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Teaching Transfer Functions Without the Laplace Transform

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Biographical Sketches

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Abstract

Transfer functions are convenient representations to analyze cause-and-effect relationships of linear time-invariant dynamic systems. Traditionally, transfer functions are introduced using the Laplace transform. In this paper, we argue that the Laplace transform method is not a necessary prerequisite to understand the topic "transfer functions". We offer an inquiry-based learning method to represent transfer functions without formally using the full machinery of Laplace transforms. The method is used in an introductory engineering course on system modeling and simulation at Aalborg University, Denmark. The paper also presents an initial assessment of the students' understanding of transfer functions without a knowledge of Laplace transforms and the experiences gained from implementing inquiry-based learning in the course. Finally, we conclude with a discussion of the impact of CAS tools on mathematics and engineering science teaching and learning.

Keywords: *Laplace transforms, inquiry-based learning, transfer functions, modeling and simulation, block diagrams, Simulink*

1 Introduction

Transfer functions are compact representations of linear time-invariant dynamic systems where the governing differential equations become algebraic expressions that can be manipulated or combined with other expressions. They are used in block diagrams to describe the cause-and-effect relationships throughout the system. Traditionally, transfer functions are introduced using the Laplace transform.

The purpose of this paper is manyfold:

1. To present a method of teaching the topic "transfer functions" without explicitly

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introducing the Laplace transform, upon which the classical definition of a transfer function depends.

2. To investigate how the students understand the concept of transfer functions.

3. To argue that the *method* of Laplace transform in *an introductory modeling course* is not a prerequisite to discern the concept of a transfer function.

Inquiry-based learning is chosen as a didactical model to make it possible for the students to discern the important features of a transfer function, using learning-by-doing computer experiments.

The main research questions of this article are

1. Given the time limitation of the course, how can we introduce the transfer functions without involving the Laplace transform theory?

2. How far did the students understand the concept of a transfer function without a background in the Laplace transform theory?

2 Educational Setting

The first author of this paper is involved in teaching the course "System Modeling and Simulation", given to 5th semester engineering students of the study program "Sustainable Design" at Aalborg university, Denmark. According to the syllabus of the study program, the course "System Modeling and Simulation" should include the following topics:

1. Introduction to mathematical modeling.

2. Modeling of mechanical, electrical, thermal and fluid systems.

3. Linearization of non-linear dynamic systems.
4. Various representations of dynamic systems, including transfer functions and state-space equations.
5. Simulation of mathematical models using MATLAB and Simulink.[1]

In the first two years of their study, the students have had introductory courses in mathematics, including linear and separable differential equations and matrix algebra, engineering mechanics and thermodynamics but not in fluid mechanics or electrical circuits.

The purpose of the course is to give the students a self-contained introduction to modeling and simulation of commonly used engineering systems. It is not meant to be a prerequisite to more advanced courses in, e.g., control systems, which may well be the case at other engineering departments. The course runs every year in the Autumn semester and is worth 5 ECTS points. It's duration is 10 weeks, where the class meets once a week.

3 Limitations of the Study

The results of this study are limited to a first, introductory course in modeling and simulation of dynamic systems. They might not be generalizable to a standard course in control systems, where the connection between the time domain and the frequency domain is crucial in the analysis and design of feedback control systems. In addition, it is not the intention of the authors to undermine the importance of the Laplace transform in mathematics, physics, and engineering. On the contrary, the Laplace transform is one of the tools used by scientists and researchers in finding the

[1]Matlab and Simulink are registered trademarks of the software company MathWorks.

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solution to their problems. In fact, [Reddy et al., 2017] reviewed 25 research papers in various disciplines and discussed how the Laplace transform was used to solve some research problems.

4 Research Methodology

Our research methodology to answer the research questions consisted of observing how the students were reasoning during the exercise sessions and listening to their discussions when working in groups. It is mainly an exploratory and interview survey, where we used unstructured group interviews as well as participant observations [Kvale, 1994] of how the students worked with the exercises.

The observations were made in Autumn 2019 before the COVID-19 pandemic, where the course was offered for the first time, and again in Autumn 2020, where the university had switched to emergency remote teaching due to the COVID-19 pandemic.

The participants of the study are 32 5th semester students, all enrolled in the "Sustainable Design" engineering study program at the university.

The students were divided into 8 groups. Each group observation lasted about 15 minutes. A group consisted typically of three to five students.

In Autumn 2019, 8 groups were interviewed for about 20 minutes each. The interviews were documented and audio recorded.

In Autumn 2020, only group observations were made possible using breakout rooms in Microsoft Teams during the emergency remote teaching.

The students' responses were written in short formulations and long statements are summarized, such that the main points are reformulated in a few words. They allowed us to understand the link between that reasoning and the instructional approach used. Listening to the students' discussions could provide us with a great

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deal of information about how well the teaching method is successful in allowing the students to acquire new knowledge about the topic.

