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MUCKENHOUPT MATRIX WEIGHTS

MORTEN NIELSEN AND HRVOJE ŠIKIĆ*

Dedicated to Guido L. Weiss

Abstract. We study matrix weights defined on the multivariate torus $\mathbb{T}^d$. Sufficient conditions for a matrix weight to be in the Muckenhoupt $\mathbf{A}_2$-class are studied, and two such sufficiency results obtained by S. Bloom for $d = 1$ are generalized to the multivariate setting. As an application, an A_2-decomposition property is introduced for matrix weights and a BMO distance theorem for matrix weights is considered.

1. Introduction

Rooted in the studies of the Hardy-Littlewood maximal operator and the work on prediction theory by Kolmogorov and Krein and by Helson and Szegö, the theory of weighted norm inequalities developed with a very fast pace some four decades ago. The Analysis School at Washington University in St. Louis, of which Guido Weiss is a central figure, played one of the main roles in this development. It is not possible for us to give a full historic account here, but we shall mention only several of the classical papers from this period. The work of Muckenhoupt [13] on the boundedness of the Hardy-Littlewood maximal operator introduces a notion of the A_p-weight. Its importance is emphasized further in the papers by Hunt, Muckenhoupt and Wheeden [10], and by Coifman and Fefferman [3]. Among others, these papers deal with the boundedness of the conjugation operator. As it turned out, there is yet another interesting approach to this problem; using the Fefferman duality theorem and the class of BMO functions, it deals with the boundedness of the commutator operator (see the fundamental paper by Coifman, Rochberg and Weiss [5] for details).

The above mentioned papers influenced a rather pioneering work of S. Bloom, whose PhD dissertation was defended in August of 1981 at Washington University in St Louis. In his dissertation [1], as well as in several publications (for us here the most important being [2]), Bloom develops a theory of matrix-valued weights, which he successfully applies to the weighted norm inequalities for vector-valued functions. It took almost twenty years for the full realization of the significance of Bloom's approach, but the theory was pushed forward through the seminal work of Treĭl′ and Volberg [24], Nazarov and Treĭl′ [14], Volberg [25], and Goldberg [7]. It is fairly obvious to extend the notion of the A_p-weight from the scalar-valued case to the case of diagonal matrices, i.e., to the finite spectrum of the matrix. However, the extension to a wider class of matrices is far from trivial. For matrix-valued weights defined on a one-dimensional domain Bloom develops the notion of a log-preserving unitary matrix (which preserve the A_2-property)

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and shows that a wide range of unitary matrices, having Lipschitz coefficients, belong to the class LP of log-preserving matrices. Bloom's approach naturally leads toward the class of functions in BMO weighted spaces. The interest in Bloom's work and related subjects increased even more in the last several years with dozens of papers devoted to commutators, matrix weights, etc. (as an illustration we mention here [9], [16], [11]).

Our interest in the subject developed initially from the studies of shift invariant spaces in the theory of wavelets and other reproducing function systems (see [17] and [26]). The relationship between Schauder bases and Riesz bases in this context guided us naturally toward the stability of Schauder basis property (see [18]) and the Garnett-Jones distance in BMO spaces (see [19]). In particular, we developed the notion of the A_2-decomposition property for a large class of the so called Calderón-Zygmund coverings, and we show that this decomposition is essential for the Garnett-Jones distance theorem (for the original work on the subject consider the paper by Garnett and Jones [6], but also the related work by Coifman and Rochberg [4] and by F.Soria [22]).

In this paper we address the question of the A_2-decomposition property for matrix weights. The question presents us immediately with a technical problem of the extension of the Bloom method of the LP matrices to the case of matrix weights defined on a d-dimensional domain. The problem proved to be more involved than one may expect. We leave its full implementation (i.e., for functions with any Lipschitz-type condition) for future research. However, we were able to prove the desired result for functions in Lip_1. The argument is somewhat complex and we present it in detail in Section 2. In Section 3 we establish the A_2-decomposition property for matrix weights defined on a d-dimensional torus. The consequences of these results, together with the discussion about the intricacies of the BMO distance theorem for matrix weights, are presented in Section 4.

2. Muckenhoupt weights

Let $\mathcal{Q}$ denote the family of cubes in the multivariate torus $\mathbb{T}^d$, $d \in \mathbb{N}$. A (measurable) scalar weight $w : \mathbb{T}^d \to (0, \infty)$ is said to satisfy the Muckenhoupt A_p-condition, $1 < p < \infty$, provided

$$[w]_{A_p(\mathbb{T}^d)} := \sup_{Q \in \mathcal{Q}} \fint_Q w(x)\,dx \cdot \left[\fint_Q w^{-\frac{1}{p-1}}(x)\,dx \right]^{p-1} < \infty, \tag{2.1}$$

where for any measurable subset $E \subset \mathbb{T}^d$ of positive measure, we define

$$f_E := \frac{1}{|E|} \int_E f(x)\,dx,$$

and we use the notation

$$\fint_E f(x)\,dx := \frac{1}{|E|} \int_E f(x)\,dx.$$

We denote the class of such Muckenhoupt weights by $A_p(\mathbb{T}^d)$. Even though the scalar A_p-conditions are quite involved, they are still very much operational since quite large classes of, e.g., polynomial weights are known to satisfy the respective conditions, see e.g. [20].

Now consider a measurable matrix valued weight $W\colon \mathbb{T}^d \to \mathbb{C}^{N\times N}$, taking values in the positive definite $N\times N$-matrices. The matrix A_p condition was introduced and studied in [14, 24, 25] and it is considerably more complicated than the scalar condition, and there are no known straightforward sufficient conditions on a matrix weight to ensure membership in the A_p class except in very special cases (e.g., for diagonal weights and for weights with strong pointwise bounds on its spectrum).

