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by

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DICOVERINGS AS QUOTIENTS

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1. Introduction.

In [5], directed coverings were introduced as a contribution to the ongoing study of topological spaces with a preferred direction, *ditopology*. One approach to ditopology, is to develop tools inspired by the tools from ordinary algebraic topology, and covering theory is of course one such tool. This paper continues the study of such *dicoverings*.

A d-space is a topological space X with a subset $\vec{P}(X)$ of the paths, called the dipaths Def. 2.1. An lpo-space is a d-space, where the dipaths induce a closed partial order locally Def. 2.1.

In [5] dicoverings and a universal dicovering were defined. The universal dicovering of an lpo-space X is simply the set $\tilde{X}_{x_0}$ of dihomotopy classes of dipaths with a common initial point, x_0. The topology is defined much as in the non-directed case, Def. 2.6. The projection map Π is the endpoint map, it is continuous and the fibers $\Pi^{-1}(x)$ are discrete and have cardinality $|\vec{\pi}_1(X, x_0, x)|$. Hence, the cardinality of the fibers is not constant. The choice of basepoint, x_0, is essential, and changing it may give a totally different universal dicovering space. A dicovering wrt. x_0 is a map $p : Y \to X$ such that dipaths and dihomotopies initiating in x_0 lift uniquely, given an initial point $y_0 \in p^{-1}(x_0)$. There is a fiber preserving map $\Phi : \tilde{X}_{x_0} \to Y$, along which all dipaths and dihomotopies initiating in y_0 lift, but it is not necessarily continuous.

Hence, the connection between the universal dicovering space wrt. x_0 and other dicoverings wrt. x_0 needs to be explored further. In the present paper, we study quotients of the universal dicovering space under certain relations, congruences, and we see that as sets, all dicoverings are in fact such quotients. In section 5, for locally well behaved (5.3) dicoverings $p : Y \to X$, a topology on Y is defined from the topology on X and $\vec{P}(X)$, along the lines of the construction of a topology on $\tilde{X}_{x_0}$. With this topology, the map $\Phi : \tilde{X}_{x_0} \to Y$ is in fact the universal dicovering of Y and $p : Y \to X$ is a dicovering.

The topology on X is in general not the quotient topology under $\Pi : \tilde{X}_{x_0} \to X$, but the dipaths and dihomotopies in X initiating from x_0 are still continuous with the quotient topology. Similarly, considering Y with the quotient topology under Φ, the dipaths and dihomotopies in Y initiating in y_0 are still continuous in this stronger topology. Hence, to study dihomotopy classes of dipaths initiating in x_0, we may change the topology on the dicovering Y, on X or on $\tilde{X}$ to the topologies defined in section 5 or the quotient topology and still get the same result. For higher homotopy or homology of the dipath spaces, the results may depend more on the topology.

Another approach to the lack of continuity is the introduction of a weaker requirement on maps, namely that they map dipaths to dipaths and dihomotopies to dihomotopies - an extension of the set of maps considered. This approach is Def. 4.1.

It is a pleasure to thank Martin Raussen for suggesting the definition of a congruence used below and for helpful discussions.

2. d-spaces which are locally ordered

We define the "directions" on a topological space by assigning directed paths, i.e., d-spaces in the sense of Marco Grandis [4]. Moreover, we require that this induces a closed partial order on small enough subsets.

Definition 2.1. *A d-space is a topological space X with a set of paths $\vec{P}(X) \in X^I$ such that*

- *$\vec{P}(X)$ contains all constant paths.*
- *$\gamma, \mu \in \vec{P}(X)$ implies $\gamma \star \mu \in \vec{P}(X)$, where $\star$ is concatenation.*
- *If $\phi : I \to I$ is monotone, $t \leq s \Rightarrow \phi(t) \leq \phi(s)$, and $\gamma \in \vec{P}(X)$, then $\gamma \circ \phi \in \vec{P}(X)$*

The d-space is saturated if whenever $\phi : I \to I$ a monotone surjection and $\gamma \circ \phi \in \vec{P}(X)$ then $\gamma \in \vec{P}(X)$.

A d-map or dimap $f : X \to Y$ is a continuous map, such that if $\alpha \in \vec{P}(X)$ then $f \circ \alpha \in \vec{P}(Y)$.

The set of distinguished paths, $\vec{P}(X)$ are called the dipaths. *They are d-maps from the ordered interval $\vec{I}$ to X.*

For a subset $V \subset X$, $\vec{P}(V, x, y)$ denotes the dipaths in V from x to y. For subsets W, $U \subset X$, let $\vec{P}(X, W, U)$ be the set of dipaths with initial point in W and final point in U.

Definition 2.2. *Let X be a d-space. On a subset $V \subset X$, we get a relation $x \leq_V y$ if $\vec{P}(V, x, y) \neq \emptyset$*

The future *of x in V is $\uparrow_V x = \{y \in V | x \leq_V y\}$*

The d-space X is locally ordered, *if there is a cover $\mathcal{U}$ of X, which is a basis for the topology and if on each $U \in \mathcal{U}$, the relation $\leq_U$ is a partial order.*

Remark 2.3. A locally ordered d-space is an lpo-space if there is a cover as above, such that $(U, \leq_U)$ is closed for all $U \in \mathcal{U}$. See [2]

Definition 2.4. *Let $x \in X$ and $U \subset X$ and let $\gamma_1, \gamma_2 \in \vec{P}(X, x, U)$. Then $\gamma_1 \sim_U \gamma_2$, if there is a dimap, $H : I \times \vec{I} \to X$ s.t. $H(0, t) = \gamma_1(t)$, $H(1, t) = \gamma_2(t)$, $H(s, 0) = x$ and $H(s, 1) \in U$. If $U = \{y\}$, a single point, then $\gamma_1 \sim_U \gamma_2$ is written*

$\gamma_1 \sim \gamma_2$. *Given two basepoints, we define* $\vec{\pi}_1(X,x,y) = \vec{P}(X,x,y)/\sim$, *and we let* $[\gamma]$ *denote the equivalence class of a dipath* γ.

