Combinatorial Conditions for Directed Collapsing

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June 4, 2021

Abstract

The purpose of this article is to study directed collapsibility of directed Euclidean cubical complexes. One application of this is in the nontrivial task of verifying the execution of concurrent programs. The classical definition of collapsibility involves certain conditions on a pair of cubes of the complex. The direction of the space can be taken into account by requiring that the past links of vertices remain homotopy equivalent after collapsing. We call this type of collapse a *link-preserving directed collapse*. In this paper, we give combinatorially equivalent conditions for preserving the topology of the links, allowing for the implementation of an algorithm for collapsing a directed Euclidean cubical complex. Furthermore, we give conditions for when link-preserving directed collapses preserve the contractability and connectedness of directed path spaces, as well as examples when link-preserving directed collapses do not preserve the number of connected components of the path space between the minimum and a given vertex.

1 Introduction

Edsger W. Dijkstra proposed the PV-model [4] as a way to study properties of concurrent programs. One way to interpret this model when the program does not contain loops is to consider the states of such a PV program as a subset of $\mathbf{R}^n$ that consist of unions of unit cubes called a Euclidean cubical complex. Each axis represents a sequence of actions a process completes in the program execution. A directed path (nondecreasing in all coordinates) represents a (partial) program execution.

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Such executions are equivalent if the corresponding directed paths are *directed homotopic*, i.e., if the space of such paths is connected. We give a well-known example of a Euclidean cubical complex with directed paths in Figure 1. This example is commonly known as the *Swiss Flag*. Using Euclidean cubical complexes as topological models for concurrent programming has proven to be beneficial. One example is that verifying properties of one execution in each connected component of the space of such paths in a Euclidean cubical complex verifies those properties for the entire concurrent program. A more in depth introduction to this view of concurrency, corresponding geometry, and benefits can be found in [5,6].

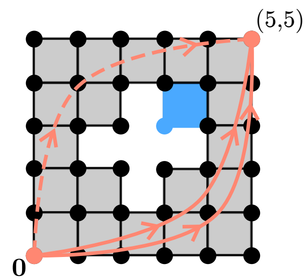


Figure 1: The Swiss Flag and Three Directed Paths. The gray squares are the two-cubes of a Euclidean cubical complex. The bi-monotone increasing paths are directed paths starting at $(0,0)$ and ending at $(5,5)$. The solid directed paths are directed homotopic while the dashed directed path is not directed homotopic to either of the other directed paths since there is a hole in the middle of the complex. Each point highlighted in blue is *unreachable*, meaning that we cannot reach any point highlighted in blue without breaking bi-monotonicity in a path starting at $(0,0)$.

A non-trivial Euclidean cubical complex contains uncountably many directed paths and more information than we need for understanding the topology of the spaces of directed paths. The main question we ask is, *How can we simplify a directed Euclidean cubical complexes while still preserving spaces of directed paths*? In the realm of concurrent computing, answering this question can help simplify concurrent programs and speed up the program verification process.

Past links are local representations of a Euclidean cubical complex at vertices. They were introduced in [18] as a means to show that any finite homotopy type can be realized as a connected component of the space of execution paths for some PV-model. In [1], we found conditions for when the local information of past links preserve the global information on the homotopy type of spaces of directed paths. Because of these relationships between past links and directed path spaces, we define collapsing in terms of past links. We call this type of collapsing *link-preserving directed collapse* (LPDC). We aim to compress a Euclidean cubical complex by LPDCs before attempting to answer questions about directed path spaces.

The main result of this paper is Theorem 4 and is a very simple criterion for such a collapsing to be allowed: *A pair of cubes,* (σ, τ)*, is an LPDC pair if and only if it is a collapsing pair in the non-directed sense and* τ *does not contain the minimum vertex of* σ*.* This condition greatly simplifies the definition of LPDC and is easy to add to a collapsing algorithm for Euclidean cubical complexes in the undirected setting. Algorithms and implementations in this setting already exist such as in [13]. Furthermore, we provide conditions for when LPDCs preserve the contractability and connectedness of directed path spaces (Section 4) along with a discussion of some of the limitations (Section 5). This work provides a start at the mathematical foundations for developing polynomial time algorithms that collapse Euclidean cubical complexes and preserve directed path spaces.

2 Background

This paper builds on our prior work [1], as well as work by others [6,8,9,14, 18]. In this section, we recall the definitions of directed Euclidean cubical complexes, which are the objects that we study in this paper. Then, we discuss the relationship between spaces of directed paths and past links in directed Euclidean cubical complexes. For additional background on directed topology (including generalizations of the definitions below), we refer the reader to [5]. We also assume the reader is familiar with the notion of homotopy equivalence of topological spaces (denoted using $\simeq$ in this paper) and homotopy between paths as presented in [10].

2.1 Directed Spaces and Euclidean Cubical Complexes

Let n be a positive integer. A *(closed) elementary cube* in $\mathbb{R}^n$ is a product of closed intervals of the following form:

$$[v_1 - j_1, v_1] \times [v_2 - j_2, v_2] \times \ldots \times [v_n - j_n, v_n] \subsetneq \mathbb{R}^n, \tag{1}$$

where $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{Z}^n$ and $\mathbf{j} = (j_1, j_2, \ldots, j_n) \in \{0,1\}^n$. We often refer to elementary cubes simply as *cubes.* The dimension of the cube is the number of unit entries in the vector $\mathbf{j}$; specifically, the dimension of the cube in Eq. (1) is the sum: $\sum_{i=1}^{n} j_i$. In particular, when $\mathbf{j} = (0, \ldots, 0)$, the elementary cube is a single point and often denoted using just $\mathbf{v}$. If τ and σ are elementary cubes such that $\tau \subseteq \sigma$, we say that τ is a face of σ and that σ is a coface of τ. Cubical sets were first introduced in the 1950s by Serre [15] in a more general setting; see also [2, 7, 11].

Elementary cubes stratify $\mathbb{R}^n$, where two points $x, y \in \mathbb{R}^n$ are in the same stratum if and only if they are members of the same set of elementary cubes; we call this the *cubical stratification of* $\mathbb{R}^n$. Each stratum in the stratification is either an open cube or a single point. A *Euclidean cubical complex* $(K, \mathcal{K})$ is a subspace $K \subsetneq \mathbb{R}^n$ that is equal to the union of a set of elementary cubes, together with the stratification $\mathcal{K}$ induced by the cubical stratification of $\mathbb{R}^n$; see Fig. 2. We topologize K using the

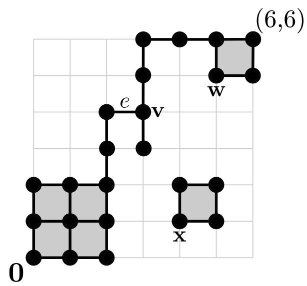


Figure 2: Euclidean cubical complex in $\mathbb{R}^2$ with 24 zero-cubes (vertices), 28 one-cubes (edges), and six two-cubes (squares). By construction, all elementary cubes in a directed Euclidean cubical complex are axis aligned. Consider the vertex $\mathbf{v} = (3, 4)$. The edge $e = [(2, 4), (3, 4)]$ (written $e = [2, 3] \times [4, 4]$ in the notation of Eq. (1)) is one of the two lower cofaces of $\mathbf{v}$. Since e is not a face of any two-cube, e is a maximal cube. The points $\mathbf{0}$, $\mathbf{w}$, and $\mathbf{x}$ are the minimal vertices of this directed Euclidean cubical complex.

subspace topology with the standard topology on $\mathbb{R}^n$. Note that if $\sigma \in \mathcal{K}$, then, since K is a union of elementary cubes, all of its faces are necessarily in $\mathcal{K}$ as well. If $\sigma \in \mathcal{K}$ with no proper cofaces, then we say that σ is a *maximal cube* in K. We denote the set of closed cubes in $(K, \mathcal{K})$ by $\overline{\mathcal{K}}$; the set of closed cubes in $\overline{\mathcal{K}}$ is in one-to-one correspondence with the open cubes in $\mathcal{K}$. Specifically, vertices in $\overline{\mathcal{K}}$ correspond to vertices in $\mathcal{K}$ and all other elementary cubes in $\overline{\mathcal{K}}$ correspond to their interiors in $\mathcal{K}$. Throughout this paper, we denote the set of zero-cubes in $\mathcal{K}$ by $\mathrm{verts}(K)$ and note that $\mathrm{verts}(K) \subsetneq \mathbb{Z}^n$, since all cubes in $(K, \mathcal{K})$ are elementary cubes.

The *product order on* $\mathbb{R}^n$, denoted $\preceq$, is the partial order such that for two points $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ in $\mathbb{R}^n$, we have $\mathbf{p} \preceq \mathbf{q}$ if and only if $p_i \leq q_i$ for each coordinate i. Using this partial order, we define the interval of points in $\mathbb{R}^n$ between $\mathbf{p}$ and $\mathbf{q}$ as

$$[\mathbf{p}, \mathbf{q}] := \{\mathbf{x} \mid \mathbf{p} \preceq \mathbf{x} \preceq \mathbf{q}\}.$$

The point $\mathbf{p}$ is the minimum vertex of the interval and $\mathbf{q}$ is the maximum vertex of the interval, with respect to $\preceq$. Notationally, we write this as $\min([\mathbf{p}, \mathbf{q}]) := \mathbf{p}$ and $\max([\mathbf{p}, \mathbf{q}]) := \mathbf{q}$. When $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} = \mathbf{q} + \mathbf{j}$, for some $\mathbf{j} \in \{0, 1\}^n$, the interval is an elementary cube as defined in Eq. (1). If, in addition, $\mathbf{j}$ is not the zero vector, then we say that $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ is a *lower coface* of $\mathbf{v}$.

