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# Orthonormal bases for anisotropic $\alpha$-modulation spaces

by

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# ORTHONORMAL BASES FOR ANISOTROPIC $\alpha$-MODULATION SPACES

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ABSTRACT. In this article we construct orthonormal bases for bi-variate anisotropic $\alpha$-modulation spaces. The construction is based on generating a nice anisotropic $\alpha$-covering and using carefully selected tensor products of univariate brushlet functions with regards to this covering. As an application, we show that $n$-term nonlinear approximation with the orthonormal bases in certain anisotropic $\alpha$-modulation spaces can be completely characterized.

## 1. INTRODUCTION

The construction of unconditional bases for a given smoothness space is important as it often leads to simple characterizations of the space. For example, smoothness measured in a Besov space is equivalent to a certain sparseness of a wavelet expansion [18]. More generally, norm characterizations allow us to identify certain smoothness spaces as nonlinear approximation spaces (see e.g. [10] and [15]). As a consequence we gain better understanding of how sufficiently smooth functions can be compressed by thresholding the expansion coefficients for a sparse representation of the function [5,6].

The $\alpha$-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$, $\alpha \in [0,1]$, were introduced by Gröbner [11] and include the Besov and modulation spaces as special cases corresponding to $\alpha = 1$ and $\alpha = 0$, respectively. They are part of a much more general construction introduced by Feichtinger and Gröbner called decomposition spaces [8], [7]. Decomposition spaces are based on structured coverings of the frequency space $\mathbb{R}^d$ and in the case of the $\alpha$-modulation spaces the $\alpha$-parameter determines the nature of the covering. The Besov spaces ($\alpha = 1$) correspond to a dyadic covering, the modulation spaces ($\alpha = 0$) correspond to a uniform covering and the intermediate cases correspond to "polynomial type" coverings of the frequency space. So far frames have been constructed for a broad subclass of the decomposition spaces [4], but the author is not aware of any general method for constructing bases for decomposition spaces. On the other hand, a orthonormal basis for bi-variate $\alpha$-modulation spaces was constructed in [19].

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The goal of this article is to construct an orthonormal basis for bi-variate anisotropic $\alpha$-modulation spaces. Building on the work in [19] the orthonormal basis is constructed by using carefully selected tensor products of univariate brushlet functions. Brushlets are the image of a local trigonometric basis under the Fourier transform, and such systems were introduced by Laeng [16]. Later Coifman and Meyer used brushlets as a tool for image compression [17]. By using the constructed orthonormal basis, we also identity certain anisotropic $\alpha$-modulation spaces as approximation spaces associated with nonlinear $n$-term approximation.

The outline of the article is as follows. In Section 2 univariate brushlets are defined, and bi-variate brushlet bases are constructed for a flexible covering of $\mathbb{R}^2$. In Section 3 anisotropic $\alpha$-modulation spaces are defined, and an anisotropic $\alpha$-covering is constructed. Furthermore, by applying the constructed $\alpha$-covering to the bi-variate brushlet bases from Section 2, we show that unconditional bases for the anisotropic $\alpha$-modulation spaces are generated. In Section 4 we apply the constructed basis to nonlinear $n$-term approximation. Finally, there is an appendix where we prove that anisotropic $\alpha$-modulation spaces are independent of the $\alpha$-covering used.

## 2. Brushlet bases

In this section we introduce orthonormal brushlet bases for $L_2(\mathbb{R})$, and use them to construct bi-variate brushlet bases associated with a flexible covering of the frequency space $\mathbb{R}^2$ (see e.g. [2]). In the following section, by choosing a covering that fits to the anisotropic $\alpha$-modulation spaces, we will then be able to show that the constructed bi-variate brushlet bases form unconditional bases for the $\alpha$-modulation spaces.

Each univariate brushlet basis is associated with a partition of the frequency axis. The partition can be chosen with almost no restrictions, but in order to have good properties of the associated basis we need to impose some growth conditions on the partition.

**Definition 2.1.**
A family of intervals $\mathbb{I}$ is called a *disjoint covering* of $\Omega = [\omega, \omega') \subseteq \mathbb{R}$, $\omega < \omega'$, if it consists of a countable set of pairwise disjoint half-open intervals $I = [\alpha_I, \alpha'_I)$, $\alpha_I < \alpha'_I$, such that $\cup_{I \in \mathbb{I}} I = \Omega$. If, furthermore, each interval in $\mathbb{I}$ has a unique adjacent interval in $\mathbb{I}$ to the left and to the right, and there exists a constant $A > 1$ such that

$$A^{-1} \leq \frac{|I|}{|I'|} \leq A, \quad \text{for all adjacent } I, I' \in \mathbb{I}, \tag{2.1}$$

we call $\mathbb{I}$ a *moderate disjoint covering* of $\Omega$. $\diamond$

Given a moderate disjoint covering $\mathbb{I}$ of $\Omega$, we can easily assign to each interval $I \in \mathbb{I}$ a cutoff radius $\varepsilon_I > 0$ at the left endpoint and a cutoff radius $\varepsilon'_I$ at the right endpoint, satisfying

$$\left.\begin{array}{c} \varepsilon_I' = \varepsilon_{I'}, \text{whenever } \alpha_I' = \alpha_{I'} \\ \varepsilon_I + \varepsilon_I' \le |I| \\ \varepsilon_I \ge C|I|, \end{array}\right\} \tag{2.2}$$

with $C > 0$ independent of $I$.

