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Feedback Stabilization Robust Against Communication Constraints for Disturbed Linear Plants via Sparse Packetized Predictive Control ⋆

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Abstract: This paper investigates closed-loop stability of linear discrete-time plants subject to bounded disturbances when controlled according to packetized predictive control (PPC) policies. In the considered feedback loop, the controller is connected to the actuator via a digital communication channel imposing bounded dropouts. Two PPC strategies are taken into account. In both cases, the control packets are generated by solving sparsity-promoting optimization problems. One is based upon an ℓ^2-constrained ℓ^0 optimization problem. Such problem is relaxed by an ℓ^1-ℓ^2 optimization problem in the other sparse PPC setting. We show that the ℓ^2-constrained ℓ^0 sparse PPC and unconstrained ℓ^1-ℓ^2 sparse PPC render system states bounded if the design parameters satisfy certain conditions. The bounds we derive on states are increasing with respect to the disturbance magnitude. We illustrate the results via simulation.

Keywords: Networked Control Systems, Sparse Packetized Predictive Control, Disturbance.

1. INTRODUCTION

Packetized predictive control (PPC) is proposed to attain robustness towards channel uncertainties such as packet dropouts in networked control systems (NCSs) Bemporad (1998). Basically, PPC is a model predictive control (MPC) strategy wherein at each time instant, the controller sends the entire sequence of control inputs as a data packet to the plant. In an early attempt, Quevedo and Nešić (2012) derives the conditions of stochastic stability for a nonlinear plant controlled via PPC over a channel with Markovian data loss. The PPC is utilized recently for the remote control of balancing robots in Branz et al. (2021).

Sparse PPC is a packetized predictive control strategy in which sparsity-promoting cost functions are utilized to generate the control packets Nagahara et al. (2012). Besides the aforementioned robustness properties, the advantage of sparse PPC for NCSs is saving the communication resources Gommans and Heemels (2015). Stability of linear disturbance-free plants under sparse PPC over delay-free communication channels with bounded dropouts is investigated in Nagahara et al. (2014). The corresponding sparsity-promoting optimization problems in Nagahara et al. (2014) are ℓ^2-constrained ℓ^0 and a relaxed version of it; unconstrained ℓ^1-ℓ^2 problems. In our recent contribution Barforooshan et al. (2019), we extended the results of Nagahara et al. (2014) to the case with channel delays.

In this paper, we analyze the stability of an NCS wherein a single-input discrete-time linear plant exchanges its entire states with a controller operating based on sparse PPC. The plant is subject to bounded disturbances. Data packets produced by the controller might drop in the channel before reaching the actuator. We consider a PPC strategy based on ℓ^2-constrained ℓ^0 sparsity-promoting optimization. As another control policy, we consider unconstrained ℓ^1-ℓ^2 sparse PPC which performs based on a relaxation of ℓ^2-constrained ℓ^0 optimization. We show in each case that under certain conditions, practical stability can be achieved. Motivated by their strength, we inspire from the approaches employed in Nagahara et al. (2014) when analyzing the stability and solving the involved optimization problems. Nevertheless, as opposed to Nagahara et al. (2014), the plant is disturbed here. So, the first contribution of our work is that we propose sufficient conditions under which the ℓ^2-constrained ℓ^0 sparse PPC and unconstrained ℓ^1-ℓ^2 sparse PPC render the system practically stable, despite the presence of plant disturbances and channel data loss. Certainly, this leads to a control design approach. The second contribution is to shed lights on the performance loss introduced by the existence of plant disturbance. This is nailed by showing that the upper bounds derived on the ℓ^2 norm of the plant state are increasing functions with respect to the upper bound on the disturbance ℓ^2 norm. We illustrate our findings via

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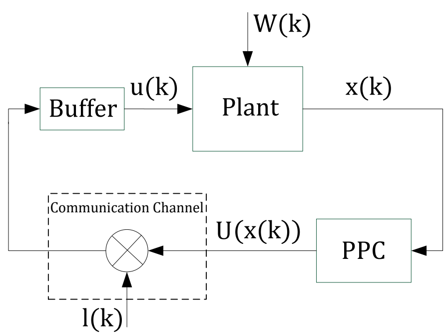

Fig. 1. Considered PPC system

a simulation example. In this example, we demonstrate that the practical stability is obtained with sparse control inputs by the utilized ℓ^2-constrained ℓ^0 sparse PPC and unconstrained ℓ^1-ℓ^2 sparse PPC. Moreover, we observe that while stability is intact, system performance degrades when disturbance magnitude grows.

2. NOTATION

The set of natural numbers is symbolized by $\mathbb{N}$ based on which $\mathbb{N}_0$ is defined as $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$. We denote the transpose of matrix M by $M^\top$. $I_{n\times n}$ represents an identity matrix with dimensions $n \times n$ where $n \in \mathbb{N}$. For the vector $z = [z_1, \ldots, z_n]^\top \in \mathbb{R}^n$ in the euclidean space, ℓ^1 and ℓ^2 norms are defined as $\|z\|_1 \triangleq |z_1| + \cdots + |z_n|$ and $\|z\|_2 \triangleq \sqrt{z^\top z}$, respectively. The ℓ^0 norm of the vector z is the number of its non-zero elements. Furthermore, $\|z\|_W$ presents the weighted norm of the vector z with respect to the positive definite matrix $W > 0$ and is defined as $\|z\|_W \triangleq \sqrt{z^\top W z}$. We symbolize the minimum and maximum eigenvalues of the Hermitian matrix W by $\lambda_{\min}(W)$ and $\lambda_{\max}(W)$, respectively. Moreover, $\sigma_{\max}(W)$ is defined as $\sigma_{\max}(W) \triangleq \sqrt{\lambda_{\max}(W^\top W)}$.

