| > r_0$ then counts of points are uncorrelated when they are observed in regions separated by a distance of r_0.

Following the idea that R should depend on some range property of the point process we therefore suggest to replace the constraint $\|u - v\| < R$ in (4) by the constraint

$$\frac{|g(u, v; \theta) - 1|}{M(u, v; \theta)} > \varepsilon,$$

for a small ε. If e.g. $\varepsilon = 1\%$ this means that we only consider pairs of points (u, v) so that the difference between $g(u, v; \theta)$ and the limiting value 1 is within 1% of the maximal value $M(u, v; \theta)$. Note that this choice of pairs of points is adaptive in that it depends on θ.

We then modify the function f_R to be

$$f_{\text{adap}}(u, v; \theta) = w\left(\varepsilon \frac{M(u, v; \theta)}{g(u, v; \theta) - 1}\right) \frac{\nabla_\theta \rho^{(2)}(u, v; \theta)}{\rho^{(2)}(u, v; \theta)} \tag{5}$$

where w is some weight function of bounded support $[-1, 1]$. Later on, when establishing asymptotic results, we will also assume that w is differentiable. A common example of admissible weight function is $w(r) = e^{1/(r^2-1)}$ for $-1 \leqslant r \leqslant 1$, while $w(r) = 0$ otherwise. The user needs to specify a value of ε but in contrast to the original tuning parameter R, ε has an intuitive meaning independent of the underlying point process. We choose $\varepsilon = 1\%$.

We emphasize that choosing $\varepsilon = 1\%$ is not necessarily optimal. An optimal ε might be found by maximizing the Godambe information as a function of ε but this is not straightforward and the computational advantages of our approach would be lost. In fact, if Godambe optimality is key, we suggest to consider the previously mentioned approach by Deng *et al.* (2017) to identify an optimal second order estimating function.

3 Asymptotic results for determinantal point processes

A point process X is a determinantal point process (DPP for short) with kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ if for all $n \geqslant 1$, the joint intensity $\rho^{(n)}$ exists and is of the form

$$\rho^{(n)}(u_1, \ldots, u_n) = \det[K](u_1, \ldots, u_n)$$

for all $\{u_1, \ldots, u_n\} \subset \mathbb{R}^d$, where $[K](u_1, \ldots, u_n)$ is the matrix with entries $K(u_i, u_j)$. The intensity function is thus $\rho(u) = K(u, u)$, $u \in \mathbb{R}^d$. If a determinantal point process with kernel K exists it is unique. General conditions for existence are presented in Lavancier *et al.* (2015). In particular, if K admits the form

$$K(u, v) = \sqrt{\rho(u)\rho(v)}C(u - v) \tag{6}$$

for a function $C : \mathbb{R}^d \to \mathbb{R}$ with $C(0) = 1$, then a sufficient condition for existence of a DPP with kernel K is that ρ is bounded and that C is a square integrable continuous covariance function with spectral density bounded by $1/\|\rho\|_\infty$. The normalization $C(0) = 1$ ensures that ρ is the intensity of the DPP.

We now consider a parametric family of DPPs on $\mathbb{R}^d$ with kernels K_θ where $\theta \in \Theta$ and $\Theta \subseteq \mathbb{R}^p$ (see Lavancier *et al.*, 2015; Biscio and Lavancier, 2016, for examples of such families). Henceforth, we assume that K_θ is symmetric, continuous and the DPP with kernel K_θ exists for all $\theta \in \Theta$. Note that in general, it is possible that two different kernels generate the same DPP distribution. This identifiability issue especially arises in the case of a discrete state space, where the distribution of a DPP is only identified up to flips of the signs of the rows and columns of its matrix kernel (see Engel and Schneider (1980) or Rising *et al.* (2015)). However, in the continuous case, corresponding to our framework, the kernel of a DPP is uniquely determined whenever the intensity function is positive, see Proposition S.3.1 and its corollary in the supplementary material. Assuming a positive intensity function is not restrictive for statistical applications of DPPs.

An expression for the likelihood of a DPP on a bounded window is provided in Lavancier *et al.* (2015), where likelihood based inference for stationary DPPs is discussed. However, the expression depends on a spectral representation of K which is rarely known in practice and must be approximated numerically. Letting n denote the number of observed points, the likelihood further requires the computation of an $n \times n$ dense matrix which

can be time consuming for large n. As an alternative, minimum contrast estimation is considered in Biscio and Lavancier (2016), based on the pair correlation function or Ripley's K-function, but only for stationary DPPs. In the following, we consider general non-stationary DPPs and the estimator $\hat{\theta}_n$ obtained by solving $e_n(\theta) = 0$ where e_n is given by (2). Note that the distribution for any classical parametric DPP model (showcased in Lavancier *et al.*, 2015; Biscio and Lavancier, 2016) is uniquely determined by its first two order intensity functions, in the sense that $\theta \mapsto (\rho(.;\theta), \rho^{(2)}(.,.;\theta))$ is injective. This justifies the use of second order estimating functions for DPPs.

We establish in Section 3.1 using Theorem 2.1 the asymptotic properties of the estimate $\hat{\theta}_n$ where e_n is given by (2) for a wide class of test functions f. In Section 3.2, we focus on a particular case of the DPP model, where the parameter $\theta = (\beta, \psi)$ can be separated into a parameter β only appearing in the intensity function and a parameter ψ only appearing in the pair correlation function. Following Waagepetersen and Guan (2009), it is natural to consider a two-step estimation procedure where in a first step β is estimated by a Poisson likelihood score estimating function, which provides a consistent estimate of the intensity Schoenberg (2005), and in a second step the remaining parameter ψ is estimated by a second order estimating function as in (2), where β is replaced by $\hat{\beta}_n$ obtained in the first step. The asymptotic properties of this two-step procedure again follow as a special case of Theorem 2.1.

3.1 Second order estimating functions for DPPs

In this part and in the rest of the document, we consider the following notation. For any set $W \subset \mathbb{R}^d$ and $r > 0$, we write $W \oplus r := \bigcup_{x \in W} B(x, r)$ and $W \ominus r := \{x \in W, B(x, r) \subset W\}$ for the dilation and erosion of the set W where $B(x, r)$ denotes the ball centered in x with radius r.