We are now ready to define the brushlet system. For each $I \in \mathbb{I}$, we first construct a smooth bell function localized in a neighborhood of $I$. Take a non-negative ramp function $\rho \in C^\infty(\mathbb{R})$ satisfying

$$\rho(\xi) = \begin{cases} 0 \text{ for } \xi \le -1 \\ 1 \text{ for } \xi \ge \quad 1, \end{cases} \tag{2.3}$$

with the property that

$$\rho(\xi)^2 + \rho(-\xi)^2 \equiv 1. \tag{2.4}$$

Define for each $I = [\alpha_I, \alpha_I')\in \mathbb{I}$ the *bell function*

$$b_I(\xi) := \rho\left(\frac{\xi - \alpha_I}{\varepsilon_I}\right)\rho\left(\frac{\alpha_I' - \xi}{\varepsilon_I'}\right). \tag{2.5}$$

Notice that $\operatorname{supp}(b_I) \subset (\alpha_I - \varepsilon_I, \alpha_I' + \varepsilon_I')$ and $b_I(\xi) = 1$ for $\xi \in [\alpha_I + \varepsilon_I, \alpha_I' - \varepsilon_I']$. Let $\hat{f}(\xi) := \mathcal{F}(f)(\xi) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$, $f \in L_2(\mathbb{R}^d)$. Now if $\mathbb{I}$ is a moderate disjoint covering of $\mathbb{R}$ then the set of local cosine functions

$$\hat{w}_{m,I}(\xi) := \sqrt{\frac{2}{|I|}} b_I(\xi)\cos\left(\pi(m+\tfrac{1}{2})\frac{\xi - \alpha_I}{|I|}\right),\ m \in \mathbb{N}_0,\ I \in \mathbb{I}, \tag{2.6}$$

constitute an orthonormal basis for $L_2(\mathbb{R})$, see e.g. [1]. We call the collection $\{w_{m,I}\}_{m\in\mathbb{N}_0, I\in\mathbb{I}}$ a *brushlet system*. There is also a more explicit representation of brushlets in direct space. Define $\hat{g}_I(\xi) := b_I(|I|\xi + \alpha_I)$ and $e_{m,I} := \pi(m+\frac{1}{2})|I|^{-1}$, we then have

$$w_{m,I}(x) = \sqrt{\frac{|I|}{2}} e^{i\alpha_I x}\left[g_I\big(|I|(x+e_{m,I})\big) + g_I\big(|I|(x-e_{m,I})\big)\right]. \tag{2.7}$$

It can easily be verified that for $r \ge 1$ there exists $C > 0$ such that

$$|g_I(x)| \le C(1+|x|)^{-r} \tag{2.8}$$

independent of $I \in \mathbb{I}$.

To later generate bi-variate brushlet bases, we define the operator $\mathcal{P}_I : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ as

$$\widehat{\mathcal{P}_I f}(\xi) := b_I(\xi)[b_I(\xi)\hat{f}(\xi) + b_I(2\alpha_I - \xi)\hat{f}(2\alpha_I - \xi) - b_I(2\alpha_I' - \xi)\hat{f}(2\alpha_I' - \xi)]. \tag{2.9}$$

By straight forward calculations it can be verified that $\mathcal{P}_I$ is an orthogonal projection, mapping $L_2(\mathbb{R})$ onto $\overline{\operatorname{span}}(\{w_{m,I}\}_{m\in\mathbb{N}_0})$. We shall list some properties of $\mathcal{P}_I$ here and refer to [12, Chap. 1] for a more detailed discussion of local trigonometric bases.

If $I$ and $J$ are two adjacent intervals in $\mathbb{I}$ then for $f \in L_2(\mathbb{R})$,

$$\widehat{\mathcal{P}_I f}(\xi) + \widehat{\mathcal{P}_J f}(\xi) = \hat{f}(\xi),\ \xi \in [\alpha_I + \varepsilon_I, \alpha'_J - \varepsilon'_J]. \tag{2.10}$$

Furthermore,

$$\mathcal{P}_I + \mathcal{P}_J = \mathcal{P}_{I \cup J} \tag{2.11}$$

with the $\varepsilon$-values $\varepsilon_I$ and $\varepsilon'_J$. It follows that $\{w_{m,I'}\}_{m\in\mathbb{N}_0, I'\in\{I,J\}}$ is an orthonormal basis for functions bandlimited to $[\alpha_I + \varepsilon_I, \alpha'_J - \varepsilon'_J]$ on $L_2(\mathbb{R}^2)$, and by repeating the argument, a basis for all functions in $L_2(\mathbb{R})$ can be constructed by using a moderate disjoint covering of $\mathbb{R}$. This will be the key idea for constructing nice bi-variate brushlet bases.

For later use, we introduce $P_Q := \mathcal{P}_I \otimes \mathcal{P}_J$, $Q := I \times J \subset \mathbb{R}^2$. By using the univariate case, we have that $P_Q$ is an orthogonal projection, mapping $L_2(\mathbb{R}^2)$ onto $\overline{\operatorname{span}}(\{w_{m_1,I} \otimes w_{m_2,J}\}_{m_1,m_2\in\mathbb{N}_0})$.

2.1. **Construction of bi-variate brushlet bases.** A simple way of constructing bi-variate brushlet bases is to use the tensor product on a univariate brushlet basis. Although this gives us a basis for $L_2(\mathbb{R}^2)$, we lose the ability to generate a structured anisotropic covering of the frequency plane. An example of this in the isometric case can be seen with tensor products of orthonormal wavelets. Here we end up with hyperbolic bi-variate wavelet systems which offer no characterizations of isotropic smoothness spaces. Instead we take the tensor product of two brushlet bases, extract the brushlets on the diagonal with regards to the frequency index, and then repopulate this subsystem in a structured way.

We saw earlier that the univariate brushlet bases were constructed from an moderate disjoint covering of $\mathbb{R}$, and the operator $\mathcal{P}_{\mathcal{I}}$ could be seen as a building block associated with the bandlimited functions on $I$. We shall use the same idea here, and first construct a covering of $\mathbb{R}^2$.