3. PPC FOR DISTURBED LINEAR PLANTS

The control setting of our interest is depicted by Fig. 1. One component of the NCS of Fig. 1 is a linear discrete-time plant which is modeled as

$$x(k+1) = Ax(k) + Bu(k) + w(k), \qquad k \in \mathbb{N}_0. \quad (1)$$

In (1), $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}$ are the plant state and the control input, respectively. Moreover, $w(k) \in \mathbb{W} \subset \mathbb{R}^n$ symbolizes an uncertain disturbance signal where $\mathbb{W}$ is a compact set for which $0 \in \mathbb{W}$ holds. It stems from the properties of $\mathbb{W}$ that

$$\|w(k)\|_2 \le W_m, \qquad k \in \mathbb{N}_0, \quad (2)$$

where $W_m \triangleq \max_{\eta\in\mathbb{W}} \|\eta\|_2$. Matrices A and B are are of appropriate dimensions, time-invariant and reachable.

The channel in the actuation path in the feedback loop of Fig. 1 is assumed to impose dropouts. We model such data packet dropouts by the binary random sequence $l(k)$, that is, if $l(k) = 1$, the data packet transmitted at time k is received by the actuator at time k, $k \in \mathbb{N}_0$. Otherwise, $l(k) = 0$ and the packet is lost. The value of $l(k)$ is unknown to the controller. The packetized predictive controller generates a sequence of control inputs at each time step $k \in \mathbb{N}_0$ described by

$$U(x(k)) = [u_0(x(k)) \ldots u_{N-1}(x(k))]^\top. \quad (3)$$

Afterwards, it sends $U(x(k))$ as a data packet to the plant. Suppose that $U(x(k))$ arrives at the plant side of the channel at time $k \in \mathbb{N}_0$. First, $U(x(k))$ is stored in a buffer next to the actuator in a way that the previous contents of the buffer are completely overwritten. Then, the actuator applies $u_0(x(k))$ to the plant. If the packet $U(x(k+1))$ is received one time instant later, then $U(x(k+1))$ replaces $U(x(k))$ in the buffer and the actuator applies $u_0(x(k+1))$ to the plant. Otherwise, $u_1(x(k))$ will be the control input. Selecting the remaining elements of $U(x(k))$ ($u_n(x(k))$, $n \ge 2$) in a successive manner and applying them as the control inputs continues until the arrival of a new control packet ($U(x(k+n))$, $n \ge 2$).

Assumption 1. The maximum number of the consecutive packet dropouts is equal to $N-1$. Moreover, the first control packet, $U(x(0))$, is received at the plant side.

As already mentioned, in PPC, the control packets are solutions to the problem of minimizing a finite-horizon cost function at each time step. Suppose that the time is $k \in \mathbb{N}_0$ and $x = x(k)$ is received by the controller. The PPC cost function has the following formulation:

$$J(x,U) = T(\tilde{x}_N) + \sum_{i=0}^{N-1} S(\tilde{x}_i, u_i), \quad (4)$$

in which $\tilde{x}_i$ denotes a prediction of $x(k+i)$ with the horizon N and $\{u_i\}_{i=0}^{N-1}$ are tentative future control inputs. The state prediction is carried out based on the update rule $\tilde{x}_{i+1} = A\tilde{x}_i + Bu_i$ for every $i \in \{0 \ldots N-1\}$ where $\tilde{x}_0 = x(k)$. So, we use the disturbance-free (nominal) model of the plant for predicting the future states. The functions T and S in (4) are said to be terminal cost and stage cost, respectively.

We seek sparse solutions to the problem of optimizing $J(x,U)$ in (4). Towards this goal, we set J together with a constraint in a way that an ℓ^2-constrained ℓ^0 sparsity-promoting optimization gives the control packets in the following section. The cost function $J(x,U)$ for ℓ^2-constrained ℓ^0 sparse PPC is specified by the terminal cost $T(\tilde{x}_N) = 0$ and the stage cost S which satisfies $S(\tilde{x}_i, u_i) = 0$ if $u_i = 0$ and $S(\tilde{x}_i, u_i) = 1$ if $u_i \neq 0$. The argument U of the latter cost function is constrained to remaining in the following set:

$$\upsilon(x) = \{U \in \mathbb{R}^N : \|\tilde{x}_N\|_P^2 + \sum_{i=0}^{N-1} \|\tilde{x}_i\|_Q^2 \le \|x\|_\Pi^2\}, \quad (5)$$

where $x = \tilde{x}_0$ and $U = [u_0 \ldots u_{(N-1)}]^\top$. Moreover, matrices P, Q, and Π are assumed to be positive definite and certify that for all $x \in \mathbb{R}^n$, $\upsilon(x)$ is non-empty. So, the control packets in ℓ^2-constrained ℓ^0 sparse PPC are computed as as follows:

$$\begin{aligned} U(x) &= \arg\min_{U\in\upsilon(x)} \|U\|_0 \\ \upsilon(x) &= \{U \in \mathbb{R}^N : \|MU - Kx\|_2^2 \le \|x\|_\Pi^2\}. \end{aligned} \quad (6)$$

