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Published in:
Communications in Partial Differential Equations

DOI (link to publication from Publisher):
10.1080/03605302.2020.1760884

Publication date:
2020

Document Version
Accepted author manuscript, peer reviewed version

Citation for published version (APA):
Garde, H. (2020). Reconstruction of piecewise constant layered conductivities in electrical impedance tomography. *Communications in Partial Differential Equations*, *45*(9), 1118-1133. https://doi.org/10.1080/03605302.2020.1760884

RECONSTRUCTION OF PIECEWISE CONSTANT LAYERED CONDUCTIVITIES IN ELECTRICAL IMPEDANCE TOMOGRAPHY

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Abstract. This work presents a new *constructive* uniqueness proof for Calderón's inverse problem of electrical impedance tomography, subject to local Cauchy data, for a large class of piecewise constant conductivities that we call *piecewise constant layered conductivities* (PCLC). The resulting reconstruction method only relies on the physically intuitive monotonicity principles of the local Neumann-to-Dirichlet map, and therefore the method lends itself well to efficient numerical implementation and generalization to electrode models [18, 17]. Several direct reconstruction methods exist for the related problem of inclusion detection, however they share the property that "holes in inclusions" or "inclusions-within-inclusions" cannot be determined. One such method is the monotonicity method of Harrach, Seo, and Ullrich [25, 26], and in fact the method presented here is a modified variant of the monotonicity method which overcomes this problem. More precisely, the presented method abuses that a PCLC type conductivity can be decomposed into nested layers of positive and/or negative perturbations that, layer-by-layer, can be determined via the monotonicity method. The conductivity values on each layer are found via basic one-dimensional optimization problems constrained by monotonicity relations.

Keywords: electrical impedance tomography, partial data reconstruction, piecewise constant coefficient, monotonicity principle.

2010 Mathematics Subject Classification: 35R30, 35Q60, 35R05, 47H05.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with piecewise $\mathscr{C}^\infty$-smooth boundary $\partial\Omega$ (without cusps), for which $\mathbb{R}^d \setminus \overline{\Omega}$ is connected. We denote by ν an outer unit normal on $\partial\Omega$, and $\Gamma \subseteq \partial\Omega$ is a non-empty relatively open subset whose role is to employ local Cauchy data. For an electrical conductivity coefficient

$$\sigma \in L^\infty_+(\Omega) := \{\varsigma \in L^\infty(\Omega;\mathbb{R}) \mid \operatorname{ess\,inf}\varsigma > 0\}$$

and boundary current density

$$f \in L^2_\diamond(\Gamma) := \{g \in L^2(\Gamma) \mid \int_\Gamma g\,\mathrm{d}S = 0\}$$

we consider the *partial data conductivity problem*

$$\nabla\cdot(\sigma\nabla u) = 0 \quad \text{in } \Omega, \qquad \nu\cdot\sigma\nabla u|_{\partial\Omega} = \begin{cases} f & \text{on } \Gamma, \\ 0 & \text{on } \partial\Omega\setminus\Gamma. \end{cases} \tag{1.1}$$

From standard elliptic theory there is a unique solution $u = u_f^\sigma$ to (1.1), representing the interior electric potential, belonging to the "Γ-mean free" Sobolev space

$$H^1_\diamond(\Omega) := \{w \in H^1(\Omega) \mid \int_\Gamma w|_\Gamma\,\mathrm{d}S = 0\}.$$

This gives rise to a well-defined local Neumann-to-Dirichlet (ND) operator $\Lambda(\sigma) : f \mapsto u|_\Gamma$ which in this work is interpreted as a compact self-adjoint operator in $\mathscr{L}(L^2_\diamond(\Gamma))$, the space of bounded linear operators on $L^2_\diamond(\Gamma)$.

The inverse problem of electrical impedance tomography (EIT), in the sense of Calderón's formulation [10], is:

Reconstruct σ from knowledge of $\Lambda(\sigma)$.

In the practical setting, this corresponds to finding the conductivity coefficient in the interior of an object from indirect measurements of current–voltage pairs (injected current and measured voltage) recorded at electrodes placed on the object's surface. Hence, $\Lambda(\sigma)$ represents the ideal datum for such a problem. This paper will provide a new simple reconstruction method for recovering a large class of piecewise constant conductivities from their corresponding local ND map. However, first we review some known results on uniqueness and reconstruction in EIT.

For full boundary data ($\Gamma = \partial\Omega$) unique recovery of σ from $\Lambda(\sigma)$, i.e. injectivity of $\sigma \mapsto \Lambda(\sigma)$, has been solved in high generality. See e.g. [2] for general $L^\infty_+(\Omega)$-conductivities in dimension two, and [12] for Lipschitz conductivities in dimension three and beyond. For full boundary data there are also reconstruction methods, based on the works of e.g. [42, 43, 7], such as the $\bar{\partial}$-method which has received much attention regarding theoretical development and practical implementation [46, 39, 40, 14, 21, 47, 28]. The motivation behind this paper stems from the expectation that, with enough restrictions on the considered class of conductivities, more straightforward and intuitive reconstruction methods will emerge. This expectation is supported by recent promising computational results in [3], based on shape optimization for piecewise constant conductivities on polygonal partitions.

For the different types of partial data problems in EIT (partial Dirichlet and/or Neumann data on various parts of the boundary) we refer to the review paper [37] and the references therein. Here we will focus on local Cauchy data, in the sense of the local ND map defined above. The uniqueness problem is treated in [32, 33] in two dimensions and for certain three-dimensional geometric shapes in [34, 36]. Although for piecewise analytic conductivities the uniqueness result holds in all reasonable geometric shapes via [41, 24]. Even when uniqueness holds for the partial data problem, exact reconstruction methods are scarce. In fact to the author's knowledge, the only other proven reconstruction method (besides the one given in this paper) is found in [44] which does not apply to local Cauchy data, but requires Dirichlet and Neumann data to be applied on a (slightly overlapping) partition of $\partial\Omega$.

We refer to the review papers [4, 5, 13, 49] and references therein for more information on the theoretical and practical aspects of EIT, and refer to the list of references in section 3 on the related problem of inclusion detection.