Let $\{I_n\}_{n\in\mathbb{Z}}$ and $\{J_n^l\}_{n\in\mathbb{N}_-, k_n\le l\le 0} \cup \{J_0^0\} \cup \{J_n^l\}_{n\in\mathbb{N}_+, 0\le l\le k_n}$ be moderate disjoint coverings of $\mathbb{R}$ such that $n < n'$ implies $\alpha_{I_n} < \alpha_{I_{n'}}$ and $\alpha_{J_n^l} < \alpha_{J_{n'}^{l'}}$, and $l < l'$ implies $\alpha_{J_n^l} < \alpha_{J_n^{l'}}$. This gives us the diagonal part of our covering and works as a "scaffold" for the rest of the covering, see Figure 1. Next, let $\{I_{n,i}\}_{1\le i\le m_n^I}$, $n \ge 1$, be moderate disjoint coverings of $\cup_{n'=-n}^{n} I_{n'}$ with the same constant $A$ from (2.1) as the covering $\{I_n\}_{n\in\mathbb{Z}}$; furthermore, we require that $I_{n,1} = I_{-n}$ and $I_{n,m_n^I} = I_n$. Define $\{J_{n,j}^0\}_{1\le j\le m_n^J}$ similarly. We introduce a covering of $\mathbb{R}^2$ with the help of the hollow rectangles $\cup_{Q\in\mathbb{P}_n} Q$,

$\mathbb{P}_n := \mathbb{P}_n^b \cup \mathbb{P}_n^t \cup \mathbb{P}_n^l \cup \mathbb{P}_n^r$, $n \geq 1$,

$$\mathbb{P}_n^b = \left\{ I_{n,i} \times J_{-n}^l | 1 \leq i \leq m_n^I, k_{-n} \leq l \leq 0 \right\}$$
$$\mathbb{P}_n^t = \left\{ I_{n,i} \times J_n^l | 1 \leq i \leq m_n^I, 0 \leq l \leq k_n \right\}$$
$$\mathbb{P}_n^l = \left\{ I_{-n} \times J_{n,j}^0 | 2 \leq j \leq m_n^J - 1 \right\}$$
$$\mathbb{P}_n^r = \left\{ I_n \times J_{n,j}^0 | 2 \leq j \leq m_n^J - 1 \right\},$$

and the center rectangle $\mathbb{P}_0$,

$$\mathbb{P}_0 = \{ I_0 \times J_0^0 \}.$$

It follows that $\cup_{Q \in \mathbb{P}} Q = \mathbb{R}^2$, $\mathbb{P} := \cup_{n=0}^{\infty} \mathbb{P}_n$ and the sets in $\mathbb{P}$ are disjoint.

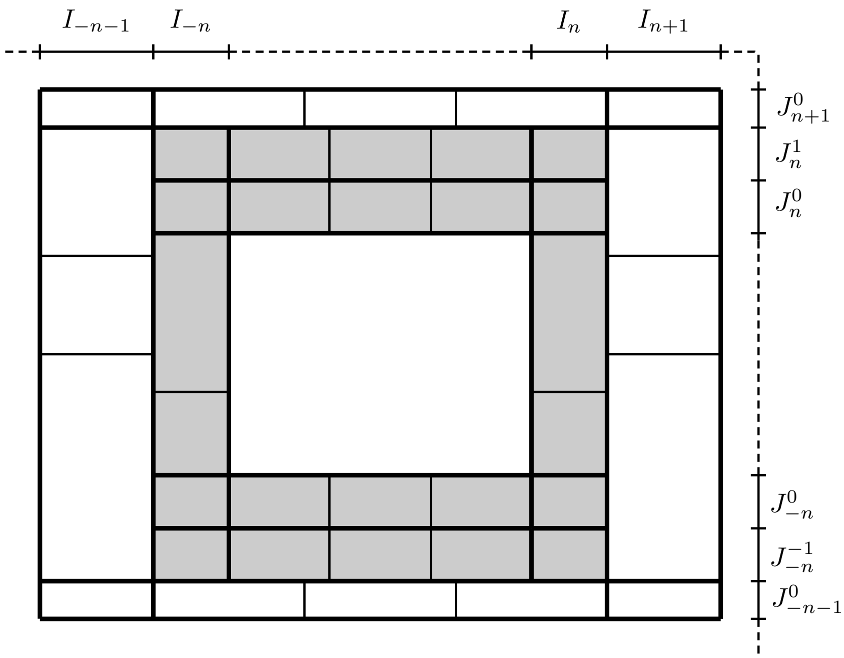


**Figure 1.** *Covering of $\mathbb{R}^2$ by $\mathbb{P}$. The shaded area is the sets in $\mathbb{P}_n$.*

With the covering $\mathbb{P}$, we can now define our bi-variate brushlet system $\{w_{m,Q}\}_{m \in \mathbb{N}_0^2, Q \in \mathbb{P}}$,

(2.12) $$w_{m,Q}(x,y) := w_{m_1,I}(x) w_{m_2,J}(y), \; m = (m_1, m_2), \; Q = I \times J,$$

where $w_{m_1,I}$ was defined in (2.6). With this notation, we have that $P_Q$ denotes the orthogonal projection onto $\overline{\text{span}}(\{w_{m,Q}\}_{m \in \mathbb{N}_0^2})$,

(2.13) $$P_Q f = \sum_{m \in \mathbb{N}_0^2} \langle f, w_{m,Q} \rangle w_{m,Q}, \; f \in L_2(\mathbb{R}^2).$$

Next, we use the orthogonal projections $P_Q$ to prove that $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ is an orthonormal basis for $L_2(\mathbb{R}^2)$.

**Proposition 2.2.**
The system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ is an orthonormal basis for $L_2(\mathbb{R}^2)$.