S. Roudenko introduced an equivalent matrix A_p condition in [21] which is often more straightforward to verify. Let $1<p<\infty$ and let q be the dual exponent, $1/p+1/q=1$. The matrix A_p condition holds if and only if $W\colon \mathbb{T}^d \to \mathbb{C}^{N\times N}$ is measurable and positive definite a.e. and satisfies

$$[W]_{\mathbf{A}_p(\mathbb{T}^d)} := \sup_{Q\in\mathcal{Q}} \int_Q \left(\int_Q \|W^{1/p}(x)W^{-1/p}(t)\|^q \frac{dt}{|Q|}\right)^{p/q} \frac{dx}{|Q|} < \infty. \tag{2.2}$$

The norm $\|\cdot\|$ appearing in the integral is any matrix norm on the $N\times N$ matrices. The family of such matrix weights is denoted by $\mathbf{A}_p(\mathbb{T}^d)$. In the special case $N=1$ (1×1-matrices), one can verify that $\mathbf{A}_p(\mathbb{T}^d)=A_p(\mathbb{T}^d)$.

We will also need weighted vector-valued L^p-spaces, $1<p<\infty$. For $W\colon \mathbb{T}^d \to \mathbb{C}^{N\times N}$ a matrix-valued function, which is measurable and positive definite a.e., let $L^p(W)$ denote the family of measurable functions $f\colon \mathbb{T}^d \to \mathbb{C}^N$ with

$$\|f\|_{L^p(W)} := \left(\int_{\mathbb{T}^d} |W^{1/p}f|^p\,dx \right)^{1/p} < \infty.$$

In order to turn $L^p(W)$ into a Banach space, one has to factorize over $\{f\colon \mathbb{T}^d \to \mathbb{C}^N; \|f\|_{L^p(W)}=0\}$.

2.1. **A sufficient condition based on commutators.** We now turn to our first sufficient condition for membership in $\mathbf{A}_p(\mathbb{T}^d)$. We will primarily focus on $\mathbf{A}_2(\mathbb{T}^d)$-weights. Weighted BMO spaces and commutators will play an essential part in the sufficient condition, together with the fact that boundedness of the (discrete) Riesz transforms on $L^2(W)$ is equivalent to $W\in\mathbf{A}_2(\mathbb{T}^d)$.

Definition 2.1. Let $\nu\in A_2(\mathbb{T}^d)$. A function $b:\mathbb{T}^d\to\mathbb{C}$ is said to be in weighted $BMO_\nu(\mathbb{T}^d)$ provided

$$\sup_{Q\in\mathcal{Q}} \frac{1}{\nu(Q)} \int_Q |b(x)-b_Q|\,dx < +\infty,$$

where $\nu(Q) := \int_Q \nu(y)dy$.

For later use, we notice that the more conventional (unweighted) BMO is given by $BMO(\mathbb{T}^d) := BMO_1(\mathbb{T}^d)$.

We can now state the result. The univariate case, $d=1$, is due to Bloom [1]. The new contribution is the multivariate case.

Theorem 2.2. *Suppose the matrix weight $W\colon \mathbb{T}^d \to \mathbb{C}^{N\times N}$, $\mathbb{T}^d\subset\mathbb{R}^d$, admits a factorisation $W(t)=U^*(t)D(t)U(t)$, with U unitary a.e. and*

$$D(t) = diag(\lambda_1(t),\lambda_2(t),\ldots,\lambda_N(t)),$$

with $\lambda_j \in A_2(\mathbb{T}^d)$, $j = 1, \ldots, N$. Suppose that $U = (u_{ij})$, with

$$u_{ij} \in BMO_{\sqrt{\lambda_i \lambda_k^{-1}}}, \qquad k = 1, 2, \ldots, N.$$

Then $W \in \mathbf{A}_2(\mathbb{T}^d)$.

Proof. The case $d = 1$ has been proved by Bloom [1]. For $d \geq 2$, we let R_j denote the discrete Riesz transform, i.e., the periodized version of $\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$. For $f \in L^2_W$, we have

$$\begin{aligned}\|R_j f\|^2_{L^2(W)} &= \langle U^* D U R_j f, R_j f\rangle \\ &= \langle D U R_j f, U R_j f\rangle \\ &= \sum_{k=1}^N \int_{\mathbb{T}^d} |(U R_j f)_k(x)|^2 \lambda_k(x)\, dx.\end{aligned}$$

Notice that

$$\begin{aligned}U R_j f &= R_j(Uf) + U R_j f - R_j(Uf) \\ &= R_j(Uf) + U R_j U^*(Uf) - U U^* R_j U f.\end{aligned}$$

Hence, with $U = (u_{lk})$,

$$(U R_j f)_k = R_j(Uf)_k + \sum_{m=1}^N u_{km}(R_j U^*(Uf)_m - U^* R_j(Uf)_m) \tag{2.3}$$

Moreover,

$$\begin{aligned}R_j U^*(Uf)_m - U^* R_j(Uf)_m &= \sum_{r=1}^N \Big(R_j \bar{u}_{rm}(Uf)_r - \bar{u}_{rm} R_j(Uf)_r \Big) \\ &= \sum_{r=1}^N [R_j, M_{\bar{u}_{rm}}]\,(Uf)_r,\end{aligned}$$

where $[\cdot, \cdot]$ denotes the commutator. The fact that U is unitary implies that $|u_{lk}| \leq 1$, so using Eq. (2.3) we obtain

$$|(U R_j f)_k| \leq |R_j(Uf)_k| + \sum_{r=1}^N \sum_{m=1}^N \big| [R_j, M_{\bar{u}_{rm}}]\,(Uf)_r \big|.$$

