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COMPACTLY SUPPORTED FRAMES FOR DECOMPOSITION SPACES

KENNETH N. RASMUSSEN AND MORTEN NIELSEN

Abstract. In this article we study a construction of compactly supported frame expansions for decomposition spaces of Triebel-Lizorkin type and for the associated modulation spaces. This is done by showing that a finite linear combination of shifts and dilates of a single function with sufficient decay in both direct and frequency space can constitute a frame for Triebel-Lizorkin type spaces and the associated modulation spaces. First, we extend the machinery of almost diagonal matrices to Triebel-Lizorkin type spaces and the associated modulation spaces. Next, we prove that two function systems which are sufficiently close have an almost diagonal "change of frame coefficient" matrix. Finally, we approximate to an arbitrary degree an already known frame for Triebel-Lizorkin type spaces and the associated modulation spaces with a single function with sufficient decay in both direct and frequency space.

1. Introduction

Smoothness spaces such as the Triebel-Lizorkin (T-L) and Besov spaces play an important role in approximation theory and harmonic analysis. Often they are characterized by (or at least imply) some decay or sparseness of an associated discrete expansion. For example, a certain sparseness of a wavelet expansion is equivalent to smoothness measured in a Besov space [16]. A consequence of this is that a sufficiently smooth function can be compressed by thresholding the expansion coefficients of a sparse representation of the function [5, 6]. More generally in non-linear approximation, the coefficient norm characterization leads to better understanding of the approximation spaces (see e.g. [12] and [13]).

The T-L and Besov spaces are special cases of T-L type spaces and the associated modulation spaces which again form a broad subclass of the decomposition spaces defined on $\mathbb{R}^d$. Decomposition spaces were introduced by Feichtinger and Gröbner [8] and Feichtinger [7], and are based on structured coverings of the frequency space $\mathbb{R}^d$. Here the classical T-L and Besov spaces correspond to dyadic coverings (see [23]). Many authors have used modulation spaces to study pseudodifferential operators, see e.g. [19] and references therein.

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In this article we study a flexible method of generating frames for T-L type spaces and for the associated modulation spaces. Frames are redundant decomposition systems with extra structure between the expansion coefficients and the function being represented which make them useful for non-linear approximation. The advantage of redundant decomposition systems is that they provide extra flexibility compared to bases as we have more than one way of representing the function. Recently, this has lead to sparser representations of certain natural images than with wavelets; two examples of this are curvelet frames [20] and bandlets [15].

Frames for T-L type spaces and the associated modulation spaces have been considered earlier: Banach frames for α-modulation spaces in [4, 9], and Banach frames for T-L type spaces and the associated modulation spaces where constructed in [1, 2]. However, these frames were constructed using band-limited functions which rules out compact support in direct space.

The goal of this article is to construct frames with compact support for inhomogeneous T-L type spaces and the associated modulation spaces. An obvious modification produces frames for homogeneous spaces as well. The idea we employ is a perturbation principle which was first introduced in [18], further generalized in [13] and refined for frames in [14]. With this perturbation principle, finite linear combinations of shifts and dilates of a single functions with sufficient decay in direct and frequency space can be used to construct frame expansions with a prescribed nature such as compact support. These frame expansions are constructed from the already known Banach frames in [1, 2]; thereby, generating frame expansions which share the same sparseness properties as the already known representations.

Next, we discuss frames in more detail. Suppose that X is a quasi-Banach space and Y the associated sequence space. We say that a countable family of functions Ψ in the dual X^* of X is a frame for X if there exists constants $C_1, C_2 > 0$ such that for all $f \in X$,

$$C_1\|f\|_X \leq \|\{\langle f, \psi\rangle\}_{\psi\in\Psi}\|_Y \leq C_2\|f\|_X,$$

where $\langle f, \psi\rangle := \psi(f)$. In the $L_2(\mathbb{R}^d)$ case, frames have the expansion

$$f = \sum_{\psi\in\Psi} \langle f, S^{-1}\psi\rangle\psi, \tag{1.1}$$

where S is the frame operator $Sf = \sum_{\psi\in\Psi}\langle f, \psi\rangle\psi$, $f \in X$. In the general case, (1.1) is not a byproduct of the theory, but we show that the frame condition is the key to proving that (1.1) holds and that $\{S^{-1}\psi\}_{\psi\in\Psi}$ is a frame. This also proves that $\{\psi\}_{\psi\in\Psi}$ is a Banach frame.

As the general setup requires a great deal of notation, we give an example of what is proven for α-modulation spaces, $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$.

Theorem 1.1.
Choose $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q < \infty$, $0 \leq \alpha < 1$, and $\delta > 0$. Let $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ be the α-modulation spaces, $r := \min(1, p, q)$, and $1/\beta := \alpha/(1-\alpha)$. If $g \in C^1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, $\hat{g}(0) \neq 0$, satisfies

$$|g^{(\kappa)}(x)| \leq C(1+|x|)^{-2\left(\frac{d}{r}+\delta\right)-1}, \ |\kappa| \leq 1,$$
$$|\hat{g}(\xi)| \leq C(1+|x|)^{-2\left(\frac{d}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2d}{r}+\frac{3\delta}{2}\right)-1},$$

then there exists $K \in \mathbb{N}$ and $\psi_{k,n}(x) := \sum_{i=1}^{K} a_{k,i} g(c_k x + b_{k,n,i})$, $a_{k,i} \in \mathbb{R}$, $c_k, b_{k,n,i} \in \mathbb{R}^d$, such that $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ constitutes a frame for $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ and

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f, S^{-1}\psi_{k,n}\rangle \psi_{k,n}$$

for all $f \in M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ with convergence in $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$.

□

The outline of the article is as follows. In Section 2 we introduce homogeneous type spaces on $\mathbb{R}^d$ which are used to generate admissible coverings of the frequency space. These coverings are then used to define T-L type spaces and to construct associated frames. In Section 3 almost diagonal matrices are introduced, and we derive conditions under which the "change of frame coefficient" matrix is almost diagonal. Next, we use the machinery of almost diagonal matrices to construct new frames from old frames in Section 4 by using function systems which are sufficiently close to the frame from Section 2. Finally, in Section 5 we show that a system which consists of finite linear combinations of shifts and dilates of a single function with sufficient decay in both direct and frequency space can approximate another system with similar decay to an arbitrary degree. Thereby, creating systems which are sufficiently close to the frame from Section 2 and by using Section 4 constituting frames themselves which is our main result. We end the paper with a small discussion in Section 6 of the possible functions which can be used to construct the frames.

Throughout the article we will make use of some standard notation. We let $\hat{f}(\xi) := \mathcal{F}(f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$, $f \in L_1(\mathbb{R}^d)$, and by duality extend it uniquely from Schwartz functions, $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$, to tempered distributions, $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$. Similarly, we use $\langle f, \eta\rangle$ for the standard inner product of two functions $\int f\bar{\eta}$, and the same notation is employed for the action of a distribution $f \in \mathcal{S}'$ on $\bar{\eta} \in \mathcal{S}$. By $F \asymp G$ we mean that there exists two constants $0 < C_1 \leq C_2 < \infty$, depending only on "allowable" parameters, such that $C_1 F \leq G \leq C_2 F$. In general the constants C, C_1 and C_2 will change throughout the article. For the sake of convenience, we write $\|f_k\|$ instead of $\|\{f_k\}_{k\in K}\|$

when the index set is well-known. Finally, for $\kappa \in \mathbb{N}_0^d$ we let $|\kappa| := \kappa_1 + \cdots + \kappa_d$, and for suitably differentiable functions we define $f^{(\kappa)} := \frac{\partial^{|\kappa|} f}{\partial_{\xi_1}^{\kappa_1} \cdots \partial_{\xi_d}^{\kappa_d}}$.

2. Triebel-Lizorkin type spaces

In this section we give a brief description of T-L type spaces and the associated modulation spaces. To define T-L type spaces and the associated modulation spaces, we need a suitable resolution of the identity on $\mathbb{R}^d$ in the sense that we need a countable collection of functions $\{\varphi_k\}$ with $\sum_k \varphi_k = 1$. To construct the resolution of the identity, we use a suitable covering of the frequency space. For a much more detailed discussion of the T-L type spaces see [2], and for the associated modulation spaces see [1].

