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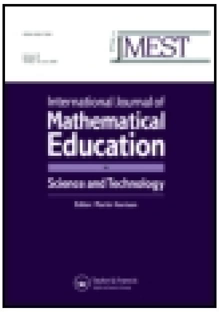

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Retention of differential and integral calculus: a case study of a university student in physical chemistry

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This paper reports a study on retention of differential and integral calculus concepts of a second-year student of physical chemistry at a Danish university. The focus was on what knowledge the student retained 14 months after the course and on what effect beliefs about mathematics had on the retention. We argue that if a student can quickly reconstruct the knowledge, given a few hints, this is just as good as retention. The study was conducted using a mixed method approach investigating students' knowledge in three worlds of mathematics. The results showed that the student had a very low retention of concepts, even after hints. However, after completing the calculus course, the student had successfully used calculus in a physical chemistry study programme. Hence, using calculus in new contexts does not in itself strengthen the original calculus learnt; they appeared as disjoint bodies of knowledge.

Keywords: calculus; retention; knowledge; three worlds of mathematics; beliefs

1. Introduction

Many students meet calculus concepts during their high school education, and some of them also meet calculus if they later study at a university. Calculus is usually mandatory for students of mathematics, science, engineering, medical, and economic study programmes. However, Maull and Berry [1] state that engineering students perceive and learn the mathematical concepts differently from students in mathematics programmes. Also Bingolbali et al. [2] argue that engineering and mathematics students develop different conceptions and preferences for such conceptions. This paper is a study of a physical chemistry student's learning of calculus, which might therefore be different from a student of mathematics, and other study programmes.

Several studies document the university students' difficulties in understanding calculus concepts. For instance, according to Artigue,[3] one reason for the problems appears to lie in the different teaching approaches used in high school and at university. She argues that in most countries, high school calculus relies on graphical, numerical exploration, and algebraic techniques, which allow students to solve simple problems, while universities demand a more formal approach. This, she argues, 'represents a tremendous jump, both conceptually and technically'.[3,p.1381] Another example is from Brandell et al. [4] who state that, generally, the tasks in Swedish upper secondary national tests do not require computational skills. Instead the tasks focus on the students' understanding of numbers

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and of arithmetic operations. The students have therefore not developed skills in arithmetic and algebraic computations, and this lack of skills becomes a problem if the students enter the university where 'routine skills and knowledge of formulas and theorems are considered necessary for the understanding of concepts and theory, as well as important tools in problem solving'.[4,p.44] Dahl [5] and Brabrand and Dahl [6,7] also argue that in Denmark it appears that for mathematics, the compulsory system, the upper secondary system, and the university system each has its own cycle of competency progression, so the next level does not directly build upon the competencies learnt at the previous level.

Even though students experience problems when transiting from one educational system/level to the next, or from one course to the next, all educational systems rely on the assumption that previously learnt mathematics will be retained to some degree some time after it has been learnt in order for later courses to build upon this mathematics. However, we still do not fully understand how mathematical knowledge is retained, which type of knowledge, and for how long it is retained,[8] especially in higher education.

In this paper, we report a study of a Danish university student's retention of knowledge of derivatives and integrals 14 months after having passed the calculus courses teaching these concepts. The student was, at the time of the present study, enrolled in a physical chemistry study programme. A year earlier, she participated in a larger survey studying the retention of all students taking a calculus course. That study took place two months after they had passed the course (see [9,10]). We will describe the study in greater details below. The study is therefore part of an overall study of student retention of calculus knowledge where Jukić and Dahl [11,12] focus on other parts of the calculus curriculum than in this paper.

2. Theoretical background

2.1. *Knowledge and three worlds of mathematics*

Knowledge that students acquire during mathematics education can be categorized in two widely accepted categories – conceptual and procedural knowledge. Conceptual knowledge is the type of knowledge rich in relationships and provides an understanding of the principles and relations between pieces of knowledge in a certain domain, while procedural knowledge is what enables us to quickly and efficiently solve problems. Procedural knowledge can be learnt with or without meaning and consists of a sequence of actions.[13] Both conceptual and procedural knowledge are dynamic in nature, either as browsing through a network consisting of concepts, principles, and rules or utilizing particular rules, algorithms, or procedures within relevant representation forms.[14] However, the procedural and conceptual knowledge show up intertwined with each other in mathematics and it is difficult to measure them validly and independently of each other, thus conceiving knowledge as either procedural or conceptual is a bit limiting for our study. In order to better describe knowledge that a person gained in some mathematics course, we refer to Tall's theory of learning mathematics. Tall [15] designed the theory which is based on perceptions of and actions on objects in the environment, and which considers three aspects of mathematics, namely geometric, symbolic, and axiomatic.

The first world is denoted as a conceptual-embodied world of human perceptions and actions, including mental images, pictures, and internal connections. This world consists of things that students can perceive and sense not only in the physical world, but also in their own mental world of meaning.[15] Perceptions are described and defined and arguments are developed to formulate conclusions typical for Euclidian geometry; therefore, in essence, this world is concerned with a student's visual–spatial imagery. For example, a triangle is embodied as a figure with three straight line segments, and this image acts as a prototype for all classes of triangles. In the conceptual-embodied world, students learn by shifting the attention from the actions to the effect of that action.[16] Here, language has a specific role. It is focused on describing properties of objects, and then used for categorizing and defining.

The second world is a proceptual-symbolic world where symbols operate flexibly either as concepts to think about or as processes to make calculations and perform symbolic manipulation. In this world, a symbol has a pivotal role and represents a process and a concept at the same time. For example, a symbol such as $5 + 2$ or *f'* represents a process to be performed or the concept obtained by that process. This kind of a combination of a symbol, process, and the concept produced from the process is called a *procept*. This dual use of symbols begins with the process as a step-by-step procedure and then after routinizing, it can be carried out without conscious attention to the details.[17] Procepts allow the human brain to switch from doing a process to thinking about the concept with minimum effort.