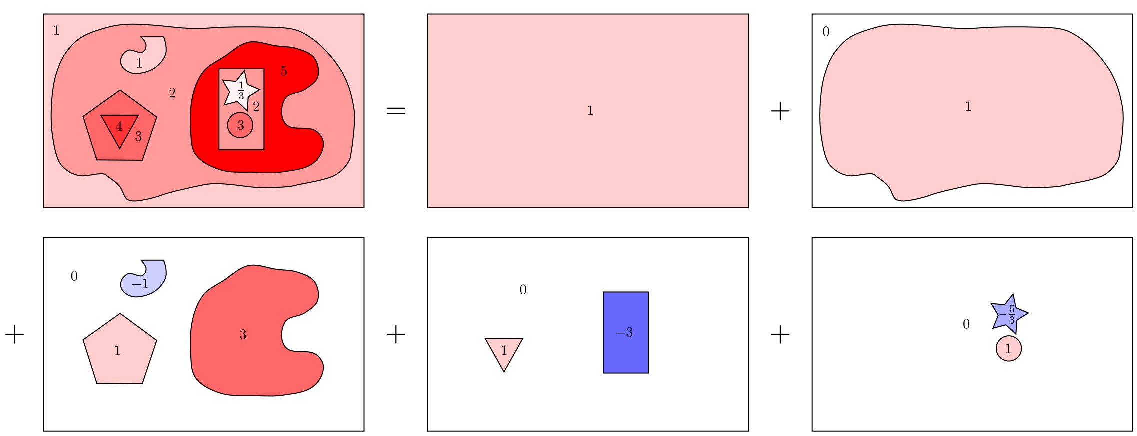

FIGURE 1.1. Decomposition of a PCLC type conductivity (top left) into each of its layers. The numbers represent function values in each of the colored regions.

In this paper, we will consider a class of piecewise constant conductivity coefficients that can be decomposed into a sum of piecewise constant functions on nested sets (layers) with connected complement. We call such a conductivity coefficient of type *piecewise constant layered conductivity* (PCLC), formally defined in Definition 2.3 in section 2. As illustrated by the example in Figure 1.1, this type of decomposition is in fact possible for many piecewise constant functions. The purpose of this paper is to provide a reconstruction method, based on a short and comparatively non-technical proof, that determines any PCLC type conductivity γ from its local ND map $\Lambda(\gamma)$ via the monotonicity principles of $\sigma \mapsto \Lambda(\sigma)$.

It is noted that [1] have used similar ideas with piecewise constant coefficients on layered sets. Their contribution is a uniqueness proof for the complicated case of *anisotropic* piecewise constant coefficients on layered sets. The result of [1] is non-constructive, each layer consists of a single connected component, and they need stronger assumptions on the boundaries of the layers. Hence the result of [1] differ considerably from the results presented here, where the main contribution is a *constructive* proof from partial boundary data.

The remainder of the paper is organized as follows. Section 2 introduces the main assumptions and the PCLC coefficients that can be reconstructed. Section 3 introduces some additional notation and mention two lemmas on the monotonicity principles of $\sigma \mapsto \Lambda(\sigma)$ and on localizing solutions to (1.1), that will be used for proving the main results. The main results Theorem 4.1 and Theorem 5.1 are stated and proved in section 4 and section 5, respectively. Section 6 summarizes the actual reconstruction method based on Theorem 4.1 and Theorem 5.1. Finally, section 7 is dedicated to illustrating that the method becomes quite straightforward if each layer only consists of a single connected component.

2. The setting

Before giving a precise definition of PCLC type conductivities, we will start by defining the (closed) τ-thinning and the outer τ-layer of a set $E \subseteq \mathbb{R}^d$ as

$$H_\tau(E) := \{x \in E \mid \mathrm{dist}(x, \partial E) \geq \tau\}, \tag{2.1}$$

$$F_\tau(E) := \{x \in E \mid \mathrm{dist}(x, \partial E) < \tau\}. \tag{2.2}$$

We now state a list of assumptions on a family of sets that will be used to represent layers of a conductivity coefficient.

Assumption 2.1. Let $\tau > 0$, $N \in \mathbb{N}$, and $\{D_j\}_{j=1}^N$ be sets in $\mathbb{R}^d$ satisfying:

(i) D_j is the closure of a non-empty open set with piecewise $\mathscr{C}^\infty$-smooth boundary.
(ii) D_j has connected complement $\mathbb{R}^d \setminus D_j$.
(iii) $D_{j+1} \subseteq H_\tau(D_j)$ for $j = 1, \dots, N-1$ and $D_1 \subset \Omega$.
(iv) Each set D_j consists of finitely many connected components $\{D_{j,n}\}_{n=1}^{N_j}$.

Before continuing, we give a few remarks on these assumptions.

Remark 2.2 (Related to Assumption 2.1).

(1) While we do not allow cusps on $\partial\Omega$, there can be cusps on ∂D_j. This is because piecewise analytic functions allow cusps on interior interfaces [41, Section 3].
(2) The case $\Gamma = \partial\Omega$ allows $D_1 \subseteq \overline{\Omega}$ with only minor modifications to the proof of Theorem 4.1.
(3) Using D_j as the closure of an open set, compared to a more general closed set, has the following immediate advantage: $B \cap D_j$ contains a non-empty open set for every open neighborhood B of $x \in D_j$. This avoids some obvious pathological cases in the proof of Theorem 4.1.
(4) Each connected component $D_{j,n}$ obviously also satisfies (i) and (ii) of Assumption 2.1 and $\mathrm{dist}(\partial D_{j,n}, D_{j+1}) \geq \tau$.
(5) We will refer to $\tau > 0$ as the *minimal thickness* related to $\{D_j\}_{j=1}^N$.
(6) The layering of the sets and $\tau > 0$ is required for the proofs to be *constructive*. Much milder conditions apply when obtaining non-constructive uniqueness and stability proofs via monotonicity-based arguments [24].

For a set $E \subseteq \mathbb{R}^d$ let χ_E denote the characteristic function on E. We now define the PCLC type conductivities.

Definition 2.3. Suppose $\{D_j\}_{j=1}^N$ satisfy Assumption 2.1 with minimal thickness $\tau > 0$, then we call γ a *piecewise constant layered conductivity* (PCLC), provided that

$$\gamma = c_0 + \sum_{j=1}^{N}\sum_{n=1}^{N_j} c_{j,n}\chi_{D_{j,n}}$$

where $c_0 > 0$ and $c_{j,n} \in \mathbb{R}\setminus\{0\}$ satisfy $0 < \beta_\mathrm{L} \leq \gamma \leq \beta_\mathrm{U}$ in Ω for scalars β_L and β_U. Here D_j is called the j'th layer of γ, with $D_0 := \overline{\Omega}$ denoting the 0'th layer.

For $k \in \{0, 1, \dots, N\}$ we define the k'th layer-truncated conductivity:

$$\gamma_k := c_0 + \sum_{j=1}^{k}\sum_{n=1}^{N_j} c_{j,n}\chi_{D_{j,n}}, \tag{2.3}$$

where in particular $\gamma = \gamma_N$. Note that Assumption 2.1 implies that γ_k is piecewise analytic (see e.g. [26, Definition 2.1] and [41, Section 3]). In the following we will devise an iterative reconstruction method that at its k'th iteration exactly reconstructs γ_k, and naturally terminates at $k = N$. Purely from a notational point of view, in the following sections we will use $D_{N+1} := \emptyset$, which naturally is the conclusion from the $(N+1)$'th iteration.

To summarize the ideas behind the proofs of the main results, consider the problem of determining γ_{k+1} from γ_k and $\Lambda(\gamma)$. This consists of two parts related to the results of Theorem 4.1 and Theorem 5.1:

(i) First we find the set D_{k+1}. In fact, we find the components of D_{k+1} inside each of the components D_{k,n_0} separately.
(ii) Afterwards we determine the constants c_{k+1,m_0}, related to each component D_{k+1,m_0}.