**Proof**:
To prove that the system is complete in $L_2(\mathbb{R}^2)$, we first observe that only adjacent rectangles in $\mathbb{P}$ overlap. It follows that there exists a family of open sets $\{U_n\}_{n\in\mathbb{Z}^2}$ such that for $f \in L_2(\mathbb{R}^2)$, $\sum_{Q\in\mathbb{P}} \widehat{P_Q f}(\xi)$, $\xi \in \bar{U}_n$, contains at most four non-zero elements and $\cup_{n\in\mathbb{Z}^2}\bar{U}_n = \mathbb{R}^2$. This can be used to show that $\sum_{Q\in\mathbb{P}} P_Q$ converges strongly to a bounded operator on $L_2(\mathbb{R}^2)$, and it suffices to prove pointwise that

$$\sum_{Q\in\mathbb{P}} \widehat{P_Q s} = \hat{s} \tag{2.14}$$

for functions $s$ in a suitable dense subset of $L_2(\mathbb{R}^2)$. Since finite linear combinations of separable functions are dense in $L_2(\mathbb{R}^2)$, we only need to verify (2.14) for a separable function $s(x,y) = g(x)h(y)$ with $g,h \in L_2(\mathbb{R})$.

We begin with the projections associated with $\mathbb{P}_0$ and $\mathbb{P}_1$. By using (2.11) on the second coordinate, we sum up the projections associated with $\mathbb{P}_1^l$ and $\mathbb{P}_1^r$,

$$\sum_{Q\in\mathbb{P}_1^l} P_Q = P_{I_{-1}\times J_0^0} \tag{2.15}$$

$$\sum_{Q\in\mathbb{P}_1^r} P_Q = P_{I_1\times J_0^0}. \tag{2.16}$$

Next, we use (2.11) on the first coordinate to sum (2.15) and (2.16) together with the projection associated with the center rectangle $I_0 \times J_0^0$,

$$\sum_{Q\in\mathbb{P}_1^l\cup\mathbb{P}_0\cup\mathbb{P}_1^r} P_Q = P_{\cup_{i=-1}^1 I_i\times J_0^0}. \tag{2.17}$$

Finally, we add the projections associated with $\mathbb{P}_1^b$ and $\mathbb{P}_1^t$ to get

$$\sum_{Q\in\cup_{n=0}^1\mathbb{P}_n} P_Q = P_{\cup_{i=-1}^1 I_i\times\cup_{j=-1}^1\cup_l J_j^l}. \tag{2.18}$$

By repeating the procedure $N$ times we end up with

$$\sum_{Q\in\cup_{n=0}^N\mathbb{P}_n} P_Q = P_{\cup_{i=-N}^N I_i\times\cup_{j=-N}^N\cup_l J_j^l}. \tag{2.19}$$

It then follows from (2.10) that as $N$ goes to infinity,

$$\sum_{Q\in\cup_{n=0}^{N}\mathbb{P}_n} \widehat{P_Q s} \tag{2.20}$$

converges pointwise to $\hat{s}$ which proves (2.14). Hence, $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$ is complete in $L_2(\mathbb{R}^2)$.

The system is orthonormal, which will follow from the fact that it consists of carefully selected tensor products of univariate brushlets. One can check that two distinct brushlets associated with the same hollow rectangle $\mathbb{P}_n$ are orthogonal. If $|n-m| \geq 2$ then two brushlets associated with $\mathbb{P}_n$ and $\mathbb{P}_m$, respectively, do not overlap in the frequency space. This leaves us with brushlets that are associated with $Q \in \mathbb{P}_n$ and $P \in \mathbb{P}_{n+1}$, respectively. If we look at the $\mathbb{R}_+ \times \mathbb{R}_+$ part of the frequency space, then the brushlets only overlap if $Q = I_n \times Q_2$, $P = I_{n+1} \times P_2$ or $Q = Q_1 \times J_n^{k_n}$, $P = P_1 \times J_{n+1}^0$ (see Figure 1). In which case we have from the univariate brushlets that the brushlets are orthogonal. The rest of the frequency space follows similarly. ■

## 3. Anisotropic $\alpha$-modulation spaces

In this section, we define the anisotropic $\alpha$-modulation spaces and show that our brushlet system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$ can constitute bases for them. To define the anisotropic $\alpha$-modulation spaces, we need a nice partition of unity and this partition is based on a covering of the frequency plane which again is based on an anisotropic quasi-norm.

First we define an anisotropic quasi-norm $|\cdot|_a$,

$$|\xi|_a := |\xi_1|^{1/a_1} + |\xi_2|^{1/a_2},\ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \tag{3.1}$$

where $a = (a_1, a_2)$, $a_1, a_2 > 0$ and $a_1 + a_2 = 2$. We also define $\langle \xi \rangle_a := (1 + |\xi|_a^2)^{1/2}$ and the balls

$$\mathcal{B}_a(\xi, r) := \{\zeta \in \mathbb{R}^2 : |\xi - \zeta|_a < r\}. \tag{3.2}$$

Notice that $|\mathcal{B}_a(\xi, r)| = r^2\lambda_a$, $\lambda_a := |\mathcal{B}_a(0,1)|$.

With such an quasi-norm $|\cdot|_a$, we can define anisotropic $\alpha$-coverings.

**Definition 3.1.**
A countable set $\mathcal{Q}$ of measurable connected subsets $Q \subset \mathbb{R}^2$ is called a *connected admissible covering* if $\mathbb{R}^2 = \cup_{Q\in\mathcal{Q}} Q$ and there exists $n_0 < \infty$ such that $\#\{Q' \in \mathcal{Q} : \bar{Q} \cap \bar{Q}' \neq \emptyset\} \leq n_0$ for all $Q \in \mathcal{Q}$. Let

$$r_Q = \sup\{r \in \mathbb{R} : \mathcal{B}_a(c_r, r) \subset Q,\ c_r \in \mathbb{R}^2\}, \tag{3.3}$$

$$R_Q = \inf\{R \in \mathbb{R} : Q \subset \mathcal{B}_a(c_R, R),\ c_R \in \mathbb{R}^2\} \tag{3.4}$$

denote the radius of the inscribed and circumscribed disc of $Q \in \mathcal{Q}$, respectively. A connected admissible covering $\mathcal{Q}$ is called an *anisotropic $\alpha$-covering* of $\mathbb{R}^2$, $0 \leq \alpha \leq 1$, if $|Q| \asymp \langle \xi \rangle_a^{2\alpha}$ for some $\xi \in Q$ and all $Q \in \mathcal{Q}$, and there exists $K < \infty$ such that $R_Q/r_Q \leq K$ for all $Q \in \mathcal{Q}$. $\diamond$

**Remark 3.2.**
Notice that $|Q| \asymp \langle \xi \rangle_a^{2\alpha}$ for some $\xi \in Q$ implies the same for all $\xi \in Q$ with constants independent of $\xi$ and $Q$. Also we have restricted ourself to connected sets to later use the general theory of decomposition spaces to show that anisotropic $\alpha$-modulation spaces are well-defined (see [4]). However, by generalizing [3, Theorem 3.1] one can drop the requirement that the sets need to be connected. $\circ$

For technical reasons we shall require our partitions of unity to satisfy the following.

