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PSEUDODIFFERENTIAL OPERATORS ON MIXED-NORM α-MODULATION SPACES

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Abstract. Mixed-norm α-modulation spaces were introduced recently by Cleanthous and Georgiadis [Trans. Amer. Math. Soc. 373 (2020), no. 5, 3323-3356]. The mixed-norm spaces $M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n)$, $\alpha \in [0,1]$, form a family of smoothness spaces that contain the mixed-norm Besov spaces as special cases. In this paper we prove that a pseudodifferential operator $\sigma(x,D)$ with symbol in the Hörmander class S^b_ρ extends to a bounded operator $\sigma(x,D)\colon M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n) \to M^{s-b,\alpha}_{\vec{p},q}(\mathbb{R}^n)$ provided $0<\alpha\leq\rho\leq 1$, $\vec{p}\in(0,\infty)^n$, and $0<q<\infty$. The result extends the known result that pseudodifferential operators with symbol in the class S^b_1 maps the mixed-norm Besov space $B^s_{\vec{p},q}(\mathbb{R}^n)$ into $B^{s-b}_{\vec{p},q}(\mathbb{R}^n)$.

1. Introduction

The α-modulation spaces is a family of smoothness spaces that contains the Besov spaces and the modulation spaces, introduced by Feichtinger [17], as special cases. In the non-mixed-norm setting, the α-modulation spaces were introduced by Gröbner [21]. Gröbner used the general framework of decomposition type Banach spaces considered by Feichtinger and Gröbner in [16, 18] to build the α-modulation spaces. The parameter α determines a specific type of decomposition of the frequency space $\mathbb{R}^n$ used to define the space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$. The α-modulation spaces contain the Besov spaces and the modulation spaces, introduced by Feichtinger [17], as special cases. The choice $\alpha = 0$ corresponds to the classical modulation spaces $M^s_{p,q}(\mathbb{R}^n)$, and $\alpha = 1$ corresponds to the Besov scale of spaces.

The applicability of α-modulation spaces to the study of pseudo-differential operators comes rather natural, and in fact the family of coverings used to construct the α-modulation spaces was considered independently by Päivärinta and Somersalo in [29]. Päivärinta and Somersalo used the partitions to extend the Calderón-Vaillancourt boundedness result for pseudodifferential operators to local Hardy spaces.

Recently, function spaces in anisotropic and mixed-norm settings have attached considerable interest, see for example [1, 4, 10–12, 19, 20, 25] and reference therein. This is in part driven by advances in the study of partial and pseudodifferential operators, where there is a natural desire to be able to better model and analyse anisotropic phenomena. In particular, pseudo-differential operators on mixed-norm Besov spaces have been studied in [14, 15].

In this paper we study pseudodifferential operators on the family of mixed-norm α-modulation spaces introduced recently by Cleanthous and Georgiadis [13]. In particular, in Section 3.2 we will study pseudodifferential operators induced by symbols in the Hörmander class $S^b_\rho(\mathbb{R}^n\times\mathbb{R}^n)$ on mixed-norm α-modulation spaces. For such symbols we prove in Theorem 3.3 that

$$\sigma(x,D)\colon M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n) \to M^{s-b,\alpha}_{\vec{p},q}(\mathbb{R}^n)$$

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provided $0 < \alpha \leq \rho \leq 1$, $\vec{p} \in (0,\infty)^n$, and $0 < q < \infty$. The case $\rho = \alpha = 1$ recovers a known result that symbols in S_1^b acts boundedly on the Besov spaces, see [19], but it is interesting to note that it is known that we have a strict inclusion $S_1^b \subset S_\rho^b$ for $0 < \rho < 1$, so a larger class of operators is covered by allowing values of $\alpha < 1$. This supports the claim that mixed-norm α-modulation spaces are well adapted for symbols in $S_\rho^b(\mathbb{R}^n \times \mathbb{R}^n)$.

In the non-mixed-norm case, pseudodifferential operators on α-modulation spaces have been considered in [7, 9, 10, 28]. Pseudodifferential operators on modulation spaces were first studied by Tachizawa [31], and later by a number of authors, see e.g. [5,6,22,23,27,32, 33]. In the mixed-norm setting, pseudodifferential operators on Besov and Triebel-Lizorkin spaces have been studied in [15, 19].

The structure of the paper is as follows. In Section 2 we introduce mixed-norm Lebesgues and α-modulation spaces based on a so-called bounded admissible partition of unity (BAPU) adapted to the mixed-norm setting. Section 2 also introduces the maximal function estimates that will be needed to prove the main result. In Section 3 give a precise definition of the class $S_\rho^b(\mathbb{R}^n \times \mathbb{R}^n)$ and the associated pseudo-differential operators. We provide some boundedness results for multiplier operators on α-modulation spaces, and then proceed to prove the main result, Theorem 3.3.

2. Mixed-norm Spaces

In this section we introduce the mixed-norm Lebesgue spaces along with some needed maximal function estimates. Then we introduce mixed-norn α-modulation spaces based on so-called bounded admissible partition of unity adapted to the mixed-norm setting.

2.1. **Mixed-norm Lebesgue Spaces.** Let $\vec{p} = (p_1, \dots, p_n) \in (0,\infty)^n$ and $f : \mathbb{R}^n \to \mathbb{C}$. We say that $f \in L_{\vec{p}} = L_{\vec{p}}(\mathbb{R}^n)$ provided

$$\|f\|_{\vec{p}} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \cdots dx_n \right)^{\frac{1}{p_n}} < \infty, \tag{2.1}$$

with the standard modification when $p_j = \infty$ for some $j = 1, \dots, n$. The quasi-norm $\|\cdot\|_{\vec{p}}$, is a norm when $\min(p_1, \dots, p_n) \geq 1$ and turns $(L_{\vec{p}}, \|\cdot\|_{\vec{p}})$ into a Banach space. Note that when $\vec{p} = (p, \dots, p)$, then $L_{\vec{p}}$ coincides with L_p. For additional properties of $L_{\vec{p}}$, see e.g. [1–3, 15].

2.2. **Maximal operators.** The maximal operator will be central to most of the estimates considered in this paper. Let $1 \leq k \leq n$. We define

$$M_k f(x) = \sup_{I \in I_x^k} \frac{1}{|I|} \int_I |f(x_1, \dots, y_k, \dots, x_n)| dy_k, \quad f \in L^1_{loc}(\mathbb{R}^n), \tag{2.2}$$

where I_x^k is the set of all intervals I in $\mathbb{R}_{x_k}$ containing x_k.