Using the fact that the partial order $(\mathbb{R}^n, \preceq)$ induces a partial order on the points in K, we define directed paths in K as the set of non-decreasing paths in K: A *path* in K is a continuous map from the unit interval $I = [0, 1]$ to K. We say that a path $\gamma \colon I \to K$ goes from $\gamma(0)$ to $\gamma(1)$. Letting K^I denote the set of all paths in K, the set of directed paths is

$$\vec{P}(K) := \{\gamma \in K^I \mid \forall i, j \text{ s.t. } 0 \leq i \leq j \leq 1, \gamma(i) \preceq \gamma(j)\}.$$

We call $\vec{P}(K)$ the set of *directed paths (dipaths)*, and we topologize it with the compact-open topology. For $\mathbf{p}, \mathbf{q} \in K$, we denote the subspace of paths from $\mathbf{p}$ to $\mathbf{q}$ by $\vec{P}_{\mathbf{p}}^{\mathbf{q}}(K)$. We refer to $(K, \vec{P}(K))$ as a *directed Euclidean cubical complex.*[1] The connected components of $\vec{P}_{\mathbf{p}}^{\mathbf{q}}(K)$ are exactly the equivalence classes of directed paths, up to dihomotopy. If two dipaths, f and g are homotopic through a continuous family of dipaths, then f and g are called *dihomotopic*.

Given a directed Euclidean cubical complex, certain subcomplexes will be of interest:

Definition 1 (Special Complexes)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\mathbf{p} \in K$ and let σ be an elementary cube (that need not be in $\mathcal{K}$).

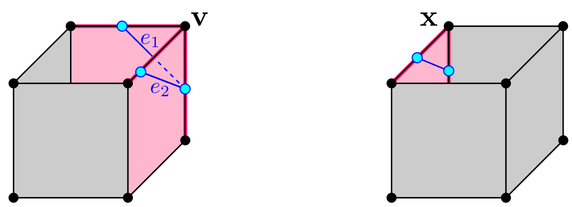
1. The complex above $\mathbf{p}$ is $K_{\mathbf{p}\preceq} := \{\mathbf{q} \in K \mid \mathbf{p} \preceq \mathbf{q}\}$.
2. The complex below $\mathbf{p}$ is $K_{\preceq\mathbf{p}} := \{\mathbf{q} \in K \mid \mathbf{q} \preceq \mathbf{p}\}$.
3. The reachable complex from $\mathbf{p}$ is $\mathrm{reach}(K, \mathbf{p}) := \{\mathbf{q} \in K \mid \vec{P}_{\mathbf{p}}^{\mathbf{q}}(K) \neq \emptyset\}$.
4. The complex restricted to σ is
$$K|_\sigma := \bigcup\{\tau \in \mathcal{K} \mid \min \sigma \preceq \min \tau \preceq \max \tau \preceq \max \sigma\}.$$
5. If $K = I^n$, then we call $(K, \mathcal{K})$ the *standard unit cubical complex* and often denote it by $(I^n, \mathcal{I})$. If $K = I^n + \mathbf{x}$ for some $\mathbf{x} \in \mathbb{Z}^n$, then K is a full-dimensional unit cubical complex.

2.2 Past Links of Directed Cubical Compelxes

An *abstract simplicial complex* is a finite collection $\mathcal{S}$ of sets that is closed under the subset relation, i.e., if $A \in \mathcal{S}$ and B is a set such that $\emptyset \neq B \subseteq A$, then $B \in \mathcal{S}$. The sets in $\mathcal{S}$ are called *simplices*. If the simplex A has $k+1$ elements, then we say that the dimension of A is $\dim(A) := k$, and we say A is a k-simplex. For example, the zero-simplices are the singleton sets and are often referred to as vertices. Since every element of a set $A \in \mathcal{S}$ gives rise to a singleton set in the finite set $\mathcal{S}$, A must be finite.

In the cubical stratification of $\mathbb{R}^n$, the link of a point $\mathbf{v} \in \mathbb{R}^n$ is the intersection of an arbitrarily small sphere around $\mathbf{v}$ with the space $\mathbb{R}^n$; that is, the link of a point is an $(n-1)$-sphere. If $\mathbf{v} \in \mathbb{Z}^n$, the link inherits the stratification as a subcomplex of $\mathbb{R}^n$, and is a simplicial complex whose i-simplices are in one-to-one correspondence with the $(i+1)$-dimensional cofaces of $\mathbf{v}$. The past link of $\mathbf{v}$ is the restriction of the link using the

[1] Directed Eulcidean cubical complexes are an example of a more general concept known as *directed space* (d-spaces). To define a d-space, we have a topological space X and we define a set of dipaths $P'(X) \subseteq X^I$ that contains all constant paths, and is closed under taking non-decreasing reparameterizations, concatenations, and subpaths. Indeed, $\vec{P}(K)$ satisfies these properties.



(a) Past Link of $\mathbf{v}$

(b) Past Link of $\mathbf{x}$

Figure 3: Past link in the Open Top Box. (a) The maximum vertex of this complex is $\mathbf{v}$. The past link $\mathrm{lk}_K^-(\mathbf{v})$ is the simplicial complex comprising three vertices and two edges (shown in blue/cyan). These simplices are in one-to-one correspondence with the set of lower cofaces of $\mathbf{v}$ (highlighted in pink). For example, the edges of $\mathrm{lk}_K^-(\mathbf{v})$, which are labeled e_1 and e_2, are in one-to-one correspondence with the elementary two-cubes that are lower cofaces of $\mathbf{v}$. These lower cofaces are $\sigma_1 = [(v_1, v_2 - 1, v_3 - 1), \mathbf{v}]$ and $\sigma_2 = [(v_1 - 1, v_2, v_3 - 1), \mathbf{v}]$, respectively. In the vector notation for simplices of $\mathrm{lk}_K^-(\mathbf{v})$, we write $e_1 = (1, 0, 1)$ and $e_2 = (0, 1, 1)$. (b) The past link of a vertex $\mathbf{x}$ that is neither a minimal nor a maximal vertex in the complex.

set of lower cofaces of $\mathbf{v}$ instead of all cofaces. Thus, we can represent each simplex in the past link as a vector in $\{0,1\}^n \setminus \{0\}^n$, where the vector $\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n$ represents the cube $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ in the simplex-cube correspondence.[2] As a simplicial complex, the past link of $\mathbf{v}$ in $\mathbb{R}^n$ has n vertices $\{x_i\}_{1 \leq i \leq n}$, and $\mathbf{j}$ represents the simplex $\{x_i | 1 \leq i \leq n, j_i = 1\}$. We are now ready to define the past link of a vertex in a Euclidean cubical complex:

Definition 2 (Past Link)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\mathbf{v} \in \mathbb{Z}^n$. The *past link* of $\mathbf{v}$ is the following simplicial complex:

$$\mathrm{lk}_K^-(\mathbf{v}) := \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid [\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K\}.$$

As a set, the past link represents all elementary cubes in K for which $\mathbf{v}$ is the maximum vertex. As a simplicial complex, it describes (locally) the different types of dipaths to or through $\mathbf{v}$ in K; see Fig. 3.

We conclude this section with a lemma summarizing properties of the past link, most of which follow directly from definitions:

Lemma 1 (Properties of Past Links)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Then, the following statements hold for all $\mathbf{v} \in \mathbb{Z}^n$:

[2] The point $\{0\}^n = (0, \ldots, 0) \in \{0,1\}^n$ is the zero vector, and will never be in the past link. We differentiate this from $\mathbf{0} \in \mathbb{R}^n$, which denotes the origin of $\mathbb{R}^n$.

1. $\mathrm{lk}^-_K(\mathbf{v}) = \bigcup_{\mathbf{p}\in\mathbb{R}^n} \mathrm{lk}^-_{K_{\mathbf{p}\preceq}}(\mathbf{v})$.
2. If $(K', \mathcal{K}')$ is a subcomplex of $(K, \mathcal{K})$, then $\mathrm{lk}^-_{K'}(\mathbf{v}) \subseteq \mathrm{lk}^-_K(\mathbf{v})$.
3. $\mathrm{lk}^-_K(\mathbf{v}) = \mathrm{lk}^-_{K_{\preceq\mathbf{v}}}(\mathbf{v})$.
4. If $(K, \mathcal{K})$ is a full-dimensional unit cubical complex and $\mathbf{v} = \max(\mathrm{verts}(K))$, then $\mathrm{lk}^-_K(\mathbf{v})$ is the complete simplicial complex on n vertices.

