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On gyroscopic stabilization

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1. Introduction

Stability properties of linear conservative gyroscopic systems of the form

$$M\ddot{x} + G\dot{x} + Kx = O \tag{1}$$

have been investigated for many years. M, G and K are real $n \times n$ matrices with $M^T = M > 0$ (positive definite), $G^T = -G$ and $K^T = K$. M is the mass matrix, G describes the gyroscopic forces and K the potential forces. The vector x represents the generalized coordinates. Systems of form (1) are important mathematical models for e.g. rotor systems, satellites and fluid-conveying pipes.

Some important existing results for conservative gyroscopic systems should be mentioned. While gyroscopic forces can never destabilize a stable conservative system, they can possibly stabilize an unstable conservative system. A classical result by Thomson and Tait [TT79] and Chetayev [Che61] states that an unstable conservative system $M\ddot{x} + Kx = 0$, $K \not\geq 0$ can be stabilized by gyroscopic forces if and only if the number of unstable degrees of freedom is even. This means e.g. that when $K < 0$, then the dimension n must be even. For this case Lakhadanov [Lak75] showed that suitable stabilizing matrices are $G = g_0 G_0$, where $\det G_0 \neq 0$ and g_0 is a sufficiently large number. Lakhadanov gave an explicit expression for one such g_0. In the general case with an indefinite K, an unstable conservative system can be decoupled by choosing modal coordinates. Hence, we only need to stabilize the subsystem which has a negative definite stiffness matrix. Gyroscopic stabilization in the case $K \leq 0$ was dealt by e.g. Merkin [Mer56]

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and Müller [Mül77]. Hagedorn [Hag75] showed that $4K - GM^{-1}G < 0$ implies instability of the system (1). On the other hand $4K - GM^{-1}G > 0$ does not generally ensure stability. For the special dimension $n = 2$, Teschner [Tes77] proved that if $K < 0$ and $4K - GM^{-1}G > 0$, then the system (1) is stable. Inman and Saggio [IS85] extended this result and showed that if $n = 2$, $K < 0$ and $\mathrm{tr}(4K - GM^{-1}G) > 0$, then the system is stable. Several investigations have been made to clarify the role of the matrix $4K - GM^{-1}G$ concerning stability in the case of arbitrary dimension n. Huseyin, Hagedorn, and Teschner [HHT83] proved the lemma that if the conditions $GM^{-1}K - KM^{-1}G \geq 0$ and $4K - GM^{-1}G > 0$ hold, then the system is stable. A theorem from the same paper states that if $GM^{-1}K = KM^{-1}G$, the relation $4K - GM^{-1}G > 0$ is necessary and sufficient for the stability of the system (1). Confined to systems with $M = I$ (identity matrix), the lemma is covered by the theorem, since the matrix $GK - KG$ can never be positive definite because it is always indefinite or the zero matrix. A system with $M = I$ can easily be established from (1) by means of the transformation $x = M^{-1/2}z$ and premultiplying by $M^{-1/2}$. Then the skew symmetry of G and the symmetry of K are transformed to $M^{-1/2}GM^{-1/2}$ and $M^{-1/2}KM^{-1/2}$ respectively. Calling these new system matrices again G and K, we get the differential equation

$$I\ddot{z} + G\dot{z} + Kz = 0. \tag{2}$$

Recently Inman [Inm88] found a sufficient condition for the stability of the system (2): if $K < 0$ and $4K - G^2 - \varepsilon I > 0$, where $2\varepsilon = \mu_{\max}(-G^2)$, then the system (2) is stable. Here $\mu_{\max}(-G^2)$ denotes the largest eigenvalue of $-G^2$.

One of the aims of the present work is to improve this last condition in a way, which makes ε independent of G, decreases the value of ε and finally results in $\varepsilon = 0$ if $GK = KG$, which is in accordance with the theorem by Huseyin, Hagedorn and Teschner. Another important task will be to investgate the behaviour of eigenvalues of the system (2) dependent on parameters in order to reveal the mechanism of transition between divergence, flutter and stability. This anlysis is based on the theory of interactions of eigenvalues developed recently by Seyranian [Sey91, Sey93].