The third world is a formal-axiomatic world, based on the properties expressed in terms of formal definitions and proofs. This world is connected to the formalism of Hilbert and uses logical deductions, formal definitions, and axioms to construct axiomatic mathematical systems. In this world, the language changes its role and also uses set-theoretic terms to become 'rigour prefixed'.[15]

From now on, for the sake of simplicity, we will refer to the three worlds as the embodied, symbolic, and formal worlds, respectively. In the process of building knowledge, learning in the embodied world is linked with learning in the symbolic world. As we develop our knowledge within the embodied world, actions that we perform on the objects become routines, which can be represented with the symbols. However, this development from the embodied into the symbolic world is not necessarily linear, nor is the transition. We can move back and forth between the worlds, developing a network of connections between embodied and symbolic worlds. This later leads to development of a formal world of mathematics with formal definitions and theorems formulated and proved about the embodied and symbolic concepts.[15] This development is not linear, with a person frequently moving between the worlds, back and forth, a number of times. Tall's theory is dynamic, and emphasizes the flexibility of movement between the worlds as a key to development of good mathematical understanding.

However, the theory of three worlds does not discard categorization of knowledge into conceptual and procedural. Instead, the terms conceptual and procedural knowledge take on refined meanings. Conceptual knowledge denotes the formation of flexible knowledge schemas and procedural knowledge denotes step-by-step actions before they are condensed into overall processes and transferred into thinkable procepts.[15] The procedural knowledge occurs in both the embodied and the symbolic worlds and is seen as part of conceptual knowledge. Since the difference between conceptual and procedural knowledge is not absolute, we decided that the use of Tall's theory of learning was more suitable for our study. We will research students' knowledge in the three worlds of mathematics 14 months after the calculus course.

2.2. *Retention*

There is no unique measure of knowledge retention, but in educational contexts, two measures are commonly used: cued recall and recognition.[18] We use the term retention in accordance with Sousa [19] who defined it as the extent to which someone can successfully access and use the information from the long-term memory. In general, and not only in mathematics, when it comes to describing the phenomenon of people forgetting what they have previously known, many forgetting graphs have been proposed, using various power functions and models of exponential decay or flat decay, with the former as the most likely model.[20] Specifically, for mathematics, research of school children up to grade 10 has shown that not only the amount of what is forgotten, but also the type of knowledge forgotten is related to student ability. Here, for instance, Krutetskii [21] found that able mathematics school children retain around 85% of the generalized relations very well, even three months after it was learnt. However, the study we report in this paper is not a study of ability versus non-ability in relation to retention. In fact, university students in Denmark studying within the STEM (science, technology, engineering, mathematics) disciplines would have been successful in mathematics in their earlier school life. In order to be enrolled at a STEM study programme at university, the student must have the highest level of high school mathematics (level A). Only level C is mandatory here. University access also requires a certain grade point average, which varies depending on programme and university. Hence, the student in this study might, as a school child, have been able to retain the above-mentioned 85% of generalized relations. Nevertheless, university students experience that they fail to recall previously learnt material in mathematics; for instance, Karsenty [8] states that the retention of high school mathematics depends on several factors such as level and total length of the courses. For instance, Engelbrecht et al. [22] investigated the retention of basic techniques from a first-year calculus course, examining the knowledge of engineering students. Comparing the pre-test results with the post-test given two years after the course, they found a significant decline in performance. However, other studies [23] show that sometimes students remember a great deal of what they learn in college courses (of psychology). Particularly, students who served as tutors retained more knowledge four months after the course than the students they tutored. This suggests that tutoring, which is a type of overlearning, positively affects long-term retention. Overlearning here refers to repetition of material after the material is completely understood. However, when it comes to mathematics, several studies show [24,25] that overlearning does not benefit long-term retention.

Other studies have also researched retention of higher education mathematics. Narli,[26] for instance, investigated and compared the long-term effects on knowledge retention of teaching Cantor set theory using a traditional and active learning approach, respectively. Investigating pre-service mathematics teachers, analysis of the data revealed that the students in the active learning environment (inquiry and problem based) showed better retention in nearly all of the concepts related to Cantor set theory than the students in the traditional class. Hence, the teaching and learning style appear to affect the quality of learning. Furthermore, Selden et al. [27–29] showed that university students who had successfully completed a traditionally taught first-year calculus course often had difficulties solving non-routine problems. This was the case for both average and good students as well as students in subsequent mathematics courses. They define a non-routine problem as a problem that not only depends on the knowledge of the course, but also involves more insight and consideration of several sub-problems. Moreover, it is something the students have not seen before. Previously, we investigated the effect of teaching style on the learning outcome of calculus. Jukić and Dahl [9] showed that two months after passing the exam in a student-centred calculus course, the non-mathematics Danish university students performed significantly better in the items belonging to the embodied and formal worlds or in the items that asked for the movement between several worlds, while non-mathematics Croatian students from the teacher-centred course performed significantly better in the items from the second (symbolic) world. However, even though the Danish students had relatively better retention across the three worlds of mathematics, they too showed a deterioration of knowledge. This result is partly in agreement with other studies where it was detected that knowledge belonging mostly to the second world is the quickest forgotten.[30]

However, instead of solely focusing on retention, we will argue that there is not necessarily a strict dichotomy between retention and non-retention, but there is an intermediate state between the two. Hence, we argue that it is not the main purpose of education that a student has all previously learnt knowledge immediately present at all times. Instead, we will argue that if a student, after a hint, fully remembers the knowledge that is being asked for, then this is just as good as retention. This argument is in line with a discussion of Karsenty [21] where he refers to works of Bartlett from 1932 and Neisser from 1984. Here it is argued that recalling something is a *reconstruction* process that yields to an altered version. Recalling is not a process of *reproduction*, i.e. details are here coded in memory and re-appear as so-called ‘copies’. Recalling is usually investigated using open-ended questions, and recognition using true–false questions.[17] Multiple-choice questions combine recall and recognition.[31] This, we find, is a further justification of the need for a follow-up study one year after the first study, where we had used multiple-choice questions which aimed more at the reproduction and less at recalling. By using an in-depth interview with one student, we can examine whether the student can reconstruct/recall the knowledge after some hints. We also argue that this view of retention still fits well with above-mentioned Sousa’s [18] definition of retention, as the extent to which someone can successfully access and use information from long-term memory.