Hence, there exists a constant C such that

$$|(U R_j f)_k|^2 \leq C |R_j(Uf)_k|^2 + \sum_{r=1}^N \sum_{m=1}^N \big| [R_j, M_{\bar{u}_{rm}}]\,(Uf)_r \big|^2,$$

where we used that the ℓ^1 and ℓ^2 norms are equivalent on $\mathbb{R}^{N^2+1}$. Now, since we have the scalar condition $\lambda_k \in A_2(\mathbb{T}^d)$, and R_j is a standard singular integral operator,

$$\int_{\mathbb{T}^d} |(R_j(Uf)_k|^2 \lambda_k\, dx \leq C \int_{\mathbb{T}^d} |(Uf)_k|^2 \lambda_k\, dx.$$

By the generalized Bloom commutator result, see [9], using that $\bar{u}_{rm} \in BMO_{\sqrt{\lambda_r \lambda_k^{-1}}}$,

$$\int_{\mathbb{T}^d} \left| [R_j, M_{\bar{u}_{rm}}]\,(Uf)_r \right|^2 \lambda_k\, dx \le C \int_{\mathbb{T}^d} |(Uf)_r|^2 \lambda_r\, dx.$$

So for $j = 1, \ldots, d$,

$$\|R_j f\|_{L^2(W)}^2 \le C \sum_{r=1}^{N} \int_{\mathbb{T}^d} |(Uf)_r|^2 \lambda_r\, dx = C\|f\|_{L^2(W)}^2.$$

It now follows from [15, Corollary 4.2] that boundedness of the discrete Riesz transforms in $L^2(W)$ imply that W is in $\mathbf{A}_2(\mathbb{T}^d)$, which completes the proof. □

2.2. **Lipschitz continuous matrix weights.** The conditions $u_{ij} \in BMO_{\sqrt{\lambda_i \lambda_k^{-1}}}$ in Theorem 2.2 may be somewhat challenging to verify in specific cases. We shall now derive a more straightforward sufficient condition to be in $\mathbf{A}_2(\mathbb{T}^d)$.

Recall that a function $f : \mathbb{T}^d \to \mathbb{C}$ is said to be in $\mathrm{Lip}_\gamma(\mathbb{T}^d)$, $0 < \gamma \le 1$, provided there exists a Lipschitz constant $C_\gamma < \infty$ such that

$$|f(x) - f(y)| \le C_\gamma |x - y|^\gamma, \qquad x, y \in \mathbb{T}^d.$$

We can now state the result. The univariate case, $d = 1$, is due to Bloom [1]. The multivariate case is, as far as we know, new.

Theorem 2.3. *Let $W = U^* D U$, with $D = diag(e^{\lambda_1}, \ldots, e^{\lambda_N})$, where $e^{\lambda_i} \in A_2$, and $U = (u_{ij})$ a unitary matrix function with entries $u_{ij} \in Lip_1(\mathbb{T}^d)$, $i, j \in \{1, 2, \ldots, N\}$. Then there exists $\eta > 0$ such that $W^\eta = U^* D^\eta U \in \mathbf{A}_2(\mathbb{T}^d)$, where the value of η and the A_2 constant of W^η depend only on the Lipschitz constants of $\{u_{ij}\}_{ij}$ and on the A_2 constants of the scalar functions $\{e^{\lambda_i}\}_i$.*

We will prove the proposition by analyzing certain averaging operators. As before, we let $\mathcal{Q}$ denote the collection of cubes in $\mathbb{T}^d$. For $Q \in \mathcal{Q}$, and a locally integrable vector-valued function $f : \mathbb{T}^d \to \mathbb{C}^N$, we consider the vector-valued operator

$$A_Q f(x) := \chi_Q(x) \frac{1}{|Q|} \int_Q f(t)dt,$$

where the integral is applied coordinate-wise.

The following fundamental property of the operators $\{A_Q\}_{Q \in \mathcal{Q}}$ was proved by Nazarov and Treĭl′ in [14], see also [7, Proposition 2.1].

Theorem 2.4. *Let $W\colon \mathbb{T}^d \to \mathbb{C}^{N \times N}$ be a matrix weight. Then $W \in \mathbf{A}_p(\mathbb{T}^d)$ if and only if the operators $\{A_Q\}_{Q \in \mathcal{Q}}$ are uniformly bounded on $L^p(W)$. Moreover, the supremum of the operator norms of $\{A_Q\}_{Q \in \mathcal{Q}}$ is equivalent to the A_p-constant $[W]_{\mathbf{A}_p(\mathbb{T}^d)}$ given by (2.2).*

A well-know useful observation is that for any linear operator T, we have

$$\|Tf\|_{L^p(W)} = \|W^{1/p} T f\|_{L^p(Id)} = \|W^{1/p} T W^{-1/p} g\|_{L^p(Id)},$$

with $g = W^{1/p} f$ satisfying $\|f\|_{L^p(W)} = \|g\|_{L^p(Id)}$. Hence, boundedness of T on $L^p(W)$ is equivalent to boundedness of the conjugate operator $W^{1/p} T W^{-1/p}$ on the unweighted space $L^p(\mathbb{C}^N) := L^p(Id)$. We now turn to the proof of Theorem 2.3.