Proof. Statement *1*: If $K = \emptyset$, then all past links are empty and the equality trivially holds. If $K \neq \emptyset$, then $\mathrm{verts}(K)$ is a finite non empty set. Thus, there exists $\mathbf{q} \in \mathbb{R}^n$ such that, for all $\mathbf{w} \in \mathrm{verts}(K)$, $\mathbf{q} \preceq \mathbf{w}$. Let $\mathbf{j} \in \mathrm{lk}^-_K(\mathbf{v})$. Then, $[\mathbf{v}-\mathbf{j}, \mathbf{v}] \subseteq K$ and so $\mathbf{v}-\mathbf{j} \in \mathrm{verts}(K)$. Hence, $\mathbf{q} \preceq \mathbf{v}-\mathbf{j}$, which means that $\mathbf{j} \in \mathrm{lk}^-_{K_{\mathbf{q}\preceq}}(\mathbf{v}) \subseteq \bigcup_{\mathbf{p}\in\mathbb{R}^n} \mathrm{lk}^-_{K_{\mathbf{p}\preceq}}(\mathbf{v})$. The reverse inclusion follows from the fact that each of these statements holds if and only if.

Statement *2*: Observe that if $\mathbf{j} \in \mathrm{lk}^-_{K'}(\mathbf{v})$, then, by definition of the past link, $[\mathbf{v}-\mathbf{j}, \mathbf{v}] \subseteq K'$. Since $K' \subseteq K$, we have $[\mathbf{v}-\mathbf{j}, \mathbf{v}] \subseteq K' \subseteq K$. Therefore, we can conclude that $\mathbf{j} \in \mathrm{lk}^-_K(\mathbf{v})$.

Statement *3*: By Statement *2* (which we just proved), we have the following inclusion $\mathrm{lk}^-_{K_{\preceq\mathbf{v}}}(\mathbf{v}) \subseteq \mathrm{lk}^-_K(\mathbf{v})$. To prove the inclusion in the other direction, let $\mathbf{j} \in \mathrm{lk}^-_K(\mathbf{v})$. Since $\mathbf{v}-\mathbf{j} \preceq \mathbf{j}$, then $[\mathbf{v}-\mathbf{j}, \mathbf{v}] \subseteq K_{\preceq\mathbf{v}}$. Therefore, we conclude that $\mathrm{lk}^-_K(\mathbf{v}) \subseteq \mathrm{lk}^-_{K_{\preceq\mathbf{v}}}(\mathbf{v})$.

Statement *4*: Let $\mathbf{v} = \max(\mathrm{verts}(K))$. Then, since K is full-dimensional, for all $\mathbf{j} \in \{0,1\}^n$, $[\mathbf{v}-\mathbf{j}, \mathbf{v}] \subseteq K$. Thus, by definition of past link, we have that the past link of v is: $\mathrm{lk}^-_K(\mathbf{v}) := \{0,1\}^n \setminus \{0\}^n$, which is the complete simplicial complex on n vertices. □

2.3 Relationship Between Past Links and Path Spaces

The topology of the past links is intrinsically related to the one of the spaces of directed paths. Specifically, in [1] we prove that the contractability and/or connectedness of past links of vertices in directed Euclidean cubical complexes with a minimum vertex[3] implies that all spaces of directed paths with $\mathbf{w}$ as initial point are also contractible and/or connected.

Theorem 1 (Contractability [1, Theorem 1])**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. If, for all vertices $\mathbf{v} \in \mathrm{verts}(K)$, the past link $\mathrm{lk}^-_K(\mathbf{v})$ is contractible, then the space $\vec{P}^{\mathbf{k}}_{\mathbf{w}}(K)$ is contractible for all $\mathbf{k} \in \mathrm{verts}(K)$.

An analogous theorem for connectedness also holds.

Theorem 2 (Connectedness [1, Theorem 2])**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. Suppose that, for all $\mathbf{v} \in \mathrm{verts}(K)$, the past link $\mathrm{lk}^-_K(\mathbf{v})$ is connected. Then, for all $\mathbf{k} \in \mathrm{verts}(K)$, the space $\vec{P}^{\mathbf{k}}_{\mathbf{w}}(K)$ is connected.

[3] In [1], the minimum (initial) vertex was often assumed to be $\mathbf{0}$ for ease of exposition. We restate the lemmas and theorems here using more general notation, where K has a minimum vertex $\mathbf{w}$.

Furthermore, we proved a partial converse to Theorem 2. Specifically, the converse holds only if K is a reachable directed Euclidean cubical complex as defined in Statement *3* of Definition 1. This is expected: Properties of a part of the directed Euclidean complex which are not reachable from $\mathbf{w}$, will not influence the path spaces from $\mathbf{w}$.

Theorem 3 (Realizing Obstructions [1, Theorem 3])**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\mathbf{w} \in \text{verts}(K)$, and let $L = \text{reach}(K, \mathbf{w})$. Let $\mathbf{v} \in \text{verts}(L)$. If the past link $\text{lk}^-_L(\mathbf{v})$ is disconnected, then the space $\vec{P}^{\mathbf{v}}_{\mathbf{w}}(K)$ is disconnected.

3 Directed Collapsing Pairs

Although simplicial collapses preserve the homotopy type of the underlying space [12, Proposition 6.14] and hence of all path spaces, this type of collapsing in directed Euclidean cubical complexes may not preserve topological properties of spaces of directed paths. In this section, we study a specific type of collapsing called a link-preserving directed collapse. We define link-preserving directed collapses in Section 3.1 and give properties of link-preserving directed collapses in Section 3.2.

3.1 Link-Preserving Directed Collapses

Since we are interested in preserving the directed path spaces through collapses, the results from Section 2.3 motivate us to study a type of directed collapse (DC) via past links, introduced in [1]. However, we will call it *link-preserving directed collapse* (LPDC) (as opposed to *directed collapse*) since we show in the last sections of this paper that when the spaces of directed paths starting from the minimum vertex are not connected, the following definition of collapse does not preserve the number of components.

Definition 3 (Link Preserving Directed Collapse)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\sigma \in \mathcal{K}$ be a maximal cube, and let τ be a proper face of σ such that no other maximal cube contains τ (in this case, we say that τ is a *free face* of σ). Then, we define the (τ, σ)-collapse of K as the subcomplex obtained by removing everything in between τ and σ:

$$K' = K \setminus \{\gamma \in \mathcal{K} \mid \overline{\tau} \subseteq \overline{\gamma} \subseteq \overline{\sigma}\}, \tag{2}$$

and let $\mathcal{K}'$ denote the stratification of the set K' induced by the cubical stratification of $\mathbb{R}^n$ (thus, $\mathcal{K}' \subsetneq \mathcal{K}$).

We call the directed Euclidean cubical complex $(K', \mathcal{K}')$ a *link-preserving directed collapse (LPDC)* of $(K, \mathcal{K})$ if, for all $\mathbf{v} \in \text{verts}(K')$, the past link $\text{lk}^-_K(\mathbf{v})$ is homotopy equivalent to $\text{lk}^-_{K'}(\mathbf{v})$ (denoted $\text{lk}^-_K(\mathbf{v}) \simeq \text{lk}^-_{K'}(\mathbf{v})$). The pair (τ, σ) is then called an *LPDC pair*.

Remark 1 (Simplicial Collapses)**.** The study of simplicial collapses is known as *simple homotopy theory* [3, 17], and traces back to the work of Whitehead in the 1930s [16]. The idea is very similar: If C is an abstract simplicial complex and $\alpha \in C$ such that α is a proper face of exactly one maximal simplex β, then the following complex is the *α-collapse of C in the simplicial setting*:

$$C' = C \setminus \{\gamma \in C \mid \alpha \subseteq \gamma \subseteq \beta\}.$$

Note that we use only the free face (α) when defining a simplicial collapse, as doing so helps to distinguish between discussing a simplicial collapse and a directed Euclidean cubical collapse. In addition, we always explicitly state "in the simplicial setting" when talking about a simplicial collapse.

Applying a sequence of LPDCs to a directed Euclidean cubical complex can reduce the number of cubes, and hence can more clearly illustrate the number of dihomotopy classes of directed paths within the directed Euclidean cubical complex. For an example, see Fig. 4. However, it is not necessarily true that LPDCs preserve directed path spaces. We discuss the relationship between directed path spaces and LPDCs in Section 4.

3.2 Properties of LPDCs

We give a combinatorial condition for a collapsing pair (τ, σ) to be an LPDC pair; namely, the condition is that τ does not contain the vertex $\min(\sigma)$. From the definition of an LPDC, we see that finding an LPDC pair requires computing the past link of *all* vertices in $\mathrm{verts}(K')$. In [1], we discussed how we can reduce the check down to only the vertices in σ since no other vertices have their past links affected. In this paper, we prove we need to only check *one* condition to determine if we have an LPDC pair. The one simple condition dramatically reduces the number of computations we need to perform in order to verify we have an LPDC. This result given in Theorem 4 depends on the following lemmas about the properties of past links on vertices.