We assume a realization of a DPP X with kernel K_{θ^*}, $\theta^* \in \mathrm{Int}(\Theta)$, is observed on a bounded window $W_n \subset \mathbb{R}^d$. We estimate the unknown parameter θ^* by the solution $\hat{\theta}_n$ of $e_n(\theta) = 0$ where $e_n(\theta)$ is given by (2) for a given $\mathbb{R}^p$-valued function f. Therefore, we are in a special case of the set-up in Section 2.2 with $l = 1$, $q_1 = 2$, $k_1 = p$ and we assume that $f_1 = f$ satisfies the assumptions (F1) through (F3) (or (F3')) listed in Appendix A. The condition (F1) in this case demands that $\theta \mapsto f(u, v; \theta)$ is twice continuously differentiable in a neighbourhood of θ^* and for θ in this neighbourhood, the

derivatives are bounded with respect to (u, v) uniformly in θ. Moreover, from (F2), there exists $R > 0$ such that for all θ in a neighbourhood of θ^*,

$$f(u, v; \theta) = 0 \quad \text{if} \quad \|u - v\| > R. \tag{7}$$

Concerning (F3) (or (F3')), this condition controls the asymptotic behaviour of the matrix $H_n(\theta)$ given by

$$H_n(\theta) = \frac{1}{|W_n|} \int_{W_n^2} f(u, v; \theta) \nabla_\theta \rho^{(2)}(u, v; \theta)^T \mathrm{d}u\mathrm{d}v,$$

where we recall that in this setting

$$\rho^{(2)}(u, v; \theta) = K_\theta(u, u)K_\theta(v, v) - K_\theta(u, v)^2. \tag{8}$$

The assumptions (F3) and (F3') are technical and needed for the consistency of the estimation procedure. When H_n is a symmetric matrix, assumption (F3) seems simpler to verify than (F3'). As an important example, when f is defined as in (5), we prove in Lemmas 3.2 and 3.3 that (F3) is generally satisfied even if X is not stationary.

Finally, as shown in the proof of Theorem 3.1 below, the assumptions (X1) through (X3) in Theorem 2.1 are implied by the following:

(D1) $\theta \mapsto K_\theta(u, v)$ is twice continuously differentiable in a neighborhood of θ^*, for all $u, v \in \mathbb{R}^d$. Moreover, the first and second derivative of K_θ with respect to θ are bounded with respect to $u, v \in \mathbb{R}^d$ uniformly in θ in a neighborhood of θ^*.

(D2) The kernel $K_{\theta*}$ satisfies, for some $\varepsilon > 0$,

$$\sup_{\|u-v\|>r} K_{\theta*}(u, v) = o(r^{-(d+\varepsilon)/2}).$$

(D3) $\liminf_n \lambda_{\min}(|W_n|^{-1}\Sigma_n) > 0$ where $\Sigma_n := \mathrm{Var}(e_n(\theta^*))$ and $\lambda_{\min}(|W_n|^{-1}\Sigma_n)$ denotes the smallest eigenvalue of $|W_n|^{-1}\Sigma_n$.

(W) $\exists \varepsilon > 0$ s.t. $|\partial W_n \oplus (R + \varepsilon)| = o(|W_n|)$, where ∂ in this context denotes the boundary of a set, R is defined in (7), and $|W_n| \to \infty$, as $n \to \infty$.

Let us briefly comment on these assumptions. (D1) is a standard regularity assumption. Condition (D2) is not restrictive since all standard parametric kernel families satisfy $\sup_{\|u-v\|>r} K_\theta(u,v) = O(r^{-(d+1)/2})$, including the most repulsive stationary DPP (see Lavancier *et al.*, 2015; Biscio and Lavancier, 2016). Condition (D3) ensures that the asymptotic variance in the central limit theorem below is not degenerated. Finally, Assumption (W) makes specific the fact that W_n is not too irregularly shaped and is not bounded in any direction. It is for instance fulfilled if W_n is a Cartesian product of d intervals whose lengths tends to infinity.

Theorem 3.1. *Under Assumptions* (D1) *and* (D2)*, if assumptions* (F1) *through* (F3) *(or* (F3')*) are satisfied for $f_1 = f$, with a probability tending to one as $n \to \infty$, there exists a $|W_n|^{1/2}$-consistent sequence of roots $\hat{\theta}_n$ of the estimating equations $e_n(\theta) = 0$. If moreover* (W) *and* (D3) *holds true, then*

$$|W_n|\Sigma_n^{-1/2} H_n(\theta^*)(\hat{\theta}_n - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I_p).$$

Proof. We deduce from (8) that (D1) implies (X1). Moreover, it was shown in Poinas *et al.* (2017) that (X2) is a consequence of (D2) and that (X3) is a consequence of (D2), (D3) and (W). Thus, we can conclude by applying Theorem (2.1) in the case $l = 1$ and $q_1 = 2$. □

In the case of a stationary X and f given by (5), the following lemma shows that (F3) is satisfied under mild assumptions that are violated only in degenerate cases. For instance, if $p = 1$, the last assumption boils down to $\nabla_\theta \rho^{(2)}(0, t; \theta^*) \neq 0$ for some $t \neq 0$ such that $|K_{\theta^*}(t)| > \sqrt{\varepsilon} K_{\theta^*}(0)$. In particular it is not difficult to verify these assumptions for the stationary parametric kernels considered in our simulation study of Section 4, namely the Bessel-type and the Gaussian kernels, see the supplementary material.

Lemma 3.2. *Assume* (W) *and* (D2)*, suppose X is stationary and let f be as in* (5)*. Assume that w is positive on $[0,1)$, vanishes on $[1,\infty)$ and is differentiable on $\mathbb{R}_+$. If $t \mapsto f(0,t;\theta^*)$ is integrable and* $\operatorname{span}\{\nabla_\theta \rho^{(2)}(0,t;\theta^*) : |K_{\theta^*}(t)| > \sqrt{\varepsilon} K_{\theta^*}(0)\} = \mathbb{R}^p$*, then* (F3) *is satisfied.*

Proof. By definition of w and (D2), there exists $R > 0$ such that $f(0,t;\theta^*) = 0$ when $\|t\| \geqslant R$. By Lemma A.1, since $t \mapsto f(0,t;\theta^*)$ is integrable then $H_n(\theta^*)$ converges towards the positive semi-definite matrix $H(\theta^*) = \int_{\|t\|<R} h(t)\mathrm{d}t$

where the function $h : \mathbb{R}^d \to \mathbb{R}^{p\times p}$ is defined by

$$h(t) = w\left(\frac{\varepsilon K_{\theta*}(0)^2}{K_{\theta*}(t)^2}\right)\frac{\nabla_\theta \rho^{(2)}(0,t;\theta^*)\nabla_\theta \rho^{(2)}(0,t;\theta^*)^T}{\rho^{(2)}(0,t;\theta^*)}.$$

In this case, proving (F3) is equivalent to showing that $\phi^T H(\theta^*)\phi = 0$ only if $\phi = 0$. For this, let A be the set of t such that $|K_{\theta*}(t)| > \sqrt{\varepsilon}K_{\theta*}(0)$, $\phi \in \mathbb{R}^p$ and note that since $w(\varepsilon K_{\theta*}(0)^2/K_{\theta*}(t)^2) > 0$ for $t \in A$ and $h(t)$ is continuous and positive semi-definite,

$$\begin{aligned}
\phi^T H(\theta^*)\phi = 0 \quad &\Leftrightarrow \quad \forall t \in A,\ \phi^T h(t)\phi = 0 \\
&\Leftrightarrow \quad \forall t \in A,\ \nabla_\theta \rho^{(2)}(0,t;\theta^*)^T\phi = 0 \\
&\Leftrightarrow \quad \phi \in \left(\operatorname{span}\{\nabla_\theta \rho^{(2)}(0,t;\theta^*)\ :\ t \in A\}\right)^\perp.
\end{aligned}$$

By assumption $\operatorname{span}\{\nabla_\theta \rho^{(2)}(0,t;\theta^*)\ :\ t \in A\} = \mathbb{R}^p$ whereby $\phi = 0$, which concludes the proof. □

Similarly, we can show that even in the non-stationary case, condition (F3) is satisfied for the function in (5) but under slightly stronger assumptions on $\nabla_\theta \rho^{(2)}(u,v;\theta^*)$. Namely, we demand that all functions $v \mapsto \nabla_\theta \rho^{(2)}(u,v;\theta^*)$ are not contained in a single hyperplane of $\mathbb{R}^p$ nor confined around 0. This is similar in essence to what we have assumed in the previous corollary but with the need of a uniform condition with respect to u. Functions that do not satisfy these requirements are arguably degenerate. In particular, a straightforward calculus carried out in the supplementary material shows that the non-stationary Bessel-type kernel used in our simulation study satisfies these assumptions.