Proof. We first assume that the elements of D have been normalized such that $\|e^{\lambda_i}\|_{L^1(\mathbb{T}^d)} = 1$ for $i = 1, 2 \ldots, N$. For $\tau > 0$, we have $W^\tau = U^* D^\tau U$, with

$$D^\tau = \operatorname{diag}(e^{\tau\lambda_1}, \ldots, e^{\tau\lambda_N}).$$

For use later, we notice that for any linear operator T on $L^p(W^\tau)$, with $f \in L^p(W^\tau)$,

$$\begin{aligned}
\|Tf\|^p_{L^p(W^\tau)} &= \int_{\mathbb{T}^d} \langle W^{2\tau/p} Tf, Tf\rangle^{p/2}\, dx \\
&= \int_{\mathbb{T}^d} \left\langle \exp\left(\frac{2\tau}{p} D\right) UTf, UTf \right\rangle^{p/2} dx \\
&= \int_{\mathbb{T}^d} \left(\sum_k |(UTf)_k|^2 \exp\left(\frac{2\tau\lambda_k}{p}\right)\right)^{p/2} dx \\
&\geq \int_{\mathbb{T}^d} |(UTf(x))_k|^p \exp(\tau\lambda_k)\, dx, \qquad k = 1, 2, \ldots, N. \qquad (2.4)
\end{aligned}$$

Now fix $Q \in \mathcal{Q}$ and let f be any smooth vector-valued function. For any fixed $2 \leq p < \infty$, we notice that for any $0 < \tau \leq 1$ the scalar Muckenhoupt conditions $e^{\tau\lambda_k} \in A_p(\mathbb{T}^d)$ for $k = 1, \ldots, N$ are satisfied, see [8, Chap. 7]. For use later, we notice that for a scalar-valued function g, we can use Theorem 2.4 in the scalar case to conclude that

$$\int_{\mathbb{T}^d} |A_Q g|^p e^{\tau\lambda_r}\, dx \leq K \int_{\mathbb{T}^d} |g|^p e^{\tau\lambda_r}\, dx, \qquad r = 1, 2, \ldots, N, \qquad (2.5)$$

where $K^{1/p}$ is any upper bound on the A_p-norms of $e^{\tau\lambda_1}, \ldots, e^{\tau\lambda_N}$. Notice, in particular, that K does not depend on Q.

We now write

$$U(x)A_Q f(x) = U(x)\frac{\chi_Q(x)}{|Q|}\int_Q f(t)\, dt = \frac{\chi_Q(x)}{|Q|}\int_Q U(x)U^*(t)(Uf)(t)\, dt,$$

where the matrix $U(x)U^*(t)$ is continuous in t, and equals the identity at $t = x$. With δ_{ik} the Kronecker delta, and recalling that $|u_{ij}| \leq 1$ since U is unitary,

$$\begin{aligned}
|(U(x)U^*(t))_{ik} - \delta_{ik}| &= \left| \sum_r u_{ir}(x)\bar{u}_{kr}(x) - \sum_r u_{ir}(x)\bar{u}_{kr}(x) \right| \\
&\leq \sum_r |u_{ir}(x)||\bar{u}_{kr}(t) - \bar{u}_{kr}(x)| \\
&\leq \sum_r |\bar{u}_{kr}(t) - \bar{u}_{kr}(x)| \\
&\leq M|x - t|,
\end{aligned}$$

with M any upper bound for the Lipschitz constants of the collection $\{\bar{u}_{ij}\}_{ij}$. Consequently,

$$|(U(x)U^*(t))_{ik}| \leq \delta_{ik} + M|x - t|. \qquad (2.6)$$

With a view towards (2.4) with $T = A_Q$, we now employ the estimate (2.6) to deduce the following

$$\begin{aligned}
|(U(x)A_Qf)(x))_r| &= \frac{\chi_Q(x)}{|Q|}\left|\int_Q \sum_k (U(x)U^*(t))_{rk}(Uf)_k(t)\,dt\right| \\
&\leq \frac{\chi_Q(x)}{|Q|}\int_Q \sum_k (\delta_{rk} + M|x-t|)|(Uf)_k(t)|\,dt \\
&\leq A_Q(|(Uf)_r|)(x) + M\sum_k \frac{1}{|Q|}\int_Q |x-t||(Uf)_k(t)|\,dt \\
&\leq A_Q(|(Uf)_r|)(x) + cM\sum_k \int_Q |x-t|^{1-d}|(Uf)_k(t)|\,dt \\
&\leq A_Q(|(Uf)_r|)(x) + cM\sum_k \int_{\mathbb{T}^d} |x-t|^{1-d}|(Uf)_k(t)|\,dt, \qquad (2.7)
\end{aligned}$$

where we used that $|Q| \asymp \operatorname{diam}(Q)^d \geq |x-t|^d$ for $x, t \in Q$.

If $d = 1$, we may skip the following step and continue the estimate (2.7) directly as indicated below in (2.8) with any $p > 2$. For $d \geq 2$, however, we need an additional estimate. We fix $q > 1$ such that $q(d-1) < d$ and we define p by $1/p + 1/q = 1$, where we notice that $p > 2$. Pick $s > 1$ such that $sq(d-1) < d$ and let $1/s + 1/s' = 1$. Then

$$\int_{\mathbb{T}^d} |x-t|^{q(1-d)} \exp(-\tau q/p\lambda_k)\,dt \leq \left(\int_{\mathbb{T}^d} |x-t|^{sq(1-d)}\,dt\right)^{1/s}\left(\int_{\mathbb{T}^d} \exp\left(-\frac{s'\tau q}{p}\lambda_k\right)dt\right)^{1/s'}.$$

