Sparse Packetized Predictive Control Over Communication Networks with Packet Dropouts and Time Delays

Barforooshan, Mohsen; Nagahara, Masaaki; Østergaard, Jan

Published in:
2019 IEEE 58th Conference on Decision and Control (CDC)

DOI (link to publication from Publisher):
10.1109/CDC40024.2019.9030023

Publication date:
2020

Document Version
Accepted author manuscript, peer reviewed version

2019 IEEE 58th Conference on Decision and Control (CDC)
Palais des Congrès et des Expositions Nice Acropolis
Nice, France, December 11-13, 2019

Sparse Packetized Predictive Control Over Communication Networks with Packet Dropouts and Time Delays

Mohsen Barforooshan, Masaaki Nagahara, and Jan Østergaard

Abstract— **This paper studies sparse packetized predictive control (PPC) for a feedback loop closed over a digital communication channel with time delay and bounded packet dropouts. In the considered networked control system (NCS), the channel is located between the controller and the actuator of a linear time-invariant (LTI) plant. We analyze the system under two PPC strategies. In one case, the controller computes each control packet by solving a sparsity-promoting unconstrained ℓ^1-ℓ^2 optimization problem. In the other case, the optimization based on which the controller performs is an ℓ^2-constrained ℓ^0 problem. We utilize effective approaches for solving these optimization problems. Moreover, we establish practical and asymptotic stability conditions for unconstrained ℓ^1-ℓ^2 and ℓ^2-constrained ℓ^0 sparse PPCs, respectively. We show that to maintain stability while increasing the channel delay, the proposed sparse PPC strategies necessitate increasing the upper bound on size of the control packet sequences. We demonstrate, through simulation, that when the channel delay is higher, the controllers designed according to the proposed methods can bring the expected stability properties to the system but with worse performance.**

I. INTRODUCTION

Networked control systems (NCSs) are feedback loops whose components are linked via communication channels. The imperfections associated with communication networks introduce new constraints to control problems [1]. For instance, stability cannot be achieved if the probability of the packet loss or the value of transmission delay is greater than a specific upper bound [2], [3]. Designing control strategies that guarantee attaining certain performance levels despite the communication constraints is one of the main goals in the theory and application of NCSs.

Minimizing the control effort is another goal in practical control design which is of high necessity. This is motivated by various environmental and economical merits which control effort minimization brings about [4], [5]. One way to minimize the control effort is maximizing the number of time intervals over which the control input is equal to zero. This approach is pursued in maximum hands-off control [6], [7].

M. Barforooshan and J. Østergaard are with the Department of Electronic Systems, Aalborg University, DK-9220, Aalborg, Denmark (email: mob@es.aau.dk; jo@es.aau.dk).Their work has received funding from VILLUM FONDEN Young Investigator Programme, under grant agreement No. 19005.

M. Nagahara is with the Institute of Environmental Science and Technology, The University of Kitakyushu, Fukuoka 808-0135, Japan (email: nagahara@kitakyu-u.ac.jp).His work is supported in part by the JSPS KAKENHI Grant Numbers JP15H02668 and JP16KK0134, the Ministry of Internal Affairs and Communications of Japan under the Strategic Information and Communications R& D Promotion Programme, 2017-2019, No. 3620, and JST-Mirai Program Grant Number JPMJMI17B4, Japan.

Sparse packetized predictive control (PPC) lies in the intersection of maximum hands-off control and networked control. In a sparse PPC setup, the controller generates a packet containing tentative future control inputs and sends it over the channel to a buffer installed at the plant. Each control packet is computed based on a finite-horizon model predictive control (MPC) strategy. The cost function associated with this MPC is a sparsity-promoting cost function [8], [9] which is commonly used in maximum hands-off control. Sparse PPC causes robustness against channel imperfections [10]–[12] while reducing the size of data transmitted over the channel. This is due to the fact that vector symbols with many zero elements are easier to be compressed. In [13], the unconstrained ℓ^1-ℓ^2 and ℓ^2-constrained ℓ^0 problems are considered as sparsity-promoting optimization problems. The packetized predictive controllers proposed in [13] solve the unconstrained ℓ^1-ℓ^2 and ℓ^2-constrained ℓ^0 optimizations online by employing fast iterative shrinkage-thresholding algorithm (FISTA) and orthogonal matching pursuit (OMP) approaches, respectively. Moreover, the authors of [13] show the practical stability for unconstrained ℓ^1-ℓ^2 PPC and asymptotic stability for ℓ^2-constrained ℓ^0 PPC over a delay-free channel subject to bounded dropouts.

In this paper, we study sparse PPC for an NCS comprised of a fully observable discrete-time noiseless linear time-invariant (LTI) plant. The control packets sent from the controller to the plant are subject to bounded dropouts and a constant time delay. We analyze the system under two sparse PPC policies. One is unconstrained ℓ^1-ℓ^2 sparse PPC for which we derive conditions of practical stability. The other is ℓ^2-constrained ℓ^0 sparse PPC. In this case, we show that asymptotic stability is guaranteed under certain tuning conditions. We utilize the methodology followed in [13] for stability analysis and solving the corresponding sparsity-promoting optimization problems. This is due to the advantages such as simplicity of the stability analysis and effectiveness (in sense of speed and convergence) of solving algorithms proposed in [13]. However, unlike [13], we consider a channel which induces a known constant delay, say h steps, on transmitted packets. Accordingly, we take a different regime of control packet production into account. Based on this regime, the controller uses the received states of the plant and some memory at each time instant to make a number of predictions for the plant states at h time steps later. Then for each prediction, the controller solves the corresponding unconstrained ℓ^1-ℓ^2 or ℓ^2-constrained ℓ^0 optimization problem and gives a sequence of control packets. Upon the arrival of a sequence at the other side of the