2. The behaviour of eigenvalues

In this section we consider the system (2) when G and K contain parameters. Assuming solutions of the form $z = u\exp(\lambda t)$, the respective eigenvalue problem is expressed by

$$(\lambda^2 I + \lambda G + K)u = 0. \tag{3}$$

Notice, that the $2n$ eigenvalues in the complex plane are placed symmetri-

Figure 1
Impossible behaviour of eigenvalues.

cally with respect to the real as well as to the imaginary axis: together with λ are $\bar{\lambda}$, $-\lambda$ and $-\bar{\lambda}$ eigenvalues. Studying the dependence of the eigenvalues on the parameters, this symmetry implies that real and purely imaginary eigenvalues, as long as they are distinct, can move only on their respective axes. Leaving an axis can not happen in the way as sometimes indicated in the literature by Fig. 1, but only as a result of a special eigenvalue collision, called strong interaction (Fig. 2), see also [Sey91, Sey93].

This symmetry also implies that stability of the system (2) can occur only if all eigenvalues are purely imaginary (marginal stability).

First we investigate the system

$$I\ddot{z} + pG_0\dot{z} + Kz = 0, \qquad G = pG_0, \tag{4}$$

with one load parameter $p \geq 0$. Notice, that all real eigenvalues of (4) are bounded, since real eigenvalues have real eigenvectors u and $u^TG_0u = 0$ such that $\lambda^2 u^T I u + u^T K u = 0$ leads to

$$\mu_{\min}(K) \leq -\lambda^2 = \frac{u^TKu}{u^Tu} \leq \mu_{\max}(K). \tag{5}$$

We now want to study the behaviour of the eigenvalues of the system (4) with n even and $K < 0$ (statically unstable) under the stabilizing process. Much can be seen by investigating the case $n = 2$ with

$$G = pG_0 = p\begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}. \tag{6}$$

Then $K < 0$ means $c_{11} < 0$, $c_{22} < 0$, $\det K > 0$. The characteristic equation $|\lambda^2 I + \lambda G + K| = 0$ results in the four symmetrically placed eigenvalues satisfying

$$\left.\begin{aligned} \lambda_2 &= -\tfrac{1}{2}(c_{11} + c_{22} + \beta^2p^2) \pm \tfrac{1}{2}\sqrt{D}, \\ D &= (c_{11} - c_{22})^2 + 4c_{12}^2 + 2(c_{11} + c_{22})\beta^2p^2 + \beta^4p^4. \end{aligned}\right\} \tag{7}$$

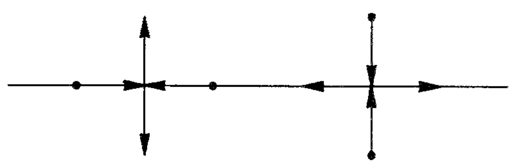

Figure 2
Strong interaction of eigenvalues.

One finds the zeros of D as

$$\left.\begin{aligned} \beta^2 p_{f_1}^2 &= -(c_{11}+c_{22}) - 2\sqrt{\det K}, \\ \beta^2 p_{f_2}^2 &= -(c_{11}+c_{22}) + 2\sqrt{\det K}. \end{aligned}\right\} \tag{8}$$

The roots of the discriminant D determines the boundaries of divergence, flutter and stability. This results in

$0 \le p \le p_{f_1}$: divergence

$p_{f_1} < p \le p_{f_2}$: flutter

$p_{f_2} < p$: stability.

Hence, gyroscopic stabilization will always take place for a sufficiently large load parameter, but the inspection of the definiteness of $4K - G^2$ alone will usually not reveal the stabilization value p_{f_2}.