Definition 2.5. *A locally ordered d-space* $(X, \vec{P}(X), \mathcal{U})$ *is locally relatively diconnected wrt.* $x_0 \in X$ *if*

- *For all* $U \in \mathcal{U}$ *and all* $x, y \in U$, $|\vec{\pi}_1(U,x,y)| \leq 1$
- *For all* $x \in X$, *there is a* $U \in \mathcal{U}$ *s.t. for* $\gamma_i \in \vec{P}(X, x_0, x)$, $\gamma_1 \sim_U \gamma_2$ *if and only if* $[\gamma_1] = [\gamma_2]$.

We define a universal dicovering space as in [5]

Definition 2.6. *For a locally ordered d-space* $(X, \vec{P}(X), \mathcal{U})$, *locally relatively connected wrt.* $x_0 \in X$, *we define the universal dicovering wrt.* x_0.

$$\tilde{X}_{x_0} = \{[\gamma] \in \vec{\pi}_1(X, x_0, -)\}$$

with topology $\mathcal{U}_{[\gamma]}$ *generated by the sets*

$$U_{[\gamma]} = \{[\mu] | \mu \in \vec{P}(X, x_0, U), \mu \sim_U \gamma\}$$

for $U \in \mathcal{U}$ *and* $\gamma \in \vec{P}(X, x_0, U)$.

The d-structure is $\vec{P}(\tilde{X}, [\gamma], -) = \{\eta(t) = [\gamma \star \mu(\frac{t+1}{2})]$, *where* $\mu \in \vec{P}(X, \gamma(1), -)\}$.

Remark 2.7. The strange choice of parameter value $\eta(t) = [\gamma \star \mu(\frac{t+1}{2})]$ makes the dipath run from $[\gamma]$ to $[\gamma \star \mu]$ by definition of concatenation of paths. This does define a *d*-structure. It is not hard to see that the dipaths are continuous and the other properties are inherited from $\vec{P}(X)$. For a proof of continuity see [5]

The universal dicovering space has the following properties

Proposition 2.8. *Let* $\Pi : \tilde{X}_{x_0} \to X$ *be the universal dicovering of a locally ordered d-space, then*

(1) $\tilde{X}_{x_0}$ *is a locally ordered d-space and* Π *is a d-map.*
(2) *Dipaths and dihomotopies initiating in* $y \in \uparrow x_0$ *lift uniquely given an initial point in* $\Pi^{-1}(y)$
(3) Π *has discrete fibers*
(4) $\Pi : \uparrow_{U_{[\eta]}} [\gamma] \to \uparrow_U \gamma(1)$ *is a bijection.*
(5) $\tilde{X}_{x_0}, \mathcal{U}_{[\gamma]}$ *is locally relatively diconnected with respect to* $[x_0]$.

Proof. For the proof of 1, see [5, Prop. 3.8], for 2, 3, and 4, see [5, Prop. 3.11]. For 5, see [5, Cor. 3.13] □

We let $[\gamma(t)]$ denote the unique lift $\eta(t) = [\gamma_{|[0,t]}]$ of the dipath γ initiating in x_0 to the universal dicovering. In particular, all dipaths μ in $\tilde{X}$ initiating in x_0 can be written $[\Pi \circ \mu(t)]$, so they are all of the form $[\gamma(t)]$ for some γ.

Definition 2.9. *A dimap* $p : Y \to X$ *is a dicovering wrt.* $x_0 \in X$ *if* $\uparrow_X x_0 = X$, $\uparrow_Y p^{-1}(x_0) = Y$, *p is surjective and dipaths and dihomotopies initiating in* x_0 *lift uniquely given an initial point in* $p^{-1}(x_0)$.

In [5] we prove that lifting of dipaths and dihomotopies initiating at other points follow:

Lemma 2.10. *Let $p : Y \to X$ be a dicovering with respect to x_0. Let $\gamma : \vec{I} \to X$ and let $y \in p^{-1}(\gamma(0))$. Then there is a unique lift $\hat{\gamma}$ of γ with $\hat{\gamma}(0) = y$*

Proof. See [5] Lem. 4.3. □

Corollary 2.11. *Let $p : Y \to X$ be a dicovering with respect to x_0. Let $H : I \times \vec{I} \to X$ have a fixed initial point $H(s, 0) = x$ and let $p(y) = x$. Then there is a unique lift $\hat{H}$ of H initiating in y.*

Proof. See [5] Cor. 4.4. □

Definition 2.12. *Let $p : Y \to X$ be a dicovering wrt. x_0. Then $p : Y \to X$ is a simple dicovering if $p^{-1}(x_0) = \{y_0\}$*

Remark 2.13. The universal dicovering wrt. x_0 is simple if and only if there is no loop at x_0 so that $\Pi^{-1}(x_0)$ is just the constant path $[x_0]$. This may be obtained by attaching an edge leading into x_0, and then take the cover with respect to the initial point of that edge.

We will often leave out the subscript x_0 in the following and assume that all dicoverings are simple.

Proposition 2.14. *Let $p : Y \to X$ be a simple dicovering wrt. $x_0 \in X$ and suppose X has a universal dicovering construction. Then there is an induced map $\Phi : \tilde{X} \to Y$ given by $\Phi([\gamma]) = \hat{\gamma}(1)$, where $\hat{\gamma}$ is the lift of γ to Y with initial point y_0.*

The map Φ may not be continuous, since the topology on Y is not revealed through the lifting properties - there may be open sets in Y, which do not interfere with the lifting of dipaths and dihomotopies. For an example, see [5].

3. Congruences induce dicoverings.

The classical covering theory describes all coverings as quotients of the universal covering space under deck transformations, which are fiberpreserving group actions. In the directed setting, deck transformations are replaced by congruences. A congruence is a special kind of equivalence relation on $\tilde{X}$, but it is not given via a group action. In this section we see that the quotient of a universal dicovering under a congruence relation is a dicovering, if the topology is sufficiently well behaved. In the next section, we give the reverse, namely a dicovering Y defines a congruence $\approx$ on $\tilde{X}$ such that Y is the quotient $\tilde{X}/\approx$ - again provided the topology fits.