We now adjust the value of τ so $\tau s' = 1$, which ensures that $e^{\tau s'\lambda_k} \in A_p$, $k = 1, \ldots, N$, and implies that $\exp(-s'\tau q\lambda_k/p) \in L^1(\mathbb{T}^d)$, $k = 1, \ldots, N$, with norms that depend only on the A_2-constants of $\{e^{\lambda_k}\}$. Consequently, with these choices,

$$\int_{\mathbb{T}^d} |x-t|^{q(1-d)} \exp(-\tau q/p\lambda_k)\,dt \leq C < \infty.$$

We now continue from the estimate (2.7),

$$\begin{aligned}
|(U(x)A_Qf)(x))_r| &\leq A_Q(|(Uf)_r|)(x) + cM\sum_k \int_{\mathbb{T}^d} |x-t|^{1-d}|(Uf)_k(t)|e^{\tau\lambda_k/p}|e^{-\tau\lambda_k/p}\,dt \\
&\leq A_Q(|(Uf)_r|)(x) + cM\sum_k \left(\int_{\mathbb{T}^d} |(Uf)_k(t)|^p e^{\tau\lambda_k}\,dt\right)^{1/p} \\
&\qquad\qquad \times \left(\int_{\mathbb{T}^d} |x-t|^{q(1-d)} e^{-\tau q\lambda_k/p}\,dt\right)^{1/q} \\
&\leq A_Q(|(Uf)_r|)(x) + cMC^{1/q}\sum_k \left(\int_{\mathbb{T}^d} |(Uf)_k(t)|^p e^{\tau\lambda_k}\,dt\right)^{1/p} \\
&\leq A_Q(|(Uf)_r|)(x) + \tilde{C}\|f\|_{L^p(W)}, \qquad (2.8)
\end{aligned}$$

where we used the estimate (2.4). Finally, using (2.8) once more,

$$\begin{aligned}\|A_Q f\|^p_{L^p(W^\tau)} &= \int_{\mathbb{T}^d}\left(\sum_r |(UA_Q f(x))_r|^2 \exp\left(\frac{2\tau}{p}\lambda_r\right)\right)^{p/2} dx\\ &\le \int_{\mathbb{T}^d}\left(\sum_r (A_Q(|(Uf)_r|)(x) + \tilde{C}\|f\|_{L^p(W)})^2 \exp\left(\frac{2\tau}{p}\lambda_r\right)\right)^{p/2} dx\\ &\le \int_{\mathbb{T}^d}\left(\sum_r (2A_Q(|(Uf)_r|)(x)^2 + 2\tilde{C}^2\|f\|^2_{L^p(W)}) \exp\left(\frac{2\tau}{p}\lambda_r\right)\right)^{p/2} dx\end{aligned}$$

Now we use Hölder's inequality with parameter $p/2 > 1$ in the finite sum over r,

$$\begin{aligned}&\le C(p)\int_{\mathbb{T}^d}\left(\sum_r (A_Q(|(Uf)_r|)(x)^p + \tilde{C}^p\|f\|^p_{L^p(W)}) \exp(\tau\lambda_r)\right) dx\\ &\le C(p)\int_{\mathbb{T}^d} A_Q(|(Uf)_r|)(x)^p \exp(\tau\lambda_r)\, dx + C(p)\tilde{C}^p\|f\|^p_{L^p(W)}\sum_r \|e^{\tau\lambda_r}\|_{L^1(\mathbb{T}^d)}\\ &\le C(p)K\int_{\mathbb{T}^d} |(Uf)_r|^p \exp(\tau\lambda_r)\, dx + C(p)\tilde{C}^p\|f\|^p_{L^p(W)}\sum_r \|e^{\tau\lambda_r}\|_{L^1(\mathbb{T}^d)},\end{aligned}$$

where we used (2.5) in the last step. We use the estimate (2.4) to conclude that

$$\|A_Q f\|^p_{L^p(W^\tau)} \le C'\|f\|^p_{L^p(W^\tau)}, \tag{2.9}$$

with

$$C' := C(p)\left(K + \tilde{C}^p\sum_r \|e^{\tau\lambda_r}\|_{L^1(\mathbb{T}^d)}\right)$$

independent of Q. Hence, A_Q extends to a uniformly bounded family of bounded operators on $L^p(W^\tau)$. We now follow Bloom [1] and use Stein's interpolation theorem, see [23],to cover the case $L^2(W^\eta)$ for a suitable value of $\eta > 0$. Define the analytic class

$$T_z := W^{\tau z/p} A_Q W^{-\tau z/p}, \qquad \{z : 0 \le \mathrm{Re}(z) \le 1\},$$

and notice that (2.9) implies that T_1 is bounded on the (unweighted) vector-valued $L^p(\mathbb{C}^N)$. Also notice that $T_0 = A_Q$ is bounded (uniformly in Q) on $L^q(\mathbb{C}^N)$, which follows from Theorem 2.4 in the scalar case. Using the observation that for $z = x + iy$,

$$W^{\tau(x+iy)/p} A_Q W^{-\tau(x+iy)/p} = W^{i\tau y/p} T_x W^{-i\tau y/p},$$

with $W^{\pm i\tau y/p}$ unitary matrices, one can check that the conditions in Stein's interpolation theorem are met. Hence, by Stein's interpolation theorem, $T_{\frac{1}{2}}$ is bounded on $L^2(\mathbb{C}^N)$ with an operator norm that does not depend on Q. Therefore, $W^{\tau/p} \in \mathbf{A}_2(\mathbb{T}^d)$, and we may put $\eta = \tau/p$. We note that the specific choice of p only depends on d and the choice of τ only depends on d and p. The operator norm of $T_{1/2}$ depends only on p and the A_2 constants of $\{e^{\lambda_i}\}_i$.