We now turn to the system (4) with $K < 0$ and arbitrary (even) dimension n. According to Lakhadanov [Lak75], the system will be stabilized for sufficient large value of p if $\det G_0 \neq 0$. But Lakhadanov also showed by an example that $\det G_0 \neq 0$ is not necessary for stabilization. Anyhow, it is obvious that the picture of stabilization is similar to that of Fig. 3: pairs of real eigenvalues have to collide, interact strongly and become complex conjugate values λ and $\bar{\lambda}$. After the flutter phase, λ and $-\bar{\lambda}$ meet in a strong interaction on the imaginary axis. According to this mechanism all eigenvalues will finally end on the imaginary axis and the system is gyroscopically stabilized. This illustrates clearly the above mentioned theorem of Thomson–Tait–Chetayev: In the case of $K < 0$ and odd number n there is a single eigenvalue left on the real axis such that no interaction with following leave of the axis is possible. Also in the case where K is indefinite the mechanism of stabilization is similar to the one shown in Fig. 3, if stabilization can be achieved at all. The only possible

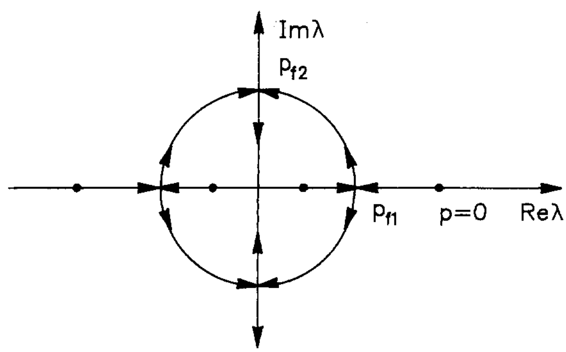

Figure 3
The mechanism of gyroscopic stabilization for the system (4), (6).

difference is that if K has zero eigenvalues, then a collision at $\lambda = 0$ will take place.

Now we shall study the case, where both the gyroscopic and the stiffness matrix contains parameters. Consider a simplified model of an elastic rotor. A massless non-circular elastic shaft carries a disk with mass and is subjected to an axial compression force. Such a model is according to e.g. Huseyin [Hus78] described by a system of form (2) with

$$G = 2p\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} c_1 - \eta - p^2 & 0 \\ 0 & c_2 - \eta - p^2 \end{bmatrix}. \tag{9}$$

Here, p is the angular velocity of the shaft, η represents the axial force and c_1 and c_2 are stiffness coefficients of the shaft in two principal directions. Using p, $a_1 = c_1 - \eta$ and $a_2 = c_2 - \eta$ as the parameters, the characteristic equation becomes

$$\lambda^4 + \lambda^2(a_1 + a_2 + 2p^2) + (a_1 - p^2)(a_2 - p^2) = 0 \tag{10}$$

with solutions

$$\left.\begin{aligned} \lambda_{1,2}^2 &= \frac{-(a_1 + a_2 + 2p^2) \pm \sqrt{D}}{2} \\ D &= (a_1 - a_2)^2 + 8(a_1 + a_2)p^2. \end{aligned}\right\} \tag{11}$$

For $p = 0$ we get $\lambda_1^2 = -a_1$ and $\lambda_2^2 = -a_2$. A zero eigenvalue $\lambda = 0$ appears for $p^2 = a_1$ and $p^2 = a_2$, i.e. only for positive values of a_1 and a_2. Splitting up the $a_1 a_2$-plane into appropriate regions, elementary calculations studying the behaviour of the roots of (10) provides the following results:

(α) $a_1 \leq 0$, $a_2 \leq 0$:

(α_1)	$0 \leq p^2 \leq p_f^2 = -\dfrac{(a_1 - a_2)^2}{8(a_1 + a_2)}$	:	divergence
(α_2)	$p_f^2 < p^2$	:	flutter

(β) $a_1 > 0$, $a_2 > 0$:

(β_1)	$0 \leq p^2 < \min(a_1, a_2)$	:	stability
(β_2)	$\min(a_1, a_2) \leq p^2 \leq \max(a_1, a_2)$	:	divergence
(β_3) and (β_4)	$\max(a_1, a_2) < p^2$	:	stability

(γ) $a_1 < 0$, $a_2 > 0$, $a_1 + a_2 < 0$:

(γ_1)	$3a_2 + a_1 > 0$, $0 \leq p^2 < a_2$	:	divergence
(γ_2)	$3a_2 + a_1 > 0$, $a_2 \leq p^2 \leq p_f^2$	:	stability
(γ_3)	$3a_2 + a_1 > 0$, $p_f^2 \leq p^2$	:	flutter
(γ_4)	$3a_2 + a_1 \leq 0$, $0 \leq p^2 \leq p_f^2$	:	divergence
	$p_f^2 < p^2$	:	flutter

(δ) $a_1 < 0$, $a_2 > 0$, $a_1 + a_2 > 0$:

(δ_1)	$0 \le p^2 \le a_2$	:	divergence
(δ_2)	$a_2 < p^2$	:	stability

This is a complete four-parameter (c_1, c_2, η, p) stability analysis of the system (9). As an example we mention the mechanism of stabilization in case δ: an eigenvalue collision at $\lambda = 0$ changes divergence into stability.

It is instructive to follow Inman [Inm88, Inm89, p. 84], fixing the values of η and p and asking for the stability regions in the $a_1 a_2$-plane or in the $c_1 c_2$-plane ($c_1 > 0$, $c_2 > 0$). For $\eta = 3$ and $p = 2$, Fig. 4 shows the result of the present investigation. Notice, that this picture does not entirely agree with Inman [Inm88] and is more complete than the picture given by Inman [Inm89].

The areas for α_1 and γ_1 correspond to negative stiffness coefficients c_1 and c_2 and are hence not shown.

Since η and p are fixed, it would be possible to use the system (4), (6) to get Fig. 4. One may compare with the stability regions according to the above mentioned theorems:

(α_2)	$4K - G^2 < 0$. Hagedorn [Hag75]: unstable.
(β_1)	$K > 0$. Thomson and Tait [TT79]: stable.
(β_2), (δ_1)	$n = 2$, K indefinite. Thomson and Tait [TT79]: unstable.
(β_3)	$n = 2$, $4K - G^2 > 0$. Teschner [Tes77]: stable.
(β_4)	n arbitrary, $4K - G^2 - \varepsilon I > 0$, $\varepsilon = \frac{1}{2}\mu = 8$. Inman [Inm88]: stable.

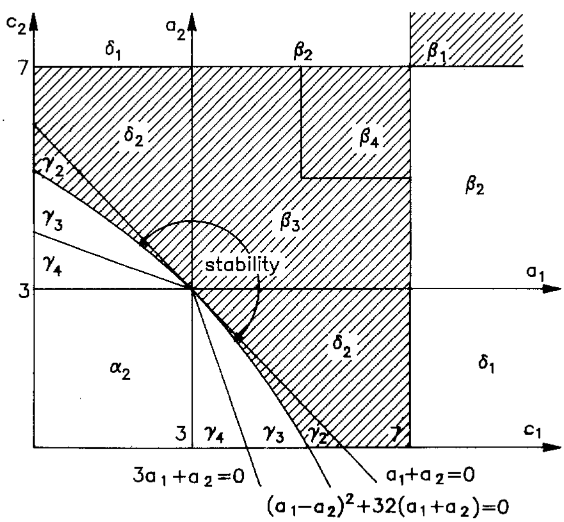

Figure 4
Stability map of the system (9) for $\eta = 3$ and $p = 2$.

(γ_2), (γ_3), (γ_4) no decision.

(β_3), (β_4) $n = 2$, $K < 0$, $\operatorname{tr}(4K - G^2) > 0$. Inman and Saggio [IS85]: stable.