2.3. *Attitudes, beliefs, and knowledge*

In order to describe students’ thoughts and decisions towards mathematics, we apply the terms attitudes and beliefs. Researchers (e.g. [32]) give different definitions for each term: some regard attitude as a collection of beliefs, while others classify beliefs as one component of attitude, but generally there is no universal agreement in mathematics education about the definition. Furthermore, attitudes, beliefs, and other related words are usually applied as synonyms.[33] Nevertheless, Goldin et al. [32] also state that beliefs do indeed have a crucial influence on the process of learning mathematics; for instance, the rather negative influences of beliefs have been very well documented. Several studies [34–39] have shown that beliefs have a high impact on students’ effective learning and use of mathematics, shaping the students to become active or passive learners. In the latter case, rote learning and remembering is more stressed by the students than understanding.

The learning outcomes are strongly related to students’ beliefs and attitudes about mathematics.[37] Therefore, in order to examine in depth a student’s understanding and retention of some mathematics, we argue that it is also essential to examine the beliefs and learning approaches of the student. In this study, we will therefore refer to Furinghetti and Pehkonen [37] who described beliefs as an individual’s subjective knowledge and gave a characterization on the view of mathematics as a mixture of knowledge, beliefs, conceptions, attitudes, and feelings. Beliefs consist of four main components: beliefs about mathematics, beliefs about oneself as a learner and as a user of mathematics, beliefs about mathematics teaching, and beliefs about mathematics learning.[40]

Hence, we particularly focus on the student’s beliefs about calculus in relation to the chosen study programme, and we investigate the student’s beliefs about the learning of calculus.

2.4. *Research questions*

In order to investigate the retention of a student's mathematical knowledge in a student-centred calculus course, we formed the following research questions: What knowledge did the student retain (with or without hints) 14 months after the course instructions and examination? And, what was the effect of beliefs and attitudes about mathematics on this retention?

3. Methodology

3.1. *Mixed method approach*

In this study, we used a mix of research strategies, a questionnaire supplemented with a qualitative interview. The questionnaire gave core information about the condition of the student's retention, and the interview gave information on how exactly the student perceived the calculus course and the concepts. Cohen et al. [41] argue that such a mixed method approach has significant advantages, since it provides triangulation of the data and hence helps in validating the conclusions.

Our paper is based on a case study of one student. In such a qualitative study, the notion of generalizability, i.e. 'external validity', is therefore replaced by 'fittingness', 'the degree to which the situation matches other situations in which we are interested'.[42, p.207] Goetz and LeCompte [43] use the notion 'translatability' to denote if the theoretical frames and research techniques are understood by other researchers in the same field, and the notion of 'comparability' to mean if a situation has been 'sufficiently well described and defined that other researchers can use the results of the study as a basis for comparison with other studies addressing related issues'.[43, p.228] 'Thick descriptions' are therefore vital for others to be able to determine if the attributes compared are relevant.[44] Therefore, we aimed to make the process transparent by providing a detailed study.

3.2. *The calculus course*

The Danish student was from a university where the courses Calculus 1 and 2 are mandatory first-year courses for all students in STEM study programmes. Calculus 1 and 2 are mainly student-centred courses [45] in which the learner is at the centre of the teaching/learning process. This means that while lectures are given to a large group of students, the students have a mathematics laboratory afterwards where they sit in groups of 5 or 10, and solve problems by themselves. Two experienced mathematics professors are present to assist the students. The students also have theoretical exercises with a teaching assistant consisting of a combination of problem solving, discussion, teacher presentation, and students solving tasks at the blackboard. The exercises are performed in small groups, and are based on problem solving. The content of these courses includes topics on differential and integral calculus of functions of one variable, and of several variables. The process of evaluating students' knowledge starts after Calculus 1, where students take a multiple-choice test, which determines whether or not the student can take the final written exam after Calculus 2. The grade obtained in the final exam is a joint grade for Calculus 1 and 2. The concepts chosen for the present study are central to the calculus course. The calculus courses use the US textbook 'Calculus: Concepts and Contexts' by James Stewart, wherefore the students are very familiar with mathematical terms in English.

3.3. *Previous study and finding the participant for this study*

The study reported in this paper involved one female student from a Danish university. The female student will be called Ann. The mathematical content we research in this paper was part of Calculus 1, which Ann had passed (October 2009) 14 months before this study was carried out in December 2010. However, as stated above, we already met Ann two months after Calculus 1 (December 2009). In December 2009, we gave a two-page questionnaire with calculus tasks in derivatives and integrals to 193 students taking Calculus 1 and 2 in the academic year of 2009–2010. The results from this are reported in Jukić and Dahl,[9,10] but for the convenience of the reader, we will summarize the main findings below.

Approximately 800 students take Calculus 1 and 2 each year, and they are divided into three cohorts. Initially, the cohorts are divided according to the study programme, but the students are free to attend any lecturer they want. We got access to two of these cohorts. The third lecturer did, unfortunately, not want to participate. When we gave the questionnaire, it was not pre-announced, so the participants were the students who attended that day; attendance was not mandatory. The students were free to fill out (or not) any part of the questionnaire they wished; hence, the number of data varied slightly from task to task. Students did not use calculators when they solved items in the questionnaire. The response rate was 97% among the students attending the lectures. The students also had to indicate their study programme, which showed that students originally in the third cohort had been attending the two lectures we had access to.

The questionnaire was first piloted to a group of chemistry students at another university. Furthermore, prior to presenting it to the two cohorts of students two months after their Calculus 1 examination in our university, we consulted one of the three lecturers and the department head in the pilot university about the relevance of the tasks, the formulations, and the appropriateness of the options for answers. The questionnaire was given during the last week of Calculus 2 which follows immediately after Calculus 1, except for a brief examination period. One of the questions asked the students to state what grade they got in Calculus 1. If the student could not remember his/her grade, the student was asked to write his/her study number so we could later find the grade in the university system. 27 students did that. When we, one year later, wanted to track some students who had participated in the first study, we used the study number of these 27 students to reach them via email, and asked if they were interested in a follow-up study. Only one student answered positively – Ann. Ann's final grade in the calculus courses was 12, the highest grade in the Danish grading system. We considered emailing all 800 students but this would mean disturbing around 600 students who had not participated at all. Furthermore, the 27 students still represented a rather large group, and we did not assume that they, as a group, varied from the larger group of 800 students; hence, volunteers from this group would be just as relevant to research as volunteers from the larger group.