Finally, for D without L^1-normalization, we introduce $V = U^*BU$, where

$$B = \mathrm{diag}(\|e^{\lambda_1}\|^{-1}_{L^1(\mathbb{T}^d)}, \dots, \|e^{\lambda_N}\|^{-1}_{L^1(\mathbb{T}^d)}).$$

Notice that $W' = VW = U^*(BD)U$, where BD has normalized elements. Using Lemma 2.5 below, $(W')^{\tau/p} \in \mathbf{A}_2(\mathbb{T}^d)$ if and only if $W^{\tau/p} \in \mathbf{A}_2(\mathbb{T}^d)$. This completes the proof. □

The following lemma, which may be of independent interest, was used in the proof of Theorem 2.3. We let $L^\infty_{N\times N}$ denote the family of $N \times N$-matrices defined on $\mathbb{T}^d$ with entries in $L^\infty(\mathbb{T}^d)$.

Lemma 2.5. *Suppose $V \in \mathbf{A}_2(\mathbb{T}^d)$ and $B = \exp(B')$ with $B' \in L^\infty_{N\times N}$ satisfying $[V, B] = 0$, then $W = VB \in \mathbf{A}_2(\mathbb{T}^d)$.*

Proof. By the result of Roudenko [21], $V \in \mathbf{A}_2(\mathbb{T}^d)$ is equivalent to the uniform estimate over cubes Q,

$$\int_Q \int_Q \|V^{1/2}(x)V^{-1/2}(t)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|} \leq C_V < \infty.$$

Now, we notice that by the uniform bound on the operator norm of B, using that $V^{1/2}$ and $B^{1/2}$ commute,

$$\int_Q \int_Q \|(BV)^{1/2}(x)V^{-1/2}(t)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|} \leq C_Q \int_Q \int_Q \|V^{1/2}(x)V^{-1/2}(t)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|}.$$

However, using the fact that $(BV)^{1/2}$ and $V^{-1/2}$ are self-adjoint, and that for self-adjoint matrices of the same size C and D, $\|CD\| = \|(CD)^*\| = \|DC\|$,

$$\int_Q \int_Q \|(BV)^{1/2}(x)V^{-1/2}(t)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|} = \int_Q \int_Q \|V^{-1/2}(t)(BV)^{1/2}(x)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|}.$$

Now we use the uniform bound on the operator norm of B^{-1}, and that $V^{-1/2}$ and $B^{-1/2}$ commute,

$$\begin{aligned}\int_Q \int_Q \|(BV)^{-1/2}(t)(BV)^{1/2}(x)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|} &\leq C_{B^{-1}} \int_Q \int_Q \|V^{-1/2}(t)(BV)^{1/2}(x)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|} \\ &= C_{B^{-1}} \int_Q \int_Q \|(BV)^{1/2}(x)V^{-1/2}(t)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|}.\end{aligned}$$

By combining the above estimates, we obtain

$$\int_Q \int_Q \|(BV)^{-1/2}(t)(BV)^{1/2}(x)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|} \leq C_Q C_{B^{-1}} \int_Q \int_Q \|V^{1/2}(x)V^{-1/2}(t)\|^2 \frac{dt}{|Q|}\frac{dx}{|Q|},$$

and we conclude that $(VB)^{-1} \in \mathbf{A}_2(\mathbb{T}^d)$ so $VB \in \mathbf{A}_2(\mathbb{T}^d)$ and the lemma follows. □

3. The matrix A_2-decomposition property

In the recent paper [19], the present authors studied the BMO distance theorem of Garnett and Jones and certain decomposition properties of scalar Muckenhoupt weights related to the Jones factorization theorem, see [12]. In particular, it was proven that the BMO distance theorem holds if and only if a certain A_2-decomposition property holds.

Here we consider a corresponding A_2-decomposition property for matrix weights.

Definition 3.1. Let $\mathcal{F}$ be a family of $\mathbf{A}_2(\mathbb{T}^d)$ matrix weights. We say that $\mathcal{F}$ has the matrix A_2-decomposition property provided there exist constants $K = K(\mathcal{F})$, and $\delta := \delta(\mathcal{F}) > 0$, such that for $W \in \mathcal{F}$, there exist commuting matrix weights V, B, satisfying

$$W^\delta = VB,$$

with $V \in \mathbf{A}_2(\mathbb{T}^d)$ such that $[V]_{\mathbf{A}_2(\mathbb{T}^d)} \leq K$, and B is a matrix with spectrum that is (essentially) bounded and bounded from below on $\mathbb{T}^d$.

Remark 3.2. The A_2-decomposition property is trivially satisfied for any finite family of matrix weights in $\mathbf{A}_2(\mathbb{T}^d)$. In the scalar case it is known, see [19], that the full A_2 class has the A_2-decomposition property.

It is not known at present whether the full class $\mathbf{A}_2(\mathbb{T}^d)$ has the matrix A_2-decomposition property. We therefore restrict our attention to certain subsets of $\mathbf{A}_2(\mathbb{T}^d)$. We need the following definition of matrix BMO.

Definition 3.3. We say that a matrix weight $W\colon \mathbb{T}^d \to \mathbb{C}^{N\times N}$, $W = (w_{ij})$, is in BMO provided that each entry $w_{ij} : \mathbb{T}^d \to \mathbb{C}$, $i, j \in \{1, \dots, N\}$, is a scalar valued BMO function. The BMO-norm of such W is defined by

$$\max_{i,j} \|w_{ij}\|_{BMO(\mathbb{T}^d)}.$$

Bloom proves in [1], in the case $d = 1$, that $W \in \mathbf{A}_2(\mathbb{T}^d)$ implies that $\log(W) \in BMO$. One can easily verify that Bloom's proof generalizes verbatim to the multivariate case. However, Bloom also constructs a symmetric $B \in BMO$ for which $\exp(tB) \notin \mathbf{A}_2(\mathbb{T}^d)$ for any $t > 0$. To avoid such ill-conditioned matrices, we follow Bloom [1] and define so-called log-preserving unitary matrices.