We will use extensively the following iterated maximal function:

$$\mathcal{M}_\theta f(x) := \Big(M_n(\cdots (M_1 |f|^\theta) \cdots) \Big)^{1/\theta} (x), \ \theta > 0, \ x \in \mathbb{R}^n. \tag{2.3}$$

Remark 2.1. If Q is a rectangle $Q = I_1 \times \cdots \times I_n$, it follows easily that for every locally integrable f

$$\int_Q |f(y)| dy \leq |Q| \mathcal{M}_1 f(x) = |Q| \mathcal{M}_\theta^\theta |f|^{1/\theta}(x), \ \theta > 0, \ x \in \mathbb{R}^n. \tag{2.4}$$

We shall need the following mixed-norm version of the maximal inequality, see [2, 25]: If $\vec{p} = (p_1, \dots, p_n) \in (0,\infty)^n$ and $0 < \theta < \min(p_1, \dots, p_n)$ then there exists a constant C such that

$$\|\mathcal{M}_\theta f\|_{L_{\vec{p}}(\mathbb{R}^n)} \leq C \|f\|_{L_{\vec{p}}(\mathbb{R}^n)} \tag{2.5}$$

An important related estimate is a Peetre maximal function estimate, which will be one of our main tools in the sequel. For $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $\vec{b} = (b_1, \ldots, b_n) \in (0,\infty)^n$, consider the corresponding rectangle $R = [a_1 - b_1, a_1 + b_1] \times \cdots \times [a_n - b_n, a_n + b_n]$, which will be denote $R[\vec{a}, \vec{b}]$.

For every $\theta > 0$, there exists a constant $c = c_\theta > 0$, such that for every $R > 0$ and f with $\mathrm{supp}(\hat{f}) \subset c_f + R[-2,2]^n$, see [19],

$$\sup_{y\in\mathbb{R}^n} \frac{|f(y)|}{\langle R(x-y)\rangle^{n/\theta}} \leq c\mathcal{M}_\theta f(x),\ x \in \mathbb{R}^n. \tag{2.6}$$

In particular, the constant c is independent of the point $c_f \in \mathbb{R}^n$.

2.3. Mixed-norm Modulation spaces.

In this section we recall the definition of mixed-norm α-modulation spaces as introduced by Cleanthous and Georgiadis [13]. The α-modulation spaces form a family of smoothness spaces that contain modulation and Besov spaces as special "extremal" cases. The spaces are defined by a parameter α, belonging to the interval $[0, 1]$. This parameter determines a segmentation of the frequency domain from which the spaces are built.

Definition 2.2. A countable collection $\mathcal{Q}$ of subsets $Q \subset \mathbb{R}^n$ is called an admissible covering of $\mathbb{R}^n$ if

i. $\mathbb{R}^n = \cup_{Q\in\mathcal{Q}} Q$

ii. There exists $n_0 < \infty$ such that $\#\{Q' \in \mathcal{Q} : Q \cap Q' \neq \emptyset\} \leq n_0$ for all $Q \in \mathcal{Q}$.

An admissible covering is called an α-covering, $0 \leq \alpha \leq 1$, of $\mathbb{R}^n$ if

iii. $|Q| \asymp \langle x\rangle^{\alpha d}$ (uniformly) for all $x \in Q$ and for all $Q \in \mathcal{Q}$,

iv. There exists a constant $K < \infty$ such that

$$\sup_{Q\in\mathcal{Q}} \frac{R_Q}{r_Q} \leq K,$$

where $r_Q := \sup\{r \in [0,\infty) : \exists c_r \in \mathbb{R}^n : B(c_r, r) \subseteq Q\}$ and $R_Q := \inf\{r \in (0,\infty) : \exists c_r \in \mathbb{R}^n : B(c_r, r) \supseteq Q\}$, where $B(x, r)$ denotes the Euclidean ball in $\mathbb{R}^n$ centered at x with radius r.

We will need a mixed-norm bounded admissible partition of unity adapted to α-coverings. We let $\mathcal{F}(f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\, dx$, $f \in L_1(\mathbb{R}^n)$, denote the Fourier transform, and let $\hat{f}(\xi) = \mathcal{F}(f)(\xi)$.

Definition 2.3. Let $\mathcal{Q}$ be an α-covering of $\mathbb{R}^n$. A corresponding mixed-norm bounded admissible partition of unity ($\vec{p}$-BAPU) $\{\psi_Q\}_{Q\in\mathcal{Q}}$ is a family of functions satisfying

- $\mathrm{supp}(\psi_Q) \subset Q$
- $\sum_{Q\in\mathcal{Q}} \psi_Q(\xi) = 1$
- $\sup_{Q\in\mathcal{Q}} |Q|^{-1} \|\chi_Q\|_{L_{\tilde{\vec{p}}}} \|\mathcal{F}^{-1}\psi_Q\|_{L_{\tilde{\vec{p}}}} < \infty$,

where $\tilde{p}_j := \min\{1, p_1, \ldots, p_j\}$ for $j = 1, 2, \ldots, n$ and $\tilde{\vec{p}} := (\tilde{p}_1, \ldots, \tilde{p}_n)$.

The results in Sections 3 and 3.2 rely on the known fact that it is possible to construct a smooth $\vec{p}$-BAPU with certain "nice" properties. We summarise the needed properties in the following proposition proved in [13], see also [8]. Let $\langle x\rangle := (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^n$.

Proposition 2.4. *For $\alpha \in [0, 1)$, there exists an α-covering of $\mathbb{R}^n$ with a corresponding $\vec{p}$-BAPU $\{\psi_k\}_{k\in\mathbb{Z}^n\setminus\{0\}} \subset \mathcal{S}(\mathbb{R}^n)$ satisfying:*

i. *$\xi_k \in Q_k$, $k \in \mathbb{Z}^n \setminus \{0\}$, where $\xi_k := k\langle k\rangle^{\alpha/(1-\alpha)}$.*

ii. *The following estimate holds,*

$$|\partial^\beta \psi_k(\xi)| \leq C_\beta \langle\xi\rangle^{-|\beta|\alpha}, \qquad \xi \in \mathbb{R}^n,$$

for every multi-index β with C_β independent of $k \in \mathbb{Z}^n \setminus \{0\}$.

iii. *Define* $\widetilde{\psi}_k(\xi) = \psi_k(|\xi_k|^\alpha \xi + \xi_k)$. *Then for every* $\beta \in \mathbb{N}^d$ *there exists a constant* C_β *independent of* $k \in \mathbb{Z}^n \setminus \{0\}$ *such that*

$$|\partial_\xi^\beta \widetilde{\psi}_k(\xi)| \leq C_\beta \chi_{B(0,r)}(\xi).$$

iv. *Define* $\mu_k(\xi) = \psi_k(a_k \xi)$, *where* $a_k := \langle \xi_k \rangle$. *Then for every* $m \in \mathbb{N}$ *there exists a constant* C_m *independent of* k *such that*

$$|\hat{\mu}_k(y)| \leq C_m a_k^{(m-n)(1-\alpha)} \langle y \rangle^{-m}, \qquad y \in \mathbb{R}^n.$$

Remark 2.5. A closer inspection of the construction presented in [13] reveals that the BAPU is in fact $\vec{p}$-independent and only depends on α through the geometry of the α-covering.