$GK = KG$ is equivalent to $c_1 = c_2$. Huseyin, Hagedorn and Teschner [HHT83]. $c_1 = c_2 > 3$: stable, $c_1 = c_2 < 3$: unstable.

In the next section we will prove a stability condition, which will reveal the region (γ_2) as stable wit^hout inspection of the eigenvalues as done in the previous investigation.

Now, let us consider a generalization of the system (9) with the form

$$M\ddot{x} + pG\dot{x} + (C - p^2B)x = 0, \tag{12}$$

where $p \geq 0$ is a load parameter and C and B are symmetric, positive definite $n \times n$ matrices. We introduce the Rayleigh quotients

$$\begin{aligned} m &= (Mu, u)/(u, u) > 0, \qquad & c &= (Cu, u)/(u, u) > 0, \\ b &= (Bu, u)/(u, u) > 0, \qquad & ig &= (Gu, u)/(u, u). \end{aligned} \tag{13}$$

Choosing eigenvectors in the Rayleigh quotients, we find that every eigenvalue λ is a root of the quadratic equation

$$m\lambda^2 + ipg\lambda + c - p^2b = 0. \tag{14}$$

Flutter appears when the discriminant of (14) is positive:

$$p^2(4bm - g^2) - 4cm > 0. \tag{15}$$

A necessary condition for flutter is therefore

$$4bm - g^2 > 0. \tag{16}$$

Since the Rayleigh quotients (13) are limited by the smallest respectively by the largest eigenvalue of the associated matrix, (16) is satisfied, if

$$4\mu_{\min}(B)\mu_{\min}(M) > \mu_{\max}(-G^2). \tag{17}$$

Assuming (17), the critical flutter parameter p_f can now be evaluated as

$$\begin{aligned} p_1^2 = \min\left(\frac{c}{b}\right) \leq \frac{c}{b} \leq p_f^2 = \frac{cm}{bm - \frac{1}{4}g^2} &\leq \frac{m_{\max}(C)\mu_{\max}(M)}{\mu_{\min}(B)\mu_{\min}(M) - \frac{1}{4}\mu_{\max}(-G^2)} \\ &= p_2^2. \end{aligned} \tag{18}$$

Here p_1^2 is the smallest eigenvalue of the problem

$$Cu = p^2Bu, \tag{19}$$

and corresponds to the eigenvalue $\lambda = 0$ of the original problem (12). So, it is obvious that the system is stable for $0 \leq p^2 < p_1^2$ and unstable for $p^2 \geq p_2^2$.

If p_1^2 is a *simple* eigenvalue of (19), then only one eigenvector u_1 corresponds to the double eigenvalue $\lambda = 0$ of (12). This means that at $p^2 = p_1^2$ a *strong interaction* of eigenvalues takes place with the result that the system (12) loses stability by *divergence* for $p^2 \geq p_1^2$.

If p_1^2 is a *double* eigenvalue of (19), the double eigenvalue $\lambda = 0$ of (12) has two linearly independent eigenvectors. In this case *weak interaction* of eigenvalues takes place and the system (12) remains stable for $p^2 \geq p_1^2$ at least in some neighbourhood of p_1^2. It is interesting that this is valid for arbitrary gyroscopic matrices G, such that the gyroscopic stabilization can be achieved by arbitrary small gyroscopic forces.

The considered case of gyroscopic stabilization in the vicinity of p_1^2 can be generalized in the following way: with an even multiplicity of the smallest eigenvalue p_1^2 of the problem (19) stability is maintained for $p^2 \geq p_1^2$, while an odd multiplicity of p_1^2 leads to a loss of stability for $p^2 \geq p_1^2$ by divergence.