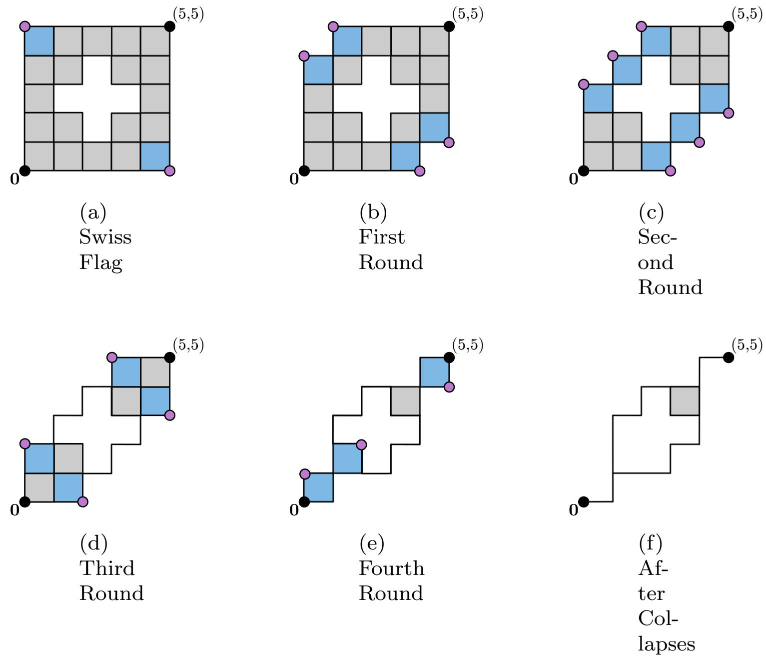
Lemma 2 (Properties of Past Links in a Vertex Collapse)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\sigma \in \mathcal{K}$ and $\tau, \mathbf{v} \in \mathrm{verts}(\sigma)$ such that $\tau \preceq \mathbf{v}$. If τ is a free face of σ and K' is the (τ, σ)-collapse, then the following two statements hold:

1. $\mathrm{lk}^-_{K|_\sigma}(\mathbf{v}) = \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \min(\sigma) \preceq \mathbf{v} - \mathbf{j}\}$.
2. $\mathrm{lk}^-_{K'|_\sigma}(\mathbf{v}) = \mathrm{lk}^-_{K|_\sigma}(\mathbf{v}) \setminus \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \mathbf{v} - \mathbf{j} \preceq \tau\}$.

Proof. To ease notation, we define the following two sets:

$$J := \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \min(\sigma) \preceq \mathbf{v} - \mathbf{j}\}$$
$$I := \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \mathbf{v} - \mathbf{j} \preceq \tau\}.$$

First, we prove Statement *1* (that $\mathrm{lk}^-_{K|_\sigma}(\mathbf{v}) = J$). We start with the forward inclusion. Let $\mathbf{j} \in \mathrm{lk}^-_{K|_\sigma}(\mathbf{v})$. By the definition of past links (see Definition 2), we know that $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K|_\sigma$. By the definition of $K|_\sigma$ (see Definition 1), $\min(\sigma) \preceq \min([\mathbf{v} - \mathbf{j}, \mathbf{v}]) = \mathbf{v} - \mathbf{j}$. This implies $\mathbf{j} \in J$. Therefore, $\mathrm{lk}^-_{K|_\sigma}(\mathbf{v}) \subseteq J$. For the backward inclusion, let $\mathbf{j} \in J$. Then, since $\mathbf{v} \in \mathrm{verts}(\sigma)$ and σ is an elementary cube by assumption, and $\min(\sigma) \preceq \mathbf{v} - \mathbf{j}$ by definition of J, we have $\mathbf{v} - \mathbf{j} \in \mathrm{verts}(\sigma)$. Since $\sigma \in \mathcal{K}$,



(a) Swiss Flag

(b) First Round

(c) Second Round

(d) Third Round

(e) Fourth Round

(f) After Collapses

Figure 4: Collapsing the Swiss Flag. A sequence of vertex collapses is presented from the top left to bottom right. At each stage, the faces and vertices shaded in blue and purple represent the vertex collapsing pairs with the blue Euclidean cube being σ and the purple vertex being τ. The result of the sequence of LPDCs is shown in (f) is a one-dimensional directed Euclidean cubical complex and one two-cube. Observe that this directed Euclidean cubical complex clearly illustrates the two dihomotopy classes of $\vec{P}_{\mathbf{0}}^{(5,5)}(K)$.

all faces must be in $\mathcal{K}$; hence, $[\mathbf{v}-\mathbf{j},\mathbf{v}] \subseteq K|_\sigma$. Therefore, $\mathbf{j} \in \mathrm{lk}^-_{K|_\sigma}(\mathbf{v})$, and so $\mathrm{lk}^-_{K|_\sigma}(\mathbf{v}) \supseteq J$. Since we have both inclusions, then Statement *1* holds.

Now, we prove Statement *2* (that $\mathrm{lk}^-_{K'|_\sigma}(\mathbf{v}) = J \setminus I$). Again, we prove the inclusions in both directions. For the forward inclusion, let $\mathbf{j} \in \mathrm{lk}^-_{K'|_\sigma}(\mathbf{v})$. By Statement *2* of Lemma 1, we have $\mathrm{lk}^-_{K'|_\sigma}(\mathbf{v}) \subseteq \mathrm{lk}^-_{K|_\sigma}(\mathbf{v})$, and so $\mathbf{j} \in \mathrm{lk}^-_{K|_\sigma}(\mathbf{v}) = J$. Next, we must show that $\mathbf{j} \notin I$. Assume, for a contradiction, that $\mathbf{j} \in I$. Then, by definition of I, $\mathbf{v}-\mathbf{j} \preceq \tau$. Since $\tau \preceq \mathbf{v}$, we obtain the partial order $\mathbf{v}-\mathbf{j} \preceq \tau \preceq \mathbf{v}$. This implies that $[\tau,\mathbf{v}] \subseteq [\mathbf{v}-\mathbf{j},\mathbf{v}]$. Since $[\mathbf{v}-\mathbf{j},\mathbf{v}]$ is an elementary cube in $K'|_\sigma$, then its face $[\tau,\mathbf{v}]$ must also be an elementary cube in $K'|_\sigma$. Setting $\overline{\gamma} = [\tau,\mathbf{v}]$ and observing $\tau = \overline{\tau} \subseteq \overline{\gamma} \subseteq \overline{\sigma}$, we observe that γ is not an elementary cube in K' by Eq. (2). This gives us a contradiction and so $\mathbf{j} \notin I$. Therefore,

$\mathrm{lk}^-_{K'|_\sigma}(\mathbf{v}) \subseteq J \setminus I$.

Finally, we prove the backward inclusion of Statement *2*. Let $\mathbf{j} \in J \setminus I$. Then, by Statement *1*, $\mathbf{j} \in \mathrm{lk}^-_{K|_\sigma}(\mathbf{v})$ and either $\tau \prec \mathbf{v} - \mathbf{j}$ or τ is not comparable to $\mathbf{v} - \mathbf{j}$ under $\preceq$. Thus, by Eq. (2), $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ is an elementary cube of $K'|_\sigma$. Thus, by Definition 2, we have that $\mathbf{j} \in \mathrm{lk}^-_{K'|_\sigma}(\mathbf{v})$. Hence, $J \setminus I \subseteq \mathrm{lk}^-_{K'|_\sigma}(\mathbf{v})$, and so Statement *2* holds. $\square$

Using Lemma 2, we see why τ cannot be the vertex $\min(\sigma)$ when performing an LPDC. If $\tau = \min(\sigma)$, then

$$\begin{aligned}
\mathrm{lk}^-_{K'|_\sigma}(\mathbf{v}) &= \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \min(\sigma) \preceq \mathbf{v} - \mathbf{j}\} \\
&\quad \setminus \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \mathbf{v} - \mathbf{j} \preceq \min(\sigma)\} \\
&= \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \min(\sigma) \preceq \mathbf{v} - \mathbf{j} \text{ and } \mathbf{v} - \mathbf{j} \succ \min(\sigma)\} \\
&= \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \min(\sigma) \prec \mathbf{v} - \mathbf{j}\} \\
&= \{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \mathbf{j} \prec \mathbf{v} - \min(\sigma)\}
\end{aligned}$$

If $\mathbf{v}$ is the maximum vertex of σ, then we obtain $\mathrm{lk}^-_{K'|_\sigma}(\mathbf{v}) = \{0,1\}^n \setminus \{\{0\}^n, \mathbf{v} - \min(\sigma)\}$. This computation gives us the following corollary, which we illustrate in Fig. 5 when K is a single closed three-cube.

Corollary 1 (Caution for a $(\min(\sigma), \sigma)$-Collapse)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\sigma \in \mathcal{K}$, $\tau = \min(\sigma)$, and $\mathbf{v} \in \mathrm{verts}(\sigma)$. If τ is a free face and K' is the (τ, σ)-collapse. Then, the past link of $\mathbf{v}$ in $K'|_\sigma$ is:

$$\{\mathbf{j} \in \{0,1\}^n \setminus \{0\}^n \mid \mathbf{j} \prec \mathbf{v} - \min(\sigma)\}$$

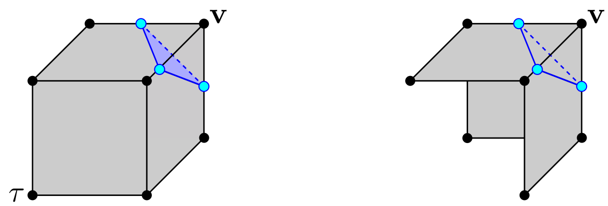
In particular, if $\mathbf{v} = \max(\sigma)$ and $k = \dim(\sigma)$, then the past link is the complete complex on k elements before the collapse, and, after the collapse, it is homeomorphic to $\mathbb{S}^{k-2}$. Thus, (τ, σ) is not an LPDC pair.