Since only a small number of engineering students participated in the study, the research results will not necessarily be generalizable: The goal is the transferability of the results, such that they could be useful in similar situations [Schofield, 1993].

5 Literature Review

The research literature on Laplace transforms and transfer functions is still scarce. However, we became acquainted with three papers that are relevant to our study.

In the paper [Carstensen and Bernhard, 2004], the authors mentioned some obstacles that could arise in the teaching and learning of Laplace transforms and concluded that it is one of the most difficult topics for electrical engineering students to grasp when learning electric circuit theory.

In the paper [Holmberg and Bernhard, 2008], the authors interviewed 22 university teachers regarding the difficulties involved in learning, and the relevance of, the Laplace transform in engineering education. They concluded that the teachers did not have a unified view on either the difficulties involved in learning, or the importance of, the Laplace transform. Moreover, the paper points to the importance of studying the conceptions of instructors themselves, in spite of the fact that educational research often focuses on the students' conceptions and misconceptions.

In [Lundberg et al., 2007], the authors, who all taught similar courses, concluded that students find the Laplace transform difficult mainly because there is significant confusion in its definition, as presented in many of the standard textbook presentations of the subject.

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6 On the Laplace Transform

The Laplace transform is a widely used integral transform that has important applications in many areas of mathematics, such as differential equations and probability theory. In physics and engineering, it is used in the analysis of linear time-invariant dynamic systems such as electrical circuits, harmonic oscillators, optical devices and mechanical systems.

The Laplace transform is named after the French mathematician and astronomer Pierre-Simon Laplace (1749-1827), who used a similar transform (now called the z-transform) in his work on probability theory [Deakin, 1981].

In 1809, Laplace extended his z-transform to find solutions of linear differential equations, giving the world the Laplace transform as we know it today [Struik, 2012]. However, the transform was not given a true physical meaning until the English mathematician and electrical engineer Oliver Heaviside (1850-1925) invented operational calculus, which is a new method using the operator D notation, that allowed him to transform difficult differential equations into simple algebraic equations.

The current widespread use of the Laplace transform came about soon after World War II, although it had been used in the 19th century [Deakin, 1982].

In introductory engineering courses on modeling, simulation and control of dynamic systems, the starting point is usually the development of appropriate mathematical models. These models usually lead to linear differential equations with constant coefficients. The method of Laplace transforms is a powerful tool for solving these equations, since it relies on algebra rather than calculus-based methods.

A Laplace transform is a mapping between the time domain and the domain of the complex variable s. It is defined by [Dorf and Bishop, 2011, p. 80]

$$F(s) = \mathscr{L}(f(t)) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \tag{1}$$

Laplace transforms are commonly used to solve the modeled differential equations, that involve functions of time t, by transforming them into algebraic equations, which involve the complex variable s. The algebraic equations, being much easier to solve than the original differential equations, are then transformed back to the time domain, using the inverse Laplace transform.

The Laplace transform methods provide a systematic approach for solving an ordinary differential equation and obtaining the dynamic response of the system which the differential equation represents. This approach consists of the following steps [Schiff, 2013, p. 60]:

1. Take the Laplace transform of every term of the differential equation and incorporate the initial conditions.

2. Using the result from step 1, solve for the Laplace transform of the dynamic variable $Y(s)$.

3. Obtain the system's dynamic response by taking the *inverse* Laplace transform $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

The critical step 3 usually yields an expression $Y(s)$ that seldom appears in Laplace transform tables. The procedure required here is to decompose the function $Y(s)$ into so-called partial fractions in order to determine the time-response function $y(t)$. This is yet another technique to be mastered in order to solve a differential equation by the Laplace transform.

The Laplace transform method offers an alternative approach for solving linear differential equations, even though it could prove tedious and time-consuming. In fact, the usual approach to solve linear differential equations is to assume a solution in the time domain. This is typically the first approach presented to most engineering students.

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Besides being a method to solve differential equations, the Laplace transform is also used in the classical definition of a transfer function [Dorf and Bishop, 2011, p. 87]:

Definition. The transfer function of a linear system is defined as the ratio of the Laplace transform of the output and the Laplace transform of the input, *assuming zero initial conditions*.

The transfer function is thus an algebraic representation of the differential equation describing the system. It is used to calculate the response of a system to a given input. Referring to the block diagram in Fig. 1, the output $Y(s)$ is equal to the product of the input $R(s)$ and the transfer function $G(S)$, i.e.,

$$Y(s) = R(s) \cdot G(s) \tag{2}$$

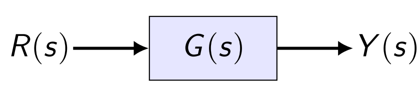


Figure 1: *A block diagram representing a transfer function*

7 Didactical Challenges in Teaching Transfer Functions

The aim of this section is to identify and analyze some of the theoretical and didactical challenges in teaching the topic "transfer functions of linear dynamic systems", given the time limitations of the course and the various backgrounds of the students.