Part (i) focuses on reconstructing the components of D_{k+1} that reside inside a component D_{k,n_0}. By finding certain upper bounds $D_{k+1}\cap D_{k,n_0} \subseteq C$, we may subsequently shrink C until we exactly capture the set $D_{k+1}\cap D_{k,n_0}$. The monotonicity principles of $\sigma \mapsto \Lambda(\sigma)$, combined with a localization result, characterize when C is an upper bound. This is done by explicitly constructing two families of operators $T^+_{k,n_0}(C)$ and $T^-_{k,n_0}(C)$, only based on γ_k and $\Lambda(\gamma)$, such that $D_{k+1}\cap D_{k,n_0} \subseteq C$ if and only if both $T^+_{k,n_0}(C)$ and $T^-_{k,n_0}(C)$ are positive semi-definite. $D_{k+1}\cap D_{k,n_0}$ can comprise several connected components, some related to positive parts of $\gamma_{k+1}-\gamma_k$ and others related to negative parts. Therefore we need both operators $T^+_{k,n_0}(C)$ (handles positive parts) and $T^-_{k,n_0}(C)$ (handles negative parts) in order to find $D_{k+1}\cap D_{k,n_0}$. The construction of $T^+_{k,n_0}(C)$ and $T^-_{k,n_0}(C)$ is such that only the components of D_{k+1} inside D_{k,n_0} influence the positive semi-definiteness, i.e. other components of D_{k+1} can be marginalized in this regard.

The ideas of part (ii) are actually very similar to those of part (i). Now we focus on a single component D_{k+1,m_0}, and construct two new families of operators $S^+_{k,m_0}(s)$ and $S^-_{k,m_0}(t)$, only based on γ_k, $\Lambda(\gamma)$, and D_{k+1}. The two families of operators are characterized by considering either a positive or negative perturbation to γ_k on the outer τ-layer of D_{k+1,m_0} (hence the need for the τ-thickness between the layers), while simultaneously marginalizing other components of D_{k+1} and the τ-thinned part of D_{k+1,m_0} when it comes to positive semi-definiteness of $S^+_{k,m_0}(s)$ and $S^-_{k,m_0}(t)$. Monotonicity principles of $\sigma \mapsto \Lambda(\sigma)$ can first determine the sign of c_{k+1,m_0}, and afterwards find its value via a one-dimensional optimization problem constrained by positive semi-definiteness of either $S^+_{k,m_0}(s)$ (for positive sign) or $S^-_{k,m_0}(t)$ (for negative sign).

For this reconstruction method the following is assumed known/unknown a priori:

- The following are assumed to be *known* a priori: Ω, Γ, $\Lambda(\gamma)$, c_0, and γ is of type PCLC with known lower and upper bounds β_L and β_U and minimal thickness τ.
- The following are *unknown* a priori: $c_{j,n}$, $D_{j,n}$, N_j, and N.

Remark 2.4. Here we assume c_0 is known a priori. Such an assumption is also often imposed on other reconstruction methods such as the $\bar{\partial}$-method, which can be circumvented by first applying another method to reconstruct γ on Γ, see e.g. [45].

3. Notational remarks and lemmas

For brevity we denote the essential infimum/supremum $\operatorname{ess\,inf}\varsigma$ and $\operatorname{ess\,sup}\varsigma$ of a function $\varsigma \in L^\infty(\Omega;\mathbb{R})$ by $\inf(\varsigma)$ and $\sup(\varsigma)$, respectively. $\langle\cdot,\cdot\rangle$ will always denote the usual $L^2(\Gamma)$-inner product.

Let $\mathscr{L}(X,Y)$ be the space of bounded linear operators between Banach spaces X and Y, with the shorthand notation $\mathscr{L}(X) := \mathscr{L}(X,X)$. For a self-adjoint operator $T \in \mathscr{L}(L^2_\diamond(\Gamma))$ then $T \geq 0$ denotes that T is a positive semi-definite operator, i.e. $\langle Tf, f\rangle \geq 0$ for all $f \in L^2_\diamond(\Gamma)$.

We will often use the symbols "+"/"−" to associate sets and operators to positive/negative perturbations. To avoid excessive repetition, "±" will indicate that a statement holds for both the "+" and "−" version of the set/operator. For example, $T^\pm_{k,n_0} \geq 0$ means that both $T^+_{k,n_0} \geq 0$ and $T^-_{k,n_0} \geq 0$ hold true.

The reconstruction method will be derived based on the following two results, the *monotonicity principle* and *localized potentials* (which is related to the Runge approximation property), both of which are well-known results for monotonicity-based reconstruction of the support of perturbations (inclusion detection) and for non-constructive uniqueness and stability proofs in EIT, cf. e.g. [35, 29, 48, 19, 25, 26, 27, 24, 11, 17, 18, 15].

Lemma 3.1 (Monotonicity principle). *For $f \in L^2_\diamond(\Gamma)$ and $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, it holds*

$$\int_\Omega \frac{\sigma_2}{\sigma_1}(\sigma_1 - \sigma_2)|\nabla u_f^{\sigma_2}|^2 \,\mathrm{d}x \leq \langle(\Lambda(\sigma_2) - \Lambda(\sigma_1))f, f\rangle \leq \int_\Omega (\sigma_1 - \sigma_2)|\nabla u_f^{\sigma_2}|^2 \,\mathrm{d}x.$$

Proof. This type of result goes back to [35, 29]. See [26, Lemma 3.1] or [25, Lemma 2.1] for a proof of this version of the result, that is readily modified to the local ND map using the variational form of (1.1). See also [26, Section 4.3] for remarks on such extensions. □

Lemma 3.2 (Localized potentials). *Let $U \subset \overline{\Omega}$ be a relatively open connected set, which intersects Γ, and has connected complement. Let $B \subset U$ be an open non-empty set and $\sigma \in L^\infty_+(\Omega)$ piecewise analytic, then there are sequences $(f_i) \subset L^2_\diamond(\Gamma)$ and $(u_i) \subset H^1_\diamond(\Omega)$ with $u_i = u^\sigma_{f_i}$ satisfying*

$$\lim_{i\to\infty} \int_B |\nabla u_i|^2 \,\mathrm{d}x = \infty \qquad \text{and} \qquad \lim_{i\to\infty} \int_{\Omega\setminus U} |\nabla u_i|^2 \,\mathrm{d}x = 0. \tag{3.1}$$

Proof. This result and its generalizations, ultimately based on unique continuation, is the main topic of [19]. Furthermore, this result is a special case of [26, Theorem 3.6 and Section 4.3], which is also stated for locally supported Neumann conditions in [24, Lemma 2.7]. □

It is also expected that other inclusion detection methods, such as the factorization method [8, 9, 38, 22, 23, 20] or the enclosure method [30, 31, 6], can lead to similar reconstruction methods under stronger assumptions on the constants $c_{j,n}$ and sets D_j.