Lemma 3.3. *Assume* (W), (D2) *and that* $K_{\theta*}$ *is bounded. Let* f *be as in* (5) *and define* $h : (\mathbb{R}^d)^2 \to \mathbb{R}^{p\times p}$ *by*

$$h(u,v) = w\left(\frac{\varepsilon K_{\theta*}(u,u)K_{\theta*}(v,v)}{K_{\theta*}(u,v)^2}\right)\frac{\nabla_\theta \rho^{(2)}(u,v;\theta^*)\nabla_\theta \rho^{(2)}(u,v;\theta^*)^T}{\rho^{(2)}(u,v;\theta^*)}.$$

Assume that w *is positive on* $[0,1)$, *vanishes on* $[1,\infty)$ *and is differentiable on* $\mathbb{R}_+$. *If* $\sup_{u\in\mathbb{R}^d} \| \int_{\mathbb{R}^d} h(u,v)\mathrm{d}v\| < +\infty$ *and if there exists* $\mu > 1$ *and* $\delta > 0$ *such that for all* $u \in \mathbb{R}^d$ *and for all unit vectors* ϕ *of* $\mathbb{R}^p$ *there exists a subset* A *of* $\{v\ :\ K_{\theta*}(u,v)^2 > \mu\varepsilon K_{\theta*}(u,u)K_{\theta*}(v,v)\}$ *of positive Lebesgue measure* $|A| > 0$ *and satisfying*

$$\forall v \in A,\ |\phi^T \nabla_\theta \rho^{(2)}(u,v;\theta^*)| > \delta$$

then (F3) *is satisfied.*

Proof. By definition of w, (D2) and the fact that $K_{\theta*}$ is bounded, there exists $R > 0$ such that $h(u, v) = 0$ when $\|v - u\| \geqslant R$. The integral in (F3) writes

$$H_n(\theta^*) = \frac{1}{|W_n|}\int_{W_n^2} h(u,v)\mathbb{1}_{\|u-v\|\leqslant R}\mathrm{d}v\mathrm{d}u = \frac{1}{|W_n|}\int_{W_n\ominus R}\int_{W_n} h(u,v)\mathbb{1}_{\|u-v\|\leqslant R}\mathrm{d}v\mathrm{d}u + \varepsilon_n$$

where

$$\varepsilon_n = \frac{1}{|W_n|}\int_{W_n\setminus(W_n\ominus R)}\int_{W_n} h(u,v)\mathbb{1}_{\|u-v\|\leqslant R}\mathrm{d}v\mathrm{d}u.$$

By (W), we have

$$\begin{aligned}\|\varepsilon_n\| &\leqslant \frac{|W_n\backslash W_n\ominus R)|}{|W_n|}\sup_{u\in\mathbb{R}^d}\int_{\mathbb{R}^d}\|h(u,v)\|\mathrm{d}v\\ &\leqslant \frac{|\partial W_n\oplus R|}{|W_n|}\sup_{u\in\mathbb{R}^d}\int_{\mathbb{R}^d}\|h(u,v)\|\mathrm{d}v \to 0,\end{aligned}$$

and for all ϕ,

$$\phi^T\left(\int_{W_n\ominus R}\int_{W_n} h(u,v)\mathbb{1}_{\|u-v\|\leqslant R}\mathrm{d}u\mathrm{d}v\right)\phi = \int_{W_n\ominus R}\left(\int_{\|u-v\|\leqslant R}\phi^T h(u,v)\phi\mathrm{d}v\right)\mathrm{d}u.$$

By our assumption on $\nabla_\theta\rho^{(2)}$, there exists a set A of positive Lebesgue measure such that

$$\forall v\in A,\ |\phi^T\nabla_\theta\rho^{(2)}(u,v;\theta^*)| > \delta \text{ and } w\left(\frac{\varepsilon K_{\theta*}(u,u)K_{\theta*}(v,v)}{K_{\theta*}(u,v)^2}\right) > \inf_{x\in[0,1/\mu]} w(x).$$

Hence for $\|\phi\| = 1$,

$$\begin{aligned}&\frac{1}{|W_n|}\phi^T\left(\int_{W_n\ominus R}\int_{W_n} h(u,v)\mathbb{1}_{\|u-v\|\leqslant R}\mathrm{d}u\mathrm{d}v\right)\phi\\ \geqslant&\frac{\inf_{x\in[0,1/\mu]} w(x)}{|W_n|\|\rho^{(2)}(.,.;\theta*)\|_\infty}\int_{W_n\ominus R}\left(\int_A|\phi^T\nabla_\theta\rho^{(2)}(u,v;\theta^*)|^2\mathrm{d}v\right)\mathrm{d}u\\ \geqslant&\frac{|W_n\ominus R||A|\delta^2\inf_{x\in[0,1/\mu]} w(x)}{|W_n|\|\rho^{(2)}(.,.;\theta*)\|_\infty}\\ =&\left(\frac{|W_n| - |W_n\cap(\partial W_n\oplus R)|}{|W_n|}\right)\frac{|A|\delta^2\inf_{x\in[0,1/\mu]} w(x)}{\|\rho^{(2)}(.,.;\theta*)\|_\infty}\\ \to&\frac{|A|\delta^2\inf_{x\in[0,1/\mu]} w(x)}{\|\rho^{(2)}(.,.;\theta*)\|_\infty} > 0\end{aligned}$$

where the limit is a consequence of (W). Since the limit does not depend on ϕ, then (F3) is satisfied. □

3.2 Two-step estimation for a separable parameter

We consider a family of kernels

$$K_\theta(u,v) = \sqrt{\rho(u;\beta)}C(u,v;\psi)\sqrt{\rho(v;\beta)},$$

where $\theta := (\beta^T, \psi^T)^T \in \Theta \subset \mathbb{R}^{p+q}$ with $\beta \in \mathbb{R}^p$ and $\psi \in \mathbb{R}^q$, $\rho(.;\beta)$ are non-negative functions, and $C(\cdot,\cdot;\psi)$ are correlation functions, in particular $C(u,u;\psi) = 1$ for any ψ. Note that in this case the DPP with kernel K_θ has intensity $\rho(.;\beta)$ and its pair correlation function is $g(u,v;\psi) = 1 - C^2(u,v;\psi)$.