978-1-7281-1398-2/19/$31.00 ©2019 IEEE

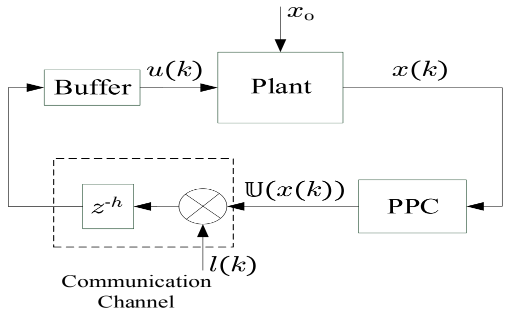

Fig. 1. Considered PPC system

channel, the actuator selects the packet associated with the precise prediction of the current plant states. So as the first contribution, we propose how to design sparse ℓ^1-ℓ^2 and ℓ^2-constrained ℓ^0 PPC schemes that stabilize the system despite the existing channel delay. As the second contribution, we reveal insights to the trade-off between the channel delay, size of the control packets and system performance. We do so by showing that at any time instant, there is an upper bound on the number of packets to be sent over the channel and this upper bound will be larger as the channel delay increases. Moreover, in the particular case of unconstrained ℓ^1-ℓ^2 sparse PPC for unstable plants, the ℓ^2-norm of system states will be bounded from above by a larger value as channel delay grows. We verify, through a numerical example, the expected stability, performance and sparsity properties.

The paper is organized as follows. Section II introduces the notation. Section III presents the problem formulation. Stability analysis is provided in Section IV. In Section V, a numerical example is given. Finally, Section VI concludes the paper.

II. NOTATION AND PRELIMINARIES

We denote the set of natural numbers by $\mathbb{N}$ and define the set $\mathbb{N}_0$ as $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$. An identity matrix with dimensions n is represented by $I_{n\times n}$ where $n \in \mathbb{N}$. Moreover, $M^\top$ symbolizes the transpose of the matrix (vector) M. Considering the vector $z = [z_1, \dots, z_n]^\top \in \mathbb{R}^n$, we define $\|z\|_1 \triangleq |z_1| + \cdots + |z_n|$, $\|z\|_2 \triangleq \sqrt{z^\top z}$ and $\|z\|_\infty \triangleq \max\{|z_1|, \dots, |z_n|\}$. Moreover, the positive definite matrix $W > 0$ defines $\|z\|_W$ as $\|z\|_W \triangleq \sqrt{z^\top W z}$. By $\text{supp}(z) = \{i : z_i \neq 0\}$, we denote the support set of the vector z based on which the ℓ^0-norm of z is defined as $\|z\|_0 \triangleq |\text{supp}(z)|$ where $|\text{supp}(z)|$ is the cardinality of the set $\text{supp}(z)$. Therefore, ℓ^0-norm of a vector is the number of its non-zero elements. The minimum and maximum eigenvalues of the Hermitian matrix W are denoted by $\lambda_{\min}(W)$ and $\lambda_{\max}(W)$, respectively. Moreover, we define $\sigma_{\max}(W)$ as $\sigma_{\max}(W) \triangleq \lambda_{\max}(W^\top W)$.

III. PROBLEM FORMULATION

We consider a discrete-time LTI plant with the following state-space representation:

$$x(k+1) = Ax(k) + Bu(k), \qquad k \in \mathbb{N}_0, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}$. Moreover, A and B are time-invariant matrices and assumed to be reachable.

The plant described through (1) is controlled in the feedback loop of Fig. 1 where a digital communication channel connects the controller to the actuator. The channel imposes a known constant delay which is an integer multiple of the system sampling period. We denote the delay by $h \in \mathbb{N}_0$. Moreover, data packet dropouts occur across the channel. The binary random process $l(k)$ represents the packet loss. If $l(k) = 0$, then the sequence of packets generated at time instant k will be dropped. If $l(k) = 1$, the sequence will arrive at the actuator h time steps later. Let assume that the packetized predictive controller in the NCS of Fig. 1 is not aware of $l(k)$ at each time instant $k \in \mathbb{N}_0$. This controller generates a sequence of control packets, which is specified by

$$\mathbb{U}(x(k)) = [U(\hat{x}_1(k+h;k)) \dots U(\hat{x}_{s(k)}(k+h;k))]^\top, \tag{2}$$