At the time of the study, Ann was enrolled in the second year of chemistry, where she chose to specialize in physical chemistry where calculus was applied. Physical chemistry is the study of macroscopic, atomic, subatomic, and particulate phenomena in chemical systems in terms of laws and concepts of physics. It applies the principles, practices, and concepts of physics such as motion, energy, force, time, thermodynamics, quantum chemistry, statistical mechanics and dynamics, and equilibrium. In his book 'Mathematics for Physical Chemistry: Opening Doors',[46] physical chemist Donald A. McQuarrie states: 'I advise my students to listen carefully the moment they decide to take no more mathematics courses. They might be able to hear the sound of closing doors' (preface). Furthermore, a review of his book stated: 'Physical chemistry is difficult because of the mathematics, but it is impossibly difficult without it'. Mathematics is needed and applied in physical chemistry. Calculus, as a branch of mathematics, is a major part of the above-mentioned book and this scientific discipline.

3.4. *Questionnaire on calculus concepts given 14 months after the course*

The questionnaire was written in English, and the interview was also held in English since the interviewer (the first author) did not know Danish. We will argue that this did not cause any problems in communication, since the Danish university uses US calculus textbooks wherefore the students are very familiar with mathematical terms in English, even though lectures are held in Danish. Ann had unlimited time for solving the tasks. She was given enough empty space to calculate and write her answers. She was not given any tables of basic derivatives or integrals, nor was she given any piece of technology such as a graphic calculator or a computer.

The questionnaire contained the same calculus items as the questionnaire the year before. However, in the first study, the questionnaire was given as multiple choice (see Appendix 2 for options of answers) but for the study reported in this paper, the items were given in an open-ended form to allow for a more in-depth study of the student's knowledge. All items can be seen in Appendix 1. There were four items related to differential calculus. The *Tangent* was concerned with a geometric interpretation of the derivative of a function at a given point. In *Quotient*, our intention was to test the student's knowledge of the differentiation rule of a simple rational function. *Composition* examined how the student responded to a composite function. *Slope* incorporated several key concepts from differential calculus: the slope of tangent line as the derivative of the function *f* at the given point, and the process of differentiation. There were also four items related to integral calculus. *Area* was concerned with the geometric interpretation of the definite integral. *Antiderivative* asked for the antiderivative of some function. *Method* required a computation of certain indefinite integral, which is usually done by using integration by parts. *Basic integrals* consisted of two indefinite integrals that are usually given in the tables of basic integrals. Each indefinite integral had the same integrand: a rational function with the number 'one' in the numerator. Our intention was to see whether the student would generalize solutions in these similar cases or not. A comment is needed about the first item, *Area*: we did not find it necessary to emphasize that the function f in the item is integrable on the closed interval $[a, b]$, since formulating the task this way gave the student an opportunity to write deeper and more creative answers, for instance, to discuss more subtle cases of non-strictly positive functions on the interval $[a, b]$.

There were two items, *Derivative Application* and *Integral Application*, which had not been on the questionnaire with multiple-choice items, since they could be solved in more than one way, and therefore were not so suited for a multiple-choice questionnaire. One reason for choosing multiple choice in the previous study was to keep the size of the questionnaire down to two pages, not taking too much time from the lecture. Another reason was that multiple-choice questions are more easily analysed and compared, particularly for larger surveys, and furthermore, they can be more specific, and the response rate is often higher. In *Derivative Application*, several conditions had to be connected in order to sketch a graph of the function. In *Integral Application*, an area between a curve and two lines had to be calculated, but the equations of the curve and the line were not given in the usual standard form.

3.5. *Interview questions on calculus concepts and beliefs 14 months after the course*

Just after finishing with the questionnaire, Ann participated in a semi-structured interview. This interview style was chosen to balance two purposes: first, to make sure the student answered certain questions (see below), and second (as we wanted to uncover, explore, and describe the students' experience), the interview also needed some unstructured qualitative parts with open-ended questions. Our interview style was therefore to aim at listening and delaying the next item, use something the student had previously said or written or reformulate her answers to get a more elaborate explanation.[44] The interviews were audio taped and transcribed.

Ann got the following questions during the interview: she was asked to define the derivative of a function $f$at some point x, to define a definite integral of a function f:[a, b]$\rightarrow$ R, and to list some applications of derivatives and integrals in mathematics and non-mathematics courses. Ann was also asked to elaborate on how she had solved the items in the questionnaire, and respond to the items given on a four-point Likert scale, ranging from 1 = strongly disagree, 2 = disagree, 3 = agree to 4 = strongly agree (see Appendix 1). These items aimed at exposing her beliefs and attitudes towards mathematics and the application of mathematics in her future profession. The neutral option was omitted since we wanted Ann to make a decision. Apart from choosing the level on the Likert scale for the given item, she had to elaborate in her own words what influenced her choice. The first five of these statements (numbered 5–9 in Appendix 1) were translated and adapted from a previous study of students at the Technical University of Denmark.[47] This study examined how engineering students view mathematics, how they take part in study activities, and how they make use of knowledge resources during the introductory first year course of mathematics. We used the questions of Rattleff et al. [47] since this study involved non-mathematics university students early in their studies in Denmark, just like in our study. We therefore assumed that building on their questionnaire would be more relevant than using one of the other existing measures, such as ASSIST (Approaches and Study Skills Inventory for Students), which is a questionnaire that tests general study approaches.[48] The last three statements (numbered 10–12 in Appendix 1) were designed by us to get specific information about the student's attitudes towards her profession in relation to derivatives and integrals in particular.

3.6. *Classification of mathematical items*

The mathematical items given to the students investigated various representations of derivative and integral across all three mathematical worlds. The items *Tangent* and *Area* were situated within the embodied world – requiring geometrical interpretation. The items *Quotient*, *Composition*, *Slope*, *Method*, and *Basic integrals* resided in the symbolic world, as an algebraic manipulation of symbols. The items *Antiderivative*, *Derivative Definition*, and *Integral Definition* belonged to the formal world, asking for the formal definition. *Derivative Application* and *Integral Application* asked for movement between the symbolic and embodied worlds. In *Derivative Application*, one moves from the symbolic into the embodied world. This means that after interpreting given symbols, a certain figure should be sketched. In *Integral Application*, one moves from the symbolic world into the embodied world and then back to the symbolic world. This happens as in order to determine proper integration bounds, a figure of the required area should be drawn or imagined in a person's mind, and then a calculation should be performed.