3. A stability condition

Consider the system (1) with even dimension n and $K < 0$. Since $M > 0$, we can introduce modal coordinates z by $x = Uz$ with orthogonal $U = [u_1 \cdots u_n]$, $(K - k_i M)u_i = 0$, $U^T M U = I$, $U^T K U = \tilde{K} = \operatorname{diag}\{k_i\}$, $k_i < 0$ and $U^T G U = \tilde{G} = -\tilde{G}^T$. Again calling $\tilde{G}$ and $\tilde{K}$ for G and K, the system (1) is written in form (2) with the special advantage of a purely diagonal K:

$$I\ddot{z} + G\dot{z} + Kz = 0, \qquad K = \operatorname{diag}\{k_i\}, \qquad k_i < 0, \quad n \text{ even}. \tag{20}$$

(20) is equivalent to the first order system

$$\dot{Y} = AY, \qquad A = \begin{bmatrix} 0 & I \\ -K & -G \end{bmatrix}, \qquad Y = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}. \tag{21}$$

A well-known theorem (see e.g. Müller [Mül77], p. 122) states that the system (21) is (marginally) stable if and only if the homogeneous Lyapunov equation

$$A^T P + PA = 0 \tag{22}$$

has a symmetric, positive definite solution $P = P^T > 0$. If P is partitioned into square submatrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \tag{23}$$

(22) is identical with the four equations

$$\left.\begin{aligned} KP_{12}^T + P_{12}K &= 0 \\ P_{22}G - GP_{22} - P_{12} - P_{12}^T &= 0 \\ P_{11} - P_{12}G - KP_{22} &= 0 \\ P_{11} - P_{11}^T &= 0 \end{aligned}\right\}. \tag{24}$$

Now we formulate the following theorem:

Theorem 1. Consider the system (20). Let Δ be any positive definite diagonal matrix, $\Delta = \operatorname{diag}\{\delta_i\}$, $\delta_1 > 0$, such that Δ and G commute: $\Delta G = G\Delta$. Then the system (20) is stable if

$$K - \Delta - G^2 + G(I - \Delta K^{-1})^{-1}G > 0. \tag{25}$$

Proof. It can easily be checked that the following matrix P of form (23) is a solution to (24):

$$P_{11} = -\Delta K + K^2, \qquad P_{12} = KG, \qquad P_{22} = K - \Delta - G^2. \tag{26}$$

For this check the relations $\Delta K = K\Delta$ and $\Delta G = G\Delta$ are of importance. Now, stability of (20) is ensured by $P > 0$, which is the case, according to a well known theorem (see e.g. Horn and Johnson [HJ85], p. 472), if and only if

$$P_{11} > 0 \quad \text{and} \quad P_{22} - P_{12}^T P_{11}^{-1} P_{12} > 0. \tag{27}$$

$P_{11} > 0$ is obviously satisfied since $K < 0$ and $\Delta > 0$. Since

$$GK(K^2 - \Delta K)^{-1}KG = G(I - \Delta K^{-1})^{-1}G, \tag{28}$$

the second condition in (27) directly implies (25), which completes the proof. □

To get optimal stabilization results from (25)—this means gyroscopic stabilization matrices G with $\|G\|$ as small as possible—we have to find optimal matrices Δ. The above definition of Δ makes the following considerations obvious.

(1) Assume that G couples the degrees of freedom pairwise in a perfect matching. In this case G contains only one nonzero element in each row and in each column, e.g.

$$G = \begin{bmatrix} 0 & 0 & g_{13} & 0 \\ 0 & 0 & 0 & g_{24} \\ -g_{13} & 0 & 0 & 0 \\ 0 & -g_{24} & 0 & 0 \end{bmatrix}, \tag{29}$$

where the coupling is $1 \leftrightarrow 3$, $2 \leftrightarrow 4$. Let these pairs be represented by $i \leftrightarrow j$. Choose $\Delta = \mathrm{diag}\{\delta_k\}$ such that $\delta_i = \delta_j$ and call this common value δ_{ij}. Then the matrices K, Δ, G^2 and $G(I - K^{-1})^{-1}G$ are all diagonal and condition (25) implies

$$g_{ij}^2 > \frac{(\delta_{ij} - k_i)(\delta_{ij} - k_j)}{\delta_{ij}}. \tag{30}$$