The map $\sigma \mapsto \Lambda(\sigma)$ is nonlinear, however it is Fréchet differentiable with derivative $D\Lambda(\sigma;\,\cdot\,) \in \mathscr{L}(L^\infty(\Omega;\mathbb{R}), \mathscr{L}(L^2_\diamond(\Gamma)))$; in fact the map is analytic [16, Appendix A]. For each $\sigma \in L^\infty_+(\Omega)$, $\eta \in L^\infty(\Omega)$, and $f \in L^2_\diamond(\Gamma)$ then $D\Lambda(\sigma;\eta)$ is compact, self-adjoint, and satisfies the well-known quadratic formula (cf. e.g. [24, Lemma 2.5])

$$\langle D\Lambda(\sigma;\eta)f, f\rangle = -\int_\Omega \eta|\nabla u_f^\sigma|^2 \,\mathrm{d}x. \tag{3.2}$$

While we could completely avoid $D\Lambda$ in this work by changing the conductivities used for the monotonicity principles, $D\Lambda$ does lead to a fast numerical method that may be of much higher practical value, without lengthening any of the proofs.

As additional notation, we define the index sets $I_j := \{1,\dots,N_j\}$ for $j \in \{1,\dots,N\}$ and $I_0 := \{1\}$ as $D_{0,1} = D_0 := \overline{\Omega}$. Moreover,

$$I_j^+ := \{n \in I_j \mid c_{j,n} > 0\}, \qquad D_j^+ := \cup_{n\in I_j^+} D_{j,n},$$
$$I_j^- := \{n \in I_j \mid c_{j,n} < 0\}, \qquad D_j^- := \cup_{n\in I_j^-} D_{j,n},$$

such that $D_j = D_j^+ \cup D_j^-$ decomposes the set into parts with only positive and only negative perturbations, respectively.

Since each connected component D_{j,n_0} of D_j can contain several connected components of D_{j+1}, it can swiftly become notationally demanding to have a hierarchical structure of such sets. For this reason we define a function $\mathfrak{n}_j : I_{j+1} \to I_j$, $m \mapsto n$, where $n \in I_j$ is the unique integer such that $D_{j+1,m} \subset D_{j,n}$ for given $j \in \{0,\dots,N-1\}$ and $m \in I_{j+1}$.

From this point onwards it is assumed γ_k is known for some $k \in \{0,\dots,N-1\}$ and we will obtain results that determine γ_{k+1}. Denoting the constants

$$\alpha_{k,n} := \gamma_k|_{D_{k,n}} \quad n \in I_k, \qquad \hat{\alpha}_{k,m} := \alpha_{k,\mathfrak{n}_k(m)} \quad m \in I_{k+1},$$

these constants will be used to define *conservative* upper bounds on the possible perturbations inside the connected components of D_k. Thereby we avoid having to consider the actual conductivity value on all connected components simultaneously when applying the monotonicity principles. Due to Definition 2.3 and Assumption 2.1 it clearly holds that $\beta_{\mathrm{L}} \leq \alpha_{k,n} \leq \beta_{\mathrm{U}}$ for all $n \in I_k$. Moreover, from (2.3), Definition 2.3, and Assumption 2.1(iii) we obtain the following bounds for any $n_0 \in I_k$:

$$\gamma - \gamma_k \leq \sum_{m\in I_{k+1}} (\beta_{\mathrm{U}} - \hat{\alpha}_{k,m})\chi_{D_{k+1,m}} \leq \sum_{n\in I_k\setminus\{n_0\}} (\beta_{\mathrm{U}} - \alpha_{k,n})\chi_{D_{k,n}} + (\beta_{\mathrm{U}} - \alpha_{k,n_0})\chi_{D_{k+1}\cap D_{k,n_0}}, \tag{3.3}$$

$$\gamma - \gamma_k \geq \sum_{m\in I_{k+1}} (\beta_{\mathrm{L}} - \hat{\alpha}_{k,m})\chi_{D_{k+1,m}} \geq \sum_{n\in I_k\setminus\{n_0\}} (\beta_{\mathrm{L}} - \alpha_{k,n})\chi_{D_{k,n}} + (\beta_{\mathrm{L}} - \alpha_{k,n_0})\chi_{D_{k+1}\cap D_{k,n_0}}. \tag{3.4}$$

In particular, $\beta_{\mathrm{L}} - \alpha_{k,n}$ represents the largest possible (signed) negative perturbation that can occur within $D_{k,n}$ when determining γ_{k+1} from γ_k, and likewise $\beta_{\mathrm{U}} - \alpha_{k,n}$ is the largest possible positive perturbation.

4. Reconstruction of D_{k+1} from γ_k and $\Lambda(\gamma)$

For $n_0 \in I_k$ and measurable $C \subseteq \overline{\Omega}$ we now define some operators based on γ_k and $\Lambda(\gamma)$:

$$T^+_{k,n_0}(C) := \Lambda(\gamma) - \Lambda(\gamma_k) - \sum_{n\in I_k\setminus\{n_0\}} (\beta_{\mathrm{U}} - \alpha_{k,n}) D\Lambda(\gamma_k;\chi_{D_{k,n}}) - (\beta_{\mathrm{U}} - \alpha_{k,n_0}) D\Lambda(\gamma_k;\chi_C),$$

$$T^-_{k,n_0}(C) := \Lambda(\gamma_k) - \Lambda(\gamma) + \sum_{n\in I_k\setminus\{n_0\}} \frac{\alpha_{k,n}}{\beta_{\mathrm{L}}} (\beta_{\mathrm{L}} - \alpha_{k,n}) D\Lambda(\gamma_k;\chi_{D_{k,n}}) + \frac{\alpha_{k,n_0}}{\beta_{\mathrm{L}}} (\beta_{\mathrm{L}} - \alpha_{k,n_0}) D\Lambda(\gamma_k;\chi_C).$$

In fact, we will consider sets C that belong to families of *admissible test inclusions* relative to some subset $E \subseteq \overline{\Omega}$:

$$\mathcal{A}(E) := \{C \subseteq \overline{E} \mid C \text{ is closed and } \mathbb{R}^d \setminus C \text{ is connected}\}.$$

In what follows these test inclusions will be used to determine D_{k+1} from γ_k. Note that Theorem 4.1 below essentially corresponds to a modified version of the usual monotonicity method for indefinite inclusions, applied separately on each connected component of D_k; cf. [18, Theorem 2.3] and [26, Section 4.2].

Theorem 4.1. *Let $n_0 \in I_k$, then for all $C \in \mathcal{A}(D_{k,n_0})$ it holds*

$$D_{k+1} \cap D_{k,n_0} \subseteq C \qquad \text{if and only if} \qquad T^{\pm}_{k,n_0}(C) \geq 0. \tag{4.1}$$

In particular, $D_{k+1} \cap D_{k,n_0} = \cap\{C \in \mathcal{A}(D_{k,n_0}) \mid T^{\pm}_{k,n_0}(C) \geq 0\}$.