2.1. **Homogeneous type spaces on $\mathbb{R}^d$.** Here we define homogeneous type spaces on $\mathbb{R}^d$ which will be used later to construct a suitable covering of the frequency space. These spaces are created with a quasi-norm induced by a one-parameter group of dilations.
Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$ induced by the inner product $\langle \cdot, \cdot \rangle$. We assume that A is a real $d \times d$ matrix with eigenvalues having positive real parts. For $t > 0$ define the group of dilations $\delta_t : \mathbb{R}^d \to \mathbb{R}^d$ by $\delta_t := \exp(A \ln t)$ and let $\nu := \text{trace}(A)$. The matrix A will be kept fixed throughout the paper. Some well-known properties of δ_t are (see [22]),

- $\delta_{ts} = \delta_t \delta_s$.
- $\delta_1 = Id$ (identity on $\mathbb{R}^d$).
- $\delta_t \xi$ is jointly continuous in t and ξ, and $\delta_t \xi \to 0$ as $t \to 0^+$.
- $|\delta_t| := \det(\delta_t) = t^\nu$.

According to [22, Proposition 1.7] there exists a strictly positive symmetric matrix P such that for all $\xi \in \mathbb{R}^d$,

$$[\delta_t \xi]_P := \langle P \delta_t \xi, \delta_t \xi \rangle^{\frac{1}{2}}$$

is a strictly increasing function of t. This helps use introduce a quasi-norm $|\cdot|_A$ associated with A.

Definition 2.1.
We define the function $|\cdot|_A : \mathbb{R}^d \to \mathbb{R}_+$ by $|0|_A := 0$ and for $\xi \in \mathbb{R}^d \backslash \{0\}$ by letting $|\xi|_A$ be the unique solution t to the equation $[\delta_{1/t} \xi]_P = 1$. ◇

It can be shown that:

- $|\cdot|_A \in C^\infty(\mathbb{R}^d \backslash \{0\})$.
- There exists a constant $C_A > 0$ such that

$$|\xi + \zeta|_A \leq C_A(|\xi|_A + |\zeta|_A), \; \xi, \zeta \in \mathbb{R}^d. \tag{2.1}$$

- $|\delta_t\xi|_A = t|\xi|_A$.
- There exists constants $C_1, C_2, \alpha_1, \alpha_2 > 0$ such that

$$C_1 \min(|\xi|_A^{\alpha_1}, |\xi|_A^{\alpha_2}) \leq |\xi| \leq C_2 \max(|\xi|_A^{\alpha_1}, |\xi|_A^{\alpha_2}),\ \xi \in \mathbb{R}^d. \tag{2.2}$$

Example 2.2.
For $A = \operatorname{diag}(\beta_1, \beta_2, \dots, \beta_d)$, $\beta_i > 0$, we have $\delta_t = \operatorname{diag}(t^{\beta_1}, t^{\beta_2}, \dots, t^{\beta_d})$, and one can verify that

$$|\xi|_A \asymp \sum_{j=1}^{d} |\xi_j|^{\frac{1}{\beta_j}}, \xi \in \mathbb{R}^d.$$

⋆

Finally, we define the balls $\mathcal{B}_A(\xi, r) := \{\zeta \in \mathbb{R}^d : |\xi - \zeta|_A < r\}$. It can be verified that $|\mathcal{B}_A(\xi, r)| = r^\nu \omega_d^A$, where $\omega_d^A := |\mathcal{B}_A(0, 1)|$, so $(\mathbb{R}^d, |\cdot|_A, \mathrm{d}\xi)$ is a space of homogeneous type with homogeneous dimension ν.

The transpose of A with respect to $\langle \cdot, \cdot \rangle$, $B := A^\top$, will be useful for generating coverings of the direct space $\mathbb{R}^d$. Since the eigenvalues of B have positive real parts we can repeat the above construction for the group $\delta_t^\top := \exp(B \ln t), t > 0$. We let $|\cdot|_B$ denote the quasi-norm induced by $\delta_t^\top$, $\mathcal{B}_B(x, r)$ the balls associated with $|\cdot|_B$, and C_B the equivalent of C_A in (2.1). Furthermore, we have that the constants α_1 and α_2 in (2.2) also hold with B and $\operatorname{trace}(B) = \nu$. Notice that if $g_m(x) := m^\nu g(\delta_m^\top x)$, $g \in L_2(\mathbb{R}^d)$, then $\hat{g}_m(\xi) = \hat{g}(\delta_{\frac{1}{m}} \xi)$. We use the convention that δ_t acts on the frequency space while $\delta_t^\top$ acts on the direct space.

The following adaption of the Fefferman-Stein maximal inequality to the quasi-norm $|\cdot|_B$ will be essential for showing the boundedness of almost diagonal matrices. For $0 < r < \infty$, the parabolic maximal function of Hardy-Littlewood type is defined by

$$M_r^B u(x) := \sup_{t>0} \left(\frac{1}{\omega_d^B \cdot t^\nu} \int_{\mathcal{B}_B(x,t)} |u(y)|^r \, \mathrm{d}y \right)^{\frac{1}{r}}, u \in L_{r,\mathrm{loc}}(\mathbb{R}^d), \tag{2.3}$$

where $\omega_d^B := |\mathcal{B}_B(0, 1)|$. There exists $C > 0$ so that the following vector-valued Fefferman-Stein maximal inequality holds for $r < q \leq \infty$ and $r < p < \infty$ (see [21, Chapters I&II]),

$$\Big\| \Big(\sum_{k \in \mathbb{Z}^d} |M_r^B f_k|^q \Big)^{1/q} \Big\|_{L_p} \leq C \Big\| \Big(\sum_{k \in \mathbb{Z}^d} |f_k|^q \Big)^{1/q} \Big\|_{L_p}. \tag{2.4}$$

If $q = \infty$, then the inner l_q-norm is replaced by the l_∞-norm.

2.2. **Construction of frames.** Here we first introduce admissible coverings and how to generate them (see e.g. [7]). These coverings are then used to construct a suitable resolution of the identity and next define the T-L type spaces and the associated modulation spaces. Finally, we construct a frame which will be used in the following sections to generate compactly supported frame expansions.

Definition 2.3.
A set $\mathcal{Q} := \{Q_k\}_{k\in\mathbb{Z}^d}$ of measurable subsets $Q_k \subset \mathbb{R}^d$ is called an admissible covering if $\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d} Q_k$ and there exists $n_0 < \infty$ such that $\#\{j \in \mathbb{Z}^d : Q_k \cap Q_j \neq \emptyset\} \leq n_0$ for all $k \in \mathbb{Z}^d$. ◇

To generate an admissible covering we will use a suitable collection of $|\cdot|_A$-balls, where the radius of a given ball is a so-called moderate function of its center.

Definition 2.4.
A function $h : \mathbb{R}^d \to [\varepsilon_0, \infty)$ for $\varepsilon_0 > 0$ is called moderate if there exists constants $\rho_0, R_0 > 0$ such that $|\xi - \zeta|_A \leq \rho_0 h(\xi)$ implies $R_0^{-1} \leq h(\zeta)/h(\xi) \leq R_0$. ◇

Example 2.5.
Let $0 \leq \alpha \leq 1$. Then

$$h(\xi) := (1 + |\xi|_A)^\alpha \tag{2.5}$$

is moderate. ⋆

With a moderate function h, it is then possible to construct an admissible covering by using balls (see [7, Lemma 4.7] and [2, Lemma 5]):

Lemma 2.6.
Given a moderate function h with constants $\rho_0, R_0 > 0$, there exists a countable admissible covering $\mathcal{C} := \{\mathcal{B}_A(\xi_k, \rho h(\xi_k))\}_{k\in\mathbb{Z}^d}$ for $\rho < \rho_0/2$, and there exists a constant $0 < \rho' < \rho$ such that the sets in $\mathcal{C}$ are pairwise disjoint.