where $U(\hat{x}_j(k+h;k)) = [u_0(\hat{x}_j(k+h;k)) \dots u_{N-1}(\hat{x}_j(k+h;k))]^\top$ denotes a packet containing tentative future control inputs. Each control input in $U(\hat{x}_j(k+h;k))$ is generated based on the prediction $\hat{x}_j(k+h;k)$ of $x(k+h)$ for every $j \in \{1, \dots, s(k)\}$ and $k \in \mathbb{N}_0$. The controller makes use of $\{\mathbb{U}(x(i))\}_{i=0}^{k-1}$ and x^k to carry out such predictions for $x(k+h)$. Then the controller forwards $\mathbb{U}(x(k))$ over the channel. On the plant side, the actuator is connected to a buffer that stores each newly received packet sequence over its previous content. Suppose that the time is k and $\mathbb{U}(x(k-h))$ arrives at the actuator. The buffer writes $\mathbb{U}(x(k-h))$ over whatever data is already available in it. For any $k \in \mathbb{N}_0$, let us denote by $\hat{x}_{p(k)}(k;k-h)$ the precise prediction of $x(k)$ made at time $k-h$, i.e., $\hat{x}_{p(k)}(k;k-h) = x(k)$, $p(k) \in \{1, \dots, s(k-h)\}$. The actuator selects the packet $U(\hat{x}_{p(k)}(k;k-h))$ and applies $u_0(\hat{x}_{p(k)}(k;k-h))$ to the plant as the control input. At the next time instant, if the packet sequence $\mathbb{U}(x(k-h+1))$ is received, then the buffer writes it over $\mathbb{U}(x(k-h))$ and applies $u_0(\hat{x}_{p(k+1)}(k+1;k+1-h))$ to the plant. Otherwise, the buffer selects $u_1(\hat{x}_{p(k)}(k;k-h))$ as the control input. Then until the successful arrival of the next packet sequence $(\mathbb{U}(x(k+n-h)),\ n \geq 2)$, the remaining elements of $\mathbb{U}(x(k-h))$ are applied in a successive manner to the plant as control inputs.

Assumption 3.1: In the NCS of Fig. 1, $u(k) = 0$, $\forall k \in \{0, \dots, h-1\}$ and $l(0) = 1$. Moreover, the number of consecutive packet dropouts is uniformly bounded from above by $N-1$. In other words, the buffer never becomes empty.

Lemma 3.1: Consider the NCS of Fig. 1 where Assumption 3.1 holds. Suppose that $N \geq 2$ and the controller has access to x^k and $\{\mathbb{U}(x(i))\}_{i=0}^{k-1}$ at each time instant $k \in \mathbb{N}_0$. Moreover, assume that the controller is to calculate all the possible values $\{\hat{x}_j(k+h;k)\}_{j=0}^{s(k)}$ for $x(k+h)$. Then $s(k) \leq 2^h$, $\forall k \in \mathbb{N}_0$.

Proof: According to Assumption 3.1 and the dynamics of the plant, there is no uncertainty in predicting $x(h)$ and $x(h+1)$ at time $k = 0$ and $k = 1$, respectively. At time $k = 2$, the value of $x(2+h)$ depends on the value of $l(1)$ to

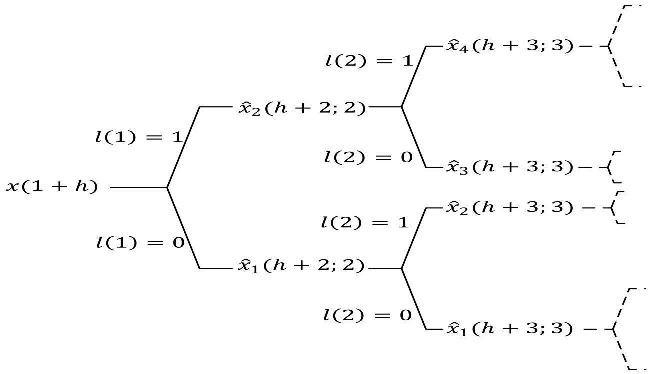

Fig. 2. The tree structure showing the possible values for $x(k+h)$ at time k

which the controller does not have access. So based on the fact that l is a binary random process and $N \geq 2$, $s(2)=2$, i.e., $x(h+2) \in \{\hat{x}_1(2+h;2), \hat{x}_2(2+h;2)\}$. At time $k=3$, the controller calculates the potential values for $x(h+3)$ based on all possible combinations of $l(1)$ and $l(2)$. However, it could be that the case where $l(1)=l(2)=0$ becomes impossible due to the boundedness of the consecutive packet dropouts. So $s(3)$ is either equal to 3 or 4. However, $s(3)$ certainly follows $s(3) \leq 2^2$. Assuming that the channel delay is large enough and by induction, we can conclude that the calculation of the possible values for $x(k+h)$ follows a tree structure as depicted in Fig. 2 where $s(k) \leq 2^k - 1$ for every $k < h+2$. From the time $k = h+2$, the controller has access to the exact value of the states for which it had calculated the future possible values. For example, at time $h+2$, the controller knows $x(h+2)$. So based upon the aforementioned tree structure, there are at most $2^{h+1}/2$ possible values for $x(2h+2)$, i.e., $s(h+2) \leq 2^h$. The same occurs for all the future time instants and this completes the proof. ■

We analyze the closed-loop system of Fig. 1 under two control strategies each specified by a sparsity-promoting optimization problem. In what follows, we formalize these optimization problems.