Thus, our study involves the design of a theoretically inspired learning environment, to directly address a local problem, namely the challenges in the teaching of transfer functions. We also want to understand the relationships among the chosen didactical model and its practice.

Thus, before designing teaching situations in the subject, we carried out a preliminary analysis, consisting of two dimensions:

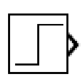
1. An epistemological and cognitive analysis of the mathematical and engineering content that will hopefully help us to find suitable teaching situations.

2. An institutional analysis of the context in which the teaching situations occur.

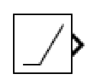
Regarding item 1, the curriculum of the course includes transfer functions of standard engineering systems, given that the concept "transfer function" relies on the terminology of the Laplace transform. Looking at some standard textbooks on modeling and control dynamic systems, such as [Ogata, 1998] and [Dorf and Bishop, 2011], they start with a somewhat comprehensive introduction to the Laplace transforms before defining transfer functions. Accordingly, the students have to learn another method of solving differential equations, in addition to the one they have had in their mathematics course, namely the method of undetermined coefficients [Boyce et al., 2017, p. 131]. Moreover, this introduction to the Laplace transform involves decomposition into partial fractions in order to eventually arrive at the time-domain solution of the differential equation. This is another didactic variable to consider, given that partial fraction decomposition is an unduly tedious process.

As mentioned, the course includes numerical simulations of dynamic systems using MATLAB and Simulink. In fact, these simulations are based on the time-domain differential equations representing the dynamic systems and not on the Laplace transform: Simulink "uses transfer functions to represent an I/O differential equation, but it does not use Laplace transform theory to obtain the system response (it uses direct numerical integration of the time-domain differential equations)." [Kluever, 2020, p. 96]

It is interesting to note that Simulink employs the icons shown in Fig. 2 to represent standard *time-domain* inputs (Fig. 3) to a system, even though the system itself is represented as a transfer function, which is a Laplace-domain concept. This clearly shows that, in Simulink, transfer functions are just *formal representations* of the underlying time-domain differential equations. In that regard, the authors believe that it is not sound, neither mathematically nor pedagogically, to mix the two domains in a block diagram, which is just a differential equation in disguise. For example, one can combine Simulink's time-domain derivative block (used to numerically differentiate an input signal with respect to time) (Fig. 4) with an integrator block or a transfer function block. Moreover, in a control system block diagram, two blocks in series can be multiplied together diagram, since every block is given in the Laplace domain. This is not valid in Simulink as one can use both domains in the same block diagram.

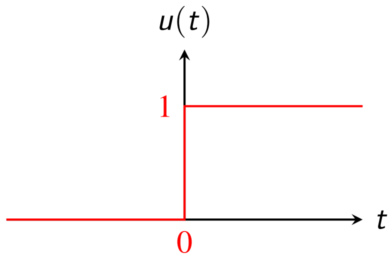


(a) *The Simulink icon for a unit step*

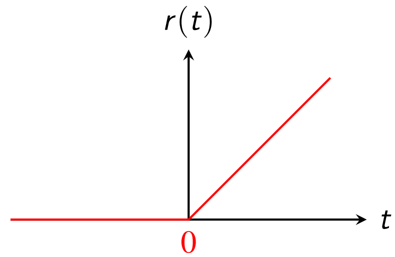


(b) *The Simulink icon for a unit ramp*

Figure 2: *Examples of input signals in Simulink*



(a) *The unit-step function,* $u(t)$



(b) *The unit-ramp function,* $r(t)$

Figure 3: *Examples of standard input time functions*

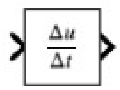


Figure 4: *The Simulink Derivative block*

Moreover, the authors find Simulink's use of the Laplace notation $\frac{1}{s}$ for the integration process (Fig. 5) unfortunate for two reasons:

1. The input signal, the numerical integration itself *and* the output signal are all in the time domain.

2. The expression $\frac{1}{s}$ is also the Laplace transform of the unit-step function (Fig. 3a):

$$u(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases} \tag{3}$$

   beside being the integral operator itself [Dorf and Bishop, 2011, p. 82]:

$$\frac{1}{s} \equiv \int_0^t \mathrm{d}t \tag{4}$$

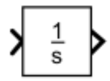


Figure 5: *The Simulink Integrator block*

In the Laplace transform theory, the differentiation theorem states that the Laplace transform of $\dot{y}$ is equal to $sY(s) - y(0)$, and the Laplace transform of $\ddot{y}$ is $s^2Y(s) - sy(0) - \dot{y}(0)$ and so on for higher derivatives. Now, since all initial conditions $y(0)$, $\dot{y}(0)$ and $\ddot{y}(0)$ are assumed to be zero when finding a transfer function of a system, we conclude that multiplying by the kth power of the Laplace variable s in the Laplace domain corresponds to the kth derivative in the time domain. We can therefore simplify the analysis of the governing mathematical model by defining the differential operator or "D operator" as