The case $\alpha = 1$, corresponding to a dyadic-covering, is not included in Proposition 2.4, but it is known that $\vec{p}$-BAPU can easily be constructed for dyadic coverings, see e.g. [14]. Using $\vec{p}$-BAPUs, it is now possible to introduce the family of mixed-norm α-modulation spaces.

Definition 2.6. Let $\alpha \in [0,1], s \in \mathbb{R}, \vec{p} \in (0,\infty)^n, q \in (0,\infty]$, and let $\mathcal{Q}$ be an α-covering with associated $\vec{p}$-BAPU $\{\psi_k\}_{k\in\mathbb{Z}^n\setminus\{0\}}$ of the type considered in Proposition 2.4. Then we define the mixed-norm α- modulation space, $M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n)$ as the set of tempered distributions $f \in S'(\mathbb{R}^n)$ satisfying

$$\|f\|_{M^{s,\alpha}_{\vec{p},q}} := \Bigg(\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \langle \xi_k \rangle^{qs} \big\| \mathcal{F}^{-1}(\psi_k \mathcal{F} f) \big\|^q_{L_{\vec{p}}(\mathbb{R}^n)} \Bigg)^{1/q} < \infty, \tag{2.7}$$

with $\{\xi_k\}_{k\in\mathbb{Z}^n\setminus\{0\}}$ defined as in Proposition 2.4. For $q = \infty$, we change of the sum to $\sup_{k\in\mathbb{Z}^n\setminus\{0\}}$.

It is proved in [13] that the definition of $M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n)$ is independent of the α-covering and of the BAPU, see also [18] for the case of general decomposition space.

3. Pseudodifferential operators on mixed-norm α-modulation spaces

We now turn to the main focus of this article, the study of pseudodifferential operators on mixed-norm α-modulation spaces. We will state and prove our main result later in this section, but let us first recall the Hörmander class $S^b_\rho(\mathbb{R}^n \times \mathbb{R}^n)$, for $b \in \mathbb{R}$ and $0 \leq \rho \leq 1$, which is the family of functions $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$|\,\sigma|^{(b)}_{N,M} := \max_{|\alpha|\leq N, |\beta|\leq M} \sup_{x,\xi\in\mathbb{R}^n} \langle \xi \rangle^{\rho|\alpha|-b} |\partial_\xi^\alpha \partial_x^\beta \sigma(\xi, x)| < \infty,$$

for all $M, N \in \mathbb{N}$.

The class $S^b_\rho(\mathbb{R}^n \times \mathbb{R}^n)$ has been studied in details in e.g. [26]. For $\rho < 1$, we have a strict inclusion $S^b_1(\mathbb{R}^n \times \mathbb{R}^n) \subset S^b_\rho(\mathbb{R}^n \times \mathbb{R}^n)$. An example of a symbol $\sigma \in S^b_{1/2}(\mathbb{R}\times\mathbb{R}) \backslash S^b_1(\mathbb{R}\times\mathbb{R})$ is the symbol associated with the convolution kernel $K(x) = e^{i/|x|}|x|^{-\gamma}$, $\gamma > 0$. It can be shown that $\hat{K}(\xi) \in S^{\gamma/2-3/4}_{1/2}(\mathbb{R}^2)$, see [30, Chap. VII].

We define the pseudodifferential operator T_σ induced by $\sigma \in S^b_\rho$ by

$$T_\sigma f(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \sigma(x,\xi) \hat{f}(\xi) e^{ix\cdot\xi} d\xi, \text{ for every } x \in \mathbb{R}^n, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{3.1}$$

where $\hat{f}$ is the Fourier transform of the test function $f \in \mathcal{S}(\mathbb{R}^n)$. We let $\mathrm{Op}S^b_\rho$ denote the family of all operators induced by S^b_ρ. Whenever convenient, we will also use the notation $\sigma(x, D) := T_\sigma$.

An important property of S^b_ρ, which we will rely on in the sequel, is the following composition result, see e.g. [26, Chap. 5],

Proposition 3.1. *Let σ_1 and σ_2 be symbols belonging to $S^{b_1}_\rho(\mathbb{R}^n \times \mathbb{R}^n)$ and $S^{b_2}_\rho(\mathbb{R}^n \times \mathbb{R}^n)$, respectively, for some $b_1, b_2 \in \mathbb{R}$. Then there is a symbol $\sigma \in S^{b_1+b_2}_\rho(\mathbb{R}^n \times \mathbb{R}^n)$ so that $T_\sigma = T_{\sigma_1} T_{\sigma_2}$. Moreover,*

$$\sigma - \sum_{|\alpha|<N} \frac{1}{i^{|\alpha|}\alpha!} D^\alpha_\xi \sigma_1 \cdot D^\alpha_x \sigma_2 \in S^{b_1+b_2-N}_\rho(\mathbb{R}^n \times \mathbb{R}^n), \quad \text{for all } N \in \mathbb{N}. \tag{3.2}$$

3.1. **Fourier multipliers.** Let us first brifely a special class of pseudodifferential operators, namely Fourier multipliers where the symbol σ is x independent. Fourier multiplies have been studied in [13] and and we will just summarize the most crucial results for our study, where we will mainly rely on the Bessel potential operator. The Bessel potential $J^b := (I - \Delta)^{b/2}$ is defined by $\widehat{J^b f}(\xi) = \langle \xi \rangle^b \hat{f}(\xi)$. It is well known that $\langle \cdot \rangle^b \in S^b_1$, so in particular $\langle \cdot \rangle^b \in S^b_\rho$ for $0 < \rho \leq 1$. It also known that for the Besov spaces we have the lifting property, $J^b B^s_{\vec{p},q}(\mathbb{R}^n) = B^{s-b}_{\vec{p},q}(\mathbb{R}^n)$, see e.g. [19], and it was proven in [13] that J^b has exactly the same lifting property when considered on $M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n)$, $0 \leq \alpha \leq 1$.