Definition 3.1. *An equivalence relation on $\tilde{X}_{x_0}$ is a congruence if $[\gamma_1] \approx [\gamma_2]$ implies $\gamma_1(1) = \gamma_2(1)$ and $[\gamma_1 \star \mu] \approx [\gamma_2 \star \mu]$ for all dipaths μ initiating in $\gamma_i(1)$.*

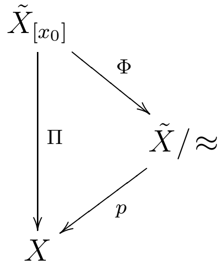

FIGURE 1. The universal dicovering and a quotient

Example 3.2. *The relation $[\gamma_1] \approx [\gamma_2]$ if $\gamma_1(1) = \gamma_2(1)$ is a congruence, and $\tilde{X}_{x_0}/\approx$ is the original space X, at least as a set.*

Example 3.3. *Let $f : X \to Y$ be an injective d-map, then the relation $[\gamma] \approx_f [\eta]$ if $[f \circ \gamma] = [f \circ \eta]$ is a congruence, since $f \circ (\gamma \star \mu) = f \circ \gamma \star f \circ \mu$ and $[f \circ \gamma] = [f \circ \eta]$ implies $f \circ \gamma(1) = f \circ \eta(1)$, so $\gamma(1) = \eta(1)$, as f is injective.*

The d-structure on the image of a surjection is defined as in [4], except that we take closure under reparametrization, which [4] does not. For a thorough discussion of reparametrization, see [1]

Definition 3.4. *Let Z be a d-space and let $f : Z \to Y$ be a surjection. The quotient d-structure, $\vec{P}(Y)$ on Y is the closure of the set of dipaths $\{f \circ \gamma | \gamma \in \vec{P}(Z)\}$ under finite concatenation and reparametrization. If f is continuous, then this provides a subset of Y^I, and it is a d-structure.*

Proposition 3.5. *Let $\Pi : \tilde{X}_{x_0} \to X$ be the universal dicovering of a locally relatively diconnected d-space. Let $\approx$ be a congruence on $\tilde{X}_{x_0}$, and let $Y = \tilde{X}_{x_0}/\approx$ with the quotient d-structure. Let $p : Y \to X$ be the map $p([[\gamma]]) = \gamma(1)$ and suppose the topology on Y is such that p and the quotient map $\Phi : \tilde{X}_{x_0} \to Y$ are continuous. Then p and Φ are dicoverings and $\vec{P}(Y) = \Phi(\vec{P}(\tilde{X}_{x_0}))$.*

Proof. Since $\vec{P}(Y)$ is the quotient d-structure and Π is a dimap, clearly a dipath in Y maps to a dipath in X, so p is a dimap.

Let $\alpha : \vec{I} \to X$, $\alpha(0) = x_0$. Then α lifts along p to the dipath $\hat{\alpha}(t) = [[\alpha(t)]]$. Hence the only problem may be uniqueness. Moreover, if there is a dipath $\beta : \vec{I} \to Y$ which does not lift, then in particular, the dipath $[p \circ \beta(t)]$ in $\tilde{X}_{x_0}$ is not a lift of β. So $p \circ \beta$ is a dipath in X with more than one lift to Y, namely β and $\Phi \circ [p \circ \beta(t)]$. Consequently, *existence* of lifts along Φ is equivalent to *uniqueness* of lifts along p. Hence it suffices to see that $\vec{P}(Y) = \Phi(\vec{P}(\tilde{X}_{x_0}))$.

Clearly $\vec{P}(Y) \supseteq \Phi(\vec{P}(\tilde{X}_{x_0}))$. The only problem may be the closure under concatenation. Suppose $\mu = \Phi \circ \gamma_1 \star \Phi \circ \gamma_2$, where $\gamma_i \in \vec{P}(\tilde{X})$. We want to see, that $\mu \in \Phi(\vec{P}(\tilde{X}_{x_0}))$.

Since $\Phi\circ\gamma_1$ can be concatenated with $\Phi\circ\gamma_2$, we have $\Phi(\gamma_1(1))=\Phi(\gamma_2(0))$, i.e. $\gamma_1(1)\approx\gamma_2(0)$.

Let $\bar{\gamma_2}$ be the unique lift to $\tilde{X}$ of $\Pi\circ\gamma_2$ with initial point $\gamma_1(1)$. The relation is a congruence, so $\gamma_2(0)\approx\gamma_1(1)=\bar{\gamma_2}(0)$ implies $\gamma_2(t)\approx\bar{\gamma_2}(t)$ for all t. Hence $\Phi\circ\gamma_2(t)=\Phi\circ\bar{\gamma_2}(t)$ and consequently $\mu=\Phi\circ\gamma_1\star\Phi\circ\gamma_2=\Phi\circ(\gamma_1\star\bar{\gamma_2})$, which is in $\Phi(\vec{P}(\tilde{X}_{x_0}))$.

Let $H: I\times\vec{I}\to Y$. Then all the dipaths lift to give a map $\tilde{H}: I\times\vec{I}\to\tilde{X}_{x_0}$, which is a lift of H. This is also the unique lift of $p\circ H$, which we already know is continuous. □

If the topology on Y is the quotient topology, then p is also continuous, and the theorem certainly holds.

But we did not need the full quotient topology on Y - just continuity of p and Φ and that $\vec{P}(Y)$ is the quotient d-structure: concatenation and reparametrization of dipaths $\Phi\circ\gamma$.

Corollary 3.6. *With notation and conditions as above, we have*

$$\vec{P}(\Phi([x_0]),[[\gamma]])=\Phi(\vec{P}([x_0],\Phi^{-1}([[\gamma]])))$$

We would of course like $\tilde{X}$ to be the universal dicovering of Y, but in general this may not be the case. The problem is the topology, not the lifting properties. For instance, we need the local structure on Y to be sufficiently well behaved to even have a universal dicovering of Y. In section 5 we will give sufficient conditions on $\approx$ for this to work. First notice, that as a set, $\tilde{X}$ corresponds to the dihomotopy classes of dipaths in $\tilde{X}/\!\approx$ and the map Φ is the endpoint map.