Although we situated the mathematical items within certain worlds, or described how they connect certain worlds, the answers for a particular item can contain representations from different worlds, and can show various relations between them. The mathematical items together with answers make a whole for detecting retention of calculus concepts.

4. Results

Ann solved all items in the original questionnaire correctly, but she was less successful answering the same items a year later. She had only one correct answer, namely the item *Area*, which resides in the embodied world. If we compare Ann's new and old results, Ann's knowledge deteriorated during the one year time span. In order to gain insight into what influenced her knowledge decline, we asked her to explain and elaborate on how she solved the items. Below is a summary of her results as well as her interview explanations.

4.1. *Ann's solution to the items in the questionnaire and in the interview*

Ann did not know how to differentiate the given functions. She tried to use rules of differentiation to get the correct answers, but she did not remember them correctly. In the case of *Quotient*, she wrote the rule for differentiating a product, but she did not use it in her answer. She began to solve it, and left the problem unfinished. She said that although she noticed that the function in *Quotient* can be simplified, she was uncertain if that was a good approach, so she tried to differentiate the function in the original form using a 'formula'.

In the case of *Composition*, she wrote a rule as if she would differentiate separately each function that constituted this composition, and then multiply the obtained derivatives (Figure 1). Even though she had written down this rule, she differentiated the function in *Composition* as if she was using the chain rule, but forgot to include one function from the composition. In the interview, we also investigated her conception of a given function and gave her hints for a correct differentiation:

A (Ann): I remembered the rule.

I (Interviewer): How many functions do you recognize?

A: Two.

I: Do you recognize which functions constitute this function?

A: sin x and $6x$.

I: What about the square [function]?

A: Oh yes... then it's three.

I: But you have noticed this [square] when you differentiated the function.

A: Yes...

I: So, is this final result [of differentiation]? Do you want to change it?

$$f \circ g'(x) = f'(x) \cdot g'(x) = 2\sin(6x) \cdot 6 = 12\sin(6x)$$

Figure 1. Ann's solution to *Composition*.

A: No, it's ok.

I: What about derivative of sine function? You missed that.

A: (laughs) I forgot.

In the item *Slope*, Ann differentiated $f(x) = (3x)^2$ as $f'(x) = 2(3x)$, and she obtained 6 by inserting $x = 1$ into the derivative. She knew that she had to differentiate the function, but she did not know if what she was doing was correct or not. She explained:

> A: The tangent is connected with derivative, so I differentiated it and put $x = 1$ into the derivative [her result was 6]. But I was in doubt if I should split it up before [refers to squaring the expression] but then it would be 9, but I do not know.

Ann did not know how to integrate the given functions. The item *Method* was solved incorrectly as was the item *Basic integrals*. Ann wrote that $\int \frac{dx}{1+x^2}$ equal In$(1 + x^2)$ (Figure 2) and that $\int \frac{dx}{x^3}$ equals In(x^3).

In the item *Tangent*, Ann wrote the answers in words and sketched a figure supporting her claim. At the beginning, the answer was written correctly but at some point she changed her mind and crossed the word 'slope' in the sentence. Her figure represented some curve denoted as $f(x)$ and a tangent line at the point $(x_0, f(x_0))$. The tangent line was labelled $f'(x)$. The item *Area* was solved correctly and Ann sketched a figure representing her written answer.

The solutions to the new items *Derivative Applications* and *Integral Applications* were incorrect. In *Derivative Application*, she described a discontinuous function as a function with 'holes', hence working in the embodied world, but did not use this in her drawing. She conceived the second derivative in condition b, as the extension of the first derivative in terms of the decrease or increase of a function, and used that in her drawing. Also, she recognized $f'(-1) = 0$ as a turning point for a function, from decreasing into increasing, and used this in the drawing too (Figure 3).

Ann was given some hints to try to reconstruct her knowledge for the application of derivative, but without success.

a. $\int \frac{1}{1+x^2}\,dx = \int \frac{1}{y}\,dx = \ln(y) = \ln(1+x^2)$

$\cancel{\frac{y}{dx} = 1+x^2\,dy}$

$\cancel{y = 1+x^2}$

$dx = 1+x^2\,dy$

I am quite sure that you have to change the boundaries, but i can't remember how

Figure 2. Ann's solution to *Basic integrals*(a).

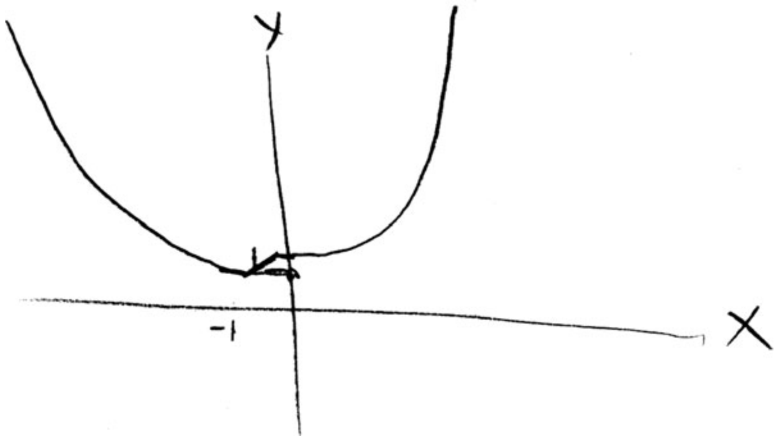

Figure 3. Ann's solution to *Derivative Application*.

I: Well, no. . .second derivative tells you something about the shape of graph [of function]. For instance, consider the shape of parabola being concave upward or downward. Can you use that?

A: . . .hm. . .the slope has to be negative but I do not know why [looking at the conditions]. . .oh, it' because when you have first derivative, it's like tangent to the line. . .if you have -2x, and take next derivative you would get -2. . .and that's getting smaller and negative. . .

In the item *Integral Application*, Ann detected what the upper and lower boundaries were and simply used them as limits of the integral, without checking if her conclusion was right or wrong. She explained that she used to do it this way all the time in her calculus course. She posed the definite integral as $\int_{2-y}^{4-y^2} y\,dy$ (Figure 4), but was not sure how to calculate this integral and she made mistakes in the solution.