$$D \equiv \frac{\mathrm{d}}{\mathrm{d}t} \tag{5}$$

Using this notation, the time derivatives in a differential equation can be written as powers of the operator D: for example, $Dy = \dot{y}$, $D^2y = \ddot{y}$. We can then replace the differential operator D in the differential equation with s to obtain the transfer function and vice versa. As an example, if a system is modeled by the differential equation,

$$\ddot{y} + 8\dot{y} + 10y = 4u \tag{6}$$

where u is the input and y is the output, then we can get the *output-to-input ratio* $\frac{y(t)}{u(t)}$ by applying the D operator

$$\frac{y(t)}{u(t)} = \frac{4}{D^2 + 8D + 10} \tag{7}$$

Finally, replacing the operator D with s yields the transfer function of the system

$$G(s) = \frac{4}{s^2 + 8s + 10} \tag{8}$$

These results are also shown in block diagram form (Fig. 6).

$$u(t) \longrightarrow \boxed{\frac{4}{D^2+8D+10}} \longrightarrow y(t)$$

(a) *A block diagram in the time-domain*

$$R(s) \longrightarrow \boxed{\frac{4}{s^2+8s+10}} \longrightarrow Y(s)$$

(b) *A block diagram in the Laplace-domain*

Figure 6: *Two representations of the differential equation*

We can therefore use transfer functions to analyze the system response using Simulink, *without* relying on the Laplace transform theory. A full coverage of the Laplace transform theory may not be necessary for the students to grasp the concept of transfer function.

Regarding item 2 in this section, our department[2] decided that the duration of a 5 ECTS course, like ours, is 10 weeks. The time constraint could entail that the instructor should acts as decision maker and problem solver, rather than an executor of the course syllabus as an algorithm, since the time allocated for the course seems to be an obstacle for a full coverage of the Laplace transform, on which the concept of transfer function relies.

Moreover, since the students were never introduced to MATLAB before, an introduction to MATLAB and Simulink in the beginning of the course is necessary. And since modeling of major engineering systems and simulation using MATLAB

[2] Department of Planning, Aalborg University.

and Simulink are core elements of the syllabus of the course, we need to find a way that avoids the classical introduction to the Laplace transform theory and, at the same time, could lead to the learning outcomes of the topic transfer functions.

8 Didactical Model: Inquiry-Based Learning

The next step in the design of the teaching situations is the choice of a didactical model, i.e. a research-based theoretical platform within which we develop and analyze the teaching situations. In general, a didactical model can be selected from many approaches to education, including deductive and inductive methods or a mix of the two methods.

Engineering and science courses are traditionally taught deductively [Prince and Felder, 2006]: The instructor introduces a topic by giving lectures followed by illustrative examples. The students are then asked to do exercises or homework that are similar to the examples shown, and finally the instructor tests their abilities to do the same kind of problems on exams. In deductive teaching, the motivation the students usually get is that the topic will be relevant later in another course or in their career.

Deductive teaching is a traditional education method where knowledge is disseminated to students, the teacher is the sole distributor of knowledge and the student is a relatively passive receiver of this knowledge. Under these conditions, the learner is the object of the learning process, but not the subject.

In contrast to this method, inductive teaching starts with observations for the students to interpret, a case study to analyze or a real-world problem to solve. In their attempts to solve the problem, the students are motivated to learn a method or principle because they generate a desire to learn and a need to know [Albanese et al., 1993]. They are *then* presented with the needed information or guided to discover it for themselves.

In inductive learning, the instructor plays many important roles such as facilitating learning, clarifying concepts and even lecturing: An inductive teaching method does not mean total absence of lectures or complete reliance on self-discovery.

It should be noted that inductive teaching and learning is a family of instructional methods that includes inquiry-based learning, problem-based learning [Dahl, 2018], project-based learning and Brousseau's theory of didactical situations [Brousseau, 2006]. All these methods can be characterized as being *constructivist* approaches in the sense that students construct new knowledge for themselves by adjusting or rejection their prior beliefs and misconceptions in the light of the evidence provided by the experiences that are orchestrated by the instructor.

Our aim in the course is to trigger the students' curiosity about the different representations of a dynamic system. Even though we have various didactical models at our disposal, including problem-based learning and variation theory [Marton et al., 2004], we decided to choose inquiry-based learning as our didactical model, since "students learn new mathematics through inquiry by engaging in mathematical discussions, posing and following up on conjectures, explaining and justifying their thinking, and solving novel problems." [Rasmussen and Kwon, 2007, p. 190]

In addition, activating a student's curiosity is, we believe, a far more important and complex goal than mere lecturing on the topic: We want the students to develop their own skills as content-area experts in MATLAB simulations and problem solving by guiding them through interactive lectures and discussions. In fact, "the only strategy that is not consistent with inquiry-guided learning is the exclusive use of traditional lecturing." [Lee, 2004, p. 10]