Definition 3.4. A unitary matrix weight $U\colon \mathbb{T}^d \to \mathbb{C}^{N\times N}$ is called log-preserving, denoted $U \in LP$, if for any diagonal matrix $\Lambda \in BMO$, there exists an $\alpha > 0$ depending only on the BMO-norm of Λ, and the matrix U, such that

$$U^* \exp(\alpha\Lambda)U \in \mathbf{A}_2(\mathbb{T}^d),$$

with the corresponding A_2-constant depending only on the the BMO-norm of Λ, and the matrix U.

We have the following immediate corollary to Theorem 2.3.

Corollary 3.5. *Suppose $U\colon \mathbb{T}^d \to \mathbb{C}^{N\times N}$, $U = (u_{ij})$, is a unitary matrix function with entries $u_{ij} \in Lip_1(\mathbb{T}^d)$, $i, j \in \{1, 2, \dots, N\}$. Then U is log-preserving.*

We now prove that certain families of matrix weights associated with log-preserving matrices satisfy the A_2-decomposition property.

Proposition 3.6. *Let $U\colon \mathbb{T}^d \to \mathbb{C}^{N\times N}$ be a log-preserving weight. Consider any family*

$$\mathcal{M} := \{U^*\Lambda_\gamma U\}_{\gamma\in F},$$

with $\Lambda_\gamma = diag(\lambda_1^\gamma, \dots \lambda_N^\gamma)$, where $\lambda_j^\gamma \in A_2(\mathbb{T}^d)$, $\gamma \in F$, $j = 1, 2, \dots, N$. Then $\mathcal{M}$ has the matrix A_2-decomposition property.

Proof. By the scalar A_2-decomposition property, there exist $K > 0$, and $\delta > 0$, such that

$$(\lambda_j^\gamma)^\delta = u_j^\gamma b_j^\gamma,$$

with $[u_j^\gamma]_{A_2} \leq K$, and b_j^γ bounded and bounded from below. Notice that by standard estimates,

$$\| \log(u_j^\gamma)\|_{BMO} \leq K, \qquad \gamma \in F, j = 1, \dots, N.$$

Let $V_\gamma = \text{diag}(u_1^\gamma, \dots, u_N^\gamma)$. Then $\log(V_\gamma) \in BMO$ with norm bounded uniformly for $\gamma \in F$. Hence there exists $\alpha := \alpha(K, U) > 0$ such that

$$U^* \exp(\alpha \log(V_\gamma))U = U^* V_\gamma^\alpha U \in \mathbf{A}_2(\mathbb{T}^d),$$

with the corresponding A_2-constant depending only on U and K. Put

$$B_\gamma = \text{diag}(b_1^\gamma, \dots, b_N^\gamma),$$

and notice that the spectrum of B_γ is bounded, and bounded from below. The decomposition

$$(U^*\Lambda_\gamma U)^\alpha = U^*\Lambda_\gamma^\alpha U = (U^* V_\gamma^\alpha U)(U^* B_\gamma^\alpha U)$$

now provided the desired A_2-decomposition. □

4. The distance to a matrix in BMO

We conclude the paper by a study of the BMO distance to L^∞ for matrices related to weights in $\mathbf{A}_2(\mathbb{T}^d)$. For a matrix weight $M : \mathbb{T}^d \to \mathbb{C}^{N\times N}$, satisfying $\exp(M) \in \mathbf{A}_2(\mathbb{T}^d)$, we define

$$\varepsilon(M) := \inf_{\lambda\in\mathbb{R}_+} \{\lambda : \exp(M/\lambda) \in A_2(\mathbb{T}^d)\},$$

where we notice that $\varepsilon(M) \leq 1$. In the scalar case, using a reverse Hölder estimate, one can deduce that ε is always strictly greater than one, but this self-improving type result is known to fail for matrix weights [1].

We introduce two notions of BMO distance to L^∞ for a matrices.

Definition 4.1. Suppose $U\colon \mathbb{T}^d \to \mathbb{C}^{N\times N}$ is a log-preserving matrix. For $M = U^*DU$, with D a diagonal matrix in BMO, we define

$$\text{dist}(M, L^\infty) := \inf\{\|D - D'\|_{BMO} : B = U^*D'U \in L^\infty_{N\times N}\},$$

where D' varies over the diagonal matrices in $L^\infty_{N\times N}$. Moreover, we define

$$\text{dist}(M, L^\infty_{N\times N}) := \inf\{\|M - B\|_{BMO} : B \in L^\infty_{N\times N} \text{ with } [M, B] = 0\}.$$

Remark 4.2. Notice that $U \in L^\infty_{N\times N}$ since U is unitary, so automatically $U^*D'U \in L^\infty_{N\times N}$ for any diagonal matrix $D' \in L^\infty_{N\times N}$. Hence, $\text{dist}(M, L^\infty_{N\times N}) \leq \text{dist}(M, L^\infty)$. At this point, however, we do not know if the two distance measures are equivalent.

Before we get to the main result, let us relate $\varepsilon(M)$ to $\varepsilon(D)$, when M and D are related through conjugation with a log-preserving weight.