**Definition 3.3.**
Given $0 \leq \alpha \leq 1$, let $Q$ be an anisotropic $\alpha$-covering. A corresponding bounded admissible partition of unity (BAPU) is a family of functions $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^2)$ satisfying:

- $\text{supp}(\psi_Q) \subseteq Q$
- $\sum_{Q \in \mathcal{Q}} \psi_Q \equiv 1$
- $\sup_{Q \in \mathcal{Q}} |Q|^{1/p-1} \|\mathcal{F}^{-1}\psi_Q\|_{L_p} < \infty$.

$\diamond$

It was proven in [4, Section 6] that an anisotropic $\alpha$-covering with a corresponding BAPU exists for every $\alpha \in [0,1]$. We define the multiplier $\psi_Q(D)f := \mathcal{F}^{-1}(\psi_Q \mathcal{F} f)$, $f \in L_2(\mathbb{R}^2)$. A standard result on band-limited multipliers [21, Proposition 1.5.1] ensures that if $\{\psi_Q\}_{Q \in \mathcal{Q}}$ is a BAPU, then $\psi_Q(D)$ extends to a bounded operator on band-limited functions in $L_p(\mathbb{R}^2)$, $0 < p \leq \infty$, uniformly in $Q \in \mathcal{Q}$.

We are now ready to define anisotropic $\alpha$-modulation spaces.

**Definition 3.4.**
Given $0 \leq \alpha \leq 1$, let $\mathcal{Q}$ be an anisotropic $\alpha$-covering of $\mathbb{R}^2$ with a corresponding BAPU $\{\psi_Q\}_{Q \in \mathcal{Q}}$. For $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q < \infty$, we define the *anisotropic $\alpha$-modulation space*, $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$, as the set of distributions $f \in \mathcal{S}'(\mathbb{R}^2)$ satisfying

$$\|f\|_{M^{s,\alpha}_{p,q}(\mathbb{R}^2)} := \Big( \sum_{Q \in \mathcal{Q}} \langle \xi_Q \rangle_a^{qs} \|\psi_Q(D) f\|_{L_p}^q \Big)^{1/q} < \infty, \tag{3.5}$$

where $\xi_Q \in Q$. $\diamond$

We show in the Appendix that anisotropic $\alpha$-modulation spaces are independent of which $\alpha$-covering is used. Furthermore, it can be shown that $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ is a quasi-Banach space (Banach space for $p,q \geq 1$), and $\mathcal{S}(\mathbb{R}^2)$ is dense in $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$, see [4] and [8]. For more information on quasi-Banach spaces, we refer the reader to [14] and [13].

3.1. **Orthonormal bases for anisotropic $\alpha$-modulation spaces.** With the anisotropic $\alpha$-modulation spaces in place, we need to adapt the covering $\mathbb{P}$ such that the associated brushlet system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ constitutes bases for them. The natural choice would be to make $\mathbb{P}$ an $\alpha$-covering, and as we shall see, this will suffice.

First we need to make $\mathbb{P}$ an $\alpha$-covering. We will focus on $\alpha \in [0,1)$ since $\alpha = 1$ corresponds to a dyadic covering, and we use a polynomial type covering. Without loss of generality we will also assume that $a_2 \geq a_1$. Let $I_0 := [-1,1)$, $I_n := [n^{\beta a_1}, (n+1)^{\beta a_1})$, and $I_{-n} := -I_n$, $n \geq 1$, $\beta \geq 1$. Next, we introduce the sequence $\{y_m\}_{m\in\mathbb{N}}$, $y_0 := 1$, $y_m := y_{m-1} + n^{\beta a_2 - a_2/a_1}$, where $n \in \mathbb{N}$ is chosen such that $n^{\beta a_2} \leq y_{m-1} < (n+1)^{\beta a_2}$. We can then define $J_n^l := [y_{m-1}, y_m)$, $m := n + l + \sum_{i=1}^{n-1} k_i$, $0 \leq l \leq k_i$, where $k_i \in \mathbb{N}$ are chosen such that $|J_n^l| = n^{\beta a_2 - a_2/a_1}$, see figure 2. Furthermore, let $J_0^0 := [-1,1)$ and $J_{-n}^{-l} := -J_n^l$. To make sure that $J_n^l$ is defined for all $n \in \mathbb{N}$, we notice that

$$y_m - y_{m-1} = n^{\beta a_2 - a_2/a_1} \leq n^{\beta a_2 - 1} < (n+2)^{\beta a_2} - (n+1)^{\beta a_2}.$$

In fact, we have $k_n + 1 \asymp n^{a_2/a_1 - 1}$.