An optimal choice for δ_{ij} the right hand side of (30) to be minimum, is

$$\delta_{ij} = \sqrt{k_i k_j}. \tag{31}$$

Then (30) implies the values of the stabilizing matrix G as

$$g_{ij}^2 > (\sqrt{-k_i} + \sqrt{-k_j})^2. \tag{32}$$

We now define

$$\varepsilon_{ij} = (\sqrt{-k_i} + \sqrt{-k_j})^2 + 4\min(k_i, k_j) \tag{33}$$

and introduce a matrix $E = \mathrm{diag}\{\varepsilon_k\}$ with $\varepsilon_i = \varepsilon_j$ and call this common value ε_{ij}. Then (32) is expressed by

$$4\min(k_i, k_j) + g_{ij}^2 - \varepsilon_{ij} > 0. \tag{34}$$

The stability condition (34) is the main part of the following theorem.

Theorem 2. Consider the system (20), where G couples the degrees of freedom pairwise in a perfect matching. For every pair $i \leftrightarrow j$ introduce

$$\varepsilon_{ij} = (\sqrt{-k_i} + \sqrt{-k_j})^2 + 4\min(k_i, k_j) \tag{35}$$

and place the elements ε_{ij} in a diagonal matrix E with ε_{ij} both in the i'th and the j'th position. Then the system (20) is stable if

$$4K - G^2 - E > 0. \tag{36}$$

If we want to replace E in (36) by εI, we can use $\varepsilon = \max \varepsilon_{ij}$ but this reduces the advantage of getting optimal values for the stabilizing matrix G. On the other side, (36) is then directly comparable with the condition of Inman [Inm88]. In the discussed case of G (a perfect matching), $GK = KG$ is valid if and only if $k_i = k_j$ for all pairs $i \leftrightarrow j$. Then (35) implies $\varepsilon_{ij} = 0$, which reduces the condition (36) to $4K - G^2 > 0$, in agreement with Huseyin, Hagedorn and Teschner [HHT83].

(2) If G does not couple the degrees of freedom in a perfect matching, it is convenient to work with the stability condition (25) of Theorem 1. We then choose $\Delta = \delta I$ with $\delta = \sqrt{k_{\min} k_{\max}}$, where $k_{\min} = \min k_i$, $k_{\max} = \max k_i$. An interesting special case is again $GK = KG$. Suppose that we want to stabilize with G, where all $g_{ij} \neq 0$, $i < j$. Then $GK = KG$ if and only if $K = kI$. (31) implies in this case $\delta_{ij} = -k$, such that $\Delta = -kI = -K$ and the

stability condition (25) takes the form $2K - G^2 + G(2I)^{-1}G > 0$ or $4K - G^2 > 0$, which again is in agreement with Huseyin, Hagedorn and Teschner [HHT83].

4. Examples

1. We want to apply Theorem 2 to example (9) with fixed $\eta = 3$ and $p = 2$. For $k_1 = c_1 - 7 < 0$, $k_2 = c_2 - 7 < 0$ and e.g. $c_1 < c_2 < 7$, (35) implies

$$\varepsilon_{12} = (\sqrt{7 - c_1} + \sqrt{7 - c_2})^2 + 4(c_1 - 7). \tag{37}$$

Then the stability condition (36) results in

$$4(c_1 - 7) + 16 - \varepsilon_{12} > 0, \tag{38}$$

which after elementary computations yields

$$(c_1 - c_2)^2 + 32c_1 + 32c_2 - 192 > 0. \tag{39}$$

Recalling $a_1 = c_1 - \eta = c_1 - 3$, $a_2 = c_2 - \eta = c_2 - 3$, (39) is equivalent to

$$(a_1 - a_2)^2 + 32(a_1 + a_2) > 0. \tag{40}$$

This stability condition results in the regions β_3, β_4, δ_2 and γ_2 of Fig. 4. Therefore condition (36) reveals γ_2 as a stable region, which earlier only was possible to recognize by inspection of the eigenvalues.