The restatement of $\upsilon(x)$ as in (6) stems from the recursion $\tilde{x}_{i+1} = A\tilde{x}_i + Bu_i$ used for predicting the future states $\{x(k+1), ..., x(k+N)\}$ by $\{\tilde{x}_1, ..., \tilde{x}_N\}$ at any time instant $k \in \mathbb{N}_0$. Moreover, the matrices M and K are defined as $M \triangleq \hat{Q}^{\frac{1}{2}}\Gamma$ and $K \triangleq -\hat{Q}^{\frac{1}{2}}\Lambda$, respectively, where $\hat{Q} = \mathrm{diag}\{Q,\dots,Q,P\}$ and

$$\Gamma = \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}. \tag{7}$$

Due to its combinatorial properties, we utilize orthogonal matching pursuit (OMP) algorithm to solve the optimization problem related to ℓ^2-constrained ℓ^0 sparse PPC Nagahara et al. (2014); Pati et al. (1993).

Remark 1. The formulation of the ℓ^0 optimization problem corresponding to (6) in terms of x, U, and weighting matrices is based on the disturbance-free (nominal) model of the plant. So, the structure of the sparsity-promoting optimization considered here for ℓ^2-constrained ℓ^0 sparse PPC is the same as formulation derived in (Nagahara et al., 2014, Section III-B) for ℓ^2-constrained ℓ^0 sparse PPC of disturbance-free LTI discrete-time plants. Hence, (Nagahara et al., 2014, Lemma 10) and (Nagahara et al., 2014, Lemma 11) hold here. So, defining $\upsilon^*(x)$ as

$$\begin{aligned} &\upsilon^*(x) \triangleq \{U \in \mathbb{R}^N : \|MU - Kx\|_2^2 \le \|x\|_{\Pi^*}^2\}, \\ &\Pi^\star \triangleq K^\top(I - MM^\dagger)K, \quad M^\dagger \triangleq (M^\top M)^{-1}M^\top, \end{aligned} \tag{8}$$

then $\upsilon(x) \supseteq \upsilon^*(x)$ holds if $\Pi \ge \Pi^*$, where Π is related to $\upsilon(x)$ through (6). Given $\Pi \ge \Pi^*$, the feasible set $\upsilon(x)$ will be closed, convex and non-empty over $\mathbb{R}^N$. Furthermore, if we define the matrix $\xi > 0$ as $\xi \triangleq \Pi - \Pi^*$. Then, there exists $\psi(x) \in \mathbb{R}^N$ for any feasible control packet $U \in \upsilon(x)$ in such a way that

$$U = U^* + \psi(x), \quad \|M\psi(x)\|_2^2 \le \|x\|_\xi^2, \tag{9}$$

holds where $U^* \in \upsilon^*(x)$.

Definition 1. The NCS of Fig. 1 is called practically stable if there exists $\varrho \in \mathbb{R}^+$ that satisfies $\lim_{k\to\infty} \|x(k)\|_2 \le \varrho$.

We define the i-th iterated (open-loop) mapping $f^i(\cdot)$, $1 \le i \le N$ with optimal vector U as

$$\begin{aligned} &f^i(x, \bar{w}_0^{i-1}) \triangleq \bar{f}^i(x) + g^i(\bar{w}_0^{i-1}), \\ &\bar{f}^i(x) \triangleq A^i x + \sum_{l=0}^{i-1} A^{i-1-l}Bu_l(x), \\ &g^i(\bar{w}_0^{i-1}) \triangleq \sum_{l=0}^{i-1} A^{i-1-l}\bar{w}_l, \end{aligned} \tag{10}$$

where $\bar{w}_0^{i-1} \triangleq [\bar{w}_0 \dots \bar{w}_{i-1}]^\top$. We assume that $\bar{w}_0^{i-1} \in \mathbb{W}^i$, $\forall i \in \{1,\dots,N\}$ and $f^0(\cdot) = \bar{f}^0(\cdot) = x$.

Lemma 1. For every realization of $\bar{w}_0^{i-1} \in \mathbb{W}^i$ and for every $i \in \{1,\dots,N\}$, the following holds:

$$\begin{aligned} &\|g^i(\bar{w}_0^{i-1})\|_2 \le \gamma_N(W_m), \\ &\gamma_N(W_m) \triangleq \sum_{l=0}^{N-1} \sigma_{\max}(A^{N-1-l})W_m. \end{aligned} \tag{11}$$

Proof. According to (10) and the triangle inequality,

$$\|g^i(\bar{w}_0^{i-1})\|_2 \le \sum_{l=0}^{i-1} \|A^{i-1-l}\bar{w}_l\|_2 \tag{12}$$

holds regardless of the realization of $\bar{w}_0^{i-1} \in \mathbb{W}^i$, $\forall i \in \{1,\dots,N\}$. Then, the claim follows from (2), properties of weighted norms and the fact that the right-hand-side (RHS) of (12) is increasing with respect to i.