□

By using that $\mathcal{B}_A(\xi_k, \rho' h(\xi_k))$ are disjoint it can be shown that $\mathcal{B}_A(\xi_k, 2\rho h(\xi_k))$ also give an admissible covering. Notice that the covering $\mathcal{C}$ from Lemma 2.6 is generated by a family of invertible affine transformations applied to $\mathcal{B}_A(0, \rho)$ in the sense that

$$\mathcal{B}_A(\xi_k, \rho h(\xi_k)) = T_k \mathcal{B}_A(0, \rho), \; T_k := \delta_{h(\xi_k)} \cdot + \xi_k.$$

We can now generate our resolution of the identity, and for technical reasons we shall require it to satisfy the following.

Definition 2.7.
Let $\mathcal{C} := \{T_k \mathcal{B}_A(0, \rho)\}_{k\in\mathbb{Z}^d}$ be an admissible covering of $\mathbb{R}^d$ from Lemma 2.6.

A corresponding bounded admissible partition of unity (BAPU) is a family of functions $\{\varphi_k\}_{k\in\mathbb{Z}^d} \subset \mathcal{S}$ satisfying:

- $\text{supp}(\varphi_k) \subseteq T_k\mathcal{B}_A(0,2\rho)$, $k \in \mathbb{Z}^d$.
- $\sum_{k\in\mathbb{Z}^d} \varphi_k(\xi) = 1$, $\xi \in \mathbb{R}^d$.
- $\sup_{k\in\mathbb{Z}^d} \|\varphi_k(T_k\cdot)\|_{H_2^s} < \infty$, $s > 0$,

where $\|f\|_{H_2^s} := \left(\int |\mathcal{F}^{-1}f(x)|^2(1+|x|_B)^{2s}\,\mathrm{d}x\right)^{1/2}$. ◇

A standard trick for generating a BAPU for $\mathcal{C}$ is to pick $\Phi \in C^\infty(\mathbb{R}^d)$ non-negative with $\text{supp}(\Phi) \subseteq \mathcal{B}_A(0,2\rho)$ and $\Phi(\xi) = 1$ for $\xi \in \mathcal{B}_A(0,\rho)$. One can then show that

$$\varphi_k(\xi) := \frac{\Phi(T_k^{-1}\xi)}{\sum_{j\in\mathbb{Z}^d}\Phi(T_j^{-1}\xi)}$$

defines a BAPU for $\mathcal{C}$. For later use, we also introduce

$$\phi_k(\xi) := \frac{\Phi(T_k^{-1}\xi)}{\sqrt{\sum_{j\in\mathbb{Z}^d}\Phi(T_j^{-1}\xi)^2}}, \tag{2.6}$$

which in a sense defines a square root of the BAPU.

With a BAPU in hand we can now define the T-L type spaces and the associated modulation spaces.

Definition 2.8.
Let h be a moderate function satisfying

$$C_1(1+|\xi|_A)^{\gamma_1} \le h(\xi) \le C_2(1+|\xi|_A)^{\gamma_2},\ \xi \in \mathbb{R}^d, \tag{2.7}$$

for some $0 < \gamma_1 \le \gamma_2 < \infty$. Let $\mathcal{C}$ be a admissible covering of $\mathbb{R}^d$ from Lemma 2.6, $\{\varphi_k\}_{k\in\mathbb{Z}^d}$ a corresponding BAPU and $\varphi_k(D)f := \mathcal{F}^{-1}(\varphi_k\mathcal{F}f)$.

- For $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$, we define $F^s_{p,q}(h)$ as the set of distributions $f \in \mathcal{S}'$ satisfying
$$\|f\|_{F^s_{p,q}(h)} := \Big\|\Big(\sum_{k\in\mathbb{Z}^d}|h(\cdot)^s\varphi_k(D)f|^q\Big)^{1/q}\Big\|_{L_p} < \infty.$$
- For $s \in \mathbb{R}$, $0 < p \le \infty$, and $0 < q < \infty$, we define $M^s_{p,q}(h)$ as the set of distributions $f \in \mathcal{S}'$ satisfying
$$\|f\|_{M^s_{p,q}(h)} := \Big(\sum_{k\in\mathbb{Z}^d}\big\|h(\cdot)^s\varphi_k(D)f\big\|^q_{L_p}\Big)^{1/q} < \infty.$$

If $q = \infty$, then the l_q-norm is replaced by the l_∞-norm. ◇

It can be shown that $F^s_{p,q}(h)$ depends only on h up to equivalence of the norms (see [2, Proposition 5.3]), so the T-L type spaces are well-defined and similar for the modulation spaces. Furthermore, they both constitute quasi-Banach spaces, and for $p, q < \infty$, $\mathcal{S}$ is dense in both (see [2, Proposition 5.2]).

Next, we construct a frame for the T-L type spaces and the associated modulation spaces. Consider the system $\{\phi_k\}_{k\in\mathbb{Z}^d}$ from (2.6) which in a sense is a square root of a BAPU. Let K_a be a cube in $\mathbb{R}^d$ which is aligned with the coordinate axes and has side-length $2a$ satisfying $\mathcal{B}_A(0, 2\rho) \subseteq K_a$. For the sake of convenience, put

$$t_k := h(\xi_k). \tag{2.8}$$

We then define

$$e_{k,n}(\xi) := (2a)^{-\frac{d}{2}} t_k^{-\frac{\nu}{2}} \chi_{K_a}(T_k^{-1}\xi) e^{-i\frac{\pi}{a} n\cdot T_k^{-1}\xi}, \ n, k \in \mathbb{Z}^d,$$

and

$$\hat{\eta}_{k,n} := \phi_k e_{k,n}, \ n, k \in \mathbb{Z}^d. \tag{2.9}$$

One can verify that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a tight frame for $L_2(\mathbb{R}^d)$. By defining $\hat{\mu}_k(\xi) := \phi_k(T_k\xi)$, we get an explicit representation of $\eta_{k,n}$ in direct space

$$\eta_{k,n}(x) = (2a)^{-\frac{d}{2}} t_k^{\frac{\nu}{2}} \mu_k(\delta_{t_k}^\top x - \frac{\pi}{a} n) e^{ix\cdot\xi_k}, \tag{2.10}$$

and for $\kappa \in \mathbb{N}_0^d$, $N \in \mathbb{N}$ there exists $C > 0$ such that

$$|\mu_k(x)^{(\kappa)}| \leq C(1 + |x|_B)^{-N} \tag{2.11}$$

independent of $k \in \mathbb{Z}^d$. To show that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ constitutes a frame for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$, we need associated sequence spaces. The following point sets will be useful for that,

$$Q(k, n) = \Big\{ y \in \mathbb{R}^d : \delta_{t_k}^\top y - \frac{\pi}{a} n \in \mathcal{B}_B(0, 1) \Big\}. \tag{2.12}$$

It can easily be verified that there exists $n_0 < \infty$ such that uniformly in x and k, $\sum_{n\in\mathbb{Z}^d} \chi_{Q(k,n)}(x) \leq n_0$. With this property in hand, we can define the associated sequence spaces.

Definition 2.9.
Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$. We then define the sequence space $f^s_{p,q}(h)$ as the set of sequences $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset \mathbb{C}$ satisfying

$$\|s_{k,n}\|_{f^s_{p,q}(h)} := \Big\| \Big(\sum_{k,n\in\mathbb{Z}^d} \big(t_k^{s+\frac{\nu}{2}} |s_{k,n}|\big)^q \chi_{Q(k,n)} \Big)^{1/q} \Big\|_{L_p} < \infty.$$

Let $s \in \mathbb{R}$, $0 < p \leq \infty$, and $0 < q < \infty$. We then define the sequence space $b^s_{p,q}(h)$ as the set of sequences $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset \mathbb{C}$ satisfying

$$\|s_{k,n}\|_{b^s_{p,q}(h)} := \Big\| t_k^{s+\frac{\nu}{2}-\frac{\nu}{p}} \Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|^p \Big)^{1/p} \Big\|_{l_q} < \infty.$$

If $p = \infty$ or $q = \infty$, then the l_p-norm or l_q-norm, respectively, is replaced by the l_∞-norm. ◇

Finally, we have that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ (2.9) constitutes a frame for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$ (see [1, Remark 6.5 & Lemma 6.3] and [2, Lemma 4 & Theorem 2]):

Proposition 2.10.
Assume that $s \in \mathbb{R}$, $0 < p, q \leq \infty$, $p < \infty$ for $F^s_{p,q}(h)$, and $q < \infty$ for $M^s_{p,q}(h)$. For any finite sequence $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset \mathbb{C}$, we have

$$\Big\| \sum_{k,n\in\mathbb{Z}^d} s_{k,n}\eta_{k,n} \Big\|_{F^s_{p,q}(h)} \leq C\|s_{k,n}\|_{f^s_{p,q}(h)}.$$

Furthermore, $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^s_{p,q}(h)$,

$$\|f\|_{F^s_{p,q}(h)} \asymp \|\langle f, \eta_{k,n}\rangle\|_{f^s_{p,q}(h)}, \; f \in F^s_{p,q}(h).$$

Similar results hold for $B^s_{p,q}(h)$ and $b^s_{p,q}(h)$.