Another reason of choosing inquiry-based learning is that several published research articles concluded that it is generally more effective than traditional instruction for achieving a variety of learning outcomes [Smith, 1997, Haury, 1993] and in improving the students' academic achievements and analytical skills [Shymansky et al., 1990]. Moreover, it is generally adopted and "approved" by policy makers, such as the European Commission:

"Inquiry-based science education (IBSE) has proved its efficacy at both primary and secondary levels in increasing children's and students' interest and attainment levels while at the same time stimulating teacher motivation. IBSE is effective with all kinds of students from the weakest to the most able and is fully compatible with the ambition of excellence." [European Commission, 2007, p. 2]

Finally, inquiry-based learning also allows the teachers themselves to *learn* how the students interpret and solve problems and *inquire* into student thinking [Rasmussen and Kwon, 2007]. This would enable the teachers to improve or revise the design of their didactical situations.

9 Implementation and Analysis of the Didactical Model

We now need to implement and analyze our didactical model by testing it in the classroom, using a set of exploratory questions the purpose of which is to enable the students to capture the essential aspects of the topic "transfer functions".

We started by asking the students to do computer experiment with the Simulink "Integrator" block using different input signals, such as constants (Fig. 7), ramps and sines. The purpose here is to allow the students to discover for themselves how the expression $\frac{1}{s}$ "operates" on the input signals to "transform" them to the output signals.

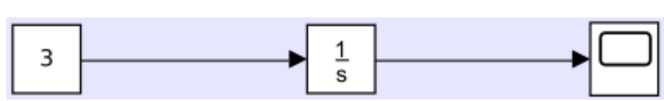


Figure 7: *Experimenting with the Simulink Integrator block*

We then started a dialogue with the students about the various definitions and representations of a function in mathematics, since different "representations are used to provide complementary information when each representation in the system contains (some) different information." [Ainsworth, 2006, p. 7]

Some students answered that a function can be given by an equation or a table. We then intervened and asked the following question: "How do you use the square root function $\sqrt{}$, found in many calculators?" The purpose of this question is to prepare the students to a *new* definition of a function: A function takes an input and *transforms* it to a single output.

The conversations in the class culminated in a block diagram (Fig. 8) where the function f *operates* on the independent variable x, now called the *input*, to produce the dependent variable y, called the *output*. The purpose of initiating the dialogue in the class is to prepare the students to acquire a new knowledge of transfer functions by building on their previous knowledge of functions in mathematics.

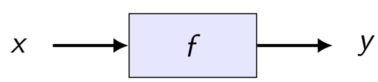


Figure 8: *A block diagram representing a function*

We then define the the "D operator" as

$$D \equiv \frac{\mathrm{d}}{\mathrm{d}t} \tag{9}$$

and used interactive lecturing to introduce the time derivatives in a differential equation as powers of the operator D: $Dy = \dot{y}$, $D^2y = \ddot{y}$, etc. (Fig. 9).

(a) *A block diagram representing the operator D*

(b) *A block diagram representing the operator D^2*

Figure 9: *Two block diagrams representing the first and second derivatives*

The students were then given a series of exercises that should be done in groups. The purpose of these exercises is to make it possible for them to "discover" the similarity between the operator D in differential equations (Fig. 10) and the complex variable s in a Simulink block diagram. Specifically, we asked the students to solve the differential equation in Fig 10b, *with pen and paper*, using the method they learned in the 1st semester, simulate the corresponding block diagram in Simulink (Fig. 11) and compare the results. We posted a video on non-homogeneous second-order linear differential equations on the course website to remind the students of the method they have learned in their mathematics course.

(a) *A block diagram representing an integral*

(b) *A block diagram representing a differential equation*

Figure 10: *Input-output block diagrams*

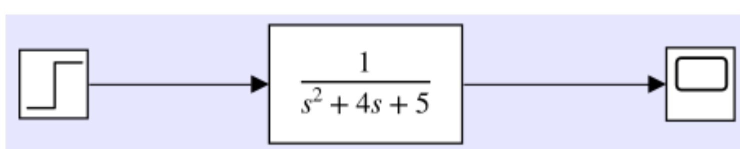


Figure 11: *The block diagram in Simulink*

Realistic problem solving improves the student's learning processes, since the student "gains *more* from solving a problem than getting to know the answer obtained" [Bosch and Winsløw, 2015]. Thus, to make it possible for the students to discover the practical relevance of the integrator block in Simulink and see it "in action", we finally gave them a problem from a real-life situation: [Kluever, 2020]

Problem. *In a cold January morning with an ambient temperature of* $T_a = -10°C$, *your instructor bought a cup of coffee at a local coffeehouse near campus. The coffee is initially at a temperature of* $80°C$ *when he received it, and it was in a capped to-go cup with a total thermal resistance* $R = 0.25°C \cdot s/J$ *(Fig. 12). The coffee has a total thermal capacitance of* $C = 1237$ *J/°C.*