A. Unconstrained ℓ^1-ℓ^2 Optimization

In this case, the controller computes control packets by minimizing $s(k)$ finite-horizon cost functions at each time instant $k \in \mathbb{N}_0$. For every $x_j = \hat{x}_j(k+h;k)$, $j \in \{1,\ldots,s(k)\}$, this cost function is described by

$$J(x_j,u_j) = \|Mu_j - Kx_j\|_2^2 + \|x_j\|_Q^2 + \nu\|u_j\|_1, \tag{3}$$

where $u_j = U(\hat{x}_j(k+h;k))$. Additionally, we have

$$\Gamma = \begin{bmatrix} B & 0 & \ldots & 0 \\ AB & B & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \tag{4}$$

upon which M in (3) is defined as $M = \hat{Q}^{\frac{1}{2}}\Gamma$ and $K = -\hat{Q}^{\frac{1}{2}}\Lambda$, where $\hat{Q} = \mathrm{diag}\{Q,\ldots,Q,P\}$. We assume that $\nu > 0$, and Q and P are positive definite matrices. The control packet associated with $\hat{x}_j(k+h;k)$ at each time instant $k \in \mathbb{N}_0$ is given by

$$u_j(x_j) = \arg\min_{u_j \in \mathbb{R}^N} \|Mu_j - Kx_j\|_2^2 + \|x_j\|_Q^2 + \nu\|u_j\|_1, \tag{5}$$

where $j \in \{1,\ldots,s(k)\}$. According to [14], the solution to the unconstrained ℓ^1-ℓ^2 optimization problem is a sparse vector that can be obtained through several methods. For such an optimization problem, simulation often gives a solution that is much sparser than a local minimum solution of the ℓ^0 problem. However, if the plant is unstable, attaining asymptotic stability is never guaranteed for the closed-loop system in the case of unconstrained ℓ^1-ℓ^2 sparse PPC [13]. Instead, practical stability can be attained in this case by choosing $Q > 0$, $P > 0$ and $\nu > 0$ appropriately; a result which we show in Section IV-A.

B. ℓ^2-Constrained ℓ^0 Optimization

In this case, the controller solves a constrained optimization problem. At each time instant $k \in \mathbb{N}_0$, the control packet $u_j = U(\hat{x}_j(k+h;k))$ pertaining to $x_j = \hat{x}_j(k+h;k)$ in the ℓ^2-constrained ℓ^0 PPC is given by

$$\begin{aligned} u_j(x_j) &= \arg\min_{u_j \in \upsilon(x_j)} \|u_j\|_0 \\ \upsilon(x_j) &= \{u_j \in \mathbb{R}^N : \|Mu_j - Kx_j\|_2^2 \leq \|x_j\|_\Pi^2\} \end{aligned} \tag{6}$$

for $j = 1,\ldots,s(k)$. We assume that weighting matrices P, Q, and Π are positive definite and selected in such away that for all $x_j \in \mathbb{R}^n$, $\upsilon(x_j)$ is non-empty.

To solve the NP hard ℓ^2-constrained ℓ^0 optimization problem [15], we use the greedy approach of OMP. This is motivated by the fact that OMP has proven to be an efficient method for solving such problems as (6) with combinatorial nature [13], [16].

IV. Stability Analysis

In this section, we derive conditions under which the considered unconstrained ℓ^1-ℓ^2 and ℓ^2-constrained ℓ^0 sparse PPC strategies render the system of Fig. 1 stable.

A. Stability of Unconstrained ℓ^1-ℓ^2 PPC

We start by defining the considered notion of stability.

Definition 4.1: The feedback loop of Fig. 1 is said to be practically stable if there exists $\varrho \in \mathbb{R}^+$ in such a way that $\lim_{k\to\infty} \|x(k)\|_2 \leq \varrho$.

To establish the conditions of practical stability, we investigate the value function V defined as

$$V(x_j) \triangleq \min_{u_j \in \mathbb{R}^N} J(x_j,u_j), \tag{7}$$

where $J(x_j,u_j)$ is as in (3).

Lemma 4.1: The value function $V(x_j)$ is bounded as

$$\lambda_{\min}(Q)\|x_j\|_2^2 \leq V(x_j) \leq \tau(\|x_j\|_2) \tag{8}$$

for any $x_j \in \mathbb{R}^n$. In (8), $\tau(y) \triangleq \alpha y + (\beta + \lambda_{\max}(Q))y^2$, $\alpha = \nu\sqrt{n}\sigma_{\max}(M^\dagger K)$ and $\beta \triangleq \lambda_{\max}(\Pi^\star)$, where matrices $M^\dagger$ and $\Pi^\star$ are specified via

$$M^\dagger = (M^\top M)^{-1}M^\top, \qquad \Pi^\star = K^\top(I - MM^\dagger)M. \tag{9}$$

Proof: The claim follows immediately from [13, Lemma 5] by noting that the cost function $J(x_j, u_j)$ in (7) is defined in the same way as for the case with delay-free channel studied in [13, Lemma 5]. ■

Recall that the controller is a model predictive controller. Hence, at each time instant $k \in \mathbb{N}_0$, the prediction of future states is based upon the update rule (1). Therefore, the recursion $x'_{(i+1)j} = Ax'_{ij} + Bu_{ij}$ gives the prediction x'_{fj} of the future state $x(k+f+h)$, $f = 1, \ldots, N$, based on $x'_{0j} = \hat{x}_j(k+h;k)$ and $[u_{0j} \ldots u_{(N-1)j}]^\top = U(\hat{x}_j(k+h;k))$. So, x'_{fj} can be stated as a function of $\hat{x}_j(k+h;k)$ and tentative future control inputs. We denote this function by $g^f(\hat{x}_j(k+h;k))$. By induction and considering the logics of the buffer, we have

$$\begin{aligned} g^f(\hat{x}_j(k+h;k)) =& A^f \hat{x}_j(k+h;k) \\ &+ \sum_{l=0}^{f-1} A^{f-1-l} B u_l(\hat{x}_j(k+h;k)), \end{aligned} \tag{10}$$

where $f = 1, \ldots, N$. The following result determines the relationship between $V(x_j)$ and $V(g^f(x_j))$.