Proof. First we prove the direction "$\Rightarrow$" in the *if and only if* statement. Assume $D_{k+1} \cap D_{k,n_0} \subseteq C$, then it holds by Lemma 3.1, (3.2), and (3.3),

$$\begin{aligned} -\langle T^+_{k,n_0}(C)f, f\rangle &\leq \int_\Omega \Big[\gamma - \gamma_k - \sum_{n\in I_k\setminus\{n_0\}} (\beta_{\mathrm{U}} - \alpha_{k,n})\chi_{D_{k,n}} - (\beta_{\mathrm{U}} - \alpha_{k,n_0})\chi_C\Big] |\nabla u_f^{\gamma_k}|^2 \,\mathrm{d}x \\ &\leq (\alpha_{k,n_0} - \beta_{\mathrm{U}}) \int_{C\setminus(D_{k+1}\cap D_{k,n_0})} |\nabla u_f^{\gamma_k}|^2 \,\mathrm{d}x \leq 0 \end{aligned}$$

for all $f \in L^2_\diamond(\Gamma)$, i.e. $T^+_{k,n_0}(C) \geq 0$.

Likewise, since $\frac{\gamma_k}{\gamma} \leq \frac{\alpha_{k,n}}{\beta_\mathrm{L}}$ in $D_{k,n}$ and $\beta_\mathrm{L} \leq \alpha_{k,n}$ then Lemma 3.1, (3.2), and (3.4) imply

$$\begin{aligned}\langle T^-_{k,n_0}(C)f,f\rangle &\geq \int_\Omega \Big[\frac{\gamma_k}{\gamma}(\gamma-\gamma_k) - \sum_{n\in I_k\setminus\{n_0\}} \frac{\alpha_{k,n}}{\beta_\mathrm{L}}(\beta_\mathrm{L}-\alpha_{k,n})\chi_{D_{k,n}} - \frac{\alpha_{k,n_0}}{\beta_\mathrm{L}}(\beta_\mathrm{L}-\alpha_{k,n_0})\chi_C\Big]|\nabla u_f^{\gamma_k}|^2\,\mathrm{d}x \\ &\geq \frac{\alpha_{k,n_0}}{\beta_\mathrm{L}}(\alpha_{k,n_0}-\beta_\mathrm{L})\int_{C\setminus(D_{k+1}\cap D_{k,n_0})} |\nabla u_f^{\gamma_k}|^2\,\mathrm{d}x \geq 0\end{aligned}$$

for all $f \in L^2_\diamond(\Gamma)$, i.e. $T^-_{k,n_0}(C) \geq 0$. This concludes the first part of the proof.

The proof of the other direction "$\Leftarrow$" of the *if and only if* statement is shown as a contrapositive, i.e. assume $D_{k+1}\cap D_{k,n_0} \not\subseteq C$ then we will in the following contradict one of the inequalities $T^\pm_{k,n_0} \geq 0$.

We now pick a relatively open connected set $U \subset \overline{\Omega}$, which intersects Γ, has connected complement, and satisfies: $D_{k+1,m_0}\cap U$ contains an open ball B for some $m_0 \in I_{k+1}$ with $\mathfrak{n}_k(m_0) = n_0$ and

$$U \cap [(D_k\setminus D_{k,n_0}) \cup C \cup (D_{k+1}\setminus D_{k+1,m_0}) \cup D_{k+2}] = \emptyset.$$

The reasoning behind the properties of U is: Assumption 2.1 and $C \in \mathcal{A}(D_{k,n_0})$ imply the set $\overline{\Omega}\setminus[(D_k\setminus D_{k,n_0})\cup C]$ is connected and contains Γ. Moreover, $(D_{k+1}\cap D_{k,n_0})\setminus C$ contains a non-empty open set due to (i) and (iii) of Assumption 2.1 (cf. Remark 2.2). Since D_{k+1} comprise finitely many closed connected components (Assumption 2.1) implies a strictly positive distance between these connected components. Thus U can be chosen to only intersect one connected component of $D_{k+1}\cap D_{k,n_0}$, and furthermore avoid D_{k+2} due to Assumption 2.1(iii).

This splits the rest of the proof into two possible cases, related to which one of the inequalities $T^\pm_{k,n_0} \geq 0$ that will be contradicted:

$$\text{(a): } m_0 \in I^+_{k+1} \qquad \text{or} \qquad \text{(b): } m_0 \in I^-_{k+1}.$$

Case (a). Note that $\gamma = \alpha_{k,n_0} + c_{k+1,m_0}$ in B with $c_{k+1,m_0} > 0$ and $\gamma \geq \gamma_k$ in U (equality holds in $U \setminus D_{k+1,m_0}$). The main idea is to construct potentials u via Lemma 3.2 where simultaneously $|\nabla u|^2$ is large inside B and small outside U, in such a way that Lemma 3.1 contradicts the inequality $T^+_{k,n_0}(C) \geq 0$. Since γ_k is piecewise analytic and by the properties of U, it follows from Lemma 3.2 that there are sequences $(f_i) \subset L^2_\diamond(\Gamma)$ of current densities and corresponding localized potentials $(u_i) \subset H^1_\diamond(\Omega)$ that solve (1.1) with conductivity γ_k, and satisfy (3.1).

Denoting

$$\hat{\gamma} := \frac{\gamma_k}{\gamma}(\gamma-\gamma_k) - \sum_{n\in I_k\setminus\{n_0\}}(\beta_\mathrm{U}-\alpha_{k,n})\chi_{D_{k,n}} - (\beta_\mathrm{U}-\alpha_{k,n_0})\chi_C,$$

we have by Lemma 3.1, (3.2), and (3.1)

$$\begin{aligned}-\langle T^+_{k,n_0}(C)f_i,f_i\rangle &\geq \int_B \frac{\gamma_k}{\gamma}(\gamma-\gamma_k)|\nabla u_i|^2\,\mathrm{d}x + \int_{U\setminus B}\frac{\gamma_k}{\gamma}(\gamma-\gamma_k)|\nabla u_i|^2\,\mathrm{d}x + \int_{\Omega\setminus U}\hat{\gamma}|\nabla u_i|^2\,\mathrm{d}x \\ &\geq \frac{\alpha_{k,n_0}c_{k+1,m_0}}{\alpha_{k,n_0}+c_{k+1,m_0}}\int_B |\nabla u_i|^2\,\mathrm{d}x + \inf(\hat{\gamma})\int_{\Omega\setminus U}|\nabla u_i|^2\,\mathrm{d}x \to \infty \text{ for } i\to\infty,\end{aligned}$$

from which we conclude $T^+_{k,n_0}(C) \not\geq 0$.