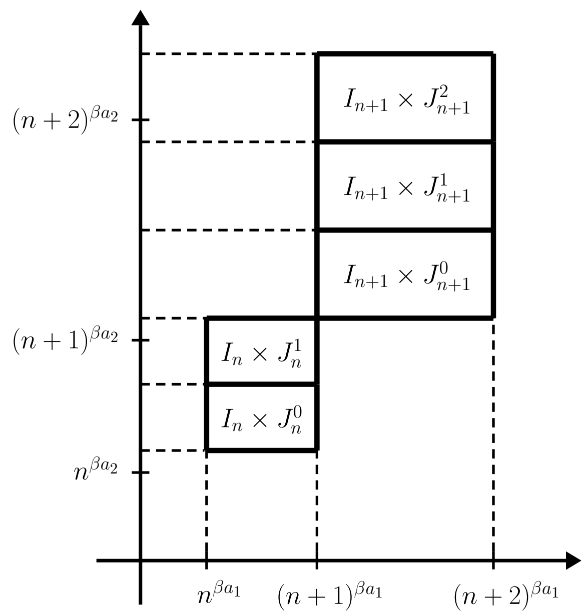


**Figure 2.** *Choosing $I_n \times J_n^l$ such that $\mathbb{P}$ is an $\alpha$-covering.*

One can check that $\{I_n\}$ and $\{J_n^l\}$ are moderate disjoint coverings of $\mathbb{R}$. To generate $\mathbb{P}$, we choose $\{I_{n,i}\}$ and $\{J_{n,j}^0\}$ such that $|I_{n,i}| \asymp |I_n|$ and $|J_{n,j}^0| \asymp |J_n^0|$. As the sets in $\mathbb{P}$ are disjoint it follows easily that $\mathbb{P}$ is a connected admissible covering of $\mathbb{R}^2$.

Next, to show that $\mathbb{P}$ is a an anisotropic $\alpha$-covering, we notice that $\mathbb{P}$ is constructed such that we only need to check the requirements for $I_n \times J_n^l$. We have

$$|I_n|^{1/a_1} \asymp n^{(\beta a_1 - 1)/a_1} = n^{(\beta a_2 - a_2/a_1)/a_2} = |J_n^l|^{1/a_2}.$$

For $Q = I_n \times J_n^l$ it follows that $r_Q \geq C|I_n|^{1/a_1}$ and $R_Q \leq |I_n|^{1/a_1} + |J_n^l|^{1/a_2}$ which gives $R_Q/r_Q \leq K < \infty$, $Q \in \mathbb{P}$. Finally, given $\alpha \in [0,1)$ we need to define $\beta$ such that $|Q| \asymp \langle \xi \rangle_a^{2\alpha}$ for some $\xi \in Q$, $Q \in \mathbb{P}$. By choosing

$$\beta := \frac{1 + \frac{a_2}{a_1}}{2(1-\alpha)}, \tag{3.6}$$

we get $2\alpha\beta = 2\beta - 1 - a_2/a_1$, and it follows that

$$|I_n \times J_n^l| \asymp n^{\beta a_1 - 1 + \beta a_2 - a_2/a_1} = n^{2\beta - 1 - a_2/a_1} = n^{2\alpha\beta} \asymp \langle \xi \rangle_a^{2\alpha},$$

where $\xi$ is the corner of $I_n \times J_n^l$ closest to origo.

We now have that $\mathbb{P}$ is an anisotropic $\alpha$-covering, and from Proposition 2.2 we know that $\{w_{m,Q}\}_{m \in \mathbb{N}_0^2, Q \in \mathbb{P}}$ is an orthonormal basis for $L_2(\mathbb{R}^2)$. Next, we show that these conditions are sufficient to prove that $\{w_{m,Q}\}_{m \in \mathbb{N}_0^2, Q \in \mathbb{P}}$ is an unconditional basis for the corresponding anisotropic $\alpha$-modulation space.

First we need the following definition and lemma.

**Definition 3.5.**
Let $\mathcal{Q}$ and $\mathcal{P}$ be coverings of $\mathbb{R}^2$ and $G$ a subset of $\mathbb{R}^2$. We define

$$A_G^{\mathcal{Q}} := \{Q \in \mathcal{Q} : \bar{Q} \cap \bar{G} \neq \emptyset\} \tag{3.7}$$

and the sets

$$\widetilde{Q} := \bigcup_{Q' \in A_Q^{\mathcal{Q}}} \bar{Q}', \ Q \in \mathcal{Q}. \tag{3.8}$$

◇

Notice that a connected admissible covering $\mathcal{Q}$ fulfills $\# A_Q^{\mathcal{Q}} \leq n_0$, $Q \in \mathcal{Q}$. One can also check that $\{\widetilde{Q}\}_{Q \in \mathbb{P}}$ is an anisotropic $\alpha$-covering.

**Lemma 3.6.**
Given $f \in L_2(\mathbb{R}^2)$, $0 \leq \alpha < 1$ and $0 < p \leq \infty$. If $\{\widetilde{\psi}_Q\}_{Q \in \mathbb{P}}$ is a partition of unity for $\{\widetilde{Q}\}_{Q \in \mathbb{P}}$ which satisfies

$$\widetilde{\psi}_Q(x) = 1, \ x \in \text{supp}(\hat{w}_{0,Q}),$$

and $\{\psi_Q\}_{Q\in\mathbb{P}}$ is a partition of unity for $\{\text{supp}(\hat{w}_{0,Q})\}_{Q\in\mathbb{P}}$, then there exists $C, C' > 0$, independent of $Q \in \mathbb{P}$, such that

$$\Big(\sum_{m\in\mathbb{N}_0^2} |\langle f, w_{m,Q}\rangle|^p\Big)^{1/p} \le C|Q|^{\frac{1}{p}-\frac{1}{2}}\|\widetilde{\psi}_Q(D)f\|_{L_p}, \text{ and}$$

$$\|\psi_Q(D)f\|_{L_p} \le C'|Q|^{\frac{1}{2}-\frac{1}{p}}\sum_{Q'\in\widetilde{Q}}\Big(\sum_{m\in\mathbb{N}_0^2}|\langle f, w_{m,Q'}\rangle|^p\Big)^{1/p}.$$

When $p = \infty$ the sum over $m \in \mathbb{N}_0^2$ is changed to sup.