Lemma 4.2: Suppose that there exists $\zeta > 0$ defining $r = \mu^2 N/\zeta$ in such a way tha the following Riccati equation holds for $P \geq 0$:

$$P = A^\top PA - A^\top PB(B^\top PB + r)^{-1} B^\top PA + Q. \tag{11}$$

Then there exists a real constant $\varphi \in (0,1)$ in such a way that $V(g^f(x_j))$ satisfies

$$V(g^f(x_j)) \leq \varphi V(x_j) + \lambda_{\min}(Q)/4 + \zeta, \tag{12}$$

for every $x_j \in \mathbb{R}^n$ where $j = 1, \ldots, s(k)$ and f belongs to the set $\{1, 2, \ldots, N\}$.

Proof: The claim can be concluded immediately from [13, Lemma 8] based upon the fact that $J(x_j, u_j)$ and update rule for future states prediction are defined identically across cases of delay-free channel and channel with delay in the NCS of Fig. 1. ■

We establish sufficient conditions for practical stability in the case of unconstrained ℓ^1-ℓ^2 sparse PPC over a channel with known constant delay and dropouts.

Theorem 4.1: Consider the NCS of Fig. 1 under the unconstrained ℓ^1-ℓ^2 sparse PPC (5) and suppose that Assumption 3.1 holds. Moreover, assume that $P > 0$ is chosen in such a way that (11) holds with $r = \mu^2 N/\zeta$, where $\zeta > 0$. Define t_n as the time instant when a packet is received by the actuator for the $(n+1)$-th time, i.e., $l(t_n - h) = 1$. Then at each time step $k \in \{t_n + 1, \ldots, t_{n+1}\}$, $n \in \mathbb{N}_0$, the ℓ^2-norm of $x(k)$ is bounded as

$$\|x(k)\|_2 \leq \sqrt{\varphi^{n+1}} \sqrt{\frac{\tau(\|x(h)\|_2)}{\lambda_{\min}(Q)}} + \Psi, \tag{13}$$

where φ and Ψ are defined as

$$\varphi \triangleq 1 - \lambda_{\min}(Q)(\alpha + \beta\lambda_{\max}(Q))^{-1} \tag{14}$$

and

$$\Psi \triangleq \sqrt{\frac{1}{1-\varphi}\left(\frac{\zeta}{\lambda_{\min}(Q)} + \frac{1}{4}\right)}, \tag{15}$$

respectively. The parameters α and β are characterized as in Lemma 4.1. Moreover, the feedback loop of Fig. 1 is practically stable as the steady-state $\|x(k)\|_2$ is bounded from above as follows:

$$\lim_{k\to\infty} \|x(k)\|_2 \leq \Psi, \tag{16}$$

Proof: Let $\mathcal{T}$ represent the set of all time instants at which a control packet sequence reaches the actuator successfully. Hence, $\mathcal{T}$ is defined as follows:

$$\mathcal{T} \triangleq \{t_n\}_{n\in\mathbb{N}_0} \subseteq \mathbb{N}_0, \tag{17}$$

where $t_{n+1} > t_n, \forall n \in \mathbb{N}_0$. Moreover, let q_n specify the number of packet dropouts between t_n and t_{n+1}. Thus, q_n is given by

$$q_n = t_{n+1} - t_n - 1, \qquad \forall n \in \mathbb{N}_0. \tag{18}$$

It is clear that $q_n \geq 0$ with equality when there is no dropout between t_n and t_{n+1}. Suppose that the current time instant is t_n. Then $U(\hat{x}_{p(t_n)}(t_n; t_n - h)) = U(x(t_n))$. Based on Assumption 3.1, q_n satisfies $q_n \leq N - 1$. Then according to the update rule for state predictions, plant dynamics (1) and Lemma 4.2, the value function of $x(k)$ is bounded as

$$V(x(k)) \leq \varphi V(x(t_n)) + \Theta \tag{19}$$

for all $k \in \{t_n+1, t_n+2, \ldots, t_n+q_n+1\}$ where Θ is defined as $\Theta \triangleq \lambda_{\min}(Q)/4 + \zeta$. It follows from $t_{n+1} = t_n + q_n + 1$ and (19) that

$$V(x(t_{n+1})) \leq \varphi V(x(t_n)) + \Theta. \tag{20}$$

Applying induction to (20) and based on Assumption 3.1, we have

$$V(x(t_n)) \leq \varphi^n V(x(h)) + (1 + \cdots + \varphi^{n-1})\Theta, \tag{21}$$

Then it follows from Lemma 4.1 that

$$V(x(t_n)) \leq \varphi^n \tau(\|x(h)\|_2) + (1-\varphi)^{-1}\Theta. \tag{22}$$

Now the inequality in (19) yields

$$V(x(k)) \leq \varphi^{n+1} \tau(\|x(h)\|_2) + (1-\varphi)^{-1}\Theta. \tag{23}$$

for any $k \in \{t_n + 1, \ldots, t_{n+1}\}$. We can use the lower bound on $V(x_j)$ in Lemma 4.1 to deduce

$$\|x(k)\|_2 \leq \sqrt{\frac{V(x(k))}{\lambda_{\min}(Q)}} \leq \left(\frac{\varphi^{n+1}\tau(\|x(h)\|_2)}{\lambda_{\min}(Q)} + \Psi^2\right)^{\frac{1}{2}}, \tag{24}$$

where Ψ is defined as in (15). The derivation in (24) together with the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $\forall a, b \geq 0$, will give

$$\|x(k)\|_2 \leq \sqrt{\varphi^{n+1}} \sqrt{\frac{\tau(\|x(h)\|_2)}{\lambda_{\min}(Q)}} + \Psi. \tag{25}$$