The following lemma shows under which condition a directed Euclidean cubical collapse induces a simplicial collapse in the past link.

Lemma 3 (Vertex Collapse Induces Simplicial Collapse)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\sigma \in \mathcal{K}$ and $\tau, \mathbf{v} \in \mathrm{verts}(\sigma)$ such that $\tau \preceq \mathbf{v}$ and $\tau \neq \min(\sigma)$. If τ is a free face of σ and K' is the (τ, σ)-collapse, then $\mathrm{lk}^-_{K'}(\mathbf{v})$ is the $(\mathbf{v} - \tau)$-collapse of $\mathrm{lk}^-_K(\mathbf{v})$ in the simplicial setting.

Proof. Consider $K_{\preceq \mathbf{v}}$. Since $\tau, \mathbf{v} \in \mathrm{verts}(\sigma)$ and σ is maximal in K, we know $[\min(\sigma), \mathbf{v}]$ and $[\tau, \mathbf{v}]$ are elementary cubes in $K_{\preceq \mathbf{v}}$. Since τ is a free face of σ, we further know that $[\min(\sigma), \mathbf{v}]$ is the only maximal proper coface of $[\tau, \mathbf{v}]$ in $K_{\preceq \mathbf{v}}$. By definition of past link (Definition 2), we then have that $\mathbf{v} - \min(\sigma)$ and $\mathbf{v} - \tau$ are simplices in $\mathrm{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$, and $\mathbf{v} - \min(\sigma)$ is the only maximal proper coface of $\mathbf{v} - \tau$ in $\mathrm{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$. Hence, $\mathbf{v} - \tau$ is free in $\mathrm{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$. Moreover, $\mathrm{lk}^-_{K'_{\preceq \mathbf{v}}}(\mathbf{v})$ is the $(\mathbf{v} - \tau)$-collapse of $\mathrm{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$. One can see this by using Statement *2* of Lemma 2 by which $\mathrm{lk}^-_{K'_{\preceq \mathbf{v}}}(\mathbf{v})$ can be characterized as the $(\mathbf{v} - \tau)$-collapse of $\mathrm{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$.

By Statement *3* of Lemma 1, we know $\mathrm{lk}^-_K(\mathbf{v}) = \mathrm{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$ and $\mathrm{lk}^-_{K'}(\mathbf{v}) = \mathrm{lk}^-_{K'_{\preceq \mathbf{v}}}(\mathbf{v})$, which concludes this proof. $\square$



(a) Initial Complex

(b) After Collapse

Figure 5: Removing the minimum vertex of a cube. Consider the directed Euclidean cubical complex in (a), which as a subset of $\mathbb{R}^3$ is a single closed three-cube; call this three-cube σ. Letting $\tau = \min(\sigma)$, we observe that the past link of $\mathbf{v} = \max(\sigma)$ is contractible before the (τ, σ)-collapse and is homeomorphic to $\mathbb{S}^1$ after the collapse. Thus, the past links before and after the collapse are not homotopy equivalent, and so this collapse is not an LPDC.

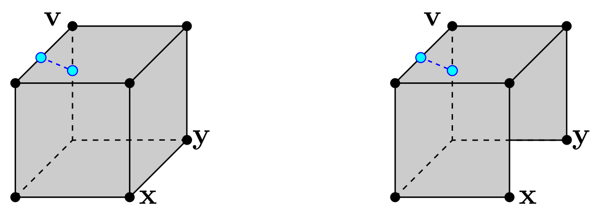
Next, we prove two lemmas concerning relationships of the past link of a vertex in the original directed Euclidean cubical complex and in the collapsed directed Euclidean cubical complex. These relationships depend on where $\mathbf{v}$ is located with respect to τ. In the first lemma, we consider the case where $\min(\tau) \not\preceq \mathbf{v}$, and we present a sufficient condition for past links in K and the (τ, σ)-collapse to be equal. See Fig. 6 for an example that illustrates the result of this lemma.

Lemma 4 (Condition for Past Links in K and K' to be Equal)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\tau, \sigma \in \mathcal{K}$ such that τ is a face of σ. If τ is a free face of σ and K' is the (τ, σ)-collapse, then, for all $\mathbf{v} \in \text{verts}(K)$ such that $\max(\tau) \not\preceq \mathbf{v}$, we have $\text{lk}^-_K(\mathbf{v}) = \text{lk}^-_{K'}(\mathbf{v})$.

Proof. By Statement *2* of Lemma 1, we have $\text{lk}^-_{K'}(\mathbf{v}) \subseteq \text{lk}^-_K(\mathbf{v})$. Thus, we only need to show $\text{lk}^-_K(\mathbf{v}) \subseteq \text{lk}^-_{K'}(\mathbf{v})$. Suppose $\mathbf{j} \in \text{lk}^-_K(\mathbf{v})$. By the definition of the past link (see Definition 2), we know that $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ is an elementary cube in K. By assumption, $\max(\tau) \not\preceq \mathbf{v}$. Thus, by Eq. (2), $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ is not removed from K and thus is an elementary cube in K'. Thus, $\mathbf{j} \in \text{lk}^-_{K'}(\mathbf{v})$. $\square$

In the following lemma, we consider the case where $\max(\tau) \preceq \mathbf{v}$, and we present a sufficient condition for past links in the (τ, σ)-collapse and the $(\min(\tau), \sigma)$-collapse to be equal. See Fig. 7 for an example that illustrates this result.

Lemma 5 (Condition for Past Links in K' and $\widehat{K}$ to be Equal)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ such that there exists cubes $\tau, \sigma \in \mathcal{K}$ with $\min(\tau)$ a free face of σ. Let K' be the (τ, σ)-



(a) Initial Complex

(b) After Collapse

Figure 6: Past link of an "uncomparable" vertex before and after a collapse. Consider the directed Euclidean cubical complex shown, comprising a single three-cube σ and all of its faces. Let $\tau = [\mathbf{x}, \mathbf{y}]$. Since $\mathbf{v}$ and $\max(\tau) = \mathbf{y}$ are not comparable, by Lemma 4, the past link of $\mathbf{v}$ is the same before and after the collapse. Indeed, we see that this is the case for this example. The past link of $\mathbf{v}$ is the complete complex on two vertices both before and after the collapse.

collapse and let $\widehat{K}$ be the $(\min(\tau), \sigma)$-collapse. If $\mathbf{v} \in \text{verts}(K')$ and $\max(\tau) \preceq \mathbf{v}$, then $\mathbf{v} \in \text{verts}(\widehat{K})$ and $\text{lk}^-_{K'}(\mathbf{v}) = \text{lk}^-_{\widehat{K}}(\mathbf{v})$.

Proof. We first show $\mathbf{v} \in \text{verts}(\widehat{K})$. If τ is a zero-cube (and hence in $\text{verts}(K)$), then $K' = \widehat{K}$, which means that $\mathbf{v} \in \text{verts}(\widehat{K})$. On the other hand, if τ is not a zero-cube, then we have $\min(\tau) \prec \max(\tau) \preceq \mathbf{v}$. In particular, $\min(\tau) \neq \mathbf{v}$. Thus, $\overline{\min(\tau)} \not\subseteq \overline{\mathbf{v}}$. And so, by definition of $\widehat{K}$ as a $(\min(\tau), \sigma)$-collapse and since $\mathbf{v} \in \mathcal{K}$, we conclude that $\mathbf{v} \in \widehat{K}$.

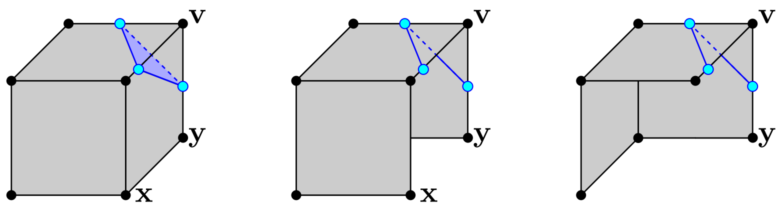
Next, we show $\text{lk}^-_{K'}(\mathbf{v}) = \text{lk}^-_{\widehat{K}}(\mathbf{v})$. By Statement *2* of Lemma 1, we have $\text{lk}^-_{\widehat{K}}(\mathbf{v}) \subseteq \text{lk}^-_{K'}(\mathbf{v})$. Thus, what remains to be proven is $\text{lk}^-_{K'}(\mathbf{v}) \subseteq \text{lk}^-_{\widehat{K}}(\mathbf{v})$. Let $\mathbf{j} \in \text{lk}^-_{K'}(\mathbf{v})$. By definition of the past link (Definition 2), we know that $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K'$. Consider two cases: $\mathbf{v} - \mathbf{j} \preceq \min(\tau)$ and $\mathbf{v} - \mathbf{j} \not\preceq \min(\tau)$.

Case 1 ($\mathbf{v} - \mathbf{j} \preceq \min(\tau)$): Since $\mathbf{v} - \mathbf{j} \preceq \min(\tau) \preceq \max(\tau) \preceq \mathbf{v}$, we know that $\overline{\tau} \subseteq [\mathbf{v} - \mathbf{j}, \mathbf{v}]$. Thus, by Eq. (2), we have $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \not\subseteq K'$, which is a contradiction. So, Case 1 cannot happen.