**Proof**:
Notice that (2.7) together with (2.8) yield the following estimates,

$$\sup_{x\in\mathbb{R}^2}\sum_{m\in\mathbb{N}_0^2}|w_{m,Q}(x)|^p \le C_p|Q|^{\frac{p}{2}} \text{ and } \sup_{m\in\mathbb{N}_0^2}\|w_{m,Q}\|_{L_p}^p \le C_p'|Q|^{\frac{p}{2}-1}. \tag{3.9}$$

Take $f \in L_2(\mathbb{R}^2)$ and let us first assume that $p \le 1$. We then have (see, e.g. [21, p. 18])

$$\begin{aligned}\sum_{m\in\mathbb{N}_0^2}|\langle f, w_{m,Q}\rangle|^p &= \sum_{m\in\mathbb{N}_0^2}|\langle\widetilde{\psi}_Q(D)f, w_{m,Q}\rangle|^p \le \sum_{m\in\mathbb{N}_0^2}\|(\widetilde{\psi}_Q(D)f)w_{m,Q}\|_{L_1}^p \\ &\le C|Q|^{1-p}\sum_{m\in\mathbb{N}_0^2}\|(\widetilde{\psi}_Q(D)f)w_{m,Q}\|_{L_p}^p \le C|Q|^{1-\frac{p}{2}}\|\widetilde{\psi}_Q(D)f\|_{L_p}^p.\end{aligned}$$

By using that $\psi_Q(D)$ is bounded on band-limited functions in $L_p$, we have the second inequality in the lemma,

$$\begin{aligned}\|\psi_Q(D)f\|_{L_p}^p &\le \sum_{Q'\in\widetilde{Q}}\sum_{m\in\mathbb{N}_0^2}|\langle f, w_{m,Q'}\rangle|^p\|w_{m,Q'}\|_{L_p}^p \\ &\le C'|Q|^{\frac{p}{2}-1}\sum_{Q'\in\widetilde{Q}}\sum_{m\in\mathbb{N}_0^2}|\langle f, w_{m,Q'}\rangle|^p.\end{aligned}$$

For $1 < p < \infty$ the lemma follows by using the two estimates in (3.9) with $p = 1$ together with Hölder's inequality (see e.g. [18, §2.5]). The case $p = \infty$ follows similar to $p \le 1$. ■

By taking the $l_q$-norm in Lemma 3.6, we can derive our main result.

**Theorem 3.7.**
Given $0 < p \le \infty$, $0 < q < \infty$, $s \in \mathbb{R}$, and $0 \le \alpha < 1$. With the system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$, we have the following characterization

$$\|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^2)} \asymp \Big(\sum_{n=0}^{\infty} n^{q\beta(s+\alpha-\frac{2\alpha}{p})}\sum_{Q\in\mathbb{P}_n}\Big(\sum_{m\in\mathbb{N}_0^2}|\langle f, w_{m,Q}\rangle|^p\Big)^{q/p}\Big)^{1/q},$$

where $\beta$ was defined in (3.6). Furthermore, $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ constitutes an unconditional basis for $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$.

**Proof**:
The norm characterization follows by taking the $l_q$-norm in Lemma 3.6 and using that $|Q| = n^{2\alpha\beta} \asymp \langle\xi_Q\rangle_a^{2\alpha}$, $\xi_Q \in Q$, $Q \in \mathbb{P}_n$. That $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ constitutes a unconditional basis for $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$, follows by standard results using the norm characterization, that $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ is a quasi-Banach space in which $S(\mathbb{R}^2)$ is dense and that $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ is an orthonormal basis for $L_2(\mathbb{R}^2)$. ■

**Remark 3.8.**
By [20, Remark 4.6], one can use $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ to construct a compactly supported basis for $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ with the same norm characterization as $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$. ○

Theorem 3.7 also shows that $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ induces a natural isomorphism between $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ and the sequence space $m^{s,\alpha}_{p,q}$ defined by:

**Definition 3.9.**
Given $0 < p \le \infty$, $0 < q < \infty$, $s \in \mathbb{R}$, $0 \le \alpha < 1$, we define the sequence space $m^{s,\alpha}_{p,q}$ as the set of sequences $c := \{c_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}} \subset \mathbb{C}$ satisfying

$$\|c\|_{m^{s,\alpha}_{p,q}} := \Big(\sum_{n=0}^{\infty} n^{q\beta(s+\alpha-\frac{2\alpha}{p})} \sum_{Q\in\mathbb{P}_n} \Big(\sum_{m\in\mathbb{N}_0^2} |c_{m,Q}|^p\Big)^{q/p}\Big)^{1/q} < \infty,$$

where $\beta$ was defined in (3.6). ◇

## 4. An application to nonlinear approximation

We finish this paper with applying $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ to $n$-term nonlinear approximation in certain anisotropic $\alpha$-modulation spaces.

First, we need some notation regarding nonlinear approximation. Let $\mathcal{D} := \{g_k\}_{k\in\mathbb{N}}$ be a Schauder basis in a quasi-Banach space $X$. We consider the collection of all possible $n$-term expansions with elements from $\mathcal{D}$:

$$\Sigma_n(\mathcal{D}) := \Big\{\sum_{i\in\Lambda} c_i g_i \Big| c_i \in \mathbb{C}, \#\Lambda \le n\Big\}.$$

The error of the best $n$-term approximation to an element $f \in X$ is then

$$\sigma_n(f,\mathcal{D})_X := \inf_{f_n\in\Sigma_n(\mathcal{D})} \|f - f_n\|_X.$$

Next, we introduce the approximation spaces $\mathcal{A}^\gamma_q(X,\mathcal{D})$ which essentially consists of the elements $f$ for which $\sigma(f,\mathcal{D})_X = \mathcal{O}(n^{-\gamma})$.