Let $P > 0$ be the solution to the following Riccati equation:

$$P = A^\top PA - A^\top PB(B^\top PB + r)^{-1}B^\top PA + Q, \tag{13}$$

where $Q > 0$ is chosen arbitrarily and $r = 0$. Then according to (Bertsekas, 1976, Chapter 3), the elements of every feasible control packet $U \in \upsilon(x)$ are obtained via

$$u_i(x) = F(A + BF)^i x + \psi_i(x), \quad i = 0,1,\dots,N-1, \tag{14}$$

where $\psi_i(x)$ denotes the $i+1$-th element of $\psi(x)$ defined in Remark 1. Moreover, the gain F is calculated as

$$F = -GA, \quad G \triangleq (B^\top PB)^{-1}B^\top P. \tag{15}$$

In this case, the i-th iterated mapping f^i follows

$$f^{i+1}(x, \bar{w}_0^i) = (A + BF)f^i(x, \bar{w}_0^{i-1}) + B\varsigma_i(x) + \varrho_i(\bar{w}_0^i), \tag{16}$$

where $\bar{w}_0^i \in \mathbb{W}^{i+1}$, $\forall i \in \{0,\dots,N-1\}$. Moreover, $f^0(\cdot) = x$ and the functions $\varsigma_i(x)$ and $\varrho_i(\bar{w}_0^i)$ are defined as follows:

$$\varsigma_i(x) \triangleq G\Gamma_i\psi(x), \quad \varrho_i(\bar{w}_0^i) \triangleq \bar{w}_i - \sum_{l=0}^{i-1} BFA^{i-1-l}\bar{w}_l, \tag{17}$$

for any $i \in \{0,\dots,N-1\}$. In (17), $\Gamma_i \in \mathbb{R}^{n\times N}$ denotes the $i+1$-th row block of matrix Γ defined in (7).

Lemma 2. For every realization of $\bar{w}_0^i \in \mathbb{W}^{i+1}$ and for every $i \in \{0,\dots,N-1\}$, the following holds:

$$\begin{aligned} &\|\varrho_i(\bar{w}_0^i)\|_2 \le \varepsilon_N(W_m), \\ &\varepsilon_N(W_m) \triangleq \big[1 + \sum_{l=0}^{N-1} \sigma_{\max}(BGA^{N-l})\big]W_m. \end{aligned} \tag{18}$$

Proof. It stems from (15), (17) and triangle inequality that

$$\|\varrho_i(\bar{w}_0^i)\|_2 \le \|\bar{w}_i\|_2 + \sum_{l=0}^{i-1} \|(BGA^{i-l})\bar{w}_l\|_2, \tag{19}$$

where $i = 0,1,\dots,N-1$ and $\bar{w}_0^i \in \mathbb{W}^{i+1}$. Now, the claim follows from (2), properties of euclidean norms and the fact that the RHS of (19) is an increasing function of i.

Lemma 3. Define $V_P(x) \triangleq \|x\|_P^2$. Choose $\Pi > 0$ in such a way that $\Pi > \Pi^*$ holds. Set an arbitrary $Q > 0$ and let $P > 0$ be the solution of the Riccati equation (13) with $r = 0$. Then one can find constants $0 \le \varphi_1 < 1$ and $c_1 > 0$ certifying that

$$\begin{aligned} \sqrt{V_P(f^{i+1}(x, \bar{w}_0^i))} \le &\Big[\varphi_1 + \frac{\sqrt{c_1\lambda_{\max}(\xi)}}{(1-\varphi_1)\sqrt{\lambda_{\min}(P)}}\Big]\sqrt{V_P(x)} \\ &+ \big[(1-\varphi_1)^{-1}\sqrt{\lambda_{\max}(P)}\big]\varepsilon_N(W_m), \end{aligned} \tag{20}$$

holds for every $x \in \mathbb{R}^n$ and every realization $\bar{w}_0^i \in \mathbb{W}^{i+1}$, $i = 0,\dots,N-1$. In (20), ξ is defined as in Remark 1.

Proof. Let us define $\Omega_i(x, \bar{w}_0^{i-1})$ as

$$\Omega_i(x, \bar{w}_0^{i-1}) \triangleq (A + BF)f^i(x, \bar{w}_0^{i-1}) + B\varsigma_i(x) \tag{21}$$

Then based on the triangle inequality of weighted norms (Poznyak, 2009, Part I-Chapter 5) and (16), we have

$$\|f^{i+1}(x, \bar{w}_0^i)\|_P \le \|\Omega_i(x, \bar{w}_0^{i-1})\|_P + \|\varrho_i(\bar{w}_0^i)\|_P. \tag{22}$$

Considering Remark 1, we can conclude from the proof of (Nagahara et al., 2014, Lemma 13) and (18) that

$$\sqrt{V_P(f^{i+1}(x,\bar{w}_0^i))} \leq \sqrt{\rho V_P(f^i(x,\bar{w}_0^{i-1})) + c_1\|x\|_\xi^2} + \left(\sqrt{\lambda_{\max}(P)}\right)\varepsilon_N(W_m) \tag{23}$$

is valid for every realization of $\bar{w}_0^i \in \mathbb{W}^{i+1}$,$\forall i \in \{0,1,\ldots,N-1\}$. In (23), ρ and c_1 are defined as follows:

$$\rho \triangleq 1 - \lambda_{\min}(QP^{-1})$$
$$c_1 \triangleq \max_{i=0,1,\ldots,N-1} \lambda_{\max}\left(\Gamma_i^\top P\Gamma_i(M^\top M)^{-1}\right). \tag{24}$$