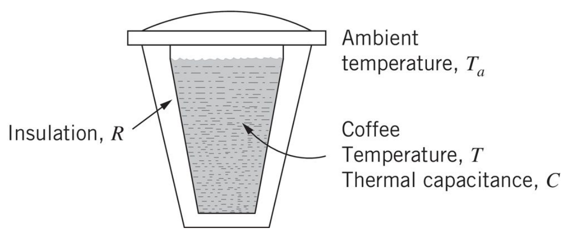


Figure 12: *A capped to-go coffee cup*

1. *Derive the mathematical model of the coffee cup.*
2. *Implement the mathematical model in Simulink.*
3. *It took your instructor 10 min to walk from the coffeehouse to his office. Use Simulink to determine the temperature of the coffee when he entered the building.*
4. *What would you suggest to increase the temperature of the coffee when he entered the building?*

The students now know that in order to use Simulink they should replace the differential operator D in the input-output ratio with the "symbol" s. Therefore, based on the students *own* experience gained from solving the exercises and the problem given, we gave the students our own *operational* definition of a transfer function, in the final wrap-up of the lesson.

Definition. The transfer function of a linear system is obtained from the input-to-output ratio by replacing the operator D in $G(D)$ (Fig. 13a) with the complex variable s (Fig. 13b).

(a) *A block diagram representing an input-to-output ratio*

(b) *A block diagram representing a transfer function*

Figure 13: *Defining the transfer function*

The motivation behind this definition is to get the students acquainted with Simulink terminology in simulating a dynamic system: Under the hood, it is precisely the time-domain input-to-output ratio (Fig. 13a) that is implemented in Simulink!

10 Evaluation of the Didactical Model

In this section, we take a step back and evaluate our didactical model. Did it fulfill our methodological ambitions and needs and was it a relevant methodological choice for solving the didactical challenge? Did it contribute to the students' learning?

Since the course is offered once per year and, to date, ran only two times, where we used inquiry-based methods in introducing transfer functions in both offerings of the course. Therefore, we have no other didactical model to compare with, in order to choose one for future use. Thus, we can only evaluate the didactical model itself and the students' learning outcomes, through internal and external validation respectively.

Internal validation refers to the comparison of the analysis of the didactical challenge of teaching the topic transfer functions with the actual implementation of the chosen didactical model. In that regard, inquiry-based learning did actually solve the problem of introducing transfer functions without a lengthy lecturing on the Laplace transforms. In fact, we just used a single teaching session to do the activities and the exercises mentioned in this paper, among the ten teaching sessions allocated to the course.

External validation is based on the observation and the documentation of the students' work on the exercises given, as wells their questions during the exercise sessions. The first author observed the students in their group rooms, while they were doing the exercises following the lecture, and acted as a "listening member" of their groups.

As an example of a group interview, we can mention the following interview that took place in October 2019. The first author has noted the student' solution, shown in Fig. 14b, of the block diagram in Fig. 10b.

Teacher: How did you come up with this formula?

Student 1: We multiplied D^2+4D+5 with y to get u.

Teacher: Okay. Is D a number?

5 sec silence

Student 2: It is a derivative, right?

Teacher: Yes, we call it an operator rather than a number.

Teacher: What is the connection with the Simulink diagram (Fig. 11), Student 3?

Student 3: I think we just substitute D with s but I really do not know why we should use two different symbols for the derivative.

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Teacher: Does any one know what s stands for?

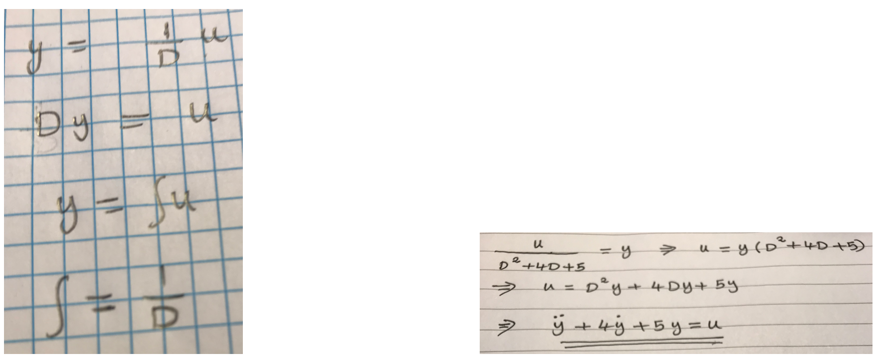
7 sec silence

Student 1: I have read at Wikipedia that s is a Laplace complex variable, but I did not understand the idea.

Teacher: Well, in mathematics, a Laplace transform converts a function of a real variable t (often time) to a function of a complex variable s (complex frequency).

Student 3: Why should we learn two symbols?

Teacher: Good question. Simulink could have used D as well. Using two symbols is really a matter of tradition and different representations.