Proposition 3.2. *Let $\alpha \in [0, 1]$, $s \in \mathbb{R}$, $\vec{p} \in (0, \infty)^n$, $q \in (0, \infty)$. Suppose $b \in \mathbb{R}$ and let $J^b = (1 - \Delta)^{b/2}$. Then we have $J^b M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n) = M^{s-b,\alpha}_{\vec{p},q}(\mathbb{R}^n)$, in the sense that*

$$\|f\|_{M^{s,\alpha}_{\vec{p},q}} \asymp \|J^b f\|_{M^{s-b,\alpha}_{\vec{p},q}} \qquad \text{for all } f \in M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n).$$

3.2. **Boundedness of pseudodifferential operators.** We can now state and prove our main result, which we believe will provides a compelling case for the use of mixed-norm α-modulation spaces with $\alpha < 1$ as the symbol classes S^b_ρ are increasing in size with ρ decreasing.

Theorem 3.3. *Suppose $b \in \mathbb{R}$, $\alpha \in (0, 1]$, $\sigma \in S^b_\rho(\mathbb{R}^n \times \mathbb{R}^n)$, $\alpha \leq \rho \leq 1$, $s \in \mathbb{R}$, $\vec{p} \in (0, \infty)^n$, and $q \in (0, \infty)$. Then*

$$\sigma(x, D) : M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n) \to M^{s-b,\alpha}_{\vec{p},q}(\mathbb{R}^n).$$

Moreover, there exist $L, N > 0$ (depending on $s, \vec{p}, q$, and ρ) such that the operator norm is bounded by $C|\,\sigma|^{(b)}_{L,N}$, with C a constant.

Let us consider an example before we turn to the proof the result.

Example 3.4. Consider the symbol associated with the convolution kernel $K(x) = e^{i/|x|}|x|^{-\gamma}$, $\gamma > 0$, $x \in \mathbb{R}^2$. As mentioned earlier, $\hat{K}(\xi) \in S^{\gamma/2-3/4}_{1/2}(\mathbb{R}^2)$. Hence, by Theorem 3.3,

$$\hat{K}(x, D) : M^{s,1/2}_{\vec{p},q}(\mathbb{R}^2) \to M^{s-\gamma/2+3/4,1/2}_{\vec{p},q}(\mathbb{R}^2),$$

for $s \in \mathbb{R}$, $\vec{p} \in (0, \infty)^n$, and $q \in (0, \infty)$.

Let us now turn to the proof of Theorem 3.3. In the Besov space case, $\alpha = 1$ [i.e., $M^{s,1}_{\vec{p},q}(\mathbb{R}^n) = B^s_{\vec{p},q}(\mathbb{R}^n)$], the proof was given by Georgiadis and the author in [19]. We will therefore only consider the case $\alpha \in (0, 1)$ below.

Proof of Theorem 3.3. Calling on Propositions 3.1 and 3.2, we have $J^{-a} M^{s,\alpha}_{\vec{p},q} = M^{s+a,\alpha}_{\vec{p},q}$, $\sigma(x, D) J^a \in \mathrm{Op}S^{b+a}_\rho$, and $J^a \sigma(x, D) \in \mathrm{Op}S^{b+a}_\rho$ when $\sigma \in S^b_\rho$, from which it follows that it is no restriction to assume that s is large and $b = 0$. Moreover, it suffices to prove that $\|\sigma(x, D) f\|_{M^{s,\alpha}_{\vec{p},q}} \leq C\|f\|_{M^{s,\alpha}_{\vec{p},q}}$ for $f \in \mathcal{S}(\mathbb{R}^n)$ since $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n)$, see [13].

Fix $f \in \mathcal{S}(\mathbb{R}^n)$. First we estimate the $L_{\vec{p}}(\mathbb{R}^n)$-norm of $\psi_k(D)\sigma(x, D) f$. Notice that for any $g \in \mathcal{S}(\mathbb{R}^n)$,

$$[\psi_k(D) g](x) = (2\pi)^{-d/2} \int_{\mathbb{R}^n} e^{ix\cdot y} \psi_k(y) \hat{g}(y)\, dy = (2\pi)^{-d/2} \int_{\mathbb{R}^n} \hat{\psi}_k(y) g(x + y)\, dy. \tag{3.3}$$

Letting $\sigma_\eta^\gamma(x,\xi) := \partial_x^\gamma \partial_\xi^\eta \sigma(x,\xi)$ and $\sigma^\gamma := \sigma_0^\gamma$, we obtain

$$\begin{aligned}
\sigma(x+y,D)f(x+y) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i(x+y)\cdot\xi}\sigma(x+y,\xi)\hat{f}(\xi)\,d\xi \\
&= (2\pi)^{-n/2}\sum_{|\gamma|\le K-1}\frac{y^\gamma}{\gamma!}\int_{\mathbb{R}^n} e^{i(x+y)\cdot\xi}\sigma^\gamma(x,\xi)\hat{f}(\xi)\,d\xi \\
&+ (2\pi)^{-n/2}\sum_{|\gamma|=K}K\frac{y^\gamma}{\gamma!}\int_{\mathbb{R}^n} e^{i(x+y)\cdot\xi}\int_0^1(1-\tau)^{K-1}\sigma^\gamma(x+\tau y,\xi)\hat{f}(\xi)\,d\tau\,d\xi \\
&:= T(x,y)+R(x,y), \qquad (3.4)
\end{aligned}$$

where we have expanded $\sigma(\cdot+y,\xi)$ in a Taylor series centered at x. We choose the order K such that $K\alpha > s + 2(1-\alpha)(1+n)/r$, where $r := \min\{1,q,p_1,\ldots,p_n\}$. Using (3.4) in (3.3), we obtain

$$\psi_k(D)\sigma(x,D)f(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\hat{\psi}_k(y)T(x,y)\,dy + (2\pi)^{-n/2}\int_{\mathbb{R}^n}\hat{\psi}_k(y)R(x,y)\,dy. \qquad (3.5)$$

We estimate each of the two terms separately. First we consider the term with $T(x,y)$. We have,

$$\begin{aligned}
\int_{\mathbb{R}^n}\hat{\psi}_k(y)T(x,y)\,dy &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\hat{\psi}_k(y)\sum_{|\gamma|\le K-1}\frac{y^\gamma}{\gamma!}\int_{\mathbb{R}^n} e^{i(x+y)\cdot\xi}\sigma^\gamma(x,\xi)\hat{f}(\xi)\,d\xi\,dy \\
&= (2\pi)^{-n/2}\sum_{|\gamma|\le K-1}\frac{1}{\gamma!}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma^\gamma(x,\xi)\hat{f}(\xi)\int_{\mathbb{R}^n} e^{iy\cdot\xi}\hat{\psi}_k(y)y^\gamma\,dy\,d\xi \\
&= \sum_{|\gamma|\le K-1}\frac{1}{\gamma!}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma^\gamma(x,\xi)\partial_\xi^\gamma\psi_k(\xi)\hat{f}(\xi)\,d\xi. \qquad (3.6)
\end{aligned}$$