Since $P \geq Q > 0$, then $\rho \in [0,1)$ and $c_1 > 0$. Define φ_1 as $\varphi_1 \triangleq \sqrt{\rho}$ and note that $f^0(\cdot) = x$. Now, based on the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $\forall a,b \geq 0$, and by mathematical induction, we can restate (23) as

$$\sqrt{V_P(f^{i+1}(x,\bar{w}_0^i))} \leq \varphi_1^i\sqrt{V_P(x)} + (1 + \cdots + \varphi_1^{i-1}) \times\left[\sqrt{c_1}\|x\|_\xi + \left(\sqrt{\lambda_{\max}(P)}\right)\varepsilon_N(W_m)\right]. \tag{25}$$

Employing $0 \leq \varphi_1 < 1$, properties of weighted norms and (25), we derive (20). The proof is complete now.

Theorem 1. Select $Q > 0$ arbitrarily and obtain $P > 0$ as the solution to the Riccati equation (13) with $r = 0$. Choose the matrix ξ in such a way that

$$\sqrt{\lambda_{\max}(\xi)} < (1-\varphi_1)^2\sqrt{\lambda_{\min}(P)}(\sqrt{c_1})^{-1} \tag{26}$$

holds where constants $\varphi_1 \in [0,1)$ and $c_1 > 0$ are defined in the proof of Lemma 3. Calculate Π^* as $\Pi^* = P - Q$ and set Π as $\Pi = \Pi^* + \xi$. Then, the ℓ^2-constrained ℓ^0 sparse PPC characterized by (6) gives control packets U in such a way that $\|x(k)\|_2$ is bounded at each time instant $k \in \mathbb{N}_0$ and

$$\lim_{k\to\infty} \|x(k)\|_2 \leq \Psi_1,$$
$$\Psi_1 \triangleq \left[(1-\varphi_1)^2\sqrt{\lambda_{\min}(P)} - \sqrt{c_1\lambda_{\max}(\xi)}\right]^{-1} \times \sqrt{\lambda_{\max}(P)}\varepsilon_N(W_m). \tag{27}$$

Proof. First, let us define φ_2 and Θ_1 as follows:

$$\varphi_2 \triangleq \varphi_1 + \sqrt{c_1\lambda_{\max}(\xi)}\left[(1-\varphi_1)\sqrt{\lambda_{\min}(P)}\right]^{-1}$$
$$\Theta_1 \triangleq (1-\varphi_1)^{-1}\sqrt{\lambda_{\max}(P)}. \tag{28}$$

Moreover, we define $\mathcal{T}$ as the set of all time instants when the control packets are not lost within transmission. The set $\mathcal{T}$ is characterized as $\mathcal{T} \triangleq \{t_n\}_{n\in\mathbb{N}_0} \subseteq \mathbb{N}_0$ where $t_{n+1} > t_n, \forall n \in \mathbb{N}_0$. We denote the number of packet dropouts between t_n and t_{n+1} by q_n. It can be implied from the definition of q_n that

$$q_n = t_{n+1} - t_n - 1, \qquad \forall n \in \mathbb{N}_0. \tag{29}$$

Assume that the current time step is t_n, $n \in \mathbb{N}_0$, and the control packet $U(x(t_n))$ arrives at the actuator. Then it follows from (1), the recursion used for the state predictions, (28) and Lemma 3 that $\sqrt{V_P(x(k))}$ is bounded as

$$\sqrt{V_P(x(k))} \leq \varphi_2\sqrt{V_P(x(t_n))} + \Theta_1\varepsilon_N(W_m) \tag{30}$$

for every $k \in \{t_n+1, t_n+2, \ldots, t_n+q_n+1\}$. Note that (20) holds for every realization $\bar{w}_0^i \in \mathbb{W}^{i+1}$, $\forall i \in \{0,1,\ldots,N-1\}$. We can conclude from from (29) and (30) that

$$\sqrt{V_P(x(t_{n+1}))} \leq \varphi_2\sqrt{V_P(x(t_n))} + \Theta_1\varepsilon_N(W_m). \tag{31}$$

According to Assumption 1, $U(x(0))$ is received by the actuator at time 0. Given this and by mathematical induction, we can deduce from (31), (26) ($0 \leq \varphi_2 < 1$) and properties of weighted norms that the following holds:

$$\sqrt{V_P(x(t_n))} \leq \varphi_2^n\sqrt{\lambda_{\max}(P)\|x(0)\|_2} + (1-\varphi_2)^{-1}\Theta_1\varepsilon_N(W_m). \tag{32}$$

Now, we can use the inequality (30) to derive

$$\sqrt{V_P(x(k))} \leq \varphi_2^{n+1}\sqrt{\lambda_{\max}(P)\|x(0)\|_2} + (1-\varphi_2)^{-1}\Theta_1\varepsilon_N(W_m). \tag{33}$$

for any $k \in \{t_n+1, \ldots, t_{n+1}\}$. Based on the properties of weighted norms, the following is extracted from (33):

$$\|x(k)\|_2 \leq \varphi_2^{n+1}\sqrt{\lambda_{\max}(P)\|x(0)\|_2(\lambda_{\min}(P))^{-1}} + \Psi_1. \tag{34}$$

where Ψ_1 is specified by (27). The inequality (34) proves the bounded-ness of the plant state at each time instant $k \in \mathbb{N}_0$. Finally, using the fact $k \to \infty$ is equivalent to $n \to \infty$ and $0 \leq \varphi_2 < 1$, we obtain (27) which completes the proof.