□

3. Almost diagonal matrices

To later generate new frame expansions for $F^s_{p,q}(h)$ and $B^s_{p,q}(h)$ from the already known frames, we introduce an associated notion of almost diagonal matrices in this section. The machinery of almost diagonal matrices was used in [11] and [10] for the Triebel-Lizorkin and Besov spaces respectively. The goal is to find a new definition for almost diagonal matrices for $F^s_{p,q}(h)$ and $B^s_{p,q}(h)$, and then show that they are bounded on the associated sequence spaces $f^s_{p,q}(h)$ and $b^s_{p,q}(h)$, and closed under compositions.

From here on we shall add some further restrictions to the moderate function h used to generate admissible coverings:

$$\left.\begin{array}{c}\text{There exists } \beta, R_1, \rho_1 > 0 \text{ such that } h^{1+\beta} \text{ is moderate and} \\ |\xi - \zeta|_A \leq ah(\xi) \text{ for } a \geq \rho_1 \text{ implies } h(\zeta) \leq R_1 ah(\xi).\end{array}\right\} \tag{3.1}$$

An abundance of functions h satisfying these conditions can be generated by using functions $s : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy $s(2b) \leq Cs(b)$, $b \in \mathbb{R}_+$, and

$$(1+b)^\gamma \leq s(b) \leq (1+b)^{\frac{1}{1+\beta}}$$

for some $\gamma > 0$. We assign $h = s(|\cdot|_A)$ and use that s is weakly sub-additive to get the results (see [7]). Notice that $s(b) = (1+b)^\alpha, 0 \leq \alpha < 1$, gives Example

2.5 and fulfills the mentioned conditions.

To motivate the definition of almost diagonal matrices, we let $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ be the frame defined in (2.10). By using (2.11) it can be verified that for fixed $N, M, L > 0$, $\eta_{k,n}$ has the following decay in direct and frequency space,

$$|\eta_{k,n}(x)| \le C t_k^{\frac{\nu}{2}} (1 + t_k |x_{k,n} - x|_B)^{-2N}, \tag{3.2}$$

$$|\hat{\eta}_{k,n}(\xi)| \le C t_k^{-\frac{\nu}{2}} (1 + t_k^{-1} |\xi_k - \xi|_A)^{-2M-2\frac{L}{\beta}}, \tag{3.3}$$

where

$$x_{k,n} = \delta^{\top}_{t_k^{-1}} \frac{\pi}{a} n, \ k, n \in \mathbb{Z}^d \tag{3.4}$$

and t_k was defined in (2.8). Let $\{\psi_{k,n}\}_{k,n\in Z^d} \subset L_2(\mathbb{R}^d)$ be a system with similar decay,

$$|\psi_{j,m}(x)| \le C t_j^{\frac{\nu}{2}} (1 + t_j |x_{j,m} - x|_B)^{-2N}, \tag{3.5}$$

$$|\hat{\psi}_{j,m}(\xi)| \le C t_j^{-\frac{\nu}{2}} (1 + t_j^{-1} |\xi_j - \xi|_A)^{-2M-2\frac{L}{\beta}}. \tag{3.6}$$

By examining $\langle \eta_{k,n}, \psi_{j,m} \rangle$ we then get the following lemma.

Lemma 3.1.
Choose $N, M, L > 0$ such that $2N > \nu$ and $2M + 2\frac{L}{\beta} > \nu$. If $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (3.2) and (3.3), and $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}^d}$ satisfies (3.5) and (3.6), we have

$$\begin{aligned} |\langle \eta_{k,n}, \psi_{j,m} \rangle| \le & C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}+L} (1 + \max(t_k, t_j)^{-1} |\xi_k - \xi_j|_A)^{-M} \\ & \times (1 + \min(t_k, t_j) |x_{k,n} - x_{j,m}|_B)^{-N}. \end{aligned}$$

Proof:
From lemma A.1 we have

$$|\langle \eta_{k,n}, \psi_{j,m} \rangle| \le C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}} (1 + \min(t_k, t_j) |x_{k,n} - x_{j,m}|_B)^{-2N}. \tag{3.7}$$

Using Lemma A.1 for $\langle \hat{\eta}_{k,n}, \hat{\psi}_{j,m} \rangle$ gives

$$|\langle \hat{\eta}_{k,n}, \hat{\psi}_{j,m} \rangle| \le C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}} (1 + \max(t_k, t_j)^{-1} |\xi_k - \xi_j|_A)^{-2M-2\frac{L}{\beta}}. \tag{3.8}$$

Next we raise the power of the first term in (3.8) at the expense of the second term. Without loss of generality assume that $t_k \le t_j$. We first consider the case

$|\xi_k - \xi_j|_A \le \rho_0 t_j^{1+\beta}$, and use that $h^{1+\beta}$ is moderate (3.1) to get

$$\frac{1}{1+t_j^{-1}|\xi_k-\xi_j|_A} \le 1 \le R_0^{\frac{\beta}{1+\beta}}\left(\frac{t_k}{t_j}\right)^{\beta}.$$

In the other case, $|\xi_k - \xi_j|_A > \rho_0 t_j^{1+\beta}$, and it follows by using $t_k \ge \varepsilon_0$ that

$$\frac{1}{1+t_j^{-1}|\xi_k-\xi_j|_A} \le \frac{1}{\rho_0\varepsilon_0^{\beta}}\left(\frac{t_k}{t_j}\right)^{\beta}.$$

Hence we have

$$|\langle \hat{\eta}_{k,n}, \hat{\psi}_{j,m}\rangle| \le C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}+2L} (1+\max(t_k,t_j)^{-1}|\xi_k-\xi_j|_A)^{-2M}. \tag{3.9}$$

The lemma follows by combining (3.7) and (3.9), and using

$$|\langle \eta_{k,n}, \psi_{j,m}\rangle| = |\langle \eta_{k,n}, \psi_{j,m}\rangle|^{\frac{1}{2}} |\langle \hat{\eta}_{k,n}, \hat{\psi}_{j,m}\rangle|^{\frac{1}{2}}.$$

■

We are now ready to define almost diagonal matrices for the T-L type spaces and show that they act boundedly on the T-L type spaces. A similar result also follows for the associated modulation spaces.

Definition 3.2.
Assume that $s \in \mathbb{R}$, $0 < p, q \le \infty$, $p < \infty$ for $f^s_{p,q}(h)$, and $q < \infty$ for $b^s_{p,q}(h)$. Let $r := \min(1, p, q)$. A matrix $\mathbf{A} := \{a_{(j,m)(k,n)}\}_{j,m,k,n \in \mathbb{Z}^d}$ is called almost diagonal on $f^s_{p,q}(h)$ and $b^s_{p,q}(h)$ if there exists $C, \delta > 0$ such that

$$\begin{aligned}|a_{(j,m)(k,n)}| \le & C\left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}}, \left(\frac{t_k}{t_j}\right)^{\frac{\delta}{2}}\right) c^{\delta}_{jk} \\ & \times (1+\min(t_k,t_j)|x_{k,n}-x_{j,m}|_B)^{-\frac{\nu}{r}-\delta},\end{aligned}$$

where

$$c^{\delta}_{jk} := \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\delta}, \left(\frac{t_k}{t_j}\right)^{\delta}\right)(1+\max(t_k,t_j)^{-1}|\xi_k-\xi_j|_A)^{-\frac{\nu}{r}-\delta}$$

with t_k defined in (2.8) and $x_{k,n}$ in (3.4). We denote the set of almost diagonal matrices on $f^s_{p,q}(h)$ and $b^s_{p,q}(h)$ by $\mathrm{ad}^s_{p,q}(h)$. ◇

The connection between almost diagonal and boundedness for matrices is an important result which will be essential for generating compactly supported frame expansions.