**Definition 4.1.**
Let $0 < \gamma, q < \infty$. We define the approximation space $\mathcal{A}_q^\gamma(X, \mathcal{D})$ as the set of distributions $f \in X$ satisfying

$$|f|_{\mathcal{A}_q^\gamma(X,\mathcal{D})} := \Big( \sum_{n=1}^{\infty} (n^\gamma \sigma_n(f, \mathcal{D})_X)^q \frac{1}{n} \Big)^{1/q} < \infty,$$

and quasi-norm it with $\|f\|_{\mathcal{A}_q^\gamma(X,\mathcal{D})} := \|f\|_X + |f|_{\mathcal{A}_q^\gamma(X,\mathcal{D})}$. $\diamond$

As Theorem 3.7 showed that $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$ induces an isomorphism between $M_{p,q}^{s,\alpha}(\mathbb{R}^2)$ and $m_{p,q}^{s,\alpha}$, we can apply [9] to get a complete characterization of certain nonlinear approximation spaces associated with anisotropic $\alpha$-modulation spaces:

**Theorem 4.2.**
Let $0 < \gamma, p < \infty$, $0 \le \alpha < 1$, $s \in \mathbb{R}$, $\tau^{-1} := \gamma + p^{-1}$ and $\rho := 2\alpha\gamma + s$. If $\mathcal{D}$ is the system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$ normalized in $M_{p,p}^{s,\alpha}(\mathbb{R}^2)$, then we have the characterization

$$\mathcal{A}_\tau^\gamma(M_{p,p}^{s,\alpha}(\mathbb{R}^2), \mathcal{D}) = M_{\tau,\tau}^{\rho,\alpha}(\mathbb{R}^2)$$

with equivalent norms. $\square$

**Remark 4.3.**
By using Remark 3.8, we can also get the characterization in Theorem 4.2 for a compactly supported basis for $M_{p,p}^{s,\alpha}(\mathbb{R}^2)$. $\circ$

## Appendix

In this appendix we show that $M_{p,q}^{s,\alpha}(\mathbb{R}^2)$ only depends on the $\alpha$-covering up to equivalence of the norms. First we extend Definition 3.5.

**Definition A.1.**
Let $\widetilde{Q}^{(0)} := \bar{Q}$, and define inductively $\widetilde{Q}^{(k+1)} := \widetilde{\widetilde{Q}^{(k)}}$, $k \ge 0$. Finally let $\widetilde{\mathcal{Q}}^{(k)} := \{\widetilde{Q}^{(k)}\}_{Q\in\mathcal{Q}}$. $\mathcal{P}$ is called *almost subordinate* to $\mathcal{Q}$ (written $\mathcal{P} \le \mathcal{Q}$) if there exists $k \in \mathbb{N}$ such that for all $P \in \mathcal{P}$, we have $P \subseteq \widetilde{Q}^{(k)}$ for some $Q \in \mathcal{Q}$. $\diamond$

Let $\mathcal{Q}$ and $\mathcal{P}$ be two anisotropic $\alpha$-coverings. If $\bar{Q} \cap \bar{P} \neq \emptyset$, $Q \in \mathcal{Q}$, $P \in \mathcal{P}$, then Definition 3.1 implies that $R_Q \asymp R_P$. This can be used to prove that there exists $d_0 < \infty$ such that

$$\#A_P^{\mathcal{Q}} \le d_0, \ P \in \mathcal{P}. \tag{A.1}$$

Lemma A.2 below then gives that $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$. By interchanging $\mathcal{Q}$ and $\mathcal{P}$, we also have that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$. From [4, Theorem 1] it then follows that $M_{p,q}^{s,\alpha}(\mathbb{R}^2)$ only depends on the $\alpha$-covering up to equivalence of the norms.

**Lemma A.2.**
Let $\mathcal{Q}$ and $\mathcal{P}$ be connected admissible coverings. Then $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$ if and only if there exists $d_0 < \infty$ such that

$$\#A_P^{\mathcal{Q}} \leq d_0, \; P \in \mathcal{P}. \tag{A.2}$$

**Proof**:
Let us first assume that $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$, and choose $P \in \mathcal{P}$. Then there exists $Q \in \mathcal{Q}$ such that $P \subseteq \widetilde{Q}^{(k)}$. One can easily prove that $\widetilde{\mathcal{Q}}^{(k)}$ is a connected admissible covering so it follows that

$$\#A_P^{\mathcal{Q}} \leq \#A_{\widetilde{Q}^{(k)}}^{\mathcal{Q}} \leq \#A_{\widetilde{Q}^{(k)}}^{\widetilde{\mathcal{Q}}^{(k)}} \leq d_0. \tag{A.3}$$

To prove the opposite way, let us assume that (A.2) is satisfied. Choose $P \in \mathcal{P}$ and $Q \in A_P^{\mathcal{Q}}$. If $A_P^{\mathcal{Q}} \backslash \{Q\} = \emptyset$, then $P \subseteq Q$, and we are done. If instead $A_P^{\mathcal{Q}} \backslash \{Q\} \neq \emptyset$ and $\bar{Q} \cap \bar{Q}' = \emptyset$ for all $Q' \in A_P^{\mathcal{Q}} \backslash \{Q\}$, then

$$P \backslash \bar{Q} = \bigcup_{Q' \in A_P^{\mathcal{Q}} \backslash \{Q\}} \bar{Q}' \cap P. \tag{A.4}$$

However, this proves that $P \backslash \bar{Q}$ is both open and closed on $P$ which contradicts that $P$ is a connected set. It follows that $Q' \subset \widetilde{Q}$ for some $Q' \in A_P^{\mathcal{Q}} \backslash \{Q\}$. Next, we use the same argument with $\widetilde{Q}$, and either $P \subseteq \widetilde{Q}$ or there exists $Q'' \in A_P^{\mathcal{Q}} \backslash \{\widetilde{Q}\}$ such that $\bar{Q}'' \cap \widetilde{Q} \neq \emptyset$. As $A_P^{\mathcal{Q}}$ contains at most $d_0$ elements, we can repeat the argument $d_0 - 1$ times to get

$$P \subseteq \bigcup_{Q' \in A_P^{\mathcal{Q}}} \bar{Q}' \subseteq \widetilde{Q}^{(d_0 - 1)} \tag{A.5}$$

which proves that $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$. ∎

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