Lemma 3.7. *Let $\approx$ be a congruence on $\tilde{X}=\tilde{X}_{x_0}$. Suppose $\Phi:\tilde{X}\to\tilde{X}/\approx$ is continuous, $p:\tilde{X}/\!\approx\,\to X$ is continuous and $\tilde{X}/\approx$ has the quotient d-structure. Then there is a bijection $F:\vec{\pi}_1(\tilde{X}/\!\approx,[[x_0]],-)\to\tilde{X}$ such that $\Phi(F([\mu]))=\mu(1)$.*

Proof. Let $\mu:\vec{I}\to\tilde{X}/\!\approx$ and assume $\mu(0)=[[x_0]]$. Then, by Cor. 3.6, $\mu=\Phi\circ\tilde{\mu}$ where $\tilde{\mu}(t)=[p\circ\mu(t)]$ is the lift to $\tilde{X}$, so $\mu(t)=\Phi([p\circ\mu(t)])=[[p\circ\mu(t)]]$. Let $F([\mu])=[p\circ\mu]\in\tilde{X}$. The map F is

- *well defined* $\mu_1\sim\mu_2$ implies $p\circ\mu_1\sim p\circ\mu_2$ and hence $[p\circ\mu_1]=[p\circ\mu_2]$
- *injective* $[p\circ\mu_1]=[p\circ\mu_2]$ if and only if $p\circ\mu_1\sim p\circ\mu_2$ and paths and dihomotopies lift uniquely along p, so $[\mu_1]=[\mu_2]$.
- *surjective* Let $[\gamma]\in\tilde{X}$. Then γ lifts uniquely to $\hat{\gamma}:\vec{I}\to\tilde{X}/\!\approx$. Now $F([\hat{\gamma}])=[p\circ\hat{\gamma}]=[\gamma]$

Now $\Phi(F([\mu]))=\Phi([p\circ\mu])$, and since $\Pi([p\circ\mu])=p\circ\mu(1)=p(\Phi([p\circ\mu]))$ and $\mu(t)$ is the unique lift of $p\circ\mu(t)$, we have $\Phi([p\circ\mu])=\mu(1)$ □

4. Dicoverings induce congruences.

Given a dicovering $p: Y\to X$ wrt. x_0, there is a congruence $\approx$ on $\tilde{X}_{x_0}$ and a bijection $\psi:\tilde{X}_{x_0}/\approx\,\to Y$ such that dipaths and dihomotopies initiating in

$[[x_0]]$ resp. y_0 lift along ψ^{-1} resp. ψ. Hence, provided all maps involved are continuous, the dicoverings wrt. x_0 are precisely the quotients of $\tilde{X}_{x_0}$ under congruences. In the last sections, we will consider what topology we may put on Y and the quotient. The following definition is certainly satisfied by dimaps, but it is a much weaker condition

Definition 4.1. *Let $f : Z \to W$ be a map of d-spaces and let $z_0 \in Z$. Then f is a z_0* point of view *or z_0-pov-map if whenever $\gamma \in \vec{P}(Z, x_0, -)$, then $f_*(\gamma) = f \circ \gamma \in \vec{P}(W, f(x_0), -)$ and for any dihomotopy $H : I \times \vec{I} \to Z$ with $H(s, 0) = x_0$, $f \circ H$ is a dihomotopy.*

Proposition 4.2. *Let $p : Y \to X$ be a simple dicovering wrt. $x_0 \in X$, and suppose that X is locally relatively diconnected wrt. x_0, with universal dicovering $\pi : \tilde{X}_{x_0} \to X$. Then there is a congruence $\approx$ on $\tilde{X}_{x_0}$ and a bijective map $\psi : \tilde{X}_{x_0}/\approx \to Y$.*

Proof. Let $\Phi : \tilde{X}_{x_0} \to Y$ be the induced map. Then define the congruence by $[\gamma] \approx [\eta]$ if $\Phi([\gamma]) = \Phi([\eta])$. We have to see, that this is a congruence: $\Phi([\gamma]) = \Phi([\eta])$ implies $\gamma(1) = \eta(1)$, since Φ preserves fibers over X. Let μ be a dipath with $\mu(0) = \gamma(1)$. Then $\Phi([\gamma \star \mu])$ is the endpoint of the lift $\widehat{\gamma \star \mu}$ of $\gamma \star \mu$ along p. Since $\Phi([\gamma]) = \Phi([\eta])$, the endpoints of $\hat{\gamma}$ and of $\hat{\eta}$ are identical. Let $\hat{\mu}$ be the unique lift of μ with initial point $\hat{\gamma}(1)$. Then $\widehat{\gamma \star \mu}(1) = \hat{\gamma} \star \hat{\mu}(1) = \hat{\eta} \star \hat{\mu}(1)$.

Since $\Phi : \tilde{X}_{x_0} \to Y$ is surjective, clearly $Y \simeq \tilde{X}_{x_0}/\approx$ via the map $\psi : \tilde{X}_{x_0}/\approx \to Y$ given by $\psi([[\gamma]]) = \Phi(\gamma)$. □

Remark 4.3. In [5], we give an example, where Φ is not continuous. Hence we cannot in general expect Y to have the quotient topology. However, Y and $\tilde{X}_{x_0}/\approx$ with the quotient topology and quotient d-structure have the same dipaths and dihomotopies:

Proposition 4.4. *With notation as above and with the quotient topology and quotient d-structure on $\tilde{X}_{x_0}/\approx$, $\gamma : \vec{I} \to \tilde{X}_{x_0}/\approx$ is a dipath if and only if, $\psi \circ \gamma$ is a dipath on Y. Similarly for dihomotopies with fixed initial point. Hence ψ is an $[[x_0]]$-pov.-map and ψ^{-1} is a y_0-pov. map.*

Proof. Use the fact that dipaths and dihomotopies lift uniquely to $\tilde{X}$ and that the image of a dipath γ in $\tilde{X}$ is a dipath in Y even when Φ is not continuous. Since it is the lift along p of $\Pi \circ \gamma$. □

Corollary 4.5. *With notation as above, Y has the quotient d-stucture under Φ, and Φ is an $[x_0]$-pov map.*

This holds even if Φ is not continuous - it is in fact just another way of stating the lifting properties of dipaths and dihomotopies along Φ. Notice that lifting is not enough, we actually use that the lifts are unique.