Define $\Psi_k := \sum_{k'}\psi_{k'}$, where the sum is taken over all $k' \in \mathbb{Z}^n\backslash\{0\}$ with $\mathrm{supp}(\psi_{k'}) \cap \mathrm{supp}(\psi_k) \neq \emptyset$. Using the fact that $\Psi_k(\xi) = 1$ on $\mathrm{supp}(\psi_k)$, and the relation $(\hat{f}\hat{g})^\vee = f * g$, we obtain for $\theta > 0$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma^\gamma(x,\xi)\partial_\xi^\gamma\psi_k(\xi)\hat{f}(\xi)\,d\xi\right| &= \left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\big(\sigma^\gamma(x,\xi)\partial_\xi^\gamma\psi_k(\xi)\big)\big(\Psi_k(\xi)\hat{f}(\xi)\big)\,d\xi\right| \\
&\le \int_{\mathbb{R}^n}\big|(\sigma^\gamma(x,\cdot)\partial_\xi^\gamma\psi_k)^\vee(y)\big||\Psi_k(D)f(x-y)|\,dy \\
&\le \int_{\mathbb{R}^n}\sup_{z\in\mathbb{R}^n}\big|(\sigma^\gamma(z,\cdot)\partial_\xi^\gamma\psi_k)^\vee(y)\big||\Psi_k(D)f(x-y)|\,dy \\
&= \int_{\mathbb{R}^n}\sup_{z\in\mathbb{R}^n}\big|(\sigma^\gamma(z,\cdot)\partial_\xi^\gamma\psi_k)^\vee(y)\big|\langle|\xi_k|^\alpha y\rangle^{n/\theta} \\
&\qquad\times\langle|\xi_k|^\alpha y\rangle^{-n/\theta}|\Psi_k(D)f(x-y)|dy,
\end{aligned}$$

where $\xi_k = k|k|^{\alpha/(1-\alpha)}$. Using the estimate (b) from Lemma 3.5 below, and the Peetre maximal function estimate (2.6), we conclude that

$$\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma^\gamma(x,\xi)\partial_\xi^\gamma\psi_k(\xi)\hat{f}(\xi)\,d\xi\right| \le C|\sigma|_{L,K}^{(0)}\mathcal{M}_\theta(\Psi_k(D)f)(x), \qquad (3.7)$$

with $C < \infty$ independent of k and f, provided we choose $L > n(1+1/\theta)$. Hence, we may also conclude that

$$\left\|\int_{\mathbb{R}^n}\hat{\psi}_k(y)T(\cdot,y)\,dy\right\|_{L_{\vec{p}}(\mathbb{R}^n)} \le C|\,\sigma|_{L,K}^{(0)}\|\Psi_k(D)f\|_{L_{\vec{p}}(\mathbb{R}^n)}, \qquad (3.8)$$

provided $0<\theta<\min\{p_1,\dots,p_n\}$. In particular, we may choose $L>n(1+1/r)$ to ensure that (3.8) holds.

We turn to the second term in (3.5). Let $\mu_k(\xi)=\psi_k(a_k\xi)$, where $a_k:=\langle k|k|^{\alpha/(1-\alpha)}\rangle$. First notice that

$$\int_{\mathbb{R}^n}\hat{\psi}_k(y)R(x,y)\,dy=\int_{\mathbb{R}^n}\hat{\mu}_k(y)R(x,a_k^{-1}y)\,dy.$$

We have,

$$\Big|\sum_{|\gamma|=K}\frac{a_k^{-K}}{\gamma!}\int_{\mathbb{R}^n}y^\gamma\hat{\mu}_k(y)\int_{\mathbb{R}^n}e^{i(x+a_k^{-1}y)\cdot\xi}\int_0^1(1-\tau)^{K-1}\sigma^\gamma(x+a_k^{-1}\tau y,\xi)\hat{f}(\xi)\,d\tau\,d\xi\,dy\Big|$$
$$\leq Ca_k^{-K}\sum_{|\gamma|=K}\int_{\mathbb{R}^n}\langle y\rangle^K\big|\hat{\mu}_k(y)\big|\Big|\int_0^1(1-\tau)^{K-1}\int_{\mathbb{R}^n}e^{i(x+a_k^{-1}y)\cdot\xi}\sigma^\gamma(x+a_k^{-1}\tau y,\xi)\hat{f}(\xi)\,d\xi\,d\tau\Big|\,dy.$$

Using Lemma 3.5 with $m=K+n+(1+n)/r$, we obtain the following estimate for the right-hand side for $0<\theta\leq r$, using that $0<r\leq 1$,

$$\begin{aligned}
&C'a_k^{-\tilde{K}}\sum_{|\gamma|=K}\int_{\mathbb{R}^n}\frac{\langle y\rangle^{-n-1}}{\langle y\rangle^{n/\theta}}\sup_{z\in\mathbb{R}^n}\big|[\sigma^\gamma(z,D)f](x+a_k^{-1}y)\big|\,dy\\
&\quad=C'a_k^{-\tilde{K}}\sum_{|\gamma|=K}\int_{\mathbb{R}^n}\langle y\rangle^{-n-1}\sup_{z\in\mathbb{R}^n}\frac{\big|[\sigma^\gamma(z,D)f](x+a_k^{-1}y)\big|}{\langle y\rangle^{n/\theta}}\,dy\\
&\quad\leq C'a_k^{-\tilde{K}}\sum_{|\gamma|=K}\int_{\mathbb{R}^n}\langle y\rangle^{-n-1}\sup_{z,\eta\in\mathbb{R}^n}\frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle a_k\eta\rangle^{n/\theta}}\,dy\\
&\quad\leq C'a_k^{-\tilde{K}}\sum_{|\gamma|=K}\int_{\mathbb{R}^n}\langle y\rangle^{-n-1}\sup_{z,\eta\in\mathbb{R}^n}\frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle\eta\rangle^{n/\theta}}\,dy,\qquad\text{since }\alpha_k\geq 1,\\
&\quad\leq C'a_k^{-\tilde{K}}\sum_{|\gamma|=K}\sup_{z,\eta\in\mathbb{R}^n}\frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle\eta\rangle^{n/\theta}},
\end{aligned}$$

where $\tilde{K}=K\alpha-(1+n)(1-\alpha)/r\geq s+\frac{n+1}{q}(1-\alpha)$, since $q\geq r$. Now,