2. Consider the system (20) with

$$K = \begin{bmatrix} -1 & & & \\ & -2 & & \\ & & -3 & \\ & & & -6 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 3 & 0.5 & 1 \\ . & 0 & 1 & 1 \\ . & . & 0 & 5.037 \\ . & . & . & 0 \end{bmatrix}. \tag{41}$$

According to the remarks in Section 3, we choose $\delta = \sqrt{k_{\min} k_{\max}} = \sqrt{6}$, $\Delta = \delta I$. Then checking condition (25) is easy by e.g. Matlab™ [Mol90]. With the matrices (41) the check turns out positive. But we change the two entries in G, g_{34} and g_{43} to 5 and -5 respectively, condition (25) is no longer satisfied. Still, (25) is sufficient but not necessary for stability, such that in case of the failure of (25) the system could be stable. But an inspection of the eigenvalues of the mentioned example with (25) not valid shows that the system really in unstable. Which is an indication of the usefulness of condition (25) with $\Delta = \delta I$, $\delta = \sqrt{k_{\min} k_{\max}}$.

The stability check by using condition (25) should also be compared with the result of Lakhadanov [Lak75]. According to this, in the present example (41) the matrix G has to be multiplied by the quite large factor $g_0 = 119$ to ensure stability.

3. To study the method for systems with large numbers of degrees of freedom, we did some computer simulations. In a typical example with a system for which $n = 20$ with a random (normally distributed) skew symmetric G_0 and a random negative definite diagonal K, we obtained that the system

$$I\ddot{z} + g_0 G_0 \dot{z} + K = 0 \tag{42}$$

was marginally stable if and only if $g_0 \geq 12.45$. Using (25), we derived the bound that the system (42) is stable for $g_0 \geq 12.86$. In comparison, the best result in literature (Müller (Mül77], p. 161, which is actually a special case of (25) with $\Delta = I$) resulted in stability for $g_0 \geq 21.8$. Finally, the bound of Lakhadanov only guaranteed stability for $g_0 \geq 140918$.

5. Conclusions

In Section 2 we have dealt with the mechanisms of transition between divergence, flutter and stability for several conservative gyroscopic systems with parameters. This investigation was based on the behaviour of the eigenvalues. Hereby, the theory of interaction of eigenvalues (see Seyranian [Sey91, Sey93]) played an essential role. As an example a stability map for a simple system with four parameters was presented in order to compare the stability regions with those derivable by existing results.

In Section 3 we proved a theorem, which states a sufficient condition for gyroscopic stabilization for conservative systems with an even dimension and with $K < 0$. The proof is based on the matrix Lyapunov equation and follows Müller [Mül77], p. 161, improved by the introduction of a convenient positive definite commutator Δ for G, where the approach of Müller corresponds to a choice of $\Delta = I$. In the case of quite general G, $\Delta = \delta I$ with $\delta = \sqrt{k_{\min} k_{\max}}$, is a good choice. If $GK = KG$, $\Delta = -K$ leads to the well known stability condition $4K - G^2 > 0$. In the special case where G couples the degrees of freedom pairwise in a perfect matching, Δ can be chosen in a certain 'optimal' way. This leads to a result, which for this special case improves a theorem by Inman [Inm88].

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Abstract

The mechanisms of transition between divergence, flutter, and stability for a class of conservative gyroscopic systems with parameters are studied. Two results are obtained which state sufficient conditions for gyroscopic stabilization of conservative systems with an even dimension and a negative definite stiffness matrix. A number of examples are given to demonstrate the feasibility of the results.

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