The item *Antiderivative* was the only item where Ann wrote she did not even know how to begin answering it. She said she considered derivatives and integrals to be the inverse of each other, so she was very puzzled by the term antiderivative.

In the interview, Ann explained what derivatives and integrals were, relying only on the geometric interpretations. When she explained how she conceived the derivative, she used

$$\int_{2-y}^{4-y^2} \cancel{y}\,dy = \left[\tfrac{1}{2}y^2\right]_{2-y}^{4-y^2}$$

cannot remember if it is + or ÷

$$= \tfrac{1}{2}(4-y^2)^2 \div \tfrac{1}{2}(2-y)^2$$

$$= 8 - \tfrac{1}{2}y^4 - 2 + \tfrac{1}{2}y^2$$

$$= \tfrac{1}{2}y^2 - \tfrac{1}{2}y^2 + 8 - 2 \qquad y=0$$

$$\Rightarrow \underline{\underline{6}}$$

Figure 4. Ann's solution to *Integral Application*.

the word ‘slope’, frequently accompanied by the derivative of the function $f(x) = x^2$ to better describe this concept. She stated:

> It’s like. . . It is a slope of the line. And if you got like x^2 then you would get a straight line. If you are on the hill and you go one step to a left and you take the derivative, and you can, by looking if it’s positive or negative, know if it’s going to be uphill instead of downhill, it tells you something about slope of the function, I think.

Ann was asked how she conceived a definite and an indefinite integral. She explained that definite and indefinite integrals were the same concepts, where one uses the indefinite to solve a definite integral and later inserts the boundaries into the obtained expression.

4.2. *Ann’s attitudes towards mathematics*

As can be seen in Appendix 1, Ann answered the Likert scale questions on attitude and she also talked about it during the interview. It is very interesting to note that her choices on the Likert scale questions in this questionnaire matched her choices in the questionnaire given a year ago. Now we will discuss Ann’s explanations for the choices she made. Ann stated that she considered the calculus courses important and useful for her future profession. She based her opinion on the rest of the courses from her study programme of physical chemistry, where she had to use calculus concepts. She saw a real application of calculus in chemistry, and was able to state examples from chemistry where derivatives and integrals were necessary for solving chemistry problems, but she did not remember any applications in mathematics. She explained the following:

> Mathematics is used in all natural sciences. It is like the language of the natural sciences. So you have to be familiar with it to know how chemistry works. I see why we have to learn it - to be able to get through other courses. We use them [derivatives and integrals] a lot in other courses. Microscopic chemistry uses them a lot. You have to use it [integral] to get to know how molecules and particles work and what the possibilities are to find them in space; this is kind of important to see how they react with new molecules.

But even though she greatly valued calculus and could easily give examples on how it was used in her study, Ann did not consider it important to understand the mathematics she was taught in the university course, especially since she believed that university mathematics was too hard for her. She stated the following:

> No, I do not agree with this statement.[7: It is important to me to understand the mathematics behind the problem] You do not have to understand mathematics behind the problem, you can use mathematics without understanding, but if you do not understand what you are doing [calculating], then a chance that you make a mistake and do not realize it is larger. But if you have an idea what you are doing, it is easier to find your own mistakes. Mostly I learned it [mathematics] in high school, and the rest [university calculus] was too hard to understand.

Having the view of mathematics as something one does not need to understand can be argued as having influenced her approach to the calculus – i.e. how much effort she put into understanding it. Furthermore, we see that she stated that the university calculus was too hard to understand, which can be argued to not push her towards trying to reach understanding:

> But lots of times I did not know what we were doing. We had just fed everything into the calculator. [. . .] But this is also our own fault because we can learn it without a calculator, but

this way is easier. We did it in high school, but here everyone uses a calculator. And I saw this was much easier. And we got the rules, but no one used them, we just used calculators.

5. Discussion and conclusion

In the research questions, we asked what knowledge Ann retained (with or without hints) 14 months after the course and what the effect of beliefs and attitudes about mathematics was on the retention.

5.1. *Retention after 14 months*

A year earlier, Ann had got the maximum grade in Calculus 1 and 2, and solved all items in the first questionnaire correctly. One year later, in the second questionnaire, she was able to solve only one item correctly – the item that asked for the representation of a definite integral in the embodied world. From the multiple-choice questionnaire given a year ago, we were not able to detect the quality of Ann's knowledge, i.e. what representations of derivative and integral Ann had in the three worlds of mathematics, and what connections existed between them. But in this study, using open-ended questions, we gained a better insight into her knowledge. Here we were able to observe how Ann recalled, i.e. reconstructed and used her knowledge belonging to different worlds of mathematics.

In the items *Quotient* and *Composition*, Ann tried to write general rules for differentiation of quotient and composite function, but her symbolic output was not right. However, she performed differentiation of the composite function based on her past experience, not according to the rules she wrote. To be able to apply the chain rule correctly, she should have the prerequisite knowledge of the composite function in the embodied and symbolic worlds, as well as the concept of derivative.[49] Therefore, referring to the results in *Quotient* and *Composition*, it seems that Ann had not the perquisite knowledge on functions necessary for the process of differentiation.

In the item *Slope*, she associated tangent line with derivative, connecting the symbolic and the embodied worlds, and similarly she established this connection in the item *Tangent*. It seems that the derivative of the function was not embodied properly, because Ann conceived derivatives sometimes as the tangent line, and sometimes as the slope, depending on the situation. The connection between the symbolic and embodied worlds was also used in the item *Derivative Application*, where Ann had embodied most symbolic representations, either in the drawn graph of the function or just described verbally. However, this connection was not correct because she did not relate the given conditions with proper graphical output. The item *Integral Application* was solved only in the symbolic world, even though Ann established connection for the definite integral in the embodied, and in the symbolic, world in the item *Area*. This indicates that Ann had a weak connection between the embodied and symbolic worlds for the definite integral and was not able to move flexibly back and forth between the worlds. Using the expressions containing y, which is a function of x, as the boundaries, implies that the concept of definite integral was not embodied entirely, meaning she did not fully understand what the boundaries represent for an area.