Case (b). In this case we have $\gamma = \alpha_{k,n_0} + c_{k+1,m_0}$ in B with $c_{k+1,m_0} < 0$ and $\gamma \leq \gamma_k$ in U. Denote

$$\tilde{\gamma} := \gamma - \gamma_k - \sum_{n\in I_k\setminus\{n_0\}} \frac{\alpha_{k,n}}{\beta_\mathrm{L}}(\beta_\mathrm{L}-\alpha_{k,n})\chi_{D_{k,n}} - \frac{\alpha_{k,n_0}}{\beta_\mathrm{L}}(\beta_\mathrm{L}-\alpha_{k,n_0})\chi_C.$$

Applying the above construction of localized potentials satisfying (3.1), we contradict the inequality $T^-_{k,n_0} \geq 0$ using Lemma 3.1 and (3.2):

$$\begin{aligned}\langle T^-_{k,n_0}(C)f_i,f_i\rangle &\leq \int_B (\gamma-\gamma_k)|\nabla u_i|^2\,\mathrm{d}x + \int_{U\setminus B}(\gamma-\gamma_k)|\nabla u_i|^2\,\mathrm{d}x + \int_{\Omega\setminus U}\tilde{\gamma}|\nabla u_i|^2\,\mathrm{d}x \\ &\leq c_{k+1,m_0}\int_B |\nabla u_i|^2\,\mathrm{d}x + \sup(\tilde{\gamma})\int_{\Omega\setminus U}|\nabla u_i|^2\,\mathrm{d}x \to -\infty \text{ for } i\to\infty,\end{aligned}$$

hence concluding $T^-_{k,n_0}(C) \not\geq 0$.

The equality $D_{k+1} \cap D_{k,n_0} = \cap\mathcal{M}$ with $\mathcal{M} := \{C \in \mathcal{A}(D_{k,n_0}) \mid T^{\pm}_{k,n_0}(C) \geq 0\}$ is satisfied via (4.1) since $D_{k+1} \cap D_{k,n_0} \subseteq C$ for each $C \in \mathcal{M}$ and that $D_{k+1} \cap D_{k,n_0}$ itself is a member of $\mathcal{M}$. □

5. Reconstruction of γ_{k+1} from γ_k, $\Lambda(\gamma)$, and D_{k+1}

Now that Theorem 4.1 gives a way of determining D_{k+1} from γ_k, the next step is to determine the constant c_{k+1,m_0} for each $m_0 \in I_{k+1}$ in order to obtain γ_{k+1}. For this purpose we define for $m_0 \in I_{k+1}$, $s \in [0, \beta_{\mathrm{U}} - \hat{\alpha}_{k,m_0}]$, and $t \in [\beta_{\mathrm{L}} - \hat{\alpha}_{k,m_0}, 0]$ the operators

$$S^+_{k,m_0}(s) := \Lambda(\gamma) - \Lambda(\gamma_{k,m_0,\beta_{\mathrm{U}}} + s\chi_{F_\tau(D_{k+1,m_0})}),$$
$$S^-_{k,m_0}(t) := \Lambda(\gamma_{k,m_0,\beta_{\mathrm{L}}} + t\chi_{F_\tau(D_{k+1,m_0})}) - \Lambda(\gamma),$$

for which $\gamma_{k,m_0,\beta}$ with $\beta \in \{\beta_{\mathrm{L}}, \beta_{\mathrm{U}}\}$ is defined as

$$\gamma_{k,m_0,\beta} := \gamma_k + \sum_{m \in I_{k+1}\setminus\{m_0\}} (\beta - \hat{\alpha}_{k,m})\chi_{D_{k+1,m}} + (\beta - \hat{\alpha}_{k,m_0})\chi_{H_\tau(D_{k+1,m_0})}.$$

Recall the definition of H_τ and F_τ in (2.1) and (2.2). As we shall see in Theorem 5.1, there are two equivalent ways of determining if $m_0 \in I_{k+1}$ belongs to I^+_{k+1} or I^-_{k+1}. Afterwards, we may find the constant $c_{k+1,m_0} \in [\beta_{\mathrm{L}} - \hat{\alpha}_{k,m_0}, 0) \cup (0, \beta_{\mathrm{U}} - \hat{\alpha}_{k,m_0}]$ via an optimization problem, by varying s and t on the outer τ-layer of D_{k+1,m_0}, constrained by positive semi-definiteness of $S^{\pm}_{k,m_0}$.

Theorem 5.1. *Let $m_0 \in I_{k+1}$, then it holds*

$$[0, \beta_{\mathrm{U}} - \hat{\alpha}_{k,m_0}] \ni s \geq c_{k+1,m_0} \quad \text{if and only if} \quad S^+_{k,m_0}(s) \geq 0, \tag{5.1}$$
$$[\beta_{\mathrm{L}} - \hat{\alpha}_{k,m_0}, 0] \ni t \leq c_{k+1,m_0} \quad \text{if and only if} \quad S^-_{k,m_0}(t) \geq 0. \tag{5.2}$$

As direct consequences,

$$m_0 \in I^+_{k+1} \quad \text{if and only if} \quad S^-_{k,m_0}(0) \geq 0 \quad \text{if and only if} \quad S^+_{k,m_0}(0) \not\geq 0,$$
$$m_0 \in I^-_{k+1} \quad \text{if and only if} \quad S^+_{k,m_0}(0) \geq 0 \quad \text{if and only if} \quad S^-_{k,m_0}(0) \not\geq 0,$$

and c_{k+1,m_0} is determined via:

$$c_{k+1,m_0} = \begin{cases} \min\{s \in (0, \beta_{\mathrm{U}} - \hat{\alpha}_{k,m_0}] \mid S^+_{k,m_0}(s) \geq 0\} & \text{if } m_0 \in I^+_{k+1}, \\ \max\{t \in [\beta_{\mathrm{L}} - \hat{\alpha}_{k,m_0}, 0) \mid S^-_{k,m_0}(t) \geq 0\} & \text{if } m_0 \in I^-_{k+1}. \end{cases}$$

Proof. Note that $\gamma = \hat{\alpha}_{k,m_0} + c_{k+1,m_0}$ in the set $F_\tau(D_{k+1,m_0})$ due to Assumption 2.1(iii). Moreover, $\gamma_k = \hat{\alpha}_{k,m_0}$ in $F_\tau(D_{k+1,m_0})$, so writing

$$\gamma - \gamma_{k,m_0,\beta_{\mathrm{U}}} = (\gamma - \gamma_{k,m_0,\beta_{\mathrm{U}}})\chi_{\overline{\Omega}\setminus F_\tau(D_{k+1,m_0})} + (\gamma - \gamma_{k,m_0,\beta_{\mathrm{U}}})\chi_{F_\tau(D_{k+1,m_0})}$$

we may apply (3.3) to bound the first term from above by 0. Likewise for $\gamma - \gamma_{k,m_0,\beta_{\mathrm{L}}}$ we obtain a lower bound using (3.4), resulting in

$$\gamma - \gamma_{k,m_0,\beta_{\mathrm{U}}} \leq c_{k+1,m_0}\chi_{F_\tau(D_{k+1,m_0})} \leq \gamma - \gamma_{k,m_0,\beta_{\mathrm{L}}}. \tag{5.3}$$

We begin by proving (5.1), hence denote the piecewise analytic $L^\infty_+(\Omega)$-function

$$\hat{\gamma} := \gamma_{k,m_0,\beta_{\mathrm{U}}} + s\chi_{F_\tau(D_{k+1,m_0})},$$

and assume $s \geq c_{k+1,m_0}$. By virtue of Lemma 3.1 and (5.3)

$$\begin{aligned} -\langle S^+_{k,m_0}(s)f, f\rangle &\leq \int_\Omega \left[\gamma - \gamma_{k,m_0,\beta_{\mathrm{U}}} - s\chi_{F_\tau(D_{k+1,m_0})}\right] |\nabla u^{\hat{\gamma}}_f|^2 \, \mathrm{d}x \\ &\leq (c_{k+1,m_0} - s)\int_{F_\tau(D_{k+1,m_0})} |\nabla u^{\hat{\gamma}}_f|^2 \, \mathrm{d}x \leq 0 \end{aligned}$$

for all $f \in L^2_\diamond(\Gamma)$, i.e. $S^+_{k,m_0}(s) \geq 0$ for $s \geq c_{k+1,m_0}$.