5. Topology on the quotient - not the quotient topology

In Ex. 5.1, we give an example, where the congruence $[\gamma] \approx [\mu]$ if $\gamma(1) = \mu(1)$, which gives $\tilde{X}_{x_0}/\approx$ equal to X via pov.maps, but the quotient topology on $\tilde{X}_{x_0}/\approx$ is not the original topology on X. The reason morally is, that the topology on $\tilde{X}$ is constructed from properties of dipaths up to variation, and hence the quotient topology will give X a topology where a set V is open if $H^{-1}(V)$ is open for all dihomotopies and $\gamma^{-1}(V)$ is open for all dipaths - initiating in x_0. But there may very well be less open sets in X than specified by this requirement.

To define the topology on $\tilde{X}$ we use a basis for the topology on X and combine this information with information about the dipaths and dihomotopies initiating in the basepoint. With this topology, the projection $\tilde{X} \to X$ is continuous, but the topology on X may not be the quotient topology under Π (as it is in the non-directed case), see Ex.5.1. In this section, a topology on quotients $\tilde{X}/\approx$ is defined in a similar way from the topology on X and the congruence relation. We restrict the allowed congruences such that both Φ and p become dicoverings and $\tilde{X}$ is the universal dicovering of $\tilde{X}/\approx$ with the constructed topology.

Example 5.1. *This example is* [5, Ex.3.17]*, but used for a different purpose. Let $X = I \times I$ with topology generated by the standard topology on $\mathbb{R}^2$ and the subsets*

$$I_a = \{(x, ax) | 0 < x < a\}$$

for $a > 0$. Let $\vec{P}(X)$ be the set of segments of lines parametrized with increasing coordinates; $\gamma(t) = (\gamma_1(t), k\gamma_1(t))$, $k \geq 0$ and $\gamma_1(t)$ increasing. Or $\gamma(t) = (0, \gamma_2(t))$ and $\gamma_2(t)$ increasing. This is certainly locally relatively diconnected wrt. $(0,0)$. A dihomotopy $H : I \times \vec{I} \to X$ with $H(s,0) = (0,0)$ has image contained in a line through $(0,0)$, since it has to be continuous, so it either reparametrizes a dipath, or stretches/shrinks it along the line. Hence $\tilde{X}_{(0,0)}$ is a wedge of the line segments $(0,0)$ to $(1,a)$, and $(0,0)$ to $(a,1)$, where $0 \leq a \leq 1$. The topology on X is not the quotient topology: Let $V = \{(x,x) | 1/4 < x < 1/2$. The $\Pi^{-1}(V)$ is open in $\tilde{X}_{(0,0)}$, but V is not open in X.

Remark 5.2. If we take Grandis' definition of dihomotopies, where $H : \vec{I} \times \vec{I} \to X$, i.e., all $H(s_0, t)$ and $H(s, t_0)$ have to be dipaths. Then in the above example, with the same d-structure, and even with the standard topology on $\mathbb{R}^n$, the universal dicovering space is again a wedge of lines, and again X does not have the quotient topology.

Definition 5.3. *Let X be a d-space and suppose $\mathcal{U}$ is a cover of X with open sets satisfying properties 2.5 wrt. $x_0 \in X$ and suppose $\Pi : \tilde{X} \to X$ is a surjection.*

A congruence $\approx$ is locally controlled*, if for any $x \in X$ and open neighborhood V of x, there is a $U \in \mathcal{U}$ such that*

(1) $x \in U \subset V$

(2) *for any* $\gamma \in \vec{P}(X, x_0, x)$, *and* $\mu \sim_U \gamma$, *if* $[\mu] \approx [\tilde{\mu}]$, *then there is a* $\tilde{\gamma} \in \vec{P}(X, x_0, x)$ *such that* $\tilde{\mu} \sim_U \tilde{\gamma}$ *and* $[\tilde{\gamma}] \approx [\gamma]$, *i.e., the diagram below can be completed.*

$$\begin{array}{ccc} [\mu] & \sim_U & [\gamma] \\ \approx & & \approx \\ [\tilde{\mu}] & \sim_U & \exists[\tilde{\gamma}] \end{array}$$

(3) *For* $y \in U$ *and* $\alpha \in \vec{P}(X, x_0, y)$ *there is* $\gamma \in \vec{P}(X, x_0, x)$ *such that* $\gamma \sim_U \alpha$

The neighborhood U *is then* controlled by x.

The following example provides a violation to condition 2) above:

Example 5.4. *Let* X *be constructed as follows: Take a directed circle* $\vec{S}^1$*; choose a point* $q \in \vec{S}^1$*; glue a directed interval* $\vec{I}$ *to* q *at the point* $1 \in \vec{I}$*; take a directed square* $\vec{I} \times \vec{I}$ *and glue that to* q *at the point* $(0, 0)$*.*

The universal dicover wrt. $0 \in \vec{I}$ *is the dihomotopy classes represented by the set of dipaths* $\vec{I} \star L^k \star \gamma$*, where* $\vec{I}$ *is a dipath from* 0 *to* 1 *in* $\vec{I}$*,* L^k *is* k *turns of the circle and* γ_x *is a directed path from* $(0, 0)$ *in the square to* $x \in \vec{I} \times \vec{I}$*.*

Now let μ_n *be a line in* $\vec{I} \times \vec{I}$ *from* $(0, 0)$ *to* $(1/2 - 1/n, 1/2 + 1/n)$ *for* $n \geq 1$*. Define a congruence relation:*

$\vec{I} \star L^n \star \mu_n \approx \vec{I} \star L^{n+1} \star \mu_n$ *generates equivalences by concatenation. The result is a congruence relation, which is not locally controlled at* $(1/2, 1/2)$*. In any neighborhood* U *with* $(1/2, 1/2) \in U \in \vec{I} \times \vec{I}$*, there will be some point* $(1/2 - 1/n, 1/2 + 1/n)$ *and thus a relation,* $\vec{I} \star L^n \star \mu_n \approx \vec{I} \star L^{n+1} \star \mu_n$*. If* $\vec{I} \star L^n \star \mu_n \sim_U \gamma$*, where* $\gamma(1) = (1/2, 1/2)$*, then* $\vec{I} \star L^{n+1} \star \mu_n \not\sim_U \gamma$*, since they differ by the number of turns of the loop. Since the congruence at* $(1/2, 1/2)$ *is trivial, we cannot satisfy condition 2).*