Ann herself seemed to be more confident in her knowledge in the embodied world than in her knowledge belonging to the other two worlds. She did not differentiate or integrate correctly, and Ann was uncertain how something should be calculated in the world of symbolic manipulations. In such a case, when the student later tries to make this knowledge of procedures re-appear, we argue that it is usually a process of reproduction since the student will try to remember the exact steps. Ann did not know formal definitions, so, we argue, her knowledge inside the formal world was inadequate. For instance, she was only able to distinguish between the definite and indefinite integrals at the symbolic level.

According to Semb et al.,[23] retained knowledge depends on the original learning; therefore, we conclude that a year before, Ann also had similar knowledge about derivative and integral, fragmented and badly connected between the worlds. This kind of knowledge can work for a short period, but for long-term retention, mathematical concepts should be fully embodied and symbolized, with good connections between the worlds. During the interview, Ann was given hints, but even this did not make her reconstruct her knowledge correctly, which is a further indicator of her knowledge not being embodied or symbolized properly when she was acquiring it.

There happens a natural decline in knowledge if it is not used regularly, but Ann actually used her calculus knowledge in other courses, as part of her study programme in physical chemistry. When the knowledge is encountered along the way, retention should be strengthened,[22] which is not the case here. However, some studies [50,51] also report that meeting calculus concepts in other contexts turns out to be difficult for many students. Although Ann was very good at giving examples of the use of calculus in chemistry, this study reveals that using calculus in other contexts does not in itself strengthen the 'original' calculus learnt in a more pure, formal mathematics course. Instead, it appears that these knowledge areas are disjoint. This also agrees with Britton et al. [52] who reported that mathematics used in other scientific disciplines differs from the 'pure' mathematics. Moreover, Hoban et al. [53] found that the mathematical difficulties which students encounter in a chemistry context may not be because of incompetence to transfer the knowledge, but due to insufficient mathematical understanding and poor knowledge of mathematical concepts relevant to chemistry.

It seems that Ann, as a non-mathematics student, desired to ground the mathematics she learnt in the embodied world, and this is strongly supported with her colourful description of derivatives and the use of integral. Further, it seems she established incorrect or weak connections between the symbolic and the embodied worlds, and could not relate the manipulation of symbols with the meaning of those actions in the embodied world. On the other hand, the formal mathematical definitions were too abstract for her, and did not fit into her understanding based on the embodied world. As mentioned earlier, flexibility in cognitive development is the key for good mathematical understanding, and therefore necessary for later knowledge reconstruction. Consequently, we argue, meeting calculus concepts in physical chemistry did not reinforce her retention.

5.2. *Ann's later university studies*

We contacted Ann a little over two years after this study was conducted (February 2013, i.e. 40 months after having passed the Calculus 1 course) to ask her about the use of the calculus knowledge in the third year of her bachelor study programme. We found that Ann had become a bachelor of physical chemistry, and was pursuing her master's degree. She explained that she met calculus concepts like derivatives and integrals in many courses 'but haven't been forced to actually use them'. She admitted that she would not be capable of using integrals or derivatives without finding some of her old notes from the time she took the mathematics courses. This might again indicate, and confirm, that her original knowledge was not well connected. However, as discussed above, looking at her notes might be regarded as something that 'triggers' her recall of such knowledge well enough for her to pass her subsequent courses. But, perhaps surprisingly, she described that, in fact, the calculus knowledge was valuable in her final project in getting a bachelor's degree:

> Hmm. . . my supervisor at my bachelor project showed me how to solve a couple of expressions with derivatives and integrals [. . .] it was sort of the theoretical background for how some of the things can be found that we wanted to find with [use of computer] different programmes.

5.3. *Attitudes on mathematics*

Ann appeared not have gained a properly embodied, symbolized, and formal understanding of the three worlds of mathematics relevant to calculus, but we also argue that she is an example of a student with a surface [54] approach to learning. A surface approach to learning means that the student is focused on rote learning rather that actual understanding. We base this on the fact that even though Ann agreed that mathematics is part of all natural sciences (answer 3 in Appendix 1) and that basic mathematical knowledge is something that all science students should have (answer 4), she disagreed to the statements on whether it was important to her not just to be able to solve the problems, but also to understand the mathematics behind it (answer 2), and if mathematics in general is exciting (answer 1) and calculus is interesting beyond the fact that she needed it in her study programme (answer 1). Furthermore, during the interview, in her own words, she, for instance, argued that 'You do not have to understand mathematics behind the problem, you can use mathematics without understanding'. The situation described by Ann is an example of Pehkonen's [40] description of the influence of beliefs and attitudes on the learning of mathematics, where the student's subjective knowledge affected the gain of mathematical knowledge.

As described above, beliefs have a high impact on students' effective learning and use of mathematics, shaping the students to become active or passive learners. Ann made the same choices on the Likert scale attitude questions in the questionnaire given a year earlier as she did in the questionnaire used in this study. This consistency shows that her attitudes towards mathematics appeared to be deeply rooted in her resistance to change. Ann's (surface) approach towards understanding mathematical problems may have affected her knowledge, the performance in the study, and her retention. Hence, we can argue that her approach led to retention of not well understood subject matter for examinations (and examinations still rewarded this behaviour), wherefore it did not promote well-connected knowledge or long-term retention of knowledge. Not even hints made her correctly reconstruct her knowledge.

5.4. *Recommendations for practice*

Mathematics, especially calculus, is important for learning physical chemistry, and it is incorporated into all physical sciences.[46] We believe that motivation for well-connected knowledge as well as better retention of calculus knowledge can be achieved by adapting calculus courses to the particular study programme. This means collaboration between mathematicians and academics from different scientific disciplines. In many cases, students from various science study programmes take the same calculus course, but they cannot relate the mathematical concepts and procedures they learn to the rest of their study programme (e.g. [2]). Separate calculus courses enable material to be presented at a practical level. Such emphasis on the applications in more meaningful contexts of the particular scientific disciplines would better enable the use of mathematical knowledge in that scientific discipline. This suggestion is also supported by Ann's comment 40 months after Calculus 1, where she still needs her notes, but she is able to see an application of calculus. This does not mean that rigour in such calculus courses is diminished. Many science students want to apply mathematics to physical problems, using some sort of intuition (the embodied world), and consider rigour unnecessary. This intuition is not always accurate, because many counterexamples can be found in calculus, and students should also be aware of these.