Proposition 3.3.
Suppose that $A \in \mathrm{ad}^s_{p,q}(h)$. Then A is bounded on $f^s_{p,q}(h)$ and $b^s_{p,q}(h)$.

Proof:
We only prove the result for $f^s_{p,q}(h)$ when $q < \infty$ as $q = \infty$ follows in a similar way with l_q replaced by l_∞, and the proof for $b^s_{p,q}(h)$ is similar to the one for $f^s_{p,q}(h)$. Let $s := \{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \in f^s_{p,q}(h)$ and assume for now that $p, q > 1$. We write $\mathbf{A} := \mathbf{A}_0 + \mathbf{A}_1$ such that

$$(\mathbf{A}_0 s)_{(j,m)} = \sum_{k:t_k\geq t_j}\sum_{n\in\mathbb{Z}^d} a_{(j,m)(k,n)} s_{k,n} \text{ and } (\mathbf{A}_1 s)_{(j,m)} = \sum_{k:t_k< t_j}\sum_{n\in\mathbb{Z}^d} a_{(j,m)(k,n)} s_{k,n}.$$

By using Lemma A.2 we have

$$\begin{aligned} |(\mathbf{A}_0 s)_{(j,m)}| &\leq C \sum_{k:t_k\geq t_j} \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}-\frac{\nu}{r}-\frac{\delta}{2}} c^\delta_{jk} \sum_{n\in\mathbb{Z}^d} \frac{|s_{k,n}|}{\left(1 + t_j\left|x_{k,n} - x_{j,m}\right|_B\right)^{\frac{\nu}{r}+\delta}} \\ &\leq C \sum_{k:t_k\geq t_j} \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}-\frac{\delta}{2}} c^\delta_{jk} M^B_r\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)(x), \end{aligned}$$

for $x \in Q(j, m)$. It then follows by Hölder's inequality and Lemma A.3 that

$$\begin{aligned} \sum_{m\in\mathbb{Z}^d} |(\mathbf{A}_0 s)_{(j,m)}\chi_{Q(j,m)}|^q &\leq C\Bigg(\sum_{k:t_k\geq t_j} \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} c^\delta_{jk} M^B_r\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Bigg)^q \\ &\leq C \sum_{k:t_k\geq t_j} c^\delta_{jk}\Bigg(\left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} M^B_r\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Bigg)^q \Bigg(\sum_{i:t_i\geq t_j} c^\delta_{ji}\Bigg)^{q-1} \\ &\leq C \sum_{k:t_k\geq t_j} c^\delta_{jk}\Bigg(\left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} M^B_r\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Bigg)^q. \end{aligned}$$

We obtain

$$\begin{aligned} \|\mathbf{A}_0 s\|_{f^s_{p,q}(h)} &\leq C\Bigg\|\Bigg(\sum_{j\in\mathbb{Z}^d}\sum_{k:t_k\geq t_j} c^\delta_{jk}\Bigg(t_k^{s+\frac{\nu}{2}} M^B_r\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Bigg)^q\Bigg)^{1/q}\Bigg\|_{L_p} \\ &\leq C\Bigg\|\Bigg(\sum_{k\in\mathbb{Z}^d}\Bigg(t_k^{s+\frac{\nu}{2}} M^B_r\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Bigg)^q\Bigg)^{1/q}\Bigg\|_{L_p}. \end{aligned}$$

Using the vector-valued Fefferman-Stein maximal inequality (2.4), we arrive at

$$\|\mathbf{A}_0 s\|_{f^s_{p,q}(h)} \leq C\Bigg\|\Bigg(\sum_{k,n\in\mathbb{Z}^d} (t_k^{s+\frac{\nu}{2}}|s_{k,n}|)^q\chi_{Q(k,n)}\Bigg)^{1/q}\Bigg\|_{L_p} = C\,\|s\|_{f^s_{p,q}(h)}.$$

The corresponding estimate for $\mathbf{A}_1$ follows from the same type of arguments resulting in both $\mathbf{A}_0$ and $\mathbf{A}_1$ being bounded on $f^s_{p,q}(h)$ and thereby $\mathbf{A}$. For the cases $q = 1$ and $p \leq 1, q > 1$ choose $0 < \tilde{r} < r$ and $0 < \tilde{\delta} < \delta$ such that $\nu/r + \delta/2 \geq \nu/\tilde{r} + \tilde{\delta}/2$ and repeat the argument with $r := \tilde{r}$, and $\delta := \tilde{\delta}$. The case $q < 1$ follows from first observing that

$$\tilde{\mathbf{A}} := \{\tilde{a}_{(j,m)(k,n)}\} := \left\{ |a_{(j,m)(k,n)}|^q \left(\frac{t_k}{t_j}\right)^{\frac{\nu}{2}-\frac{\nu q}{2}} \right\}$$

is almost diagonal on $f^{sq}_{\frac{p}{q},1}(h)$. Furthermore, if $v := \{v_{k,n}\} := \{|s_{k,n}|^q t_k^{\frac{\nu q}{2}-\frac{\nu}{2}}\}$ we have

$$\|v\|^{\frac{1}{q}}_{f^{sq}_{\frac{p}{q},1}(h)} = \Big\| \Big(\sum_{k,n\in\mathbb{Z}^d} (t_k^{s+\frac{\nu}{2}} |s_{k,n}|)^q \chi_{Q(k,n)} \Big)^{1/q} \Big\|_{L_p} = \|s\|_{f^s_{p,q}(h)}.$$

Before we can put these two observations into use we need that

$$|(\mathbf{A}s)_{(j,m)}|^q \leq \sum_k \sum_{n\in\mathbb{Z}^d} |a_{(j,m)(k,n)}|^q |s_{k,n}|^q = t_j^{\frac{\nu}{2}-\frac{\nu q}{2}} \sum_k \sum_{n\in\mathbb{Z}^d} \tilde{a}_{(j,m)(k,n)} v_{k,n}.$$

We then have

$$\|\mathbf{A}s\|_{f^s_{p,q}(h)} \leq \|\tilde{\mathbf{A}}v\|^{\frac{1}{q}}_{f^{sq}_{\frac{p}{q},1}(h)} \leq C\|v\|^{\frac{1}{q}}_{f^{sq}_{\frac{p}{q},1}(h)} = C\|s\|_{f^s_{p,q}(h)}.$$

■

The fact that almost diagonal matrices are closed under compositions is another result which will be essential for generating compactly supported frame expansions. First, we simplify the notation by defining

$$w^{s,\delta}_{(j,m)(k,n)} := \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}}, \left(\frac{t_k}{t_j}\right)^{\frac{\delta}{2}} \right) c^{\delta}_{jk} \times (1 + \min(t_k, t_j)|x_{k,n} - x_{j,m}|_B)^{-\frac{\nu}{r}-\delta},$$

where we have used the notation from Definition 3.2.