As in the preceding section, we assume a DPP X with kernel K_{θ^*}, $\theta^* \in \mathrm{Int}(\Theta)$, is observed on a bounded window $W_n \subset \mathbb{R}^d$. In the spirit of Waagepetersen and Guan (2009), we estimate θ^* in two steps. First, β^* is estimated as the solution $\hat{\beta}_n$ of $s_n(\beta) = 0$ where

$$s_n(\beta) = \sum_{u \in X \cap W_n} \frac{\nabla_\beta \rho(u;\beta)}{\rho(u;\beta)} - \int_{W_n} \nabla_\beta \rho(u;\beta) \mathrm{d}u$$

is the score function for a Poisson point process. Then, we estimate ψ^* by the solution $\hat{\psi}_n$ of $u_n(\hat{\beta}_n, \psi) = 0$ where

$$u_n(\theta) = \sum_{u,v \in X \cap W_n}^{\neq} f(u,v;\theta) - \int_{W_n^2} f(u,v;\theta)\rho^{(2)}(u,v;\theta)\mathrm{d}u\mathrm{d}v$$

for a given $\mathbb{R}^q$-valued function f and where $\rho^{(2)}(u,v;\theta) = \rho^{(2)}(u,v;\beta,\psi) = \rho(u;\beta)\rho(v;\beta)(1 - C^2(u,v;\psi))$ in this case. Here and in the following, for convenience of notation, we identify $u_n(\beta,\psi)$ with $u_n(\theta)$ when $\theta = (\beta^T, \psi^T)^T$.

This two-step procedure is a particular estimating equation procedure, since $\hat{\theta}_n := (\hat{\beta}_n^T, \hat{\psi}_n^T)^T$ is obtained as the solution of $e_n(\theta) = 0$ where $e_n(\theta) = (s_n(\beta)^T, u_n(\beta,\psi)^T)^T$. Thus, this is a particular case of the setting in Section 2.2 where $l = 2$, $q_1 = 1$, $q_2 = 2$, $f_1 = \nabla_\beta \rho(u;\beta)/\rho(u;\beta)$ and $f_2 = f$.

We assume in the following theorem the same conditions on the DPP X as in the previous section. Similarly, we assume that (F1) through (F3) (or (F3')) are satisfied for f_1 and f_2. In this particular case, the matrix H_n involved in (F3) simply writes

$$H_n(\beta,\psi) = \begin{pmatrix} H_n^{1,1}(\beta,\psi) & 0 \\ H_n^{2,1}(\beta,\psi) & H_n^{2,2}(\beta,\psi) \end{pmatrix}$$

where

$$H_n^{1,1}(\beta) = \frac{1}{|W_n|}\int_{W_n} \frac{\nabla_\beta \rho(u;\beta)\nabla_\beta \rho(u;\beta)^T}{\rho(u;\beta)}\mathrm{d}u,$$
$$H_n^{2,1}(\beta,\psi) = \frac{1}{|W_n|}\int_{W_n^2} f(u,v;\beta,\psi)\nabla_\beta \rho^{(2)}(u,v;\beta,\psi)^T \mathrm{d}u\mathrm{d}v,$$
$$H_n^{2,2}(\beta,\psi) = \frac{1}{|W_n|}\int_{W_n^2} f(u,v;\beta,\psi)\nabla_\psi \rho^{(2)}(u,v;\beta,\psi)^T \mathrm{d}u\mathrm{d}v.$$

Since it is a non symmetric matrix, condition (F3') is more applicable than (F3). Mild conditions ensuring (F3') in the stationary case are provided in Lemma 3.5.

Theorem 3.4. *Under Assumptions* (D1) *and* (D2)*, if assumptions* (F1) *through* (F3) *(or* (F3')*) are satisfied for* $f_1 = \nabla_\beta \rho(u;\beta)/\rho(u;\beta)$ *and* $f_2 = f$*, then with a probability tending to one as* $n \to \infty$*, there exists a* $|W_n|^{1/2}$*-consistent sequence of roots* $\hat{\theta}_n$ *of the estimating equations* $e_n(\theta) = 0$*. If moreover* (W) *and* (D3) *hold true, then*

$$|W_n|\Sigma_n^{-1/2} H_n(\theta^*)(\hat{\theta}_n - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I_{p+q}).$$

Proof. The proof follows the same lines as the proof of Theorem 3.1. □

The next lemma is similar to Lemma 3.2. When $q = 1$ the last technical condition boils down to $\nabla_\psi(1 - C^2(0,t;\psi^*)) \neq 0$ for some t such that $C(0,t;\psi^*) \geqslant \sqrt{\varepsilon} C(0,0;\psi^*)$. In particular, the stationary kernels in Section 4 satisfy the required assumptions, see the supplementary material.

Lemma 3.5. *Assume that for all* θ*,* $K_\theta(u,v)$ *only depends on* $u-v$*, in which case* $\rho(u;\beta) = \beta$ *with* $\beta > 0$ *and* $C(u,v;\psi) = C(0,v-u;\psi)$ *with* $\psi \in \mathbb{R}^q$*. Then the output of the first step is* $\hat{\beta}_n = N(X \cap W_n)/|W_n|$*. In the second step, assume*

$$\begin{aligned} f(u,v;\beta,\psi) =& w\left(\frac{\varepsilon}{1-g(u,v;\psi)}\right)\frac{\nabla_\psi \rho^{(2)}(u,v;\beta,\psi)}{\rho^{(2)}(u,v;\beta,\psi)} \\ =& w\left(\frac{\varepsilon}{C(0,v-u;\psi)^2}\right)\frac{\nabla_\psi(1-C^2(0,v-u;\psi))}{1-C^2(0,v-u;\psi)}. \end{aligned}$$

Assume that w *is positive on* $[0,1)$*, vanishes on* $[1,\infty)$ *and is differentiable on* $\mathbb{R}_+$*. If* $t \mapsto f(0,t;\theta^*)$ *is integrable and* $\mathrm{span}\{\nabla_\psi(1 - C^2(0,t;\psi^*)) : C(0,t;\psi^*) > \sqrt{\varepsilon}\} = \mathbb{R}^q$*, then* (F3') *is satisfied under* (W)*,* (D1) *and* (D2)*.*

Proof. By definition of w and (D2), there exists $R > 0$ such that $f(0, t; \theta^*) = 0$ when $\|t\| \geqslant R$. Since K_θ and f are invariant by translation and $t \mapsto f(0, t; \theta^*)$ is integrable then $H_n(\theta)$ converges by Lemma A.1. In particular, we have

$$H_n^{1,1}(\beta) \to \frac{1}{\beta},$$

$$H_n^{2,2}(\beta, \psi) \to \beta^2 \int_{\|t\| \leqslant R} h(t; \psi) \mathrm{d}t,$$

$$H_n^{2,1}(\beta, \psi) \to 2\beta \int_{\|t\| \leqslant R} w\left(\frac{\varepsilon}{C(0, t; \psi)^2}\right) \nabla_\psi (1 - C^2(0, t; \psi)) \mathrm{d}t,$$

where the function $h(.; \psi) : \mathbb{R}^d \to \mathbb{R}^{p \times p}$ is defined by

$$h(t; \psi) = w\left(\frac{\varepsilon}{C(0, t; \psi)^2}\right) \frac{\nabla_\psi (1 - C^2(0, t; \psi)) \nabla_\psi (1 - C^2(0, t; \psi))^T}{1 - C^2(0, t; \psi)}.$$

The limit of $H_n(\theta)$ is continuous by (D1). In this case, proving (F3') is equivalent to showing that the limit of $H_n(\theta^*)$ is invertible. Since this matrix is block triangular and $\beta > 0$ then it is invertible if and only if the limit of $H_n^{2,1}(\theta^*)$ is invertible. This is done the same way as in Lemma 3.2. □

4 Simulation study

In this section we use simulation studies to investigate the performance of our adaptive estimating function. In Section 5 of the supplementary material, we additionally compare two-step estimation, when it is feasible, with simultaneous estimation. Our recommendation is to use the two-step approach.