Furthermore, Tall [55] gives a complete framework for developing calculus as a blend of the symbolic and embodied worlds, which gives rise to the formal world. These three worlds encompass also three modes of thinking. This will surely be beneficial for non-mathematics students who feel more confident in the embodied world since this will enable them to give a meaning to the actions they perform, and allow them to perceive mathematical concepts in the physical or mental world, similar and closer to their field of interest. If insufficient time is spent on embodying a certain concept, then the focus on the symbolism, incorporated with the formal definitions, brings many misconceptions for non-mathematics students and poor long-term retention of knowledge. Moreover, we suggest that the tasks based on the movement from the symbolic world to the embodied one, and vice-versa, should have a more prominent role in calculus courses designed for non-mathematics students.

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Appendix 1. The questionnaire given to Ann, including some of her answers

Derivatives tasks

1. *Tangent*: What is a geometric interpretation of the derivative of the function $f : R \to R$ at the point x_0?
2. *Quotient*: If given $f(x) = \frac{x^2+2}{x^3}$, find $f'(x)$.
3. *Composition*: Find $f'(x)$, if given $f(x) = \sin^2 6x$.

4. *Slope*: What is the slope of the tangent line to the graph of the function $f(x) = (3x)^2$ at the point $x = 1$?
5. *Derivative Application*: Sketch the graph of the function f which satisfies the following conditions:
   a. f is discontinuous at x = 0 and $f(0) = 1$
   b. $f''(x)<0$ for all x < 0 and $f''(x)>0$ for all $x > 0$
   c. $f'(-1) = 0$ and $f'(x) \neq 0$ for $x \neq -1$

Integral tasks

1. *Area*: What is a geometric interpretation of definite integral $\int_a^b f(x)dx$?
2. *Antiderivative*: Define the antiderivative of a function $f: R \to R$.
3. *Method*: Find the given integral $\int xe^x dx$. What is the most appropriate method for computing the integral?
4. *Basic integrals*: Determine
   a. $\int \frac{dx}{1+x^2}$
   b. $\int \frac{dx}{x^3}$
5. *Integral Application*: Find the area above the x-axis bounded by the curve $x = 4 - y^2$ and lines $y = 0, x = 2 - y$.

Additional tasks and questions

1. *Derivative Definition*: State the definition of derivative of a function $f : R \to R$ at x_0. Explain this concept.
2. List some application of derivatives.
3. *Integral Definition*: State the definition of definite integral of bounded function $f : [a, b] \to R$. Explain this concept.
4. List some application of integrals.

Beliefs questions, Ann's answers in brackets

With numbers from 1 to 4 mark the extent to which you agree with the following statements, where
1 = strongly disagree 2 = disagree 3 = **agree** 4 = **strongly agree**

5. Everyone who studies natural and technical sciences ought to have knowledge of basic mathematical disciplines. (**4**)

6. Mathematics is a central part of technical and natural sciences. (**3**)

7. It is important to me not just to be able to solve a problem, but also to understand the mathematics behind it. (**2**)

8. Mathematics is an exciting subject in general. (**2**)

9. The calculus course was interesting to me beyond the fact that I had it as a part of my study programme. (**1**)

10. It was more difficult for me to understand integrals in comparison to derivatives. (**1**)

11. I think derivatives have a significant application in my profession. (**3**)

12. I think integrals have a significant application in my profession. (**4**)

Appendix 2. The questionnaire given to all the students one year before

Derivatives questions surveyed with given options for answers

1. Question *Tangent*: What is the geometric interpretation of the derivative of the function $f : R \to R$ at the point x_0? Offered answers: A: maximum/minimum of the function f at x_0; B: slope of tangent line to the curve $y = f(x)$ at x_0; C: continuity of the function f in the given point; D: none of the above.

2. Question *Quotient*: Differentiate the function $f(x) = \frac{x^2+2}{x^3}$. Offered answers:

$$\text{A}: \frac{x^3(2x) - (x^2+2)(3x^2)}{(x^3)^2}; \text{B}: \frac{x^3(2x) - (x^2+2)(3x^2)}{x^3}; \text{C}: \frac{x^3(2x) - x^2(3x^2)}{(x^3)^2}.$$

3. Question *Composite*: Differentiate the function $f(x) = \sin^2 6x$. Offered answers: A: $2\sin(6x)$; *B:* $12\sin(6x)$; C: $12\sin(6x)\cos(6x)$.
4. Question *Slope:* Calculate the slope of the tangent line to the curve $y = (3x)^2$ at the point $x = 1$. Offered answers: A: 9; B: 18; C: 6.

Integral questions surveyed with given options for answers

1. Question *Area*: What is the geometric interpretation of the definite integral $\int_b^a f(x)dx$? Offered answers: A: The area between the curve $y = f(x)$ and the x-axis for x between a and b; B: The arc length of the curve $y = f(x)$ on the interval $[a, b]$; C: continuity of the function f on interval $[a, b]$; D: none of the above.
2. Question *Antiderivative*: What is an antiderivative of a function f? Offered answers: A: $\int f(x)\mathrm{d}x$; B: every function F such that $F'(x) = f(x)$ holds; C: The set of elementary functions; D: none of the above.
3. Question *Method*: Which method is the most appropriate for computing the integral $\int xe^x dx$? Offered answers: A: substitution $t = e^x$; B: integration by parts; C: trigonometric substitution; D: none of the above.
4. Question *Basic integrals*:
   c. $\int \frac{dx}{1+x^2} = ?$ Offered answers: A: $\ln(1+x^2) + C$; B: $\arctan x + C$.
   d. $\int \frac{dx}{x^3} = ?$ Offered answers: A:$-\frac{1}{2}x^{-2} + C$; B: $\ln(x^3) + C$.

With numbers from 1 to 4 mark the extent to which you agree with the following statements, where

1 = strongly disagree, 2 = disagree, 3 = **agree, 4 = strongly agree**

1. Everyone who studies natural and technical sciences ought to have knowledge of basic mathematical disciplines.
2. Mathematics is a central part of technical and natural sciences.
3. It is important to me not just to be able to solve a problem, but also to understand the mathematics behind it.
4. Mathematics is an exciting subject in general.
5. The calculus course was interesting to me beyond the fact that I had it as a part of my study program.
6. It was more difficult for me to understand integrals in comparison to derivatives.
7. I think derivatives have significant application in my profession.
8. I think integrals have significant application in my profession.