Proposition 3.4.
Let $s \in \mathbb{R}$, $0 < r \leq 1$ and $\delta > 0$. We then have

$$\sum_{i,l\in\mathbb{Z}^d} w^{s,\delta}_{(j,m)(i,l)} w^{s,\delta}_{(i,l)(k,n)} \leq C w^{s,\delta/2}_{(j,m)(k,n)},$$

Proof:
Notice that the factors $t_i^{s+\frac{\nu}{2}}$ in the first terms of $w^{s,\delta}_{(j,m)(i,l)}$ and $w^{s,\delta}_{(i,l)(k,n)}$ cancel leaving $(t_k/t_j)^{s+\frac{\nu}{2}}$ which can be moved outside the sums. Therefore we

only need to deal with the last three terms in $w^{s,\delta}_{(j,m)(i,l)}$ and $w^{s,\delta}_{(i,l)(k,n)}$. First we consider the case $t_j \le t_k$ and split the sum over i into three parts,

$$\begin{aligned}\sum_{i,l\in\mathbb{Z}^d} w^{s,\delta}_{(j,m)(i,l)} w^{s,\delta}_{(i,l)(k,n)} &= \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} \left(\sum_{i:t_i>t_k} + \sum_{i:t_j\le t_i\le t_k} + \sum_{i:t_i<t_j}\right) \sum_{l\in\mathbb{Z}^d} \\ &= \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} (\mathrm{I}+\mathrm{II}+\mathrm{III}).\end{aligned}$$

For I, by using lemma A.4 and lemma A.5, we have

$$\begin{aligned}\mathrm{I} =& \sum_{i:t_i>t_k}\sum_{l\in\mathbb{Z}^d} \left(\frac{t_j}{t_i}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} \left(\frac{t_k}{t_i}\right)^{\frac{\delta}{2}} c^{\delta}_{ji} c^{\delta}_{ik} \\ &\times \frac{1}{(1+t_j|x_{j,m}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \frac{1}{(1+t_k|x_{k,n}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \\ \le& \frac{C}{(1+t_j|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}} \sum_{i:t_i>t_k} \left(\frac{t_j}{t_i}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} \left(\frac{t_k}{t_i}\right)^{\frac{\delta}{2}-\nu} c^{\delta}_{ji} c^{\delta}_{ik} \\ \le& C\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c^{\delta/2}_{jk} \frac{1}{(1+t_j|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}}.\end{aligned}$$

Similarly for II we get

$$\begin{aligned}\mathrm{II} =& \sum_{i:t_j\le t_i\le t_k}\sum_{l\in\mathbb{Z}^d} \left(\frac{t_j}{t_i}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} \left(\frac{t_i}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c^{\delta}_{ji} c^{\delta}_{ki} \\ &\times \frac{1}{(1+t_j|x_{j,m}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \frac{1}{(1+t_i|x_{k,n}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \\ \le& C\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c^{\delta/2}_{jk} \frac{1}{(1+t_j|x_{j,m}-x_{k,n}|)^{\frac{\nu}{r}+\delta}}.\end{aligned}$$

For III we get

$$\begin{aligned}\mathrm{III} =& \sum_{i:t_i<t_j}\sum_{l\in\mathbb{Z}^d} \left(\frac{t_i}{t_j}\right)^{\frac{\delta}{2}} \left(\frac{t_i}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c^{\delta}_{ji} c^{\delta}_{ik} \\ &\times \frac{1}{(1+t_i|x_{j,m}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \frac{1}{(1+t_i|x_{k,n}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}}\end{aligned}$$

$$\leq \sum_{i:t_i<t_j} C\left(\frac{t_i}{t_j}\right)^{\frac{\delta}{2}}\left(\frac{t_i}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{ji}^{\delta}c_{ik}^{\delta}\frac{1}{(1+t_i|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}}$$

$$\leq \frac{C}{(1+t_j|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}}\sum_{i:t_i<t_j} C\left(\frac{t_i}{t_j}\right)^{\frac{\delta}{2}-\frac{\nu}{r}-\delta}\left(\frac{t_i}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{ji}^{\delta}c_{ik}^{\delta}$$

$$\leq C\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{jk}^{\delta/2}\frac{1}{(1+t_j|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}}.$$

In the case $t_j > t_k$, we observe that $w^{s,\delta}_{(j,m)(k,n)} = w^{2\nu/r-s-\nu,\delta}_{(k,n)(j,m)}$, so applying the first case to $w^{2\nu/r-s-\nu,\delta}_{(k,n)(j,m)}$ proves the proposition for $t_j > t_k$.

■

It follows from Proposition 3.4 that for $\delta_1, \delta_2 > 0$ we have

$$\sum_{i,l\in\mathbb{Z}^d} w^{s,\delta_1}_{(j,m)(i,l)} w^{s,\delta_2}_{(i,l)(k,n)} \leq C w^{s,\min(\delta_1,\delta_2)/2}_{(j,m)(k,n)} \tag{3.10}$$

which proves that $\mathrm{ad}^s_{p,q}(h)$ is closed under composition.

4. New frames from old frames

In this section we study a system $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ which is a small perturbation of the frame $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ constructed in (2.10). The goal is first to show that a system $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ which is close enough to $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is also a frame for $F^s_{p,q}(h)$ and $B^s_{p,q}(h)$. Next to get a frame expansion, we show that $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is also a frame, where S is the frame operator

$$Sf = \sum_{k,n\in\mathbb{Z}^d} \langle f, \psi_{k,n}\rangle \psi_{k,n}.$$

The results are inspired by [14] where perturbations of frames were studied in classical Triebel-Lizorkin and Besov spaces.

Let $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset L_2(\mathbb{R}^d)$ be a system that is close to $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ in the sense that there exists $\varepsilon, \delta > 0$ such that

$$|\eta_{k,n}(x) - \psi_{k,n}(x)| \leq \varepsilon t_k^{\frac{\nu}{2}}(1+t_k|x_{k,n}-x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)}, \tag{4.1}$$

$$|\hat{\eta}_{k,n}(\xi) - \hat{\psi}_{k,n}(\xi)| \leq \varepsilon t_k^{-\frac{\nu}{2}}(1+t_k^{-1}|\xi_k-\xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)}, \tag{4.2}$$

where we have used the notation from Definition 3.2. Motivated by the fact that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a tight frame for $L_2(\mathbb{R}^d)$, we formally define $\langle f, \psi_{j,m}\rangle$ as

$$\langle f, \psi_{j,m}\rangle := \sum_{k,n\in\mathbb{Z}^d} \langle \eta_{k,n}, \psi_{j,m}\rangle\langle f, \eta_{k,n}\rangle,\ f \in F^s_{p,q}(h). \tag{4.3}$$

It follows from Lemma 3.1 and Proposition 3.3 that $\langle \cdot, \psi_{j,m} \rangle$ is a bounded linear functional on $F^s_{p,q}(h)$; in fact we have

$$\sum_{k,n\in\mathbb{Z}^d} |\langle \eta_{k,n}, \psi_{j,m}\rangle||\langle f, \eta_{k,n}\rangle| \leq \left\| \Big\{ \sum_{k,n\in\mathbb{Z}^d} |\langle \eta_{k,n}, \psi_{j,m}\rangle||\langle f, \eta_{k,n}\rangle| \Big\}_{j,m\in\mathbb{Z}^d} \right\|_{f^s_{p,q}(h)}$$

$$\leq C\|\langle f, \eta_{k,n}\rangle\|_{f^s_{p,q}(h)} \leq C\|f\|_{F^s_{p,q}(h)}. \tag{4.4}$$

Furthermore, $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a norming family for $F^s_{p,q}(h)$ as it satisfies $\|\langle f, \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} \leq C\|f\|_{F^s_{p,q}(h)}$. This can be used to show that S is a bounded operator on $F^s_{p,q}(h)$, and for small enough ε, this will be the key to showing that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^s_{p,q}(h)$.

Theorem 4.1.
There exists $\varepsilon_0, C_1, C_2 > 0$ such that if $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (4.1) and (4.2) for some $0 < \varepsilon \leq \varepsilon_0$ and $f \in F^s_{p,q}(h)$, then we have

$$C_1\|f\|_{F^s_{p,q}(h)} \leq \|\langle f, \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} \leq C_2\|f\|_{F^s_{p,q}(h)}. \tag{4.5}$$

Similarly for $B^s_{p,q}(h)$ and $b^s_{p,q}(h)$.