Case 2 ($\mathbf{v} - \mathbf{j} \not\preceq \min(\tau)$): If $\mathbf{v} - \mathbf{j} \not\preceq \min(\tau)$, then, by the definition of a $(\min(\tau), \sigma)$-collapse in Definition 3, we know $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq \widehat{K}$ and thus $\mathbf{j} \in \text{lk}^-_{\widehat{K}}(\mathbf{v})$.

Hence, $\text{lk}^-_{K'}(\mathbf{v}) \subseteq \text{lk}^-_{\widehat{K}}(\mathbf{v})$. Since we have both subset inclusions, we conclude $\text{lk}^-_{K'}(\mathbf{v}) = \text{lk}^-_{\widehat{K}}(\mathbf{v})$. $\square$

The next result states that vertex collapses result in homotopy equivalent past links as long as we are not collapsing the minimum vertex of the directed Euclidean cubical complex.



(a) Original Complex

(b) The (τ, σ)-Collapse

(c) The $(\mathbf{x}, \sigma)$-Collapse

Figure 7: Two collapses with same past links. For example, in the directed Euclidean cubical complex K shown in (a), let σ be the three-cube, and let $\tau = [\mathbf{x}, \mathbf{y}]$. We look at the past link of the vertex $\mathbf{v}$. In the original directed Euclidean cubical complex, the past link of $\mathbf{v}$ is the complete complex on three vertices. By Lemma 5, the past link of $\mathbf{v}$ is the same in both the (τ, σ)-collapse and the $(\mathbf{x}, \sigma)$-collapse since $\max(\tau) = \mathbf{y} \preceq \mathbf{v}$. By Lemma 6, we also know that the past links of $\mathbf{v}$ in K and the $(\mathbf{x}, \sigma)$-collapse are homotopy equivalent. Indeed, we see that this is the case.

Lemma 6 (Past Links in a Vertex Collapse)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\sigma \in \mathcal{K}$ and let $\tau \in \text{verts}(\sigma)$ such that $\tau \neq \min(\sigma)$. Let $\mathbf{v} \in \text{verts}(K)$ with $\mathbf{v} \neq \tau$ If τ is a free face of σ and K' is the (τ, σ)-collapse, then $\text{lk}^-_K(\mathbf{v}) \simeq \text{lk}^-_{K'}(\mathbf{v})$.

Proof. We consider three cases:

Case 1 ($\mathbf{v} \notin \text{verts}(\sigma)$): By definition of past link (Definition 2), if $\mathbf{v} \notin \text{verts}(\sigma)$, then the past links $\text{lk}^-_K(\mathbf{v})$ and $\text{lk}^-_{K'}(\mathbf{v})$ are equal.

Case 2 ($\tau \not\preceq \mathbf{v}$): By Lemma 4, if $\tau = \max(\tau) \not\preceq \mathbf{v}$, again we have equality of the past links $\text{lk}^-_K(\mathbf{v})$ and $\text{lk}^-_{K'}(\mathbf{v})$.

Case 3 ($\mathbf{v} \in \text{verts}(\sigma)$ and $\tau \preceq \mathbf{v}$): By Lemma 3, we know that $\text{lk}^-_{K'}(\mathbf{v})$ is the τ-collapse of $\text{lk}^-_K(\mathbf{v})$ in the simplicial setting. Since simplicial collapses preserve the homotopy type (see e.g., [12, Proposition 6.14]), we conclude $\text{lk}^-_K(\mathbf{v}) \simeq \text{lk}^-_{K'}(\mathbf{v})$.

□

We give an example of Lemma 6 in Fig. 7 by showing how the LPDC induces a simplicial collapse on past links.

Lastly, we are ready to prove the main result.

Theorem 4 (Main Theorem)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ such that there exist cubes $\tau, \sigma \in \mathcal{K}$ with τ a free face of σ. Then, (τ, σ) is an LPDC pair if and only if $\min(\sigma) \notin \text{verts}(\tau)$.

Proof. Let $(K', \mathcal{K}')$ be the (τ, σ)-collapse. Let $(L, \mathcal{L})$ be the cubical complex such that $L = K|_\sigma$. Since $\sigma \in K$, we know $L = \overline{\sigma}$ (i.e., L is a unit cube). Since L is a single unit cube and σ is the maximal elementary cube, all proper faces of σ, including τ and $\min(\tau)$, are free faces. Thus, let $(L', \mathcal{L}')$ be the (τ, σ)-collapse of L, and let $(\widehat{L}, \widehat{\mathcal{L}})$ be the $(\min(\tau), \sigma)$-collapse of L.

We first prove the forward direction by contrapositive (if $\min(\sigma) \notin \text{verts}(\tau)$, then (τ, σ) is not an LPDC pair. Assume $\min(\sigma) \in \text{verts}(\tau)$. Since τ is a face of σ, we know $\max(\tau) \preceq \max(\sigma)$. Since $\min(\sigma) \in \text{verts}(\tau)$ and since $\dim(\sigma) = \dim(\tau) + 1$, we know that $\max(\sigma) \neq \max(\tau)$. Thus, $\tau \neq \min(\sigma)$ and so $\min(\sigma) \in \text{verts}(L')$. Applying Lemma 5 using $\mathbf{v} = \max(\sigma)$, we obtain $\text{lk}^-_{L'}(\mathbf{v}) = \text{lk}^-_{\widehat{L}}(\mathbf{v})$. Let $d = \dim(\sigma)$. By Corollary 1, we obtain $\text{lk}^-_{L}(\mathbf{v})$ is homeomorphic to $\mathbb{B}^{d-1}$ and $\text{lk}^-_{\widehat{L}}(\mathbf{v})$ is homeomorphic to $\mathbb{S}^{d-2}$. Thus,

$$\text{lk}^-_{L}(\mathbf{v}) \simeq \mathbb{B}^{d-1} \not\simeq \mathbb{S}^{d-2} \simeq \text{lk}^-_{\widehat{L}}(\mathbf{v}) = \text{lk}^-_{L'}(\mathbf{v}),$$

and so $\text{lk}^-_{L}(\mathbf{v}) \not\simeq \text{lk}^-_{L'}(\mathbf{v})$. Since $\mathcal{K} \setminus \mathcal{L}$ and $\mathcal{K}' \setminus \mathcal{L}'$ are sets of faces of σ, the past link of $\mathbf{v}$ remains the same outside of L in both K and K'.Thus, $\text{lk}^-_{K}(\mathbf{v}) \not\simeq \text{lk}^-_{K'}(\mathbf{v})$ and so we conclude that (τ, σ) is not an LPDC pair, as was to be shown.

Next, we show the backwards direction. Suppose $\min(\sigma) \notin \text{verts}(\tau)$. Let $\mathbf{v} \in \text{verts}(K')$, and consider two cases: $\max(\tau) \not\preceq \mathbf{v}$ and $\max(\tau) \preceq \mathbf{v}$.

Case 1 ($\max(\tau) \not\preceq \mathbf{v}$): By Lemma 4, we have $\text{lk}^-_{K}(\mathbf{v}) = \text{lk}^-_{K'}(\mathbf{v})$. Hence, $\text{lk}^-_{K}(\mathbf{v}) \simeq \text{lk}^-_{K'}(\mathbf{v})$. Since $\mathbf{v}$ was arbitrarily chosen, we conclude that (τ, σ) is an LPDC pair.

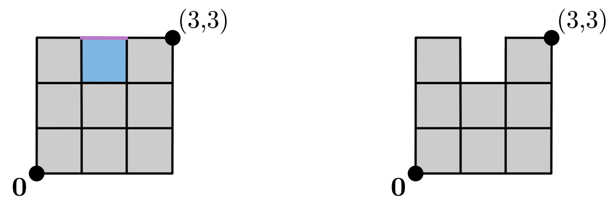
Case 2 ($\max(\tau) \preceq \mathbf{v}$): By Lemma 5, we have that $\text{lk}^-_{L'}(\mathbf{v}) = \text{lk}^-_{\widehat{L}}(\mathbf{v})$. Since $\min(\sigma) \notin \text{verts}(\tau)$, we know that $\min(\tau) \neq \min(\sigma)$. Applying Lemma 6, we obtain $\text{lk}^-_{L}(\mathbf{v}) \simeq \text{lk}^-_{\widehat{L}}(\mathbf{v})$. Again, since $\mathcal{K} \setminus \mathcal{L}$ and $\mathcal{K}' \setminus \mathcal{L}'$ are sets of faces of σ, the past link of $\mathbf{v}$ remains the same outside of L in both K and K'.Thus, $\text{lk}^-_{K}(\mathbf{v}) \simeq \text{lk}^-_{K'}(\mathbf{v})$. Since $\mathbf{v}$ was arbitrarily chosen, we conclude that (τ, σ) is an LPDC pair. $\square$

4 Preservation of Spaces of Directed Paths

We focus on LPDCs because of the relationships between past links and spaces of directed paths that were proved in [1]. We explain how those relationships extend to the LPDC setting in this section.