(a) *A student solution of the integrator block in Fig. 10a*

(b) *A student solution of the block diagram in Fig. 10b*

Figure 14: *Examples of student solution of Fig. 10*

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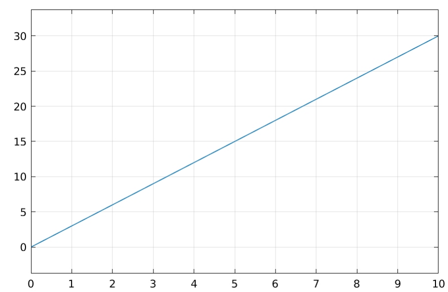
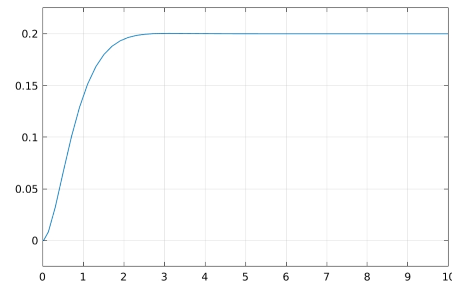


(a) *A student solution of the integrator block in Fig. 7*



(b) *A student simulation of the block diagram in Fig. 11*

Figure 15: *Examples of student simulations in Simulink*

The observation and interview of this student group could suggest that the students manipulated D as they would do for a real number in an equation. However, they recognized D as a representation of a time derivative and did in fact plot the solution in Simulink, but they failed to understand the idea that D is an (*operator*) rather than a number. Actually, this also goes for most groups that were interviewed.

Moreover, the observations of the students' work on the exercises, seem to suggest that the students have captured the simple idea that a transfer function is just *another* representation of a differential equation describing a dynamic system. Moreover, they had an *operational* understanding of Simulink inner logic in the block diagram representations of linear differential equation in the form of transfer functions.

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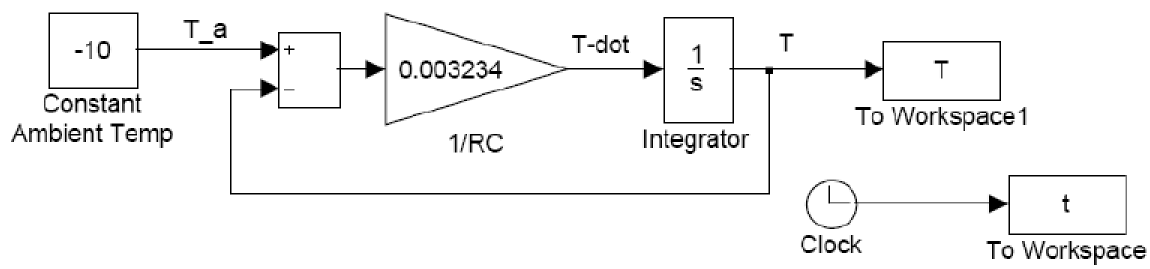

Figure 16: *A student's Simulink block diagram of the coffee cup problem*

The Simulink block diagram of the coffee cup problem (Fig. 16) shows that the student managed to implement the governing differential equation of the system *directly* in Simulink, using the integrator block. This was our intended object of learning regarding the integrator block and the implementation of differential equations in Simulink.

Even though some students needed more guidance than others, all succeeded in doing the exercises at the end. During the observations of the students' work, the first author became aware of the way the students understand the symbol s in a transfer function: They regard it as another symbol of the derivative with respect to time and not as a complex number. Moreover, the students had no knowledge of the connection between the time domain and the frequency domain. As mentioned, some students wondered why Simulink uses an s instead of D in the transfer function block. Apparently, this may be regarded as a challenge for the teacher, as the students were never introduced to Laplace transforms. In fact, it is not. It just showed that the students were still engaged in *learner inquiry* and asked the *right* question that could trigger them to seek new knowledge about the complex variable s.

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11 Discussion and Conclusion

Inquiry-based learning methods have been used extensively in the sciences [McDermott et al., 1995, Schlenker and Schlenker, 2000, Thacker et al., 1994], but, surprisingly enough, to a lesser extent in engineering [Buch and Wolff, 2000, Stahovich and Bal, 2002]. We have no evidence that the inquiry methods can be completely implemented in an engineering program or even in a single engineering course. However, this paper shows that can be *possible* to base teaching of a topic or a part of an engineering course on inquiry-based learning. The paper also shows that our "model" of the instructor as a transmitter or even a facilitator of knowledge does not completely correspond to the manifold of activities he or she carries out: The instructor is also a director of the didactic process, a problem-solver, a designer and a decision maker. Thus, even though our preliminary analysis was mainly triggered by the limited time allocated to the course, it seems that we have succeeded in designing and testing a method of tackling the topic "transfer functions" that we can use even if the teaching hours of the course were increased.