Lemma 5.5. *The* $[\tilde{\gamma}]$ *provided in 2) is unique.*

Proof. If $\tilde{\tilde{\gamma}}$ satisfies 2), then $\tilde{\tilde{\gamma}} \sim_U \tilde{\mu} \sim_U \tilde{\gamma}$, and hence $[\tilde{\tilde{\gamma}}] = [\tilde{\gamma}]$, since $U \in \mathcal{U}$ □

If a congruence is locally controlled, then the sets $U \in \mathcal{U}$ which are controlled by a point still provide a basis, and we will tacitly replace $\mathcal{U}$ with this subset.

Remark 5.6. For the trivial relation, $[\gamma] \approx [\mu]$ if $\gamma(1) = \mu(1)$, the only restriction on $\mathcal{U}$ is the last condition: If U is controlled by x, then $\Pi(\sqcup_{[\gamma] \in \vec{\pi}_1(X, x_0, x)} U_{[\gamma]}) = U$, i.e., that all points in U are reachable by a dipath which is equivalent, $\sim_U$, to a dipath to x. In Ex. 5.1, that is not satisfied. This property is what will make $p : \tilde{X}/\!\approx \to X$ continuous; when $\approx$ is the trivial relation this says, that X has the quotient topology.

Definition 5.7. *Let $\approx$ be a locally controlled congruence on $\tilde{X}$. For $U \in X$ controlled by x and $\gamma \in \vec{P}(X, x_0, x)$, let*

$$U_{[[\gamma]]} = \bigcup_{[\gamma_i] \in [[\gamma]]} \Phi(U_{[\gamma_i]}) = \Phi(\bigcup_{[\gamma_i] \in [[\gamma]]} (U_{[\gamma_i]}))$$

and let $\mathcal{U}_{[[]]}$ be the set of all such sets.

Lemma 5.8. *If $\approx$ is locally controlled, then $\mathcal{U}_{[[]]}$ is a basis for a topology on $\tilde{X}/\!\approx$. If $\approx$ is locally controlled by both $\mathcal{U}^1$ and $\mathcal{U}^2$, and they are bases for the same topology on X, then the bases $\mathcal{U}^1_{[[]]}$ and $\mathcal{U}^2_{[[]]}$ define the same topology on $\tilde{X}/\!\approx$.*

Proof. $\mathcal{U}_{[[]]}$ is a basis for a topology: Let $[[\alpha]] \in U^1_{[[\gamma]]} \cap U^2_{[[\mu]]}$. Choose $W \subseteq U^1 \cap U^2$ locally controlled by $\alpha(1) \in W$. Then we claim $W_{[[\alpha]]} \subset U^1_{[[\gamma]]} \cap U^2_{[[\mu]]}$. Let $[[\beta]] \in W_{[[\alpha]]}$, i.e., there is $[\alpha_m] \in [[\alpha]]$ and $[\beta_m] \in [[\beta]]$ with $[\alpha_m] \sim_W [\beta_m]$. Moreover, since U^1 is controlled by $\gamma(1)$, by condition 3) there is a $[\gamma_{j_m}]$ with endpoint $\gamma(1)$ s.t., $[\gamma_{j_m}] \sim_U [\alpha_m]$. But then, since $[\alpha_m] \in [[\alpha]]$, by 2) there is some $\hat{\gamma}$ with $\hat{\gamma} \sim_U \alpha$ and $[\hat{\gamma}] \approx [\gamma_{j_m}]$. On the other hand, $[[\alpha]] \in U^1_{[[\gamma]]}$, so there is some $\tilde{\gamma}$ with $\tilde{\gamma} \sim_U \alpha$ and $[\tilde{\gamma}] \in [[\gamma]]$. By uniqueness, $[\tilde{\gamma}] = [\hat{\gamma}]$ and hence $[\gamma_{j_m}] \in [[\gamma]]$. As $W \subset U^1$, we conclude $[\beta_m] \sim_{U^1} [\gamma_{j_m}]$, so $[[\beta]] \in U^1_{[[\gamma]]}$ and similarly $[[\beta]] \in U^2_{[[\mu]]}$.

Suppose now $U^1 \in \mathcal{U}^1$ and $[[\alpha]] \in U^1_{[[\gamma]]}$, where $\gamma(1)$ controls U^1. Choose $W \in \mathcal{U}^2$ such that $W \subset U^1$ and W is controlled by $\alpha(1)$. Imitate the argument above to obtain $W_{[[\alpha]]} \subseteq U^1_{[[\gamma]]}$. Hence the topologies $\mathcal{U}^i_{[[]]}$ are the same.

□

Corollary 5.9. *Suppose $(X,\mathcal{U}^1)$ and $(X,\mathcal{U}^2)$ are locally relatively diconnected wrt. x_0 and that $\mathcal{U}^1$ and $\mathcal{U}^2$ are both controlled by the trivial relation and induce the same topology on X. Then they induce the same topology on $\tilde{X}$.*

Lemma 5.10. *With conditions as above, the neighborhoods $U_{[[\gamma]]}$ satisfy 2.5, i.e., with the basis $\mathcal{U}_{[[]]}$, $\tilde{X}/\!\approx$ is locally relatively diconnected wrt. $[[x_0]]$.*

Proof. Suppose $[[\gamma(t)]]$, $[[\mu(t)]]$ are dipaths in $U_{[[\eta]]} \subset \tilde{X}/\!\approx$ from $[[\gamma(0)]] = [[\mu(0)]]$ to $[[\gamma(1)]] = [[\mu(1)]]$. Then γ and μ are dipaths in $U \subset X$ from $\gamma(0)$ to $\gamma(1)$. These are dihomotopic. The dihomotopy lifts to $\tilde{X}$ and gives a dihomotopy $\tilde{H}$ with initial point $\tilde{H}(s,0) = [\gamma(0)]$ and $\tilde{H}(0,t) = [\gamma(t)$, $\tilde{H}(1,t)$ is the lift of μ with initial point $[\gamma(0)]$, and since $[\gamma(0)] \approx [\mu(0)]$, we have $\tilde{H}(1,t) \approx [\mu(t)]$. Clearly $\Phi \circ \tilde{H}$ is a dihomotopy from $[[\gamma(t)]]$ to $[[\mu(t)]]$ which has values in $U_{[[\eta]]}$, since each point in $\tilde{H}$ can be written $[\gamma \star H_s(t)]$ where H_s is a dipath in U and $[\gamma] \sim_U [\eta_k]$ for some $\eta_k \in [[\eta]]$. So $[\gamma \star H_s(t)] \in U_{[\eta_k]}$.