Finally, since $k \to \infty$ implies $n \to \infty$ and $x(h)$ is finite for a finite h, we can conclude that

$$\lim_{k\to\infty} \|x(k)\|_2 \leq \Psi, \tag{26}$$

which completes the proof. ■

Remark 4.1: According to Assumption 3.1, plant states grow with respect to time over the interval $[0, h]$ when the plant is unstable. This means $\|x(k+1)\|_2 \geq \|x(k)\|_2$ for every $k \in \{0, \dots, h-1\}$. Therefore, increasing the channel delay h while keeping the initial states and every other component of the system intact will lead to a greater $\|x(h)\|_2$. Hence, according to (13), $\|x(k)\|_2$ is bounded by a larger value if the channel delay is greater. However, since h is finite, increasing the delay will not affect the stability of the system.

B. Stability of ℓ^2-Constrained ℓ^0 PPC

Here, we establish conditions under which the asymptotic stability is guaranteed in the ℓ^2-constrained ℓ^0 PPC case. The corresponding optimization problem is formalized by (6). First, we investigate the feasibility of this problem.

Lemma 4.3: Assume that for any $j \in \{1, \dots, s(k)\}$, $k \in \mathbb{N}_0$, $\upsilon^*(x_j)$ is defined as

$$\upsilon^*(x_j) \triangleq \{u_j \in \mathbb{R}^N : \|Mu_j - Kx_j\|_2^2 \leq \|x_j\|_{\Pi^*}^2\}, \quad (27)$$

in which Π^* is given by (9) and $x_j = \hat{x}_j(k+h;k)$. Then $\Pi \geq \Pi^*$ yields $\upsilon(x_j) \supseteq \upsilon^*(x_j)$ where $\upsilon(x_j)$ is defined as in (6). The feasible set $\upsilon(x_j)$ associated with any $\Pi \geq \Pi^*$ is closed, convex and non-empty over $\mathbb{R}^N$.

Proof: The set $\upsilon(x_j)$ (as a function of x_j) and matrices Π^* and Π are defined in the same way as $\mathcal{U}(x)$ (as a function of x), W and W^* in [13, Lemma 10] for the system with delay-free channel, respectively. Therefore, the claim immediately follows from [13, Lemma 10]. ∎

Let denote the difference between Π and Π^* in the previous lemma by ξ. Hence, ξ is described by

$$\xi = \Pi - \Pi^* > 0. \quad (28)$$

In the following lemma, ξ is used to obtain a derivation which is similar to the result stated in Lemma 4.2.

Lemma 4.4: Consider $\Pi > 0$ and Π^* satisfying $\Pi > \Pi^*$ and ξ as in (28). Let define $V_P(x_j) \triangleq \|x_j\|_P^2$. For an arbitrary $Q > 0$, assume that $P > 0$ solves the Riccati equation (11) with $r = 0$. Then for any $x_j = \hat{x}_j(k+h;k)$, $j = 1, \dots, s(k)$ and $k \in \mathbb{N}_0$, there exist constants $\varphi \in [0, 1)$ and $z > 0$ in such a way that

$$V_P(x_{ij}^{'}) \leq \varphi^i V_P(x_j) + z\|x_j\|_\xi^2, \qquad i = 1, 2, \dots, N, \quad (29)$$

where $x_{(i+1)j}^{'} = Ax_{ij}^{'} + Bu_{ij}$ and $x_{0j}^{'} = x_j$. Moreover, $u_j = [u_{0j} \dots u_{(N-1)j}]^\top$ is the optimal control packet associated with x_j defined in (6).

Proof: The recursion related to the prediction of the future states and the structure of the optimization problem are specified identically across the ℓ^2-constrained ℓ^0 PPC analyzed here and the one studied in [13]. Therefore, we can conclude the claim immediately from [13, Lemma 13]. ∎

Now, we derive the conditions for asymptotic stability in the case of ℓ^2-constrained ℓ^0 sparse PPC.

Theorem 4.2: For an arbitrary $Q > 0$, let $P > 0$ solve the Riccati equation (11) with $r = 0$. Pick a matrix ξ satisfying $0 \leq \xi \leq (1-\varphi)P/z$ where the constants $\varphi \in [0, 1)$ and $z > 0$ are calculated through (24), (25) and (26) in [13]. Suppose that Π^* and Π are set as $\Pi^* = P - Q$ and $\Pi = \Pi^* + \xi$, respectively. Then, the sparsity-promoting ℓ^2-constrained ℓ^0 optimization (6) solved by using these tuning parameters (P, Q, and Π) gives a control packet sequence $\mathbb{U}(x(k))$, at each time instant $k \in \mathbb{N}_0$, in such a way that $\lim_{k\to\infty} x(k) = 0$.