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}^n\setminus\{0\}}a_k^{sq}\Big\|\int_{\mathbb{R}^n}\hat{\psi}_k(y)R(x,y)\,dy\Big\|_{L_{\vec{p}}(dx)}^q\Big)^{1/q}\\
&\qquad\leq\Big\{C\sum_{k\in\mathbb{Z}^n\setminus\{0\}}a_k^{(s-\tilde{K})q}\Big(\sum_{|\gamma|=K}\Big\|\sup_{\eta,z\in\mathbb{R}^n}\frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle\eta\rangle^{n/\theta}}\Big\|_{L_{\vec{p}}(dx)}\Big)^q\Big\}^{1/q}.
\end{aligned}$$

We notice that $L^q:=C\sum_{k\in\mathbb{Z}^n\setminus\{0\}}a_k^{(s-\tilde{K})q}\leq C\sum_{k\in\mathbb{Z}^n\setminus\{0\}}|k|^{-n-1}$ is finite. Based on this observation, recalling that $r=\min\{1,q,p_1,\dots,p_n\}$, we use the equivalence of ℓ^τ-norms on finite dimensional spaces to estimate the right-hand side by,

$$\begin{aligned}
&L\Big(\sum_{|\gamma|=K}\Big\|\sup_{\eta,z\in\mathbb{R}^n}\frac{\big|[\sigma^\gamma(z,D)f](x+\eta)\big|}{\langle\eta\rangle^{n/\theta}}\Big\|_{L_{\vec{p}}(dx)}^r\Big)^{1/r}\\
&\qquad=L\Big(\sum_{|\gamma|=K}\Big\|\sup_{\eta,z\in\mathbb{R}^n}\frac{\big|[\sigma^\gamma(z,D)\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\psi_k(D)f](x+\eta)\big|}{\langle\eta\rangle^{n/\theta}}\Big\|_{L_{\vec{p}}(dx)}^r\Big)^{1/r}\\
&\qquad\leq L\Big(\sum_{|\gamma|=K}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\Big\|\sup_{\eta,z\in\mathbb{R}^n}\frac{\big|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)\big|}{\langle\eta\rangle^{n/\theta}}\Big\|_{L_{\vec{p}}(dx)}^r\Big)^{1/r}.
\end{aligned}$$

We now focus on the individual term $A_k := \big|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)\big|$. Put $f_k(x) := [\Psi_k(D)f](x)$, with Ψ_k defined as above. We have

$$\begin{aligned} A_k &= \left| \int_{\mathbb{R}^n} (\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(x+\eta-y) f_k(y)\,dy \right| \\ &\leq \int_{\mathbb{R}^n} |(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(x+\eta-y)||f_k(y)|\,dy \\ &\leq \sup_{u\in\mathbb{R}^n} \frac{|f_k(u)|}{\langle x-u\rangle^{n/\theta}} \int_{\mathbb{R}^n} |(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(x+\eta-y)|\langle x-y\rangle^{n/\theta}\,dy. \end{aligned}$$

Now, $\langle x-y\rangle^{n/\theta} \leq c\langle x-y+\eta\rangle^{n/\theta}\langle\eta\rangle^{n/\theta}$, so

$$\begin{aligned} \sup_{z,\eta\in\mathbb{R}^n} \frac{A_k}{\langle\eta\rangle^{n/\theta}} &\leq C \sup_{\eta\in\mathbb{R}^n} \frac{|f_k(x-\eta)|}{\langle\eta\rangle^{n/\theta}} \sup_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} |(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(u)|\langle u\rangle^{n/\theta}\,du \\ &\leq C' \sup_{\eta\in\mathbb{R}^n} \frac{|f_k(x-\eta)|}{\langle\eta\rangle^{n/\theta}} |\,\sigma|^{(0)}_{L,K}, \end{aligned}$$

provided $L > n + n/\theta$, where we have used Lemma 3.5. Hence,

$$\begin{aligned} &\sum_{|\gamma|=K} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \left\| \sup_{\eta,z\in\mathbb{R}^n} \frac{\big|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)\big|}{\langle\eta\rangle^{n/\theta}} \right\|^r_{L_{\vec{p}}(dx)} \\ &\qquad = \sum_{|\gamma|=K} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} a_k^{nr/\theta} \left\| \sup_{\eta,z\in\mathbb{R}^n} \frac{\big|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)\big|}{a_k^{n/\theta}\langle\eta\rangle^{n/\theta}} \right\|^r_{L_{\vec{p}}(dx)} \\ &\qquad \leq C'(|\,\sigma|^{(0)}_{L,K})^r \sum_{k\in\mathbb{Z}^n\setminus\{0\}} a_k^{nr/\theta} \left\| \sup_{\eta\in\mathbb{R}^n} \frac{|f_k(x-\eta)|}{\langle a_k\eta\rangle^{n/\theta}} \right\|^r_{L_{\vec{p}}(dx)}. \end{aligned}$$

We now use the Peetre maximal estimate,

$$\sup_{z\in\mathbb{R}^n} \frac{|f_k(x-z)|}{\langle a_k z\rangle^{n/\theta}} \leq C\mathcal{M}_\theta(f_k)(x),$$

and we may apply the $L_{\vec{p}}$-norms, using the maximal inequality, to obtain

$$\left\| \sup_{z\in\mathbb{R}^n} \frac{|f_k(x-z)|}{\langle a_k z\rangle^{n/\theta}} \right\|^r_{L_{\vec{p}}(\mathbb{R}^n)} \leq C'\|f_k\|^r_{L_{\vec{p}}(\mathbb{R}^n)},$$

provided $0 < \theta < \min\{p_1,\dots,p_n\}$. Putting these estimates together yields,

$$\begin{aligned} &\sum_{|\gamma|=K} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \left\| \sup_{\eta,z\in\mathbb{R}^n} \frac{\big|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)\big|}{\langle\eta\rangle^{n/\theta}} \right\|^r_{L_{\vec{p}}(dx)} \\ &\qquad \leq C''(|\,\sigma|^{(0)}_{L,K})^r \sum_{k\in\mathbb{Z}^n\setminus\{0\}} a_k^{nr/\theta} \|\Psi_k(D)f\|^r_{L_{\vec{p}}(\mathbb{R}^n)}, \end{aligned}$$