Proof:
The proof will only be given for $F^s_{p,q}(h)$ as it follows the same way for $B^s_{p,q}(h)$. That $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a norming family gives the upper bound, thus we only need to establish the lower bound. For this we notice that $\{\varepsilon^{-1}(\eta_{k,n} - \psi_{k,n})\}_{k,n\in\mathbb{Z}^d}$ is also a norming family so we have

$$\|\langle f, \eta_{k,n} - \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} \leq C\varepsilon\|f\|_{F^s_{p,q}(h)}.$$

It then follows that

$$\begin{aligned}
\|f\|_{F^s_{p,q}(h)} &\leq C\|\langle f, \eta_{k,n}\rangle\|_{f^s_{p,q}(h)} \\
&\leq C(\|\langle f, \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} + \|\langle f, \eta_{k,n} - \psi_{k,n}\rangle\|_{f^s_{p,q}(h)}) \\
&\leq C(\|\langle f, \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} + \varepsilon\|f\|_{F^s_{p,q}(h)}).
\end{aligned}$$

By choosing $\varepsilon < 1/C$ we get the lower bound.

■

As one might guess from Theorem 4.1, the boundedness of the matrix $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ on $f^s_{p,q}(h)$ is the key to showing that $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is also a frame.

Proposition 4.2.
There exists $\varepsilon_0 > 0$ such that if $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^0_{22}(h) = L_2(\mathbb{R}^d)$ and satisfies (4.1) and (4.2) for some $0 < \varepsilon \leq \varepsilon_0$, then $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$ and $b^s_{p,q}(h)$.

Proof:
The proof will only be given for $f^s_{p,q}(h)$ as it follows similarly for $b^s_{p,q}(h)$. The fact that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$ ensures that S^{-1} is a bounded operator on $L_2(\mathbb{R}^d)$. We first show that S^{-1} is bounded on $F^s_{p,q}(h)$. This will follow from showing that

$$\|(I-S)f\|_{F^s_{p,q}(h)} \le C\varepsilon\|f\|_{F^s_{p,q}(h)}, f\in F^s_{p,q}(h), \tag{4.6}$$

choosing ε small enough and using the Neumann series. Assume for the moment that $\mathcal{D} := \{d_{(j,m)(k,n)}\} := \{\langle (I-S)\eta_{k,n}, \eta_{j,m}\rangle\}$ satisfies

$$\|\mathcal{D}s\|_{f^s_{p,q}(h)} \le C\varepsilon\|s\|_{f^s_{p,q}(h)} \tag{4.7}$$

By using that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$, we have that S is self-adjoint which leads to

$$\begin{aligned}\|(I-S)f\|_{F^s_{p,q}(h)} &\le C\|\langle (I-S)f,\eta_{j,m}\rangle\|_{f^s_{p,q}(h)} = C\|\mathcal{D}\{\langle f,\eta_{k,n}\rangle\}_{k,n\in\mathbb{Z}^d}\|_{f^s_{p,q}(h)}\\ &\le C\varepsilon\|\langle f,\eta_{j,m}\rangle\|_{f^s_{p,q}(h)} \le C\varepsilon\|f\|_{F^s_{p,q}(h)}.\end{aligned}$$

So to show (4.6) it suffices to prove (4.7). Note that

$$\begin{aligned}\langle (I-S)\eta_{k,n},\eta_{j,m}\rangle &= \sum_{i,l\in\mathbb{Z}^d}\langle \eta_{k,n},\eta_{i,l}\rangle\langle \eta_{i,l},\eta_{j,m}\rangle - \sum_{i,l\in\mathbb{Z}^d}\langle \eta_{k,n},\psi_{i,l}\rangle\langle \psi_{i,l},\eta_{j,m}\rangle\\ &= \sum_{i,l\in\mathbb{Z}^d}\langle \eta_{k,n},\eta_{i,l}\rangle\langle \eta_{i,l}-\psi_{i,l},\eta_{j,m}\rangle + \sum_{i,l\in\mathbb{Z}^d}\langle \eta_{k,n},\eta_{i,l}-\psi_{i,l}\rangle\langle \psi_{i,l},\eta_{j,m}\rangle.\end{aligned}$$

By setting

$$\begin{aligned}\mathcal{D}_1 &:= \{d_{1(j,m)(i,l)}\} := \{\langle \eta_{i,l}-\psi_{i,l},\eta_{j,m}\rangle\},\\ \mathcal{D}_2 &:= \{d_{2(i,l)(k,n)}\} := \{\langle \eta_{k,n},\eta_{i,l}\rangle\},\\ \mathcal{D}_3 &:= \{d_{3(j,m)(i,l)}\} := \{\langle \psi_{i,l},\eta_{j,m}\rangle\},\\ \mathcal{D}_4 &:= \{d_{4(i,l)(k,n)}\} := \{\langle \eta_{k,n},\eta_{i,l}-\psi_{i,l}\rangle\},\end{aligned}$$

we have the decomposition

$$\mathcal{D} = \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_3\mathcal{D}_4.$$

Since $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (4.1) and (4.2), we have from Lemma 3.1 that $\varepsilon^{-1}\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \varepsilon^{-1}\mathcal{D}_4 \in \mathrm{ad}^s_{p,q}(h)$. Next, we use that $\mathrm{ad}^s_{p,q}(h)$ is closed under composition (3.10), and by Proposition 3.3,

$$\|\mathcal{D}s\|_{f^s_{p,q}(h)} \le C\varepsilon\|s\|_{f^s_{p,q}(h)}.$$

Consequently, (4.6) holds, and for sufficiently small ε, the operator S^{-1} is bounded on $F^s_{p,q}(h)$. Finally, let $s := \{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \in f^s_{p,q}(h)$ and $g :=$

$\sum_{k,n\in\mathbb{Z}^d} s_{k,n}\eta_{k,n}$. By using Proposition 2.10 we have that $g \in F^s_{p,q}(h)$, and as $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$, we have that S^{-1} is self-adjoint which gives

$$\sum_{k,n\in\mathbb{Z}^d} \langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle s_{k,n} = \sum_{k,n\in\mathbb{Z}^d} \langle S^{-1}\eta_{k,n}, \psi_{j,m}\rangle s_{k,n} = \langle S^{-1}g, \psi_{j,m}\rangle.$$

If we combine this with $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ being a norming family for $F^s_{p,q}(h)$, see (4.4), we get

$$\begin{aligned}\Big\| \sum_{k,n\in\mathbb{Z}^d} \langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle s_{k,n}\Big\|_{f^s_{p,q}(h)} &= \|\langle S^{-1}g, \psi_{j,m}\rangle\|_{f^s_{p,q}(h)} \le C\|S^{-1}g\|_{F^s_{p,q}(h)} \\ &\le C\|g\|_{F^s_{p,q}(h)} \le C\|s\|_{f^s_{p,q}(h)}\end{aligned}$$

which proves that $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$.

■

That $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^s_{p,q}(h)$ and $B^s_{p,q}(h)$ now follows as a consequence of $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ being bounded on $f^s_{p,q}(h)$ and $b^s_{p,q}(h)$. We state the following results without proofs as they follow directly in the same way as in the classical Triebel-Lizorkin and Besov spaces. The proofs can be found in [14]. First, we have the frame expansion.

Lemma 4.3.
Assume that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$ and satisfies

$$\begin{aligned}|\psi_{k,n}(x)| &\le Ct_k^{\frac{\nu}{2}}(1 + t_k|x_{k,n} - x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)}, \\ |\hat{\psi}_{k,n}(\xi)| &\le Ct_k^{-\frac{\nu}{2}}(1 + t_k^{-1}|\xi_k - \xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)},\end{aligned}$$

where we have used the notation from Definition 3.2. If $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$, then for $f \in F^s_{p,q}(h)$ we have

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f, S^{-1}\psi_{k,n}\rangle \psi_{k,n}$$

in the sense of $\mathcal{S}'$. Similarly for $B^s_{p,q}(h)$ and $b^s_{p,q}(h)$.

□

Moreover, we have that $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame.

Theorem 4.4.
Assume that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$ and satisfies

$$\begin{aligned}|\psi_{k,n}(x)| &\le Ct_k^{\frac{\nu}{2}}(1 + t_k|x_{k,n} - x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)}, \\ |\hat{\psi}_{k,n}(\xi)| &\le Ct_k^{-\frac{\nu}{2}}(1 + t_k^{-1}|\xi_k - \xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)},\end{aligned}$$

where we have used the notation from Definition 3.2. Then $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^s_{p,q}(h)$ if and only if $\{\langle\eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$. Similarly for $B^s_{p,q}(h)$ and $b^s_{p,q}(h)$.