In order to assess the adaptive test function (5) against the truncated test function (4) with a prescribed R, we consider a DPP model in $\mathbb{R}^2$ with a Bessel-type kernel

$$K(u, v) = \sqrt{\rho(u)\rho(v)}\, \frac{J_1(2\|u - v\|/\alpha)}{\|u - v\|/\alpha},$$

where J_1 denotes the Bessel function of the first kind, ρ is the intensity and α controls the range of interaction of the DPP. For existence, ρ and α must satisfy

$$\alpha^2 \|\rho\|_\infty \leqslant \frac{1}{\pi}. \tag{9}$$

This relation shows the tradeoff between the expected number of points and the strength of repulsiveness that we can obtain. This model is a particular instance of the Bessel-type DPP introduced in Biscio and Lavancier (2016). It covers a large range of repulsiveness, from the Poisson point process (when α is close to 0) to the most repulsive DPP (when $\alpha = 1/\sqrt{\pi\|\rho\|_\infty}$).

For this model, we consider three constant values of ρ, $\rho \in \{50, 100, 1000\}$, corresponding to homogeneous DPPs, and an inhomogeneous situation where $\rho(u) = \rho(x, y) = 20\exp(4x)$ when $u \in [0, 1]^2$. The latter case corresponds to a log-linear intensity function involving two parameters. For each ρ, three values of α are considered: a small one, a medium one, and a last one close to the maximal possible value satisfying (9). Examples of point patterns simulated on $[0, 1]^2$ are displayed in Figure 1. All simulations are carried out using `R` (R Core Team, 2017), in particular the library `spatstat` (Baddeley *et al.*, 2015).

We estimate ρ and α by a two-step procedure as studied in Section 3.2 from realizations of the DPP on $W = [0, 1]^2$. The alternative global approach of Section 3.1 is discussed in the next section. In the first step, the parameters arising in ρ are estimated by the score function for a Poisson point process. This gives $\hat{\rho} = N(X \cap W)/|W|$ in the homogeneous cases. In the second step, we consider the estimating equation based on (4) where θ is α in this setting and when $R \in \{0.05, 0.1, 0.25\}$, and based on the adaptive test function (5) with $\varepsilon = 0.01$ and the weight function w given at the end of Section 2.4. This yields four different estimators of α. The root mean square errors (RMSEs) of these estimators and the mean computation time estimated from 1000 replications are summarised in Table 1. Boxplots are displayed in Figure S3 in the supplementary material. Note that the codes have not been optimised, but the same computational strategy has been used for all methods, making the comparison of the mean computation time meaningful.

The Bessel-type kernel and the aforementioned test functions used in the two-step estimation procedure fulfill the assumptions of Theorem 3.4 and Lemma 3.5 (for the homogeneous case), ensuring nice asymptotic properties of the estimators considered in this section. This is confirmed by the estimated RMSE's reported in Table 1, that decrease when the intensity ρ increases (which mimics the effect of an increasing window since rescaling the window by a factor $1/k$ is equivalent to change ρ into $k^2\rho$ and α into α/k, see (2.4) in Lavancier *et al.*, 2015). Moreover, these RMSE's show that the best choice of R in the test function (4) clearly depends on the range of interaction of the underlying process. This emphasizes the importance of a

data-driven approach to choosing R since the range is unknown in practice. Fortunately, the performance of the adaptive method is, except for the case $\rho = 100, \alpha = 0.01$, always better than the worst choice of R and very close to the best R. For the exceptional case, the small differences in performance can be explained by Monte Carlo error. Further, use of the adaptive method implies only little or no extra computional effort. In presence of many points, the adaptive version is in fact much faster to compute than the estimator based on (4) with the choice of a too large R, see for instance the results for $\rho = 1000$ and $R = 0.25$.

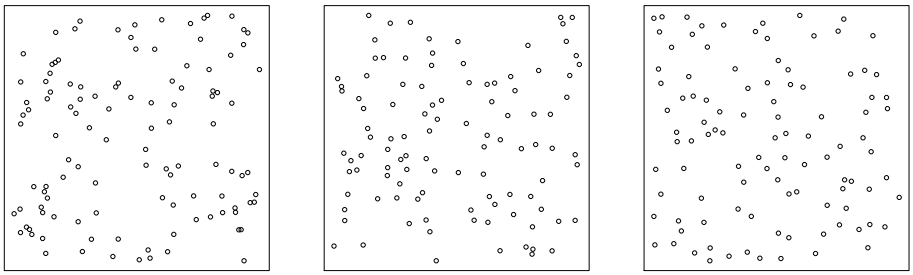
Table S2 in the supplementary material shows the root mean square errors of the adaptive estimator using $\varepsilon = 0.05$. The RMSEs obtained with $\varepsilon = 0.05$ are bigger than those obtained with $\varepsilon = 0.01$. Nevertheless, the adaptive method with $\varepsilon = 0.05$ still performs well in the sense that it usually performs better than the worst R and usually almost as good as the best R. Because the above estimation methods sometimes fail to converge, we also report in Table S1 in the supplementary material the percentages of times each method has converged in our simulation study. These percentages are similar for all methods. Note that the results in Table 1 and in Figure S3 are based on 1000 simulations where all four methods have converged.

5 Application

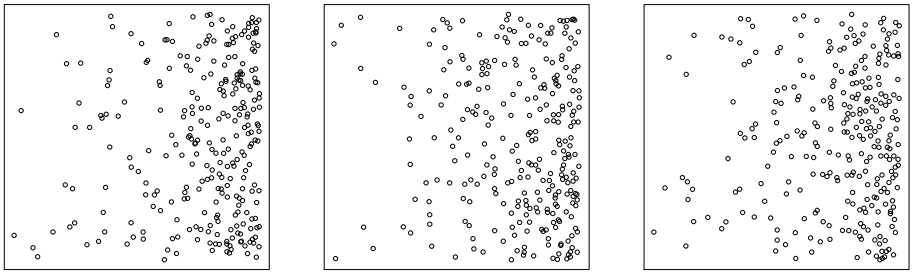
To illustrate the practical importance of our adaptive estimating function and our asymptotic results we consider the problem of fitting a DPP model to the point pattern data in the left plot of Figure 2. This dataset collected by Numata (1964) records the locations of 204 seedlings and saplings of Japanese black pines in an observation window of dimension 10m by 10m. It has previously been analysed in Ogata and Tanemura (1986) using an inhomogeneous Gibbs model and later in Lavancier *et al.* (2015) using an inhomogeneous DPP with kernel of the form (6) with a cubic polynomial in the Cartesian coordinates for the log-intensity and $C(u) = \exp(-\|u\|^2/\alpha^2)$. The estimation in Lavancier *et al.* (2015) was carried out using a two-step procedure where the intensity parameters were fitted in the first step by the Poisson likelihood method and in the second step α was estimated by minimisation of a contrast function based on the pair correlation function. This gave $\hat{\alpha} = 0.226$ and the fit was judged to be satisfying based on several goodness of fit envelope tests. However this second step relies on the arbitrary

ρ	α		$R = 0.05$	$R = 0.1$	$R = 0.25$	Adaptive	$\hat{R}$
50	0.02	RMSE:	5.84 (0.15)	5.83 (0.17)	6.29 (0.19)	5.97 (0.18)	0.047
		TIME:	0.43	0.48	0.68	0.64	(0.020)
	0.04	RMSE:	15.60 (0.44)	9.18 (0.20)	9.19 (0.22)	9.25 (0.21)	0.106
		TIME:	0.48	0.50	0.68	0.73	(0.037)
	0.07	RMSE:	13.32 (0.33)	8.25 (0.23)	8.22 (0.24)	8.15 (0.24)	0.147)
		TIME:	0.50	0.45	0.59	0.72	(0.050
100	0.01	RMSE:	2.44 (0.08)	2.45 (0.08)	2.58 (0.09)	2.63 (0.09)	0.024
		TIME:	0.44	0.57	1.22	0.70	(0.009)
	0.03	RMSE:	5.34 (0.13)	5.12 (0.13)	5.28 (0.14)	5.27 (0.13)	0.064
		TIME:	0.40	0.47	0.98	0.70	(0.019)
	0.05	RMSE:	5.78 (0.12)	4.43 (0.12)	4.50 (0.10)	4.53 (0.12)	0.139)
		TIME:	0.52	0.56	1.16	0.95	(0.022
1000	0.005	RMSE:	0.67 (0.02)	0.88 (0.02)	0.83 (0.02)	0.72 (0.02)	0.015
		TIME:	3.83	19.04	110.07	9.38	(0.003)
	0.01	RMSE:	0.57 (0.01)	0.59 (0.02)	0.61 (0.01)	0.56 (0.01)	0.028
		TIME:	2.68	10.40	60.79	6.84	(0.005)
	0.015	RMSE:	0.47 (0.01)	0.46 (0.01)	0.52 (0.01)	0.47 (0.01)	0.026
		TIME:	2.53	9.81	55.78	7.75	(0.002)
Inhom	0.005	RMSE:	1.58 (0.04)	1.65 (0.04)	1.66 (0.04)	1.61 (0.04)	0.014
		TIME:	0.89	2.50	10.30	1.19	(0.005)
	0.01	RMSE:	1.34 (0.03)	1.36 (0.03)	1.36 (0.03)	1.32 (0.03)	0.025
		TIME:	0.76	1.86	7.66	1.22	(0.008)
	0.015	RMSE:	1.43 (0.03)	1.47 (0.03)	1.48 (0.03)	1.40 (0.03)	0.030
		TIME:	0.86	1.90	7.46	1.40	(0.006)

Table 1: Estimated root mean square errors ($\times 10^3$) and mean computation time (in seconds) of $\hat{\alpha}$ for a Bessel-type DPP on $[0,1]^2$, for different values of ρ and α. The 3 first estimators use the test function (4) with $R = 0.05$, $R = 0.1$ and $R = 0.25$ respectively, while the last estimator is the adaptive version based on (5). The standard errors of the RMSE estimations are given in parenthesis. The last column gives the averages of "practical ranges" (i.e. maximal solution to $|g(r) - 1| = 0.01$) used for the adaptive estimator, along with their standard deviations in parenthesis. For each value of ρ and α, these quantities are computed from 1000 simulations where all four estimation methods have converged.



$\rho = 100$, $\alpha = 0.01$ $\rho = 100$, $\alpha = 0.03$ $\rho = 100$, $\alpha = 0.05$



$\rho = \rho(u)$, $\alpha = 0.005$ $\rho = \rho(u)$, $\alpha = 0.01$ $\rho = \rho(u)$, $\alpha = 0.015$

Figure 1: Examples of point patterns simulated from a Bessel-type DPP on $[0,1]^2$ for different values of ρ and α. For the last row, $\rho(x,y) = 20\exp(4x)$.

choice of several tuning parameters similar to R and no confidence intervals were provided. We also fit the same inhomogeneous DPP model by the two-step approach (detailed in Section 3.2) but using in the second step the test function (4) (for the non-adaptive approach) or (5) (for our adaptive version).

For the non-adaptive approach, the default choice of R provided by `spatstat` (Baddeley *et al.*, 2015) is one quarter of the smallest side length of the observation window, which in the current case gives $R = 2.5$. This choice of R is subject to criticism since it does not at all take into account the correlation properties of the data generating point process. Alternatively one could, following the references mentioned in Section 2.4, choose an R based on inspection of the kernel estimate of the pair correlation function shown in the right plot in Figure 2. This suggest using a value of R around 0.4. However, this last approach completely eludes a theoretical underpinning. First, the asymptotic properties of the kernel estimator itself are complicated and second it is not possible to handle mathematically the visual assessment of

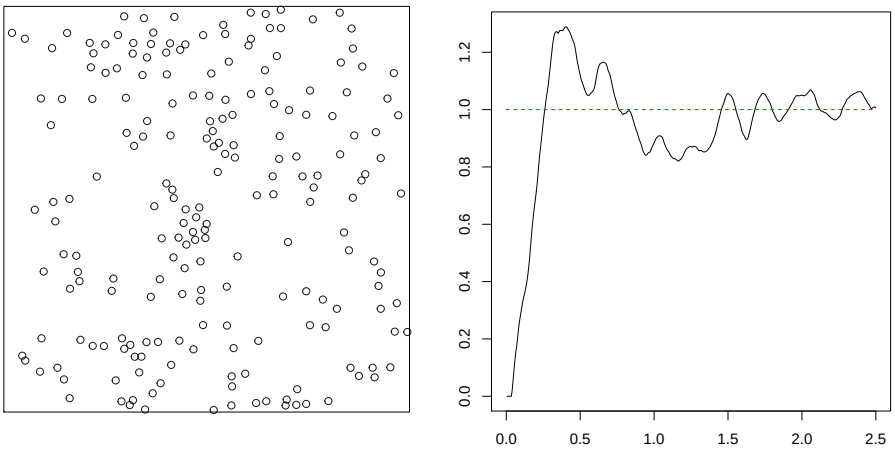


Figure 2: Japanese pines dataset and kernel estimate of its pair correlation function.

R. We use instead our adaptive procedure and obtain $\hat{\alpha} = 0.241$ for the range parameter and the adaptively chosen value of $\hat{R} = 0.37$, which is in agreement with the visual inspection of the pair correlation function.

Based on our Theorem 3.4 in Section 3.2 we further in a standard way obtain a 95% approximate confidence interval for α (estimate $\pm$ 1.96 times asymptotic standard deviation). For ease of implementation we use a parametric bootstrap to estimate the asymptotic standard deviation (alternatively one could use numerical integration to compute the asymptotic covariance matrix). More precisely, we generate 1000 realisations of the fitted model and refit the model for each simulation. The empirical standard deviation of the resulting simulated estimates is then an estimate of the standard deviation of $\hat{\alpha}$. Note that our consistency result in Theorem 3.4 is a requirement for the validity of this approach (see for instance Beran (1997)). We obtain the specific estimate 0.026 and hence the approximate 95% confidence interval $[0.19; 0.29]$ for α. This result confirms that there is significant inhibition between the Japanese pines and in particular provides strong evidence against the inhomogeneous Poisson process model ($\alpha = 0$). Note that a similar bootstrap approach is not possible in the non-adaptive case where R is chosen by visual inspection, since the variability of this choice can not be included in an automatic procedure.

6 Discussion

In this paper we provide a very general asymptotic framework for estimating function inference for spatial point processses with known joint intensities. Specific asymptotic results are obtained for determinantal point processes.

The performance of second order estimating functions depends strongly on a tuning parameter R that controls which pairs of points are used in the estimation. Although not statistically optimal, our adaptive procedure for selecting this tuning parameter is intuitively appealing and easy to implement. The method depends on a new tuning parameter ε for which it is easier to identify reasonable values than for the original tuning parameter R. The resulting estimation procedure is computationally tractable and performs well in terms of mean squared error in the simulation studies considered. It moreover seamlessly integrates with the asymptotic results where the use of the adaptive method poses no extra theoretical difficulties.

Though we focus in this paper on determinantal point processes, the adaptive method is applicable for any spatial point process with known pair correlation function. As an example we provide in Section 7 of the supplementary material a simulation study in case of a cluster process.

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References

Affandi, R. H., Fox, E., Adams, R. & Taskar, B. (2014). Learning the parameters of determinantal point process kernels. In: *International Conference on Machine Learning*, 1224–1232.

Baddeley, A. J., Rubak, E. & Turner, R. (2015). *Spatial Point Patterns: Methodology and Applications with R*. Interdisciplinary Statistics, Chapman & Hall/CRC, Boca Raton, Florida.

Bai, Y., Kang, J. & Song, P. X.-K. (2014). Efficient pairwise composite likelihood estimation for spatial-clustered data. *Biometrics* **70**(3), 661–670.

Beran, R. (1997). Diagnosing bootstrap success. *Annals of the Institute of Statistical Mathematics* **49**(1), 1–24.

Biscio, C. & Lavancier, F. (2016). Quantifying repulsiveness of determinantal point processes. *Bernoulli* **22**, 2001–2028.

Biscio, C. & Lavancier, F. (2017). Contrast estimation for parametric stationary determinantal point processes. *Scandinavian Journal of Statistics* **44**, 204–229.

Deng, C., Guan, Y., Waagepetersen, R. & Zhang, J. (2017). Second-order quasi-likelihood for spatial point processes. *Biometrics* **73**(4), 1311–1320.

Engel, G. M. & Schneider, H. (1980). Matrices diagonally similar to a symmetric matrix. *Linear Algebra and its Applications* **29**, 131–138.

Guan, Y. (2006). A composite likelihood approach in fitting spatial point process models. *Journal of the American Statistical Association* **101**, 1502–1512.

Guan, Y. & Loh, J. M. (2007). A thinned block bootstrap procedure for modeling inhomogeneous spatial point patterns. *Journal of the American Statistical Association* **102**, 1377–1386.

Heagerty, P. J. & Lele, S. R. (1998). A composite likelihood approach to binary spatial data. *Journal of the American Statistical Association* **93**(443), 1099–1111.

Heyde, C. C. (1997). *Quasi-likelihood and its application: a general approach to optimal parameter estimation*. Springer Science & Business Media.

Illian, J., Penttinen, A., Stoyan, H. & Stoyan, D. (2008). *Statistical analysis and modelling of spatial point patterns*. Statistics in Practice, Wiley, New York.

Jacod, J. & Sørensen, M. (2017). A review of asymptotic theory of estimating functions. *Statistical Inference for Stochastic Processes* , 1–20.

Lavancier, F. & Rochet, P. (2016). A general procedure to combine estimators. *Computational Statistics & Data Analysis* **94**, 175–192.

Lavancier, F. & Rochet, P. (2017). A tutorial on estimator averaging in spatial point process models. *Journal de la SFDS* **158**(3), 106–123.

Lavancier, F., Møller, J. & Rubak, E. (2015). Determinantal point process models and statistical inference. *Journal of Royal Statistical Society: Series B (Statistical Methodology)* **77**, 853–877.

Macchi, O. (1975). The coincidence approach to stochastic point processes. *Advances in Applied Probability* **7**, 83–122.

Møller, J. & Waagepetersen, R. P. (2004). *Statistical Inference and Simulation for Spatial Point Processes*. Chapman and Hall/CRC, Boca Raton.

Møller, J. & Waagepetersen, R. P. (2007). Modern statistics for spatial point processes. *Scandinavian Journal of Statistics* **34**, 643–684.

Numata, M. (1964). Forest vegetation, particularly pine stands in the vicinity of Choshi-flora and vegetation in Choshi, Chiba prefecture, VI. *Bull. Choshi Mar. Lab* **6**, 27–37.

Ogata, Y. & Tanemura, M. (1986). Likelihood estimation of interaction potentials and external fields of inhomogeneous spatial point patterns. In: *Pacific Statistical Congress* (eds. I. Francis, B. Manly and F. Lam), Elsevier, Amsterdam, 150–154.

Poinas, F., Delyon, B. & Lavancier, F. (2017). Mixing properties and central limit theorem for associated point processes. *to appear in Bernoulli (available at arXiv:1705.02276)* .

Prokešová, M. & Jensen, E. B. V. (2013). Asymptotic Palm likelihood theory for stationary point processes. *Annals of the Institute of Statistical Mathematics* **65**(2), 387–412.

Prokešová, M., Dvořák, J. & Jensen, E. (2016). Two-step estimation procedures for inhomogeneous shot-noise Cox processes. *Annals of the Institute of Statistical Mathematics* .

R Core Team (2017). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Rising, J., Kulesza, A. & Taskar, B. (2015). An efficient algorithm for the symmetric principal minor assignment problem. *Linear Algebra and its Applications* **473**, 126–144.

Schoenberg, F. P. (2005). Consistent parametric estimation of the intensity of a spatial-temporal point process. *Journal of Statistical Planning and Inference* **128**, 79–93.

Shirai, T. & Takahashi, Y. (2003). Random point fields associated with certain Fredholm determinants II: Fermion shifts and their ergodic and Gibbs properties. *Ann. Probab.* **31**(3), 1533–1564.

Sørensen, M. (1999). On asymptotics of estimating functions. *Brazilian Journal of Probability and Statistics* **13**(2), 111–136.

Soshnikov, A. (2000). Determinantal random point fields. *Russian Mathematical Surveys* **55**, 923–975.

Tanaka, U., Ogata, Y. & Stoyan, D. (2008). Parameter estimation and model selection for Neyman-Scott point processes. *Biometrical Journal* **50**(1), 43–57.

Van der Vaart, A. W. (2000). *Asymptotic statistics*, volume 3. Cambridge university press.

Waagepetersen, R. (2007). An estimating function approach to inference for inhomogeneous Neyman-Scott processes. *Biometrics* **63**, 252–258.

Waagepetersen, R. & Guan, Y. (2009). Two-step estimation for inhomogeneous spatial point processes. *Journal of the Royal Statistical Society, Series B* **71**, 685–702.

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A Appendix

Our general Theorem 2.1 depends on a number of assumptions. The setting is the same as in Section 2.2. We moreover define $\mathrm{diam}(x)$ as the largest

distance between two coordinates of x. The assumptions (F1) through (F3) are mainly related to the test functions f_i, while for X we assume (X1) through (X3).

(F1) For all $i = 1, \dots, l$ and for all $x \in (\mathbb{R}^d)^{q_i}$, $\theta \mapsto f_i(x;\theta)$ is twice continuously differentiable in a neighbourhood of θ^*. Moreover, the first and second derivative of f_i with respect to θ are bounded with respect to $x \in (\mathbb{R}^d)^{q_i}$ uniformly in θ belonging to this neighbourhood.

(F2) There exists a constant $R > 0$ such that for all θ in a neighbourhood of θ^*, all functions $x \mapsto f_i(x;\theta)$ vanish when $\mathrm{diam}(x) > R$.

Define the matrices $H_n(\theta)$ by

$$H_n(\theta) = \begin{pmatrix} H_n^1(\theta) \\ \vdots \\ H_n^l(\theta) \end{pmatrix},$$

where for all i

$$H_n^i(\theta) := \frac{1}{|W_n|} \int_{W_n^{q_i}} f_i(x;\theta) \nabla_\theta \rho^{(q_i)}(x;\theta)^T \mathrm{d}x.$$

(F3) The matrices $H_n(\theta^*)$ satisfy

$$\liminf_{n\to\infty} \left(\inf_{\|\phi\|=1} \phi^T H_n(\theta^*) \phi \right) > 0.$$

(F3') There exists a neighbourhood of θ^* such that for all n high enough and all θ in this neighbourhood, $H_n(\theta)$ is invertible and $\|H_n(\theta)^{-1}\|$ is uniformly bounded with respect to n and θ, where $\|\cdot\|$ stands for any matrix norm.

(X1) For all θ in a neighbourhood of θ^* and all q_i, $i = 1, \dots, l$, the intensity functions $x \mapsto \rho^{(q_i)}(x;\theta)$ are well-defined and bounded. Moreover, $\theta \mapsto \rho^{(q_i)}(x;\theta)$ is twice continuously differentiable in a neighbourhood of θ^*, for all $x \in (\mathbb{R}^d)^{q_i}$. Finally, the first and second derivative of $\rho^{(q_i)}$ with respect to θ are bounded with respect to $x \in (\mathbb{R}^d)^{q_i}$ uniformly in θ belonging to this neighbourhood.

(X2) For all q_i, $i = 1, \dots, l$, the intensity functions $\rho^{(q_i)}(\cdot;\theta^*), \cdots, \rho^{(2q_i)}(\cdot;\theta^*)$ of X are well-defined. Moreover, the intensity functions $\rho^{(q_i)}(\cdot;\theta^*), \cdots, \rho^{(2q_i-1)}(\cdot;\theta^*)$ are bounded and for all bounded sets $W \subset \mathbb{R}^d$ there exists a constant $C_0 > 0$, so that $\int_W \varphi_i(x_1)\mathrm{d}x_1 < C_0$, $i = 1, \dots, l$ where φ_i is the function

$$\varphi_i : x_1 \mapsto \sup_{\mathrm{diam}(x)<R} \sup_{\mathrm{diam}(y)<R} \sup_{y_1 \in W} \rho^{(2q_i)}(x_1, x_2, \cdots, x_{q_i}, y_1, \cdots, y_{q_i}; \theta^*) - \rho^{(q_i)}(x_1, x_2, \cdots, x_{q_i}; \theta^*)\rho^{(q_i)}(y_1, \cdots, y_{q_i}; \theta^*)$$

with R coming from (F2).

(X3) X satisfies the central limit theorem

$$\Sigma_n^{-1/2} e_n(\theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, I_p),$$

where e_n is defined in Section 2.2 and $\Sigma_n = \mathrm{Var}(e_n(\theta^*))$.

Assumptions (F1) and (F2) are basic regularity conditions on the f_i's. Similarly (X1) and (X2) ensure that the intensity functions of X exist and are sufficiently regular. The technical assumptions are in fact (F3) (or (F3')) and (X3). While the latter strongly depends on the underlying point process (see Waagepetersen and Guan (2009) for Cox processes and Poinas *et al.* (2017) for DPPs), the former can be simplified in some cases. For example, if $H_n(\theta^*)$ are symmetrical matrices for all n then (F3) writes $\liminf_n \lambda_{\min}(H_n(\theta^*)) > 0$ where $\lambda_{\min}(H_n(\theta^*))$ denotes the smallest eigenvalue of $H_n(\theta^*)$. If the matrices $H_n(\theta^*)$ are not symmetrical, Assumption (F3') will be preferred since (F3) does not translate well for non-symmetrical matrices. Furthermore, if X is stationary, all f_i's are invariant by translation, and the sequence of windows $\{W_n\}_{n\geqslant 1}$ satisfies (W) in Section 3.1, then $H_n(\theta)$ converges towards a matrix $H(\theta)$ explicitly given in Lemma A.1 below. Assumption (F3) thus simply becomes $\inf_{\|\phi\|=1} \phi^T H(\theta^*)\phi > 0$ and (F3') is satisfied whenever $H(\theta^*)$ is invertible by continuity of $H(\theta)$. In specific applications of Theorem 2.1, further conditions on the sequence of observation windows $\{W_n\}_{n\geqslant 1}$ may be required, see e.g. (W) in Section 3.1.

Lemma A.1. *Assume* (W), (X1), (F2) *and let* $\theta \in \mathbb{R}^p$. *Suppose that all* $\rho^{(q_i)}(\cdot;\theta)$*'s and* $f_i(\cdot;\theta)$*'s are invariant by translation, i.e.* $f_i(u_1, u; \theta) = f_i(0, u - u_1; \theta)$ *where* u *is the vector* $(u_2, \cdots, u_{q_i})$ *and* $u - u_1 = (u_2 - u_1, \cdots, u_{q_i} - u_1)$.

If $u \mapsto f_i(0, u; \theta)$ is integrable for all i such that $q_i \geqslant 2$, then $H_n(\theta)$ converges to a matrix $H(\theta)$. In particular, for all i we have

$$\lim_{n\to\infty} H_n^i(\theta) = \int_{\|t\|\leqslant R} f_i(0, t; \theta) \nabla_\theta \rho^{(q_i)}(0, t; \theta)^T \mathrm{d}t.$$

The proof of this lemma is available in the supplementary material.