provided $L > n + n/\theta$, and consequently

$$\begin{aligned} \bigg(\sum_{k\in\mathbb{Z}^n\setminus\{0\}} a_k^{sq} \left\| \int_{\mathbb{R}^n} \hat{\psi}_k(y)R(x,y)\,dy \right\|^q_{L_p} \bigg)^{1/q} &\leq C''|\,\sigma|^{(0)}_{L,K} \bigg(\sum_{k\in\mathbb{Z}^n\setminus\{0\}} a_k^{rn\theta} \|\Psi_k(D)f\|^r_{L_{\vec{p}}(\mathbb{R}^n)} \bigg)^{1/r}. \\ &= C''|\,\sigma|^{(0)}_{L,K} \bigg(\sum_{k\in\mathbb{Z}^n\setminus\{0\}} a_k^{rn\theta-sr} \cdot a_k^{sr} \|\Psi_k(D)f\|^r_{L_{\vec{p}}(\mathbb{R}^n)} \bigg)^{1/r}. \\ &\leq C'''|\,\sigma|^{(0)}_{L,K} \left\| a_k^s \|\Psi_k(D)f\|_{L_{\vec{p}}(\mathbb{R}^n)} \right\|_{\ell_q}, \end{aligned}$$

where for the last estimate, we used Hölder's inequality with parameters q/r and $q/(q-r)$ and the fact that $s > n(1+\theta)/r$, where we also notice that $n(1+\theta)/r < 3n/r$ since $0 < \theta < 2$. Finally, we can put all the estimates together to close the case $b = 0$ and $s > 3n/r$. We have, with $L > n + n/r$,

$$\begin{aligned}
&\|\sigma(x,D)f\|_{M^{s,\alpha}_{\vec{p},q}} \\
&\asymp \big\| a_k^s \|\psi_k(D)\sigma(x,D)f\|_{L_{\vec{p}}(\mathbb{R}^n)} \big\|_{\ell^q} \\
&\leq C\left\{ \left\| a_k^s \left\| \int_{\mathbb{R}^n} \hat{\psi}_k(y) T(x,y)\,dy \right\|_{L_{\vec{p}}(dx)} \right\|_{\ell_q} + \left\| a_k^s \left\| \int_{\mathbb{R}^n} \hat{\psi}_k(y) R(x,y)\,dy \right\|_{L_{\vec{p}}(dx)} \right\|_{\ell_q} \right\} \\
&\leq C'\left(|\,\sigma|^{(0)}_{L,K} \left\| a_k^s \big\| \Psi_k(D) f \big\|_{L_{\vec{p}}(\mathbb{R}^n)} \right\|_{\ell_q} + |\,\sigma|^{(0)}_{L,K} \left\| a_k^s \big\| \Psi_k(D) f \big\|_{L_{\vec{p}}(\mathbb{R}^n)} \right\|_{\ell_q} \right) \\
&\leq C''|\,\sigma|^{(0)}_{L,K} \|f\|_{M^{s,\alpha}_{\vec{p},q}}.
\end{aligned}$$

This concludes the proof of the theorem. □

The following technical lemma has been used in the proof of Theorem 3.3.

Lemma 3.5. *Let $\alpha \in [0,1)$ and let $\{\psi_k\}_{k\in\mathbb{Z}^n\setminus\{0\}}$ be the $\vec{p}$-BAPU from Proposition 2.4, depending only on α. Suppose $\sigma \in S^0_{\rho,0}$, $\alpha \leq \rho \leq 1$. Then for $|\gamma| \leq K$ and $m \geq 0$, we have*

(a) *For $|\gamma|, |\nu| \leq K$ and $J \in \mathbb{N}$ there exists a constant $C := C(K,J)$ such that*

$$M(x) := \sup_{z\in\mathbb{R}^n} \big|(\partial_x^\gamma \sigma(z,\cdot)\partial_\xi^\nu \psi_k)^\vee(x)\big| \leq C|\,\sigma|^{(0)}_{J,K} |k|^{\alpha n/(1-\alpha)} \langle |k|^{\alpha/(1-\alpha)} x\rangle^{-J},$$

for $x \in \mathbb{R}^n, k \in \mathbb{N}$.

(b) *For $|\gamma|, |\nu| \leq K$ and $m \geq 0$ there exists a constant $C' := C'(K,m)$, such that*

$$I := \int_{\mathbb{R}^n} \sup_{z\in\mathbb{R}^n} \big|(\partial_x^\gamma \sigma(z,\cdot)\partial_\xi^\nu \psi_k)^\vee(x)\big| \langle |k|^{\alpha/(1-\alpha)} x\rangle^m\,dx \leq C'|\,\sigma|^{(0)}_{M,K}, \qquad k \in \mathbb{N},$$

for any $M \in \mathbb{N}$ satisfying $M > m + n$.

Proof. First we prove (a). Let $\sigma^\gamma_\eta(x,\xi) := \partial_x^\gamma \partial_\xi^\eta \sigma(x,\xi)$ and $\sigma^\gamma := \sigma^\gamma_0$. We have the equality

$$M(x) = \sup_{z\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma^\gamma(z,\xi) \partial_\xi^\nu \psi_k(\xi)\,d\xi \right|.$$

Let $T_k = |\xi_k|^\alpha + \xi_k$, where $\xi_k = k|k|^{\alpha/(1-\alpha)}$. Then a substitution yields

$$M(x) = |\xi_k|^{n\alpha} \sup_{z\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i|\xi_k|^\alpha x\cdot\xi} \sigma^\gamma(z,T_k\xi) \partial_\xi^\nu \psi_k(T_k\xi)\,d\xi \right|. \tag{3.9}$$

Fix $J > 1$. We use the well-known estimate $\langle x\rangle^J |\hat{g}(x)| \leq C_J \sum_{|\beta|\leq J} \|\partial^\beta g\|_{L_1}$, for some finite constant C_J. We apply the estimate to (3.9) to obtain

$$M(|\xi_k|^{-\alpha}x) \leq C_J |\xi_k|^{n\alpha} \sup_{z\in\mathbb{R}^n} \sum_{|\beta|\leq J} \int_{\mathbb{R}^n} \left| \partial_\xi^\beta \Big[\sigma^\gamma(z,T_k\xi)\partial_\xi^\nu \psi_k(T_k\xi)\Big] \right| d\xi \langle x\rangle^{-J},$$

which by Leibniz's rule provides the bound

(3.10)
$$M(|\xi_k|^{-\alpha}x) \leq C'|\xi_k|^{n\alpha} \sum_{\substack{|\beta|\leq J\\ 0\leq\eta\leq\beta}} \sup_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} |\xi_k|^{\alpha|\eta|} |\partial_\xi^\eta(\sigma^\gamma(z,T_k\xi))| |\partial_\xi^{\beta-\eta}(\partial_\xi^\nu \psi_k(T_k\xi))|\,d\xi \langle x\rangle^{-J}.$$