Proof: Recall t_n and q_n from Theorem 4.1. Consider a specific t_n and note that $q_n \leq N-1$. Now it follows from selection logic of the actuator and Lemma 4.4 that

$$V_P(x(k)) \leq x(t_n)^\top(\varphi P + z\xi)x(t_n) < V_P(x(t_n)), \quad (30)$$

where $k \in \{t_n+1, \dots, t_n+q_n+1\}$. Considering $t_n+q_n+1 = t_{n+1}$, we have

$$V_P(x(t_{n+1})) < V_P(x(t_n)). \quad (31)$$

It follows from (31) that due to the positivity of $V_P(.)$, $\lim_{n\to\infty} x(t_n) = 0$. Then based on (30), we can conclude that $\lim_{k\to\infty} x(k) = 0$ which completes the proof. ∎

V. Numerical Example

We consider the model of an inverted pendulum on a cart taken from [17] as the plant model in the NCS of Fig. 1. This model is specified by the following state and input matrices:

$$A = \begin{bmatrix} 1 & 0.0498 & 0.0028 & 0.0001 \\ 0 & 0.9913 & 0.1116 & 0.0028 \\ 0 & -0.0005 & 1.0327 & 0.0508 \\ 0 & -0.0189 & 1.3062 & 1.0327 \end{bmatrix}, B = \begin{bmatrix} 0.0098 \\ 0.3908 \\ 0.0212 \\ 0.8485 \end{bmatrix}$$

It is straightforward to verify that the pair (A, B) is reachable. We simulate the feedback loop of Fig. 1 for both unconstrained ℓ^1-ℓ^2 PPC and ℓ^2-constrained ℓ^0 PPC. We set $N = 10$ and $Q = I$ in both cases. Furthermore, we set the process l in such a way that the number of consecutive packet dropouts is distributed uniformly over $[0, 1, .., N-1]$. We simulate the system for three different values of channel delay, $h \in \{0, 10, 20\}$. For the unconstrained ℓ^1-ℓ^2 PPC, we select the parameters ν and r as $\nu = 15$ and $r = 2$. We solve the corresponding sparsity-promoting optimization problem in (5) by using FISTA. For the case of ℓ^2-constrained ℓ^0 PPC, we set ξ as $\xi = (1-\varphi)P/2z$.

Figures 3 and 4 demonstrate the simulation results which are average over 200 number of 300-sample-long simulations. The ℓ^2-norm of the states is illustrated by Fig. 3 (top). The curves in Fig. 3 (top) show that the controller designed based on the ℓ^1-ℓ^2 sparse PPC strategy renders the system practically stable. Moreover, in Fig. 3 (top), the values of curves associated with higher channel delays are larger. For the ℓ^2-constrained ℓ^0 sparse PPC setting, the asymptotic stability of the resulted system is verified by the curves in Fig. 3 (bottom). As depicted in this figure, increasing the channel delay degrades the performance of the system. The ℓ^0-norm of the selected control packet $U_p(x)$ is demonstrated by Fig. 4. The curves in Fig. 4 show that for a fixed channel time delay, the unconstrained ℓ^1-ℓ^2 PPC generates sparser control commands than ℓ^2-constrained ℓ^0 PPC. This is of course caused by our choice of ν. According to (3),

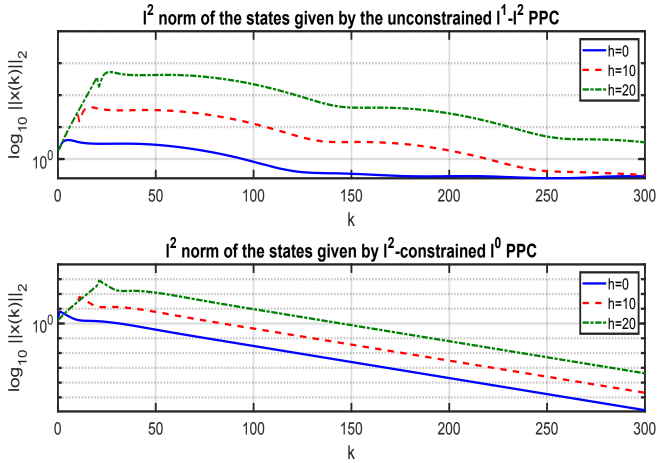

Fig. 3. Average ℓ^2 norm of the state $x(k)$ in the unconstrained ℓ^1-ℓ^2 PPC (top) and ℓ^2-constrained ℓ^0 PPC (bottom)

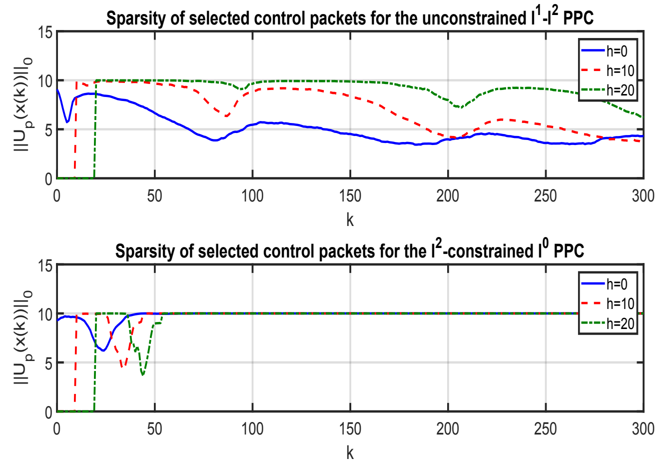

Fig. 4. Average ℓ^0 norm of the control packet $U_h(x(k))$ in the unconstrained ℓ^1-ℓ^2 PPC (top) and ℓ^2-constrained ℓ^0 PPC (bottom)

making the parameter ν smaller will reduce the sparsity of the obtained control vector. However, improving sparsity by enlarging ν comes with the cost of control performance degradation.