Similar to mathematical models, didactical models are rarely complete representations of the complex reality in the classroom, such as the students' heterogeneity and the variety of knowledge to be studied. These models should take into account the kind of help and guidance the instructor should give to the class, given that students differ in skills and knowledge and many of them need tailored guidance in order to learn.

When two models of the same didactical problem are available, the instructor may want to compare them with an eye to choosing one for future use. Such a comparison will always contain the instructor's individual interpretation of texts, the syllabuses and the teaching aims in the discipline, since there are many different aspects on which the decision of the instructor can be based. These aspects can also include the instructor's experience, beliefs and knowledge.

Based on the results of the study and the authors' experience and knowledge as instructors, we believe that using the Laplace transform method to obtain the system response does not impart new knowledge of the solution of a differential equation or the behavior of the dynamic system, just like using partial fractions to decompose an indefinite integral does not teach anything new about the concept of integration by doing it by hand. The method is cumbersome and time-consuming, given the limited time allocated to the course. Moreover, most differential equation, linear or not, that are relevant for the engineering profession, can be directly implemented in Simulink and solved numerically in the time domain, without the need of Laplace transforms or even transfer functions. Thus, the Laplace transform method to solve linear differential equations or derive the transfer function of a system, does not necessarily lead to the mastery of understanding transfer functions or the dynamics of differential equations, but rather, regarding a transfer function as a differential equation in disguise is a *fundamental concept* that unlocks the students' understanding of other concepts and builds a bridge to learning the Laplace transform itself.

All the three papers mentioned in Section 5 "Literature Review" concluded that the students find the Laplace transform difficult. For example, in [Carstensen and Bernhard, 2004], the authors pointed out that "what makes transient response difficult is that the mathematics used is rather advanced, using the Laplace Transform to solve differential equations."

We believe that this can be partially due to introducing the Laplace transform by starting with its standard mathematical definition: It may not make sense to many engineering students, since meaning and motivation are taken away from them, not to mention the reasons why other definitions failed. Moreover, "Connections among concepts, formal representations, and the real world are often lacking after traditional instruction." [Thornton, 1997]

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Therefore, instructors should strive to enable the students to establish *relations* between the formal definition of the Laplace transform and the real world, through realistic contexts that they are familiar with. This is the main reason why we chose inquiry-based learning as a didactical model to introduce transfer functions.

Since the students were introduced to time-domain differential equations in their first year, we therefore stayed in the time domain in analyzing differential equations and only used MATLAB and Simulink in numerical simulations. Furthermore, while it is true that MATLAB and Simulink often use transfer functions, which use the complex variable s, to *represent* the differential equations, their numerical solutions are based on the time-domain differential equations and *not* on the Laplace transform. For the sake of *representing* dynamics systems, we can simply interchange the D and s symbols to get the transfer function in order to use Simulink.

Moreover, if it is the case that we could find the inverse Laplace transform of an expression without the need to simplify in order to use the Laplace transform tables, engineering textbooks on the subject would not need to cover partial fractions in their introduction to Laplace transforms.

And why are we asking a student to find the inverse Laplace transform in the first place? What is being tested is the student's ability to have memorized and then apply a repetitive computational process, like a computer does. The student just reproduces the process and gets an answer. If it is the right answer, all we have done, as instructors, would be a verification that we have trained a well-prepared student to be replaced by a computer. The problem is with the question itself. We should train the students to understand when and where to apply an engineering or mathematical concept, to be able to break a question down and identify what aspects are involved and what methods need to be applied. Therefore, we should devise problems that test these skills.

To use the terminology of the anthropological approach in didactics, developed by Chevallard [Chevallard, 1999], engineering education should reconsider the study of a *domain*, taking the obsolescence of traditional techniques into account and to recognize new *techniques* as components of new *praxeologies* for that domain. Thinking of new techniques linked to the use of CAS tools and of their possible epistemic value is not easy because mathematical culture is traditionally associated with paper-and-pencil techniques and may not be accustomed to the idea that other tools can contribute to students' learning [Lagrange, 2005].

Solving linear differential equations by the Laplace transform method seems to be outdated; it is a painstaking paper-and-pencil technique that retain little pragmatic value, since this method is challenged by "mouse click" techniques in Simulink. However, paper-and-pencil techniques in solving linear differential equations in the time domain do not become obsolete because of the ease of using Simulink or any other CAS tool. On the contrary, they have an important epistemic value, and play a crucial role in the conceptualization of the *dynamics* of differential equations and in the time evolution of engineering systems.

In our course, we did not introduce the D-operator to generate suitable teaching situations only; we did it also to prepare our students for the world as it is now, not as it was. Therefore, we do not anticipate that the students solve engineering science and mathematics problems only by using CAS tools. Rather, we expect a "CAS-assisted" practice that is intertwined with paper-and-pencil techniques. Thus, we should regard the use of simulations in Simulink as calling for an integration between new techniques and paper-and-pencil methods.

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