4. UNCONSTRAINED ℓ^1-ℓ^2 SPARSE PPC

As already mentioned, the cost function based on which PPC works has the structure of (4). For the specific case of unconstrained ℓ^1-ℓ^2 sparse PPC, $T(\tilde{x}_N) = \|\tilde{x}_N\|_P^2$ and $S(\tilde{x}_i, u_i) = \|\tilde{x}_i\|_Q^2 + \nu|u_i|$. We consider $\nu > 0$, and Q and P as positive definite matrices. Remember that $\tilde{x}_{i+1} = A\tilde{x}_i + Bu_i$, $i = 0, ..., N-1$, describes the recursion used for the future states prediction. Considering this, we can reexpress the cost function J as follows:

$$J(x,U) = \|MU - Kx\|_2^2 + \|x\|_Q^2 + \nu\|U\|_1, \tag{35}$$

where matrices M and K are defined based on (7) as in the previous section. Furthermore, $x = \tilde{x}_0$ and $U = [u_0 \ldots u_{(N-1)}]^\top$. It follows from (35) that every control packet $U(x(k))$, $k \in \mathbb{N}_0$, is generated based on

$$U(x) = \arg\min_{U\in\mathbb{R}^N} \|MU - Kx\|_2^2 + \|x\|_Q^2 + \nu\|U\|_1. \tag{36}$$

The solution to an optimization problem such as the one associated with (36) is sparse and there are several approaches for solving such problem Hayashi et al. (2013). For proving the stability analysis results, we utilize the value function

$$V(x) \triangleq \min_{U\in\mathbb{R}^N} J(x,U), \tag{37}$$

where $J(x,U)$ represents the cost function in (35).

Remark 2. Based on the same line of arguments as in Remark 1, (Nagahara et al., 2014, Lemma 5) holds here. Therefore, if we consider $\Pi^\star$ and $M^\dagger$ as specified in (8) and define the function $\tau(.)$ as $\tau(y) \triangleq \alpha y + (\beta + \lambda_{\max}(Q))y^2$ with $\alpha = \nu\sqrt{n}\sigma_{\max}(M^\dagger K)$ and $\beta = \lambda_{\max}(\Pi^\star)$, then for every $x \in \mathbb{R}^n$, lower and upper bounds are derived on $V(x)$ as follows:

$$\lambda_{\min}(Q)\|x\|_2^2 \leq V(x) \leq \tau(\|x\|_2). \tag{38}$$

Lemma 4. Let $P > 0$ solve the Riccati equation (13) with $r = \mu^2 N/(4\zeta)$, $\zeta > 0$. Define $\chi \triangleq \lambda_{\max}(Q) + \lambda_{\max}(K^\top K)$. Then the following holds for every $x \in \mathbb{R}^n$ and every realization of $\bar{w}_0^{i-1} \in \mathbb{W}^i$:

$$\sqrt{V(f^i(x,\bar{w}_0^{i-1}))} \leq \sqrt{V(x)-\lambda_{\min}(Q)\|x\|_2^2} + \sqrt{\chi}\gamma_N(W_m)+\sqrt{\zeta}, \quad (39)$$

where $i=1,2,\ldots,N$.

Proof. It is deduced from Remark 2 that the formulation of $J(x,U)$ in (35) is the same across our paper and (Nagahara et al., 2014, Section III-A). Moreover, $\bar{f}^i(x)$ is equivalent to $f^i(x)$ in (Nagahara et al., 2014, Section IV-A). Having the latter statements in mind and considering $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$, $\forall a,b \geq 0$, we can deduce from (Nagahara et al., 2014, Lemma 7) that

$$\sqrt{V(\bar{f}^i(x))} \leq \sqrt{V(x)-\lambda_{\min}(Q)\|x\|_2^2}+\sqrt{\zeta} \quad (40)$$

holds for any $x\in\mathbb{R}^n$, $i=1,2,\ldots,N$. It stems from the definition of $V(x)$ that $V(x)\leq J(x,0)$, $\forall x\in\mathbb{R}^n$. Then according to (35) and properties of weighted norms, $\sqrt{V(x)}\leq\sqrt{\chi}\|x\|_2$ is deduced. Using (38), we can derive

$$\sqrt{V(f^i(x,\bar{w}_0^{i-1}))}-\sqrt{V(\bar{f}^i(x))} \leq \sqrt{\chi}\|f^i(x,\bar{w}_0^{i-1})\|_2-\sqrt{\lambda_{\min}(Q)}\|\bar{f}^i(x)\|_2 \quad (41)$$

Since $\chi\geq\lambda_{\min}(Q)$ and according to the reverse triangle inequality, Lemma 1 and (10), we derive

$$\sqrt{V(f^i(x,\bar{w}_0^{i-1}))}-\sqrt{V(\bar{f}^i(x))}\leq\sqrt{\chi}\gamma_N(W_m) \quad (42)$$

for every realization of $\bar{w}_0^{i-1}\in\mathbb{W}^i$ and every $x\in\mathbb{R}^n$. The claim follows immediately by adding up the corresponding sides of (40) and (42).