Now by 3.6, all dipaths in $\tilde{X}/\!\approx$ have the above form. □

Lemma 5.11. *If the cover $\mathcal{U}$ satisfies the conditions above and if furthermore, $(U, \leq_U)$ is a po-space, then all $U_{[[\gamma]]}$ are po-spaces as well.*

Proof. Let $[[\alpha]], [[\beta]] \in U_{[[\gamma]]}$ Suppose $[[\alpha]] \not\leq [[\beta]]$. If $\alpha(1) \neq \beta(1)$, choose U^1 and U^2 disjoint with $\alpha(1) \in U^1$ and $\beta(1) \in U^2$ and $U^i \subset U$. Then clearly $U^1_{[[\alpha]]} \cap U^2_{[[\beta]]} = \emptyset$.

If $\alpha(1) = \beta(1)$, we have $[[\alpha]] \neq [[\beta]]$. Let $[\gamma_i] \in [[\gamma]]$ with $\gamma_1 \sim_U \alpha$ and $\gamma_2 \sim_U \beta$. Let $\bar{U} \subset U$ be dominated by $\alpha(1) = \beta(1)$. Suppose $[[\mu]] \in \bar{U}_{[[\alpha]]} \cap \bar{U}_{[[\beta]]}$. Then there is $[\alpha_j] \in [[\alpha]]$ with $\mu \sim_U \alpha_j$ and $\beta_j \in [[\beta]]$ with $\mu \sim_U \beta_j$. Hence $\alpha_j \sim_U \beta_j$, a contradiction. So $\bar{U}_{[[\alpha]]} \cap \bar{U}_{[[\beta]]} = \emptyset$. □

Corollary 5.12. *If $(X, \mathcal{U})$ is an lpo-space and $\approx$ is locally controlled by $\mathcal{U}$, then $\tilde{X}/\approx$ is an lpo-space.*

Lemma 5.13. *Let $\approx$ be locally controlled in $\mathcal{U}$. Let $\mathcal{U}_{[[]]}$ define the topology on $\tilde{X}/\approx$. Then, $\Phi : \tilde{X} \to \tilde{X}/\approx$ is continuous, so the quotient d-structure gives continuous paths. With this topology and d-structure, $\Phi : \tilde{X} \to \tilde{X}/\approx$ is a dimap and p is a dimap.*

Proof. $\Phi^{-1}(U_{[[\gamma]]})$ is open: Let $[\mu] \in \Phi^{-1}(U_{[[\gamma]]})$. Then there is a $[\gamma_i] \approx [\gamma]$ and $[\eta] \in U_{[\gamma_i]}$ such that $\Phi([\mu]) = \Phi([\eta])$, i.e., $[[\mu]] = [[\eta]]$. Since U is controlled by x, by cond. 2) there is a γ_j s.t., $[\gamma_j] \approx [\gamma_i] (\approx [\gamma])$ and $[\mu] \sim_U [\gamma_j]$. Hence $[\mu] \in U_{[\gamma_j]}$, so $\Phi^{-1}(U_{[[\gamma]]}) = \bigcup_{[\gamma_i] \in [[\gamma]]} (U_{[\gamma_i]})$. Hence Φ is continuous, so the quotient construction of dipaths still gives a d-structure.

Let $U \in \mathcal{U}$ be controlled by x. Then

$$p^{-1}(U) = \bigcup_{[\gamma] \in \vec{\pi}_1(X, x_0, x)} \Phi(U_{[\gamma]}) = \bigcup_{[[\gamma]] \in \vec{\pi}_1(X, x_0, x)} U_{[[\gamma]]}$$

so p is continuous and the dipaths are as in the quotient construction, so it is a d-map.

□

Corollary 5.14. *If $\approx$ is a locally controlled congruence and $\tilde{X}/\approx$ is topologized as above, then $\Phi : \tilde{X}_{x_0} \to \tilde{X}/\approx$ and $p : \tilde{X}/\approx \to X$ are dicoverings.*

Proof. This follows from Prop. 3.5 □

Corollary 5.15. *If $\approx$ is a locally controlled congruence and $Y = \tilde{X}/\approx$ is topologized as above, then $\Phi : \tilde{X}_{x_0} \to Y$ is the universal dicovering of Y wrt. $y_0 = [[x_0]]$ and the basis $\mathcal{U}_{[[]]}$*

Proof. We only have to see, that the topology is right. Since a dipath initiating in y_0 projects to a dipath in X, which then again lifts uniquely to Y, all dipaths in Y have the form $[[\alpha(t)]]$ for some dipath α in X. Now suppose $[[\beta(t)]] \sim_{U_{[[\gamma]]}} [[\alpha(t)]]$. Then in particular, $\alpha(1), \beta(1) \in U$. Now there is $[\gamma_i], [\gamma_j] \in [[\gamma]]$ s.t. $[\alpha] \sim_U [\gamma_j]$ and $[\beta] \sim_U [\gamma_i]$.