Lemma 4.3. *Suppose U is a continuous log-preserving matrix associated with the constant $\alpha > 0$. For $M = U^*DU$, with $D = diag(\lambda_1, \dots, \lambda_N)$, satisfying* $\exp(M) \in \mathbf{A}_2(\mathbb{T}^d)$, *it holds that*

$$\alpha\varepsilon(M) \leq \inf_{\eta\in\mathbb{R}_+} \{e^{\lambda_i/\eta} \in A_2, i = 1, 2, \dots, N\} \leq \varepsilon(M).$$

Proof. Notice that $\exp(M/\eta) = U^* \exp(D/\eta)U$, where $\exp(D/\eta) = \mathrm{diag}(e^{\lambda_1/\eta}, \dots, e^{\lambda_N/\eta})$. So by Bloom's result [1, Theorem 4.9], for any $\eta > 0$ for which $\exp(M/\eta) \in \mathbf{A}_2(\mathbb{T}^d)$, it holds that $e^{\lambda_i/\eta} \in A_2$ for $i = 1, 2, \dots, N$. Hence,

$$\inf_{\eta\in\mathbb{R}_+} \{e^{\lambda_i/\eta} \in A_2, i = 1, 2, \dots, N\} \leq \varepsilon(M).$$

Conversely, if $e^{\lambda_i/\eta} \in A_2, i = 1, 2, \dots, N$, then $\lambda_i/\eta = \log(e^{\lambda_i/\eta}) \in BMO$, and it follows from the log-preserving property of U that $\exp(\alpha M/\eta) = U^* \exp(\alpha D/\eta)U \in \mathbf{A}_2(\mathbb{T}^d)$. Hence,

$$\varepsilon(M) \leq \alpha^{-1} \inf_{\eta\in\mathbb{R}_+} \{e^{\lambda_i/\eta} \in A_2, i = 1, 2, \dots, N\},$$

which completes the proof. □

Remark 4.4. Bloom's result [1, Theorem 4.9] is proved for $d = 1$, but one can verify that the proof can be adapted to the case $d > 1$. For the sake of brevity, we leave the details for the reader.

We now turn to the main result of this section.

Proposition 4.5. *Suppose $U: \mathbb{T}^d \to \mathbb{C}^{N\times N}$ is a log-preserving matrix. Then there exist constants $c_1, c_2, c_3 > 0$, depending only on U, such that for $W = \exp(M) = U^*DU \in \mathbf{A}_2(\mathbb{T}^d)$,*

$$dist(M, L^\infty_{N\times N}) \leq c_1\, dist(M, L^\infty) \leq c_2\varepsilon(M) \leq c_3\, dist(M, L^\infty).$$

Proof. Suppose $W = \exp(M) = U^*DU \in \mathbf{A}_2(\mathbb{T}^d)$. Then $M \in BMO(\mathbb{T}^d)$ by Bloom's theorem. Pick λ satisfying $\varepsilon(M) \leq \lambda \leq 2\varepsilon(M)$. Then $W^{1/\lambda} = U^*D^{1/\lambda}U \in \mathbf{A}_2(\mathbb{T}^d)$. We now use the A_2-decomposition property to write $W^{\delta/\lambda} = VB$, with $[V, B] = 0$ and $[V]_{\mathbf{A}_2(\mathbb{T}^d)} \leq K$ for some K depending only on U. Notice that $\log B \in L^\infty_{N\times N}$. We apply the matrix logarithm to conclude that

$$\frac{\delta}{\lambda}M = \log V + \log B,$$

where we used that V and B commute. Hence,

$$\left\|M - \frac{\lambda}{\delta}\log B\right\|_{BMO(\mathbb{T}^d)} = \left\|\frac{\lambda}{\delta}\log V\right\|_{BMO(\mathbb{T}^d)} \leq \frac{\lambda K}{\delta} \leq \frac{2K}{\delta}\varepsilon(M),$$

and we conclude that $\mathrm{dist}(M, L^\infty_{N\times N}) \leq \frac{2K}{\delta}\varepsilon(M)$. In fact, the proof of Proposition 3.6 actually reveals that $\mathrm{dist}(M, L^\infty) \leq \frac{2K}{\delta}\varepsilon(M)$.

For the converse, we consider $W = \exp(M) = U^* \exp(D)U \in \mathbf{A}_2(\mathbb{T}^d)$, with U log-preserving and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$. Hence, by the log-preserving property of U, there exists $\alpha > 0$, independent of W, such that for $\lambda \geq \|D\|_{BMO(\mathbb{T}^d)}$,

$$W^{\alpha/\lambda} = U^* \exp(\alpha D/\lambda)U \in \mathbf{A}_2(\mathbb{T}^d).$$

We conclude that $\varepsilon(M) \leq \alpha^{-1}\|D\|_{BMO(\mathbb{T}^d)}$. Next we pick a sequence of diagonal matrices $\{D'_n\}_{n=1}^\infty \in L^\infty_{N\times N}$, satisfying

$$\|D - D'_n\|_{BMO} \to \mathrm{dist}(M, L^\infty).$$

We have $B_n := U^* D_n' U \in L^\infty_{N\times N}$ since U is unitary and thus contained in $L^\infty_{N\times N}$. Notice that for any $B \in L^\infty_{N\times N}$ with $[\exp(M), \exp(B)] = 0$,

$$(W \exp(-B))^{1/\gamma} = W^{1/\gamma} \exp(-B/\gamma),$$

with $-B/\gamma \in L^\infty_{N\times N}$. Using Lemma 2.5 we conclude that

$$\varepsilon(M) = \varepsilon(M - B).$$

Hence,

$$\varepsilon(M) = \varepsilon(M - B_n) \le \alpha^{-1} \|D - D_n'\|_{BMO(\mathbb{T}^d)} \to \alpha^{-1} \mathrm{dist}(M, L^\infty),$$

so

$$\varepsilon(M) \le \alpha^{-1} \mathrm{dist}(M, L^\infty).$$

□

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