From Proposition 2.4, we have

$$|\partial_\xi^{\beta-\eta}(\partial_\xi^\nu \psi_k(T_k\xi))| \leq C\chi_Q(\xi), \tag{3.11}$$

with $C := C(\nu, \beta, \eta)$. We also notice that for $\xi \in Q$,

$$|\partial_\xi^\eta(\sigma^\gamma(z, T_k\xi))| \le |\,\sigma|^{(0)}_{|\eta|,K}\langle |\xi_k|^\alpha \xi + \xi_k\rangle^{-\rho|\eta|} \le C|\,\sigma|^{(0)}_{|\eta|,K}\langle \xi_k\rangle^{-\rho|\eta|}. \tag{3.12}$$

Now, by assumption $\alpha \le \rho$, so using the estimates (3.11) and (3.12) in (3.10), we obtain

$$M(|\xi_k|^{-\alpha}x) \le C''|\xi_k|^{n\alpha} \sum_{\substack{|\beta|\le J\\ 0\le\eta\le\beta}} |\,\sigma|^{(0)}_{J,K} \int_{\mathbb{R}^n} \chi_Q(\xi)\, d\xi \langle x\rangle^{-J} \le C'''|\xi_k|^{\alpha n} \cdot |\,\sigma|^{(0)}_{J,K}\langle x\rangle^{-J},$$

which proves (a), since $|\xi_k| = |k|^{1/(1-\alpha)}$.

Let us turn to (b). Pick $J > m + n$ in (a). We have

$$\begin{aligned} I &= \int_{\mathbb{R}^n} M(x)\langle |k|^{\alpha/(1-\alpha)}x\rangle^m\, dx \\ &\le C'|\,\sigma|^{(0)}_{J,K}|k|^{\alpha n/(1-\alpha)} \int_{\mathbb{R}^n} \langle |k|^{\alpha/(1-\alpha)}x\rangle^{-J}\langle |k|^{\alpha/(1-\alpha)}x\rangle^m\, dx \\ &= C'|\,\sigma|^{(0)}_{J,K} \int_{\mathbb{R}^n} \langle x\rangle^{-J}\langle x\rangle^m\, dx \\ &\le \tilde{C}|\,\sigma|^{(0)}_{J,K}, \end{aligned}$$

where we made a change of variable in the integral and used $J > n + m$, which of course implies that $m - J < -n$. This concludes the proof. □

4. Hypoelliptic pseudodifferential operators

In this final section we consider an application of the result in the previous section to hypoelliptic pseudodifferential operators based on standard machinery, see e.g. [24]. Let us introduce some notation. Let

$$S_\rho^\infty := \bigcup_{b\in\mathbb{R}} S_\rho^b, \quad \text{and} \quad S_\rho^{-\infty} := \bigcap_{b\in\mathbb{R}} S_\rho^b.$$

Assume that $b_0, b \in \mathbb{R}$ such that $b_0 \le b$. An element $\sigma \in S_\rho^b(\mathbb{R}^n \times \mathbb{R}^n)$ is called *hypoelliptic* with parameters b_0 and b if there are positive constants c and a such that

$$a\langle\xi\rangle^{b_0} \le |\sigma(x,\xi)|, \qquad \langle\xi\rangle \ge c,$$

and

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}|\sigma(x,\xi)|\langle\xi\rangle^{-\rho|\alpha|}, \qquad \langle\xi\rangle \ge c.$$

Let $HS_\rho^{b,b_0}(\mathbb{R}^n \times \mathbb{R}^n)$ the family of all such symbols. The following result is well-know, see [24, Theorem 22.1.3].

Theorem 4.1. *Suppose $\sigma \in HS_\rho^{b,b_0}$, with $0 < \rho \le 1$. Then there exists $\tau \in HS_\rho^{-b_0,-b}$ such that $I - \sigma(x,D)\tau(x,D)$ and $I - \tau(x,D)\sigma(x,D)$ are both in $Op(S_\rho^{-\infty})$.*

Let $M_{\vec{p},q}^{-\infty,\alpha}(\mathbb{R}^n) = \cup_{s\in\mathbb{R}} M_{\vec{p},q}^{s,\alpha}(\mathbb{R}^n)$. Using Theorem 4.1 and the result from the previous section we have

Theorem 4.2. *Suppose $\sigma \in HS_\rho^{b,b_0}$, with $\rho \ge \alpha > 0$, and $f \in M_{p,q}^{-\infty,\alpha}(\mathbb{R}^n)$. If $\sigma(\cdot,D)f \in M_{\vec{p},q}^{s,\alpha}(\mathbb{R}^n)$ for some $s \in \mathbb{R}$, then $f \in M_{\vec{p},q}^{s+b_0,\alpha}(\mathbb{R}^n)$.*

Proof. Let $S = \sigma(\cdot,D)$, and let $T = \tau(\cdot,D)$ be as in Theorem 4.1. Notice that $f = T(Sf) + (I - TS)f$. By Theorem 3.3, T maps $M_{\vec{p},q}^{s,\alpha}(\mathbb{R}^n)$ to $M_{\vec{p},q}^{s+b,\alpha}(\mathbb{R}^n)$ and $(I - TS)$ maps $M_{\vec{p},q}^{-\infty,\alpha}(\mathbb{R}^n)$ to $M_{\vec{p},q}^{s+b,\alpha}(\mathbb{R}^n)$. □

The following example will conclude the paper.

Example 4.3. Consider the heat operator L given by

$$L(u) := \frac{\partial u}{\partial t} - \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$

The symbol of L is given by

$$l(\tau,\xi) = (i\tau + |\xi|^2), \qquad (\tau,\xi) \in \mathbb{R} \times \mathbb{R}^n,$$

and one can easily verify that $l \in HS_1^{2,1}$. We consider an approximate inverse P to L with symbol

$$a(\tau,\xi) = (i\tau + |\xi|^2)^{-1}\eta(\tau,\xi), \qquad (\tau,\xi) \in \mathbb{R} \times \mathbb{R}^n,$$

where η is a smooth cut-off function that vanishes near the origin and equals 1 for large (τ,ξ). It is easy to check that $a(\tau,\xi) \in HS_1^{-1,-2}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$. Hence, if $u \in M_{\vec{p},q}^{-\infty,\alpha}(\mathbb{R}^{n+1})$, $\vec{p} \in (0,\infty)^n$, $0 < q < \infty$, $\alpha \in (0,1]$, and $P(u) \in M_{\vec{p},q}^{s,\alpha}(\mathbb{R}^{n+1})$, then $u \in M_{\vec{p},q}^{s-1,\alpha}(\mathbb{R}^{n+1})$.

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