Lemma 5. Suppose that $P>0$ is a solution to the Riccati equation (13) with $r=\mu^2N/(4\zeta)$ where $\zeta>0$ is an arbitrary positive real number. Then real constant $\varphi\in(0,1)$ exists in such a way that

$$\sqrt{V(f^i(x,\bar{w}_0^{i-1}))} \leq \sqrt{\varphi V(x)}+\frac{\sqrt{\lambda_{\min}(Q)}}{2} + \sqrt{\chi}\gamma_N(W_m)+\sqrt{\zeta} \quad (43)$$

holds for every $x\in\mathbb{R}^n$, every realization of $\bar{w}_0^{i-1}\in\mathbb{W}^i$ and every i belonging to the set $\{1,2,\ldots,N\}$.

Proof. Following the same steps as in the proof of (Nagahara et al., 2014, Lemma 8) leads us towards deriving the claim. This is due to the fact that we use the same formulation for the cost function and recursion for state prediction as the ones utilized in (Nagahara et al., 2014, Section III-A) for the unconstrained ℓ^1-ℓ^2 sparse PPC of disturbance-free plants.

Theorem 2. Suppose $\zeta>0$ exists in such a way that $P>0$ satisfies (13) with $r=\mu^2N/\zeta$. Then for the NCS of Fig. 1 controlled according to the unconstrained ℓ^1-ℓ^2 sparse PPC (36), the ℓ^2 norm of $x(k)$ is bounded at each time instant $k\in\mathbb{N}_0$ and

$$\lim_{k\to\infty}\|x(k)\|_2\leq\Psi,$$

$$\Psi \triangleq \frac{1}{1-\sqrt{\varphi}}\Big[\frac{1}{2}+\sqrt{\frac{\zeta}{\lambda_{\min}(Q)}}+\frac{\sqrt{\chi}\gamma_N(W_m)}{\sqrt{\lambda_{\min}(Q)}}\Big], \quad (44)$$

$$\varphi\triangleq 1-\lambda_{\min}(Q)(\alpha+\beta+\lambda_{\max}(Q))^{-1},$$

holds where α and β are characterized as in Remark 2.

Proof. Consider t_n (as defined in Theorem 1) as the current time step. Based on the recursion used for the prediction of the future states, dynamics of the plant (1) and Lemma 5, we can derive the following upper bound on $\sqrt{V(x(k))}$:

$$\sqrt{V(x(k))}\leq\sqrt{\varphi V(x(t_n))}+\Theta \quad (45)$$

for every $k\in\{t_n+1,t_n+2,\ldots,t_n+q_n+1\}$. In (45), Θ is defined as $\Theta\triangleq\sqrt{\lambda_{\min}(Q)}/2+\sqrt{\chi}\gamma_N(W_m)+\sqrt{\zeta}$. Note that (43) holds for every realization $\bar{w}_0^{i-1}\in\mathbb{W}^i$, $\forall i\in\{1,2,\ldots,N\}$. Based on $t_{n+1}=t_n+q_n+1$ and (45), we can derive

$$\sqrt{V(x(t_{n+1}))}\leq\sqrt{\varphi V(x(t_n))}+\Theta. \quad (46)$$

Based on Assumption 1 and by mathematical induction, $V(x(t_n))$ is bounded as

$$\sqrt{V(x(t_n))}\leq\sqrt{\varphi}^n\sqrt{V(x(0))}+(1+\sqrt{\varphi}+\cdots+\sqrt{\varphi}^{n-1})\Theta.$$

Then according to Remark 2, we have

$$\sqrt{V(x(t_n))}\leq\sqrt{\varphi}^n\sqrt{\tau(\|x(0)\|_2)}+(1-\sqrt{\varphi})^{-1}\Theta. \quad (47)$$

Now, it stems from (45) that

$$\sqrt{V(x(k))}\leq\sqrt{\varphi}^{n+1}\sqrt{\tau(\|x(0)\|_2)}+(1-\sqrt{\varphi})^{-1}\Theta \quad (48)$$

holds for any $k\in\{t_n+1,\ldots,t_{n+1}\}$. Considering the lower bound on $V(x)$ in Remark 2, we establish

$$\|x(k)\|_2\leq\sqrt{\varphi^{n+1}}\sqrt{\frac{\tau(\|x(0)\|_2)}{\lambda_{\min}(Q)}}+\Psi, \quad (49)$$

which implies the boundedness of the plant state at each time instant $k\in\mathbb{N}_0$. Moreover, Ψ is defined as in (44). By letting k go to infinity, which is equivalent to n going to infinity, we derive (44) and the proof will be complete.

Remark 3. According to (18), (27), (44) and (11), Ψ and Ψ_1 are increasing functions of W_m. Thus, rendering W_m larger while keeping other parameters will render the bounds on the steady-state ℓ^2 norm of the state larger.