One result from [1](Theorem 1) states that for a directed Euclidean cubical complex with a minimum vertex, if all past links are contractible, then all spaces of directed paths starting at that minimum vertex are also contractible. If we start with a directed Euclidean cubical complex with a minimum vertex that has all contractible past links, then all spaces of directed paths from the minimum vertex are contractible by this theorem. Applying an LPDC will preserve the homotopy type of past links by definition. Hence, applying the theorem again, we see that any LPDC will also have contractible directed path spaces from the minimum vertex. Notice that the minimum vertex is not removed in an LDPC, since it is a vertex and minimal in all cubes containing it (including the maximal cube). We give an example of this in Example 1.

Example 1 (3×3 filled grid)**.** Let K be the 3×3 filled grid. For all $\mathbf{v} \in \text{verts}(K)$, $\text{lk}^-_{K}(\mathbf{v})$ is contractible. By Theorem 1, this implies that all spaces of directed paths starting at $\mathbf{0}$ are contractible. Applying an LPDC such as the edge $[(1,3),(2,3)]$ results in contractible past links in K' and so all spaces of directed paths in K' are also contractible. See Fig. 8. We can generalize this example to any k^d filled grid where $k, d \in \mathbb{N}$.



(a) Initial Complex

(b) After Collapse

Figure 8: (a) The 3×3 filled grid has contractible past links and directed path spaces. The pair comprising of the purple edge $[(1,3),(2,3)]$ and the blue square $[(1,2),(2,3)]$ is an LPDC pair. (b) The result of performing the LPDC. All past links are contractible and so all directed path spaces are also contractible.

An analogous result holds for connectedness (Theorem 2). If we start with a directed Euclidean cubical complex that has all connected past links, then all directed path spaces are connected. Any LPDC will result in a directed Euclidean cubical complex that also has connected directed path spaces. See Example 2.

Example 2 (Outer Cubes of the $5 \times 5 \times 5$ Grid)**.** Let $K = [0,5]^3 \setminus [1,4]^3$, which, as an undirected complex, is homeomorphic to a thickened two-sphere. For all $\mathbf{v} \in \text{verts}(K)$, $\text{lk}^-_K(\mathbf{v})$ is connected. By Theorem 2, this implies that for all $\mathbf{v} \in \text{verts}(K)$, the space of directed paths $\vec{P}^{\mathbf{v}}_{\mathbf{0}}(K)$ is connected. Applying an LPDC such as with the vertex $(5,0,0)$ in the cube $[(4,0,0),(5,1,1)]$ results in connected past links in K' and so all spaces of directed paths $\vec{P}^{\mathbf{v}}_{\mathbf{0}}(K')$ are connected. We can generalize this example to any k^d grid where $d \geq 3$ and the inner cubes of dimension d are removed.

Both Theorem 1 and Theorem 2 have assumptions on the topology of past links and results on the topology of spaces of directed paths from the minimum vertex. We may ask if the converse statements are true. Does knowing the topology of spaces of directed paths from the minimum vertex tell us anything about the topology of past links? The converse to Theorem 1 holds. To prove this, we first need a lemma whose proof appears in [18].

Lemma 7 (Homotopy Equivalence [18, Prop. 5.3])**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$. If $\vec{P}^{\mathbf{q}-\mathbf{j}}_{\mathbf{p}}(K)$ is contractible for all $\mathbf{j} \in \text{lk}^-_K(\mathbf{q})$, then $\vec{P}^{\mathbf{q}}_{\mathbf{p}}(K) \simeq \text{lk}^-_{K_{\mathbf{p}\preceq}}(\mathbf{q})$.

Thus, we obtain:

Theorem 5 (Contractability)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. The following two statements are equivalent:

1. For all $\mathbf{v} \in \text{verts}(K)$, the space of directed paths $\vec{P}^{\mathbf{v}}_{\mathbf{w}}(K)$ is contractible.
2. For all $\mathbf{v} \in \text{verts}(K)$, the past link $\text{lk}^-_K(\mathbf{v})$ is contractible.

Proof. By Theorem 1, we obtain Statement *2* implies Statement *1*.

Next, we show that Statement *1* implies Statement *2*. Let $\mathbf{v} \in \text{verts}(K)$. For all $\mathbf{j} \in \text{lk}^-_K(\mathbf{v})$, the cube $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K$, which means that $\mathbf{v} - \mathbf{j} \in \text{verts}(K)$. Thus, by assumption, all directed path spaces $\vec{P}^{\mathbf{v}-\mathbf{j}}_{\mathbf{w}}(K)$ are contractible. By Lemma 7, we know that $\vec{P}^{\mathbf{v}}_{\mathbf{w}}(K) \simeq \text{lk}^-_{K_{\mathbf{w}\preceq}}(\mathbf{v}) = \text{lk}^-_K(\mathbf{v})$. Again, since $\mathbf{v} \in \text{verts}(K)$, the path space $\vec{P}^{\mathbf{v}}_{\mathbf{w}}(K)$ is contractible. Therefore, $\text{lk}^-_K(\mathbf{v})$ is contractible. □

As a consequence of this theorem, we know that if we start with a directed Euclidean cubical complex with contractible directed path spaces starting at the minimum vertex, then any LPDC will also result in a directed Euclidean cubical complex with all contractible directed path spaces starting at the minimum vertex, and vice versa.

Corollary 2 (Preserving Directed Path Space Contractability)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. Let $\tau, \sigma \in \mathcal{K}$ such that τ is a face of σ. If τ is a free face of σ, let $(K', \mathcal{K}')$ be the (τ, σ)-collapse. If K' is an LPDC of K, then the spaces of directed paths $\vec{P}^{\mathbf{v}}_{\mathbf{w}}(K)$ are contractible for all $\mathbf{v} \in \text{verts}(K)$ if and only if the spaces of directed paths $\vec{P}^{\mathbf{k}}_{\mathbf{w}}(K')$ are contractible for all $\mathbf{k} \in \text{verts}(K')$.

Proof. We start with the forwards direction by assuming that the spaces of directed paths $\vec{P}^{\mathbf{v}}_{\mathbf{w}}(K)$ are contractible for all $\mathbf{v} \in \text{verts}(K)$. Theorem 5 tells us that all past links $\text{lk}^-_K(\mathbf{v})$ are contractible for all $\mathbf{v} \in \text{verts}(K)$. This implies that $\text{lk}^-_{K'}(\mathbf{k})$ is contractible for all $\mathbf{k} \in \text{verts}(K')$ because K' is an LPDC of K. Applying Theorem 5 again, we see that all spaces of directed paths $\vec{P}^{\mathbf{k}}_{\mathbf{w}}(K')$ are contractible for all $\mathbf{k} \in \text{verts}(K')$.

Next we prove the backwards direction by assuming that the spaces of directed paths $\vec{P}^{\mathbf{k}}_{\mathbf{w}}(K')$ are contractible for all $\mathbf{k} \in \text{verts}(K')$. Let $\mathbf{v} \in \text{verts}(K)$. Either $\mathbf{v} \in \text{verts}(K')$ or $\mathbf{v} \notin \text{verts}(K')$.

Case 1 ($\mathbf{v} \in \text{verts}(K')$): By Theorem 5, we know that $\text{lk}^-_{K'}(\mathbf{v})$ is contractible. Since K' is an LPDC of K, then $\text{lk}^-_K(\mathbf{v})$ is also contractible.

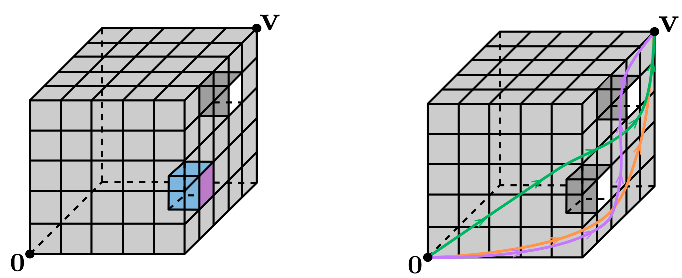
Case 2 ($\mathbf{v} \notin \text{verts}(K')$): If $\mathbf{v} \notin \text{verts}(K)$, then τ is a vertex and $\mathbf{v} = \tau$. Observe that $\text{lk}^-_{\overline{\sigma}}(\tau)$ is contractible since $\overline{\sigma}$ is an elementary cube and τ does not contain $\min(\sigma)$. Furthermore, notice that $\text{lk}^-_K(\tau) = \text{lk}^-_{\sigma}(\tau)$ because τ is a free face of σ. Hence, $\text{lk}^-_K(\tau)$ is contractible.

Therefore $\text{lk}^-_K(\mathbf{v})$ is contractible for all $\mathbf{v} \in \text{verts}(K)$. Applying Theorem 5, we get that $\vec{P}^{\mathbf{v}}_{\mathbf{w}}(K)$ is contractible for all $\mathbf{v} \in \text{verts}(K)$. □

Using Theorem 2, the partial converse to the connectedness theorem [1, Theorem 3], we get that any LPDC of a directed Euclidean cubical complex with connected directed path spaces and reachable vertices will result in a directed Euclidean cubical complex with connected directed path spaces.