VI. Conclusion

In this paper, sparse PPC over digital communication channels subject to time delays and data packet dropouts has been studied. In the considered NCS, the communication channel is located in the actuation path between the controller and a discrete-time LTI plant. We have analyzed the stability of the overall closed-loop system under unconstrained ℓ^1-ℓ^2 and ℓ^2-constrained ℓ^0 sparse PPCs. We have shown that under certain conditions, the unconstrained ℓ^1-ℓ^2 PPC will bring practical stability to the system. Moreover, we have derived conditions guaranteeing asymptotic stability in the case of utilizing ℓ^2-constrained ℓ^0 PPC. For both cases, we have shown that the number of packets generated by the controller at each time instant is bounded from above by a fixed value which becomes larger as channel time delay grows. We have demonstrated, through simulation, that in each case of unconstrained ℓ^1-ℓ^2 PPC or ℓ^2-constrained ℓ^0 PPC, the optimization problem giving the stabilizing control inputs has sparse solution. Moreover, the simulation results show that increasing the channel-induced delay worsens the system performance, though not affecting system stability.

References

[1] Xian-Ming Zhang, Qing-Long Han, and Xinghuo Yu, "Survey on recent advances in networked control systems," *IEEE Transactions on Industrial Informatics*, vol. 12, no. 5, pp. 1740–1752, 2016.

[2] Kunihisa Okano and Hideaki Ishii, "Stabilization of uncertain systems with finite data rates and markovian packet losses," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 4, pp. 298–307, 2014.

[3] Cheng Tan and Huanshui Zhang, "Necessary and sufficient stabilizing conditions for networked control systems with simultaneous transmission delay and packet dropout," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 4011–4016, 2017.

[4] Rongfang Rachel Liu and Iakov M Golovitcher, "Energy-efficient operation of rail vehicles," *Transportation Research Part A: Policy and Practice*, vol. 37, no. 10, pp. 917–932, 2003.

[5] Ching Chue Chan, "The state of the art of electric, hybrid, and fuel cell vehicles," *Proceedings of the IEEE*, vol. 95, no. 4, pp. 704–718, 2007.

[6] Masaaki Nagahara, Daniel E Quevedo, and Dragan Nešić, "Maximum hands-off control: A paradigm of control effort minimization," *IEEE Transactions on Automatic Control*, vol. 61, no. 3, pp. 735–747, 2016.

[7] Masaaki Nagahara, Jan Østergaard, and Daniel E Quevedo, "Discrete-time hands-off control by sparse optimization," *EURASIP Journal on Advances in Signal Processing*, vol. 2016, no. 1, pp. 76, 2016.

[8] Masaaki Nagahara and Daniel E Quevedo, "Sparse representations for packetized predictive networked control," *IFAC Proceedings Volumes*, vol. 44, no. 1, pp. 84–89, 2011.

[9] M. Nagahara, D. E. Quevedo, and J. Østergaard, "Packetized predictive control for rate-limited networks via sparse representation," in *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)*, Dec. 2012, pp. 1362–1367.

[10] D. E. Quevedo and D. Nešić, "Input-to-state stability of packetized predictive control over unreliable networks affected by packet-dropouts," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 370–375, 2011.

[11] Daniel E Quevedo, Jan Østergaard, and Dragan Nešić, "Packetized predictive control of stochastic systems over bit-rate limited channels with packet loss," *IEEE Transactions on Automatic Control*, vol. 56, no. 12, pp. 2854–2868, 2011.

[12] Guo-Ping Liu, Yuanqing Xia, Jie Chen, David Rees, and Wenshan Hu, "Networked predictive control of systems with random network delays in both forward and feedback channels," *IEEE Transactions on Industrial Electronics*, vol. 54, no. 3, pp. 1282–1297, 2007.

[13] Masaaki Nagahara, Daniel E Quevedo, and Jan Østergaard, "Sparse packetized predictive control for networked control over erasure channels," *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1899–1905, 2014.

[14] Kazunori Hayashi, Masaaki Nagahara, and Toshiyuki Tanaka, "A user's guide to compressed sensing for communications systems," *IEICE Transactions on Communications*, vol. 96, no. 3, pp. 685–712, 2013.

[15] Balas Kausik Natarajan, "Sparse approximate solutions to linear systems," *SIAM Journal on Computing*, vol. 24, no. 2, pp. 227–234, 1995.

[16] Y. C. Pati, R. Rezaiifar, and P. S. Krishnaprasad, "Orthogonal matching pursuit: Recursive function approximation with applications to wavelet decomposition," in *Proceedings of the 27th Asilomar Conference on Signals, Systems and Computers*, Nov 1993, vol. 1, pp. 40–44.

[17] Takashi Tanaka, Karl Henrik Johansson, and Mikael Skoglund, "Optimal block length for data-rate minimization in networked lqg control," *IFAC-PapersOnLine*, vol. 49, no. 22, pp. 133 – 138, 2016, 6th IFAC Workshop on Distributed Estimation and Control in Networked Systems NECSYS 2016.