5. SIMULATION EXAMPLE

Here, we simulate the NCS of Fig. 1 via applying the sparse PPC policies developed in the previous sections. To do so, first we need to consider a plant model satisfying (1). This model is characterized by the following state matrix and input vector

$$A=\begin{bmatrix}0.3966 & -0.4586 & -0.0250 & -0.7958\\ 0.7459 & 0.8061 & -0.0983 & 0.7943\\ -0.9451 & -0.3111 & -0.8236 & 0.2473\\ 0.1551 & -1.3821 & -1.9151 & 0.0369\end{bmatrix}, \quad B=[1.0617 \quad -0.1986 \quad -0.3184 \quad 0.5562]^\top, \quad (50)$$

respectively, and a disturbance signal with i.i.d samples taken from a uniform distribution over $[-W_m,W_m]$. The elements of A and B are samples of a normal distribution with mean 0 and variance 1. For the controller simulation, we consider the prediction horizon N as $N=10$ and the weighting matrix Q as $Q=I_{4\times4}$ in both cases of ℓ^2-constrained ℓ^0 sparse PPC and unconstrained ℓ^1-ℓ^2 sparse PPC. For simulating the ℓ^2-constrained ℓ^0 sparse PPC, we set ξ as $\xi=\big((1-\varphi_1)^4\lambda_{\min}(P)/4c_1\big)I_{4\times4}$. Moreover, we implement OMP as the algorithm for solving the optimization problem related to (6). For simulating the ℓ^1-ℓ^2 sparse PPC, the parameters ν and r are set as $\nu=200$ and $r=2$. In this case, fast iterative shrinkage-thresholding algorithm (FISTA) is implemented for solving

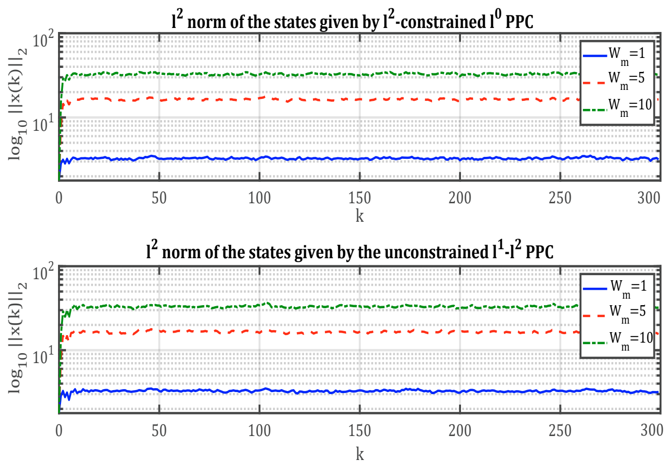

Fig. 2. Average ℓ^2 norm of the state $x(k)$

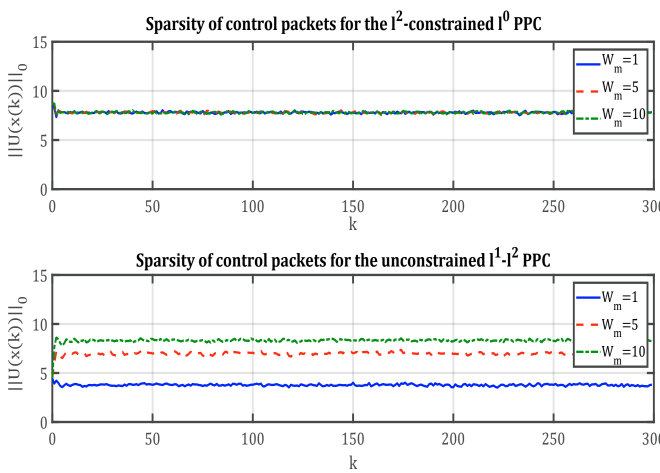

Fig. 3. Average ℓ^0 norm of the control packet $U(x(k))$

the corresponding optimization problem. We carry out the simulation for 3 different values of W_m, i.e., $W_m \in \{1, 5, 10\}$.

The results of the simulation are illustrated by Fig. 2 and Fig. 3. Such results are obtained by averaging over 200 number of 300-sample-long simulations. The vertical axis of Fig. 2 corresponds to $\|x(k)\|_2$ and the horizontal axis to time. As observed from Fig. 2, the ℓ^2 norm of the state varies over a bounded range along the simulation time in both cases of ℓ^2-constrained ℓ^0 sparse PPC and unconstrained ℓ^1-ℓ^2 sparse PPC. This shows that the proposed sparse PPC designs can render the NCS of Fig. 1 practically stable. According to Fig 2, such stability holds regardless of the magnitude of W_m. Moreover, we can observe from Fig. 2 that the states take larger values when W_m is larger. Such performance degradation by increasing W_m agrees with Remark 3. Figure 3 demonstrates the ℓ^0 norm of the control packets $U(x(k))$. As curves in Fig. 3 signify, indeed, the control packets generated based on both ℓ^2-constrained ℓ^0 sparse PPC and unconstrained ℓ^1-ℓ^2 sparse PPC are sparse. Of course by manipulating ν in (35) and Π in (6), one can regulate the sparsity of the control actions.

6. CONCLUSIONS

This paper has investigated the stability of linear discrete-time plants with bounded disturbances controlled based on sparse PPC. The control occurs in a feedback loop where the controller and the actuator communicate via a digital channel with data packet dropouts. The sparse PPC strategies taken into account perform based upon unconstrained ℓ^1-ℓ^2 and ℓ^2-constrained ℓ^0 sparsity-promoting optimizations. We have established the conditions of practical stability for both ℓ^1-ℓ^2 sparse PPC and ℓ^2-constrained ℓ^0 sparse PPC. The bounds we have derived on the plant state are increasing functions of the disturbance amplitude. Through simulation, we have illustrated that with the proposed sparse PPC design, practical stability can indeed be obtained by sparse control packets. Moreover, enlarging the amplitude of the disturbance signal leads to performance degradation without jeopardizing stability.

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