Corollary 3 (Condition for LPDCs to Preserve Connectedness of All Directed Path Spaces)**.** Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. Let $(L, \mathcal{L}) = \text{reach}(K, \mathbf{w})$. Let (τ, σ) be an LPDC pair in L, and let L' be the (τ, σ)-collapse. The spaces of directed paths in $\overrightarrow{P}^{\mathbf{k}}_{\mathbf{w}}(L)$ are connected for all $\mathbf{v} \in \text{verts}(L)$ if and only if the spaces of directed paths $\overrightarrow{P}^{\mathbf{v}}_{\mathbf{w}}(L')$ are connected for all $\mathbf{v} \in \text{verts}(L')$.

We note that reachability is a necessary condition. Below we give an example of a directed Euclidean cubical complex K that has all connected directed path spaces but an LPDC yields a directed Euclidean cubical complex with a disconnected path space.



(a) Original Complex

(b) After Collapse

Figure 9: The bowling ball before and after the collapse described in Example 3. Observe $\overrightarrow{P}^{(5,5,5)}_{\mathbf{0}}(K)$ has one connected component. Additionally, $\sigma = [(4,1,1),(5,2,2)]$ (highlighted in purple) and $\tau = [(5,1,1),(5,2,2)]$ (highlighted in blue) is an LPDC pair. After collapsing (τ, σ), $\overrightarrow{P}^{(5,5,5)}_{\mathbf{0}}(K)$ changes from having one connected component to three connected components. The three connected components are represented by the three directed paths.

Example 3 (Bowling Ball)**.** Let K be the boundary of the $5 \times 5 \times 5$ grid union $[(4,1,1),(5,2,2)]$ and $[(4,3,3),(5,4,4)] \setminus [(5,3,3),(5,4,4)]$. See Fig. 9(a). Notice that some vertices of K are unreachable. Furthermore, all past links of vertices in K are connected and so all directed path spaces starting at $\mathbf{0}$ are also connected. After performing an LPDC with $\tau = [(5,1,1),(5,2,2)]$ and $\sigma = [(4,1,1),(5,2,2)]$, the directed path space between $\mathbf{0}$ and $(5,5,5)$ changes from having one connected component to three connected components, as shown in the figure. This example shows that the reachability condition in Corollary 3 is necessary for preserving connnectedness in LPDCs.

LPDCs can also preserve dihomotopy classes of directed paths starting at the minimum vertex of many directed Euclidean cubical complexes that have disconnected past links. Recall the Swiss flag as discussed in Fig. 4. The Swiss flag has disconnected past links at $(3, 4)$ and $(4, 3)$, yet there exists a sequence of LPDCs that results in a directed Euclidean cubical complex that highlights the two dihomotopy classes of directed paths between $\mathbf{0}$ and (5.5). Example 4 gives another similar situation.

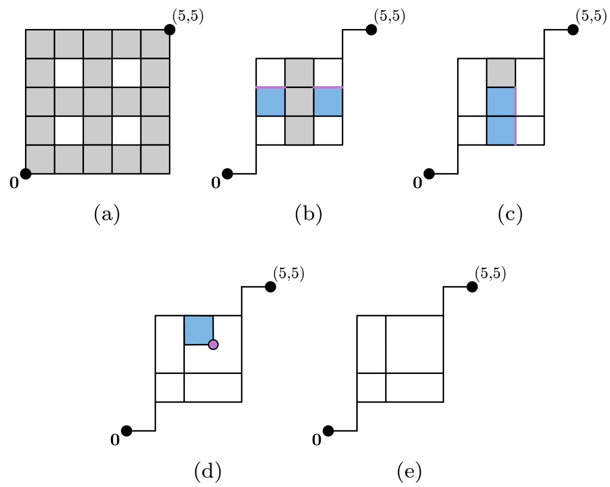


Figure 10: Link-preserving DCs of the window. A sequence of LPDCs is presented from (a)-(e). The directed Euclidean cubical complex in (b) comes from performing several vertex LPDCs to remove the two-cubes along the border of K. In (b)-(d), the LPDC pairs (τ, σ) are highlighted in purple and blue respectively. The result of the sequence of LPDCs is a graph of vertices and edges that more clearly illustrates the dihomotopy classes of directed paths in the directed path space.

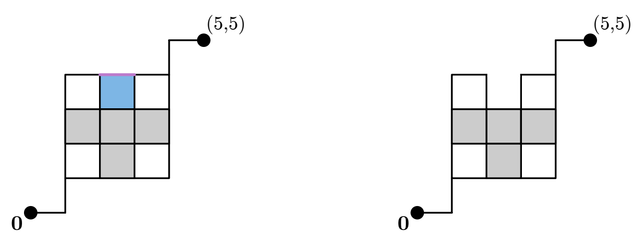
Example 4 (Window)**.** Let K be the 5×5 grid with the following two-cube interiors removed: $[(1, 1), (2, 2)]$, $[(3, 1), (4, 2)]$, $[(1, 3), (2, 4)]$, $[(3, 3), (4, 4)]$. See Fig. 10(a). K has disconnected past links at the vertices $(2, 2)$, $(4, 2)$, $(2, 4)$, $(4, 4)$ so K does not satisfy Corollary 2 or Corollary 3. Observe that $\overrightarrow{P}(K)_{\mathbf{0}}^{(5,5)}$ has six connected components. We can perform a sequence of LPDCs that preserves the dihomotopy classes of directed paths between $\mathbf{0}$ and $(5, 5)$ at each step. First, we apply vertex LPDCs to remove the two-cubes along the border. Then we can apply four edge LPDCs and one vertex LPDC to get a graph of vertices and edges. This graph more clearly illustrates the six dihomotopy classes of directed paths in $\overrightarrow{P}(K)_{\mathbf{0}}^{(5,5)}$.

In summary, LPDCs preserve the connectedness and/or contractability of directed path spaces starting at the minimum vertex as long as K has all reachable vertices and all directed path spaces starting at the minimum vertex in K connected and/or contractible to begin with. If K does not have these properties, the first step could be to remove all unreachable vertices and cubes before collapsing. In the next section, it will become clear that this will not suffice, if the path spaces are not all connected or contractible.

5 Discussion

LPDCs preserve spaces of directed paths in many examples (see Section 4), in particular, if they are all trivial in the sense of either all connected or all contractible and the directed Euclidean cubical complex is reachable for the minimum. However, LPDCs do not always preserve spaces of directed paths. We discuss some of those instances here. One limitation of LPDCs is that the number of components may increase after an LPDC as we saw in Example 3 or, as we will see in Example 5 they may decrease.

Example 5 (Window. LPDC that does not Preserve Directed Path Spaces)**.** Consider K as given in Example 4. After applying vertex LPDCs that remove the two-cubes on the border of K, we can apply an LPDC to the edge $[(2,4),(3,4)]$. Now $\overrightarrow{P}(K')_{\mathbf{0}}^{(5,5)}$ has five connected components; whereas, the path space $\overrightarrow{P}(K)_{\mathbf{0}}^{(5,5)}$ has six connected components. See Fig. 11. This example shows that there are both "good" and "bad" ways



(a) Initial Complex

(b) After Collapse

Figure 11: Link-preserving DC of the window that changes directed path space. The LPDC of the edge $[(2,4),(3,4)]$ changes the directed path space between $\mathbf{0}$ to $(5,5)$ from having six connected components to five connected components.

to apply a sequence of LPDCs to a directed Euclidean cubical complex. As illustrated in Example 4, there exists a sequence of LPDCs that preserves the six connected components in $\overrightarrow{P}(K)_{\mathbf{0}}^{(5,5)}$. However, if we perform a sequence of LPDCs that removes the edge $[(2,4),(3,4)]$ as in this example, then we get a directed Euclidean cubical complex that does not preserve the dihomotopy classes of directed paths in $\overrightarrow{P}(K)_{\mathbf{0}}^{(5,5)}$.

Example 5 illustrates the need to investigate other properties if we want to preserve directed path spaces when performing an LPDC.

In Example 3 the problem was the existence of unreachable vertices. In Example 5, the vertex $(2,4)$ is a *deadlock* after the LPDC - only trivial dipaths initiate from there - whereas before collapse, that was not the case. This seems to suggest that the introduction of new deadlocks should not be allowed - an extra, but computationally easy check on vertices of σ.

Moreover, vertex LPDCs appear to not introduce the problems of unreachability and deadlocks. These observations lead us to suspect that studying unreachability, deadlocks, and vertex LPDCs can help us better understand when LPDCs perserve and do not preserve directed path spaces between the minimum and a given vertex. We leave this as future work.

In summary, we provided an easy check for determining when we have an LPDC pair as well as various settings for when LPDCs preserve spaces of directed paths. This work provides a mathematical foundation for algorithms that simplify directed Euclidean cubical complexes that model phenomena with directed paths. Fully understanding when LPDCs preserve spaces of directed paths between a minimum and a given vertex will yield the full algorithm of how to compress a directed Euclidean cubical complex while preserving spaces of directed paths.

Acknowledgement 1. This research is a product of one of the working groups at the Women in Topology (WIT) workshop at MSRI in November 2017. This workshop was organized in partnership with MSRI and the Clay Mathematics Institute, and was partially supported by an AWM ADVANCE grant (NSF-HRD 1500481).

This material is based upon work supported by the US National Science Foundation under grant No. DGE 1649608 (Belton) and DMS 1664858 (Fasy), as well as the Swiss National Science Foundation under grant No. 200021-172636 (Ebli).

We thank the Computational Topology and Geometry (CompTaG) group at Montana State University for giving helpful feedback on drafts of this work.

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