For the opposite implication we assume $s < c_{k+1,m_0}$. In a similar way to the proof of Theorem 4.1, we pick a relatively open connected set $U \subset \overline{\Omega}$, which intersects Γ, has connected complement, satisfies $(D_{k+1} \setminus D_{k+1,m_0}) \cap U = H_\tau(D_{k+1,m_0}) \cap U = \emptyset$, and $F_\tau(D_{k+1,m_0}) \cap U$ contains an open ball B. Once again this is possible due to Assumption 2.1. Hence $\gamma - \hat{\gamma} = c_{k+1,m_0} - s > 0$ in B and $\gamma \geq \hat{\gamma}$ in U.

Now let $(f_i) \subset L^2_\diamond(\Gamma)$ and $(u_i) \subset H^1_\diamond(\Omega)$ be chosen via Lemma 3.2 with respect to the sets U and B for the conductivity $\hat{\gamma}$. Lemma 3.1 gives

$$\begin{aligned} -\langle S^+_{k,m_0}(s)f_i, f_i\rangle &\geq \int_\Omega \frac{\hat{\gamma}}{\gamma}(\gamma - \hat{\gamma})|\nabla u_i|^2 \,\mathrm{d}x \\ &\geq \frac{\hat{\alpha}_{k,m_0} + s}{\hat{\alpha}_{k,m_0} + c_{k+1,m_0}}(c_{k+1,m_0} - s)\int_B |\nabla u_i|^2 \,\mathrm{d}x + \inf(\tfrac{\hat{\gamma}}{\gamma}(\gamma - \hat{\gamma}))\int_{\Omega\setminus U} |\nabla u_i|^2 \,\mathrm{d}x. \end{aligned}$$

Since $0 \leq s < c_{k+1,m_0}$ then (3.1) implies $\lim_{i\to\infty}\langle S^+_{k,m_0}(s)f_i, f_i\rangle = -\infty$. We conclude $S^+_{k,m_0}(s) \ngeq 0$ for $s < c_{k+1,m_0}$.

Next we prove (5.2) in an analogous way. Denote the piecewise analytic $L^\infty_+(\Omega)$-function

$$\tilde{\gamma} := \gamma_{k,m_0,\beta_\mathrm{L}} + t\chi_{F_\tau(D_{k+1,m_0})}.$$

First we assume $t \leq c_{k+1,m_0}$, and since $t \in [\beta_\mathrm{L} - \hat{\alpha}_{k,m_0}, 0]$ it holds $\frac{\tilde{\gamma}}{\gamma} \geq \frac{\beta_\mathrm{L}}{\beta_\mathrm{U}}$ in Ω. Thus from Lemma 3.1 and (5.3) it holds

$$\langle S^-_{k,m_0}(t)f, f\rangle \geq \int_\Omega \frac{\tilde{\gamma}}{\gamma}(\gamma - \tilde{\gamma})|\nabla u^{\tilde{\gamma}}_f|^2 \,\mathrm{d}x \geq \frac{\beta_\mathrm{L}}{\beta_\mathrm{U}}(c_{k+1,m_0} - t)\int_{F_\tau(D_{k+1,m_0})} |\nabla u^{\tilde{\gamma}}_f|^2 \,\mathrm{d}x \geq 0$$

for all $f \in L^2_\diamond(\Gamma)$, i.e. $S^-_{k,m_0}(t) \geq 0$ for $t \leq c_{k+1,m_0}$.

For the opposite implication we assume $t > c_{k+1,m_0}$ and pick the sets U and B in exactly the same way as in the proof of (5.1). In particular, $\gamma - \tilde{\gamma} = c_{k+1,m_0} - t < 0$ in B and $\gamma \leq \tilde{\gamma}$ in U. Now let $(f_i) \subset L^2_\diamond(\Gamma)$ and $(u_i) \subset H^1_\diamond(\Omega)$ be chosen according to Lemma 3.2 for the sets U and B and with conductivity $\tilde{\gamma}$.

Applying Lemma 3.1 and (3.1) yields

$$\begin{aligned} \langle S^-_{k,m_0}(t)f_i, f_i\rangle &\leq \int_\Omega (\gamma - \tilde{\gamma})|\nabla u_i|^2 \,\mathrm{d}x \\ &\leq (c_{k+1,m_0} - t)\int_B |\nabla u_i|^2 \,\mathrm{d}x + \sup(\gamma - \tilde{\gamma})\int_{\Omega\setminus U} |\nabla u_i|^2 \,\mathrm{d}x \to -\infty \text{ for } i \to \infty, \end{aligned}$$

whence $S^-_{k,m_0}(t) \ngeq 0$ for $t > c_{k+1,m_0}$. □

Remark 5.2. Based on the proofs of Theorem 4.1 and Theorem 5.1, it is straightforward to show that the conclusion of whether $m_0 \in I_{k+1}$ belongs to I^+_{k+1} or I^-_{k+1} in Theorem 5.1 is preserved when replacing $S^\pm_{k,m_0}(0)$ with $\tilde{S}^\pm_{k,m_0}$ defined below, where $\tilde{D} := H_\tau(D_{k+1,m_0})$:

$$\tilde{S}^+_{k,m_0} := \Lambda(\gamma) - \Lambda(\gamma_k) - \sum_{m\in I_{k+1}\setminus\{m_0\}}(\beta_\mathrm{U} - \hat{\alpha}_{k,m})D\Lambda(\gamma_k;\chi_{D_{k+1,m}}) - (\beta_\mathrm{U} - \hat{\alpha}_{k,m_0})D\Lambda(\gamma_k;\chi_{\tilde{D}}),$$

$$\tilde{S}^-_{k,m_0} := \Lambda(\gamma_k) - \Lambda(\gamma) + \sum_{m\in I_{k+1}\setminus\{m_0\}}\frac{\hat{\alpha}_{k,m}}{\beta_\mathrm{L}}(\beta_\mathrm{L} - \hat{\alpha}_{k,m})D\Lambda(\gamma_k;\chi_{D_{k+1,m}}) + \frac{\hat{\alpha}_{k,m_0}}{\beta_\mathrm{L}}(\beta_\mathrm{L} - \hat{\alpha}_{k,m_0})D\Lambda(\gamma_k;\chi_{\tilde{D}}).$$

It is tempting to also use $D\Lambda$ to apply the variation of s and t on $F_\tau(D_{k+1,m_0})$ in Theorem 5.1. However, the set U for the localized potentials will intersect part of the set on which $D\Lambda$ is applied (unlike in the proof of Theorem 4.1, where this is specifically avoided), and the resulting integrals do not lead to a proof of the desired assertion.