□

It is worth noting that Proposition 4.2, Lemma 4.3 and Theorem 4.4 imply that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a Banach frame if it satisfies (4.1) and (4.2) with sufficiently small ε, and $p,q \geq 1$. Furthermore, we also have a frame expansion with $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$.

Lemma 4.5.
Assume that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$ and satisfies

$$|\psi_{k,n}(x)| \leq Ct_k^{\frac{\nu}{2}}(1+t_k|x_{k,n}-x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)},$$
$$|\hat{\psi}_{k,n}(\xi)| \leq Ct_k^{-\frac{\nu}{2}}(1+t_k^{-1}|\xi_k-\xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)},$$

where we have used the notation from Definition 3.2. If the transpose of $\{\langle\eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$, then for $f\in F^s_{p,q}(h)$ we have

$$f = \sum_{k,n\in\mathbb{Z}^d}\langle f,\psi_{k,n}\rangle S^{-1}\psi_{k,n}$$

in the sense of $\mathcal{S}'$. Similarly for $B^s_{p,q}(h)$ and $b^s_{p,q}(h)$.

□

Remark 4.6.
If we have that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is normalized in $L_2(\mathbb{R}^d)$, then $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is an orthonormal basis for $L_2(\mathbb{R}^d)$ as a consequence of $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ being a tight frame for $L_2(\mathbb{R}^d)$ with constant 1. With arguments similar to the ones used in the proof of Proposition 4.2, it can be shown that there exists ε_0 such that if $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (4.1) and (4.2) for some $\varepsilon\leq\varepsilon_0$, then $\{\langle\eta_{k,n},\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ has a bounded inverse on $f^s_{p,q}(h)$ and $b^s_{p,q}(h)$, and consequently $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is an unconditional basis for $F^s_{p,q}(h)$ and $B^s_{p,q}(h)$.

For example, by using the uncondition basis for α-modulation spaces constructed in [17], one can generate a compactly supported basis for the α-modulation spaces. ○

5. Construction of new frames

In this section we generate compactly supported frame expansions for $F^s_{p,q}(h)$ and $B^s_{p,q}(h)$. More precisely, we show that finite linear combinations $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ of shifts and dilates of a function g with sufficient decay in both direct and frequency space can fulfill (4.1) and (4.2). As a consequence of the previous section, $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ will then constitute frames for $F^s_{p,q}(h)$ and

$B^s_{p,q}(h)$. In particular, by using a generating function g with compact support one can construct a compactly supported frame expansion. This is, as far as the authors are aware, a new approach. Earlier work, as in [13], used finite linear combinations of a function with sufficient smoothness and decay in direct space and vanishing moments.

It suffices to prove that there exists a system of functions $\{\tau_k\}_{k\in\mathbb{Z}^d} \subset L_2(\mathbb{R}^d)$ which is close enough to $\{\mu_k\}_{k\in\mathbb{Z}^d}$ (2.10):

$$|\mu_k(x) - \tau_k(x)| \le \varepsilon(1+|x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)},$$
$$|\hat{\mu}_k(\xi) - \hat{\tau}_k(\xi)| \le \varepsilon(1+|\xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)}.$$

The system

$$\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d} := \left\{t_k^{\nu/2}\tau_k\left(\delta_{t_k}^\top x - \frac{\pi}{a}n\right)e^{ix\cdot\xi_k}\right\}_{k,n\in\mathbb{Z}^d}$$

will then satisfy (4.1) and (4.2). First, we take $g \in C^1(\mathbb{R}^d)\cap L_2(\mathbb{R}^d)$, $\hat{g}(0)\neq 0$, which for fixed $N, M > 0$ satisfies

$$|g^{(\kappa)}(x)| \le C(1+|x|_B)^{-N-\alpha_1},\ |\kappa|\le 1, \tag{5.1}$$

$$|\hat{g}(\xi)| \le C(1+|\xi|_A)^{-M-\alpha_2}. \tag{5.2}$$

Next for $m\ge 1$, we define $g_m(x) := C_g m^\nu g(\delta_m^\top x)$, where $C_g := \hat{g}(0)^{-1}$. It then follows that

$$\begin{aligned} |g_m^{(\kappa)}(x)| &\le Cm^{\nu+\alpha_2|\kappa|}(1+m|x|_B)^{-N-\alpha_1},\ |\kappa|\le 1,\\ \int_{\mathbb{R}} g_m(x)\,\mathrm{d}x &= 1, \\ |\hat{g}_m(\xi)| &\le Cm^{M+\alpha_2}(1+|\xi|_A)^{-M-\alpha_2}. \end{aligned} \tag{5.3}$$

To construct τ_k we also need a set of finite linear combinations,

$$\Theta_{K,m} = \{\psi : \psi(\cdot) = \sum_{i=1}^K a_i g_m(\cdot + b_i), a_i\in\mathbb{R}, b_i\in\mathbb{R}^d\}.$$

We are now ready to show that any function with sufficient decay in both direct and frequency space can be approximated to an arbitrary degree by a finite linear combination of another function with similar decay.

Proposition 5.1.
Let $N' > N > \nu$ and $M' > M > \nu$. If $g \in C^1(\mathbb{R}^d)\cap L_2(\mathbb{R}^d)$, $\hat{g}(0)\neq 0$, fulfills (5.1) and (5.2) and $\mu_k \in C^1(\mathbb{R}^d)\cap L_2(\mathbb{R}^d)$ fulfills

$$|\mu_k(x)| \le C(1+|x|_B)^{-N'},$$
$$|\mu_k^{(\kappa)}(x)| \le C,\ |\kappa|\le 1,$$

$$|\hat{\mu}_k(\xi)| \leq C(1+|\xi|_A)^{-M'},$$

then for any $\varepsilon > 0$ there exists $K, m \geq 1$ and $\tau_k \in \Theta_{K,m}$ such that

$$|\mu_k(x) - \tau_k(x)| \leq \varepsilon(1+|x|_B)^{-N}, \tag{5.4}$$

$$|\hat{\mu}_k(\xi) - \hat{\tau}_k(\xi)| \leq \varepsilon(1+|\xi|_A)^{-M}. \tag{5.5}$$

Proof:
We construct the approximation of μ_k in direct space in three steps. First, by a convolution operator $\omega_m = \mu_k * g_m$, then, by $\theta_{q,m}$ which is the integral in ω_m taken over a region Q, and finally, by a discretization over dyadic cubes $\tau_{l,q,m}$. From (5.3) we have

$$\mu_k(x) - \omega_m(x) = \int_{\mathbb{R}^d} (\mu_k(x) - \mu_k(x-y)) g_m(y)\, \mathrm{d}y. \tag{5.6}$$

Define $U := m^{\lambda/2N}$, where $\lambda := \min(\alpha_1, N' - N)$. For $|x|_B \leq U$, we use the mean value theorem to get

$$|\mu_k(x) - \mu_k(x-y)| \leq C \min(1, |y|).$$

Inserting this in (5.6) we have

$$\begin{aligned} |\mu_k(x) - \omega_m(x)| &\leq C \int_{\mathbb{R}^d} \frac{\min(1, |y|_B^{\alpha_1}) m^\nu}{(1+m|y|_B)^{N+\alpha_1}}\, \mathrm{d}y \\ &\leq Cm^{-\alpha_1} \leq \frac{Cm^{-\lambda/2}}{U^N} \leq \frac{Cm^{-\lambda/2}}{(1+|x|_B)^N}. \end{aligned} \tag{5.7}$$

For $|x|_B > U$, we split the integral over $\Omega := \{y : |y|_B \leq |x|_B/2C_B\}$ and Ω^c. If $y \in \Omega$, then $|x-y|_B \geq |x|_B/2C_B$, and we have

$$\begin{aligned} \int_\Omega |\mu_k(x) - \mu_k(x-y)||g_m(y)|\, \mathrm{d}y &\leq C(1+|x|_B)^{-N'} \\ &\leq \frac{C}{(