Let $H : I \times \vec{I} \to Y$ realize the equivalence $[[\beta(t)]] \sim_{U_{[[\gamma]]}} [[\alpha(t)]]$. Then $p \circ H$ is a dihomotopy from α to β with endpoints in U, so $[\alpha] \sim_U [\beta]$. Hence $[\gamma_i] \sim_U [\gamma_j]$, which implies $[\gamma_i] = [\gamma_j]$ and thus $[\alpha],\ [\beta] \in U_{[\gamma_i]}$. This arguments holds for all dipaths $H_s(t)$, so in fact $H_s(1) \in \Phi(U_{[\gamma_i]})$. We conclude that $(U_{[[\gamma]]})_{[[\beta]]} = U_{[\gamma_i]} = U_{[\beta]}$. □

6. The quotient topology.

Ex. 5.1 shows that the topology τ on X may not be the quotient topology under $\Pi : \tilde{X}_{x_0}$. But if we take the quotient topology τ' on X, then the identity map $(X, \tau) \to (X, \tau')$ is an x_0-pov bijection, and hence an x-pov bijection for all $x \in X$.

Proposition 6.1. *Let $\Pi : \tilde{X}_{x_0} \to X$ be the universal dicovering of X and suppose that Π is surjective. Let $(X, \tau, \vec{P}(X))$ be X with the original d-structure and let $(X, \tau', \vec{P}'(X))$ be X with the quotient d-structure. Then $\vec{P}(X) = \vec{P}'(X)$. Moreover, $H : I \times \vec{I} \to X$ with $H(s, 0)$ constant, is a dihomotopy in $(X, \tau', \vec{P}'(X))$ if and only if, it is a dihomotopy in $(X, \tau, \vec{P}(X))$.*

Proof. Let $\gamma \in \vec{P}(X)$. Then, by Lem. 2.10, γ lifts uniquely to the dipath $[\gamma(t)] \in \vec{P}(\tilde{X})$. Now $\Pi \circ [\gamma(t)]$ is in $\vec{P}'(X)$ and $\Pi \circ [\gamma(t)] = \gamma(t)$, so $\vec{P}(X) \subset \vec{P}'(X)$. Now let $\mu \in \vec{P}'(X)$ and suppose $\mu = \Pi \circ \eta_1 \star \Pi \circ \eta_2$, where $\eta_i \in \vec{P}(\tilde{X})$. Use the lifting property to get $\eta_i(t) = [\mu_i(t)]$, where $\mu_i = \Pi \circ \eta_i \in \vec{P}(X)$. Hence $\mu = \Pi \circ [\mu_1(t)] \star \Pi \circ [\mu_2(t)]$. Then $\mu_1(1) = \mu_2(0)$ and by uniqueness of lifts, $\mu = \Pi \circ [\mu_1 \star \mu_2] = \mu_1 \star \mu_2 \in \vec{P}(X)$. Similarly reparametrization does not give any new dipaths in $\vec{P}'(X)$, so $\vec{P}'(X) = \vec{P}(X)$.

Let $H : I \times \vec{I} \to X$, $H_s(0)$ constant; then by the above reasoning, all dipaths $H_s(t)$ are in $\vec{P}'(X)$ if and only if they are in $\vec{P}(X)$. Moreover, if H is continuous in τ, it lifts uniquely to $\tilde{H} : I \times \vec{I} \to \tilde{X}$, and $H = \Pi \circ \tilde{H}$ is continuous in τ'. If H is continuous in τ', it is certainly continuous in τ, since τ' is the quotient topology and Π is continuous as a map to (X, τ), so the quotient topology is finer than τ. □

Corollary 6.2. *Let $Id : (X, \tau) \to (X, \tau')$ be the identity map. Then I is an x_0-pov bijection.*

Hence, for studying $\vec{\pi}_1$ taking the (perhaps finer) quotient topology on X is no restriction.

7. Final remarks

Up to choice of topology, dicoverings $p : Y \to X$ are quotients of the universal dicovering. But, since the topology on the universal dicovering is defined via dipaths and dihomotopies of such, the base space X may not have the quotient topology under $\Pi : \tilde{X}_{x_0} \to X$, so the quotient topology is not the right choice for $Y = \tilde{X}/\approx$ either. If the congruence relation $\approx$ is locally controlled, we can construct a topology on $Y = \tilde{X}/\approx$ such that Y is locally relatively diconnected with universal dicovering space $\tilde{X}$.

The other candidates for topologies - the quotient topology and the original topology on Y support the same dipaths and dihomotopies of dipaths as the ones constructed here, so they all give commutative diagrams

$$\begin{array}{ccc} \vec{P}(\tilde{X}_{[x_0]}, [[x_0]], [\gamma]) & \xrightarrow{\Phi} & \vec{P}(\tilde{X}/\approx, [[x_0]], [[\gamma]]) \\ \downarrow \Pi & \swarrow p & \\ \vec{P}(X, x_0, \gamma(1)) & & \end{array}$$

$$\begin{array}{ccc} \vec{\pi}_1(\tilde{X}_{[x_0]}, [[x_0]], [\gamma]) & \xrightarrow{\Phi} & \vec{\pi}_1(\tilde{X}/\approx, [[x_0]], [[\gamma]]) \\ \downarrow \Pi & \swarrow p & \\ \vec{\pi}_1(X, x_0, \gamma(1)) & & \end{array}$$

All the maps are surjections. By uniqueness of lifts, we have similar diagrams with $[x_0], [[x_0]], x_0$ replaced by $[\mu], [[\mu]], \mu(1)$.

If the space X is the geometric realization of a locally final $\square$-set, and we only consider dicoverings $Y \to X$ induced by $\square$-maps, then the choice of topology on Y and $\tilde{X}$ is given, X has the quotient topology and the topology constructed on Y is also the quotient topology. We will study this special case in a subsequent paper.

One may argue, that with this paper, we have provided an example, where generalization from geometric realizations of $\square$ complexes and $\square$-maps to topological spaces with direction does not pay off. However, even in classical topology, covering theory requires locally "nice" spaces, so it is to be expected that we find this here as well.

In relativity theory, the problem of finding the right topology for a space time has similarities to our problems her: Is the manifold topology the right one, if the objects of study are the increasing broken geodesics, the worldlines. See for instance [6, 3]. There is a metric involved and hence, ditopology cannot use the results verbatim, but similarities should definitely be explored. The common point is, that the topology (and d-structure) on a space defines the (di)paths, (and for our purposes, the dihomotopies) but not necessarily vice versa.

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