6. Monotonicity-based reconstruction of PCLC conductivities

We can now summarize the reconstruction method based on Theorem 4.1 and Theorem 5.1 in the following way:

(1) Let γ_k for some $k \in \{0, 1, \ldots, N\}$ be given (initially $\gamma_0 = c_0$ with $D_0 := \overline{\Omega}$).
(2) Determine D_{k+1} via: for each $n_0 \in I_k$ using Theorem 4.1 we find

$$D_{k+1} \cap D_{k,n_0} = \cap\{C \in \mathcal{A}(D_{k,n_0}) \mid T^\pm_{k,n_0}(C) \geq 0\}.$$

(3) For each $m_0 \in I_{k+1}$ we employ Theorem 5.1/Remark 5.2 to determine if $m_0 \in I_{k+1}^+$ or $m_0 \in I_{k+1}^-$ by the positive semi-definiteness (or lack thereof) of either

$$S^+_{k,m_0}(0), \qquad S^-_{k,m_0}(0), \qquad \tilde{S}^+_{k,m_0}, \qquad \textit{or} \qquad \tilde{S}^-_{k,m_0}.$$

(4) Theorem 5.1 determines c_{k+1,m_0} as:

$$c_{k+1,m_0} = \begin{cases} \min\{s \in (0, \beta_\mathrm{U} - \hat{\alpha}_{k,m_0}] \mid S^+_{k,m_0}(s) \geq 0\} & \text{if } m_0 \in I^+_{k+1}, \\ \max\{t \in [\beta_\mathrm{L} - \hat{\alpha}_{k,m_0}, 0) \mid S^-_{k,m_0}(t) \geq 0\} & \text{if } m_0 \in I^-_{k+1}. \end{cases}$$

(5) The above steps determine γ_{k+1}. Repeat the above steps iteratively, until we reach $\gamma_{N+1} = \gamma_N$ by finding $D_{N+1} = \emptyset$ in step (2), hence concluding the reconstruction method.

Remark 6.1. Note that numerical implementation of step (2) above can be handled, both in terms of regularization theory and practical implementation, via a layer peeling approach [18, Theorem 3.1 and Algorithm 1]. For other considerations in this direction see also [17, 27, 15]. Step (4) can be handled straightforwardly via bisection due to (5.1) and (5.2) in Theorem 5.1.

7. Simplifications when each layer only has a single connected component

This section will illustrate the considerable simplifications to the reconstruction method, in the special case when each layer D_j only consists of a single connected component. Hence the complicated expressions dedicated to marginalizing other components are no longer required.

In this situation, we may name the constants c_j rather than $c_{j,n}$ and write

$$\gamma_k := \sum_{j=0}^{k} c_j \chi_{D_j}, \quad k \in \{0, 1, \dots, N\}, \tag{7.1}$$

using the convention that $D_0 := \overline{\Omega}$. Again we have $\gamma = \gamma_N$. Recall that $c_0 > 0$ is assumed known, and for each $k \in \{0, 1, \dots, N-1\}$ we must reconstruct the set D_{k+1} and constant $c_{k+1} \in \mathbb{R} \setminus \{0\}$ based on knowledge of γ_k and $\Lambda(\gamma)$.

Define $\alpha_k := \sum_{j=0}^k c_j$ for each $k \in \{0, 1, \dots, N\}$, and for measurable $C \subseteq \overline{\Omega}$ we define the operators

$$\begin{aligned} T_k^+(C) &:= \Lambda(\gamma) - \Lambda(\gamma_k) - (\beta_\mathrm{U} - \alpha_k) D\Lambda(\gamma_k; \chi_C), \\ T_k^-(C) &:= \Lambda(\gamma_k) - \Lambda(\gamma) + \frac{\alpha_k}{\beta_\mathrm{L}}(\beta_\mathrm{L} - \alpha_k) D\Lambda(\gamma_k; \chi_C). \end{aligned}$$

For $s \in [0, \beta_\mathrm{U} - \alpha_k]$ and $t \in [\beta_\mathrm{L} - \alpha_k, 0]$ we define the operators

$$\begin{aligned} S_k^+(s) &:= \Lambda(\gamma) - \Lambda\left(\gamma_k + (\beta_\mathrm{U} - \alpha_k)\chi_{H_\tau(D_{k+1})} + s\chi_{F_\tau(D_{k+1})}\right), \\ S_k^-(t) &:= \Lambda\left(\gamma_k + (\beta_\mathrm{L} - \alpha_k)\chi_{H_\tau(D_{k+1})} + t\chi_{F_\tau(D_{k+1})}\right) - \Lambda(\gamma). \end{aligned}$$

Hence, in this situation, the reconstruction method is as follows:

(1) Let γ_k for some $k \in \{0, 1, \dots, N\}$ be given (initially $\gamma_0 = c_0$ with $D_0 := \overline{\Omega}$).
(2) Determine D_{k+1} via:

$$D_{k+1} = \cap\{C \in \mathcal{A}(D_k) \mid T_k^\pm(C) \geq 0\}.$$

(3) The sign of c_{k+1} is determined via either of:

$$c_{k+1} < 0 \Leftrightarrow S_k^+(0) \geq 0, \qquad c_{k+1} > 0 \Leftrightarrow S_k^-(0) \geq 0.$$

(4) Find c_{k+1} via:

$$c_{k+1} = \begin{cases} \min\{s \in (0, \beta_\mathrm{U} - \alpha_k] \mid S_k^+(s) \geq 0\} & \text{if } c_{k+1} > 0, \\ \max\{t \in [\beta_\mathrm{L} - \alpha_k, 0) \mid S_k^-(t) \geq 0\} & \text{if } c_{k+1} < 0. \end{cases}$$

(5) The above steps determine γ_{k+1}. Repeat the steps iteratively, until we reach $\gamma_{N+1} = \gamma_N$ by finding $D_{N+1} = \emptyset$ in step (2), hence concluding the reconstruction method.

Remark 7.1. Note that in step (3) we may also use operators of the form in Remark 5.2 that involve $D\Lambda$. If there is the further simplification that all the constants c_j have the same sign, then only one of the operators $T_k^{\pm}$ is needed for step (2) and only one of the operators $S_k^{\pm}$ is needed for step (4).

Acknowledgments. This work was supported by the Academy of Finland (decision 312124) and the Aalto Science Institute (AScI). HG thanks Nuutti Hyvönen for encouraging discussions during a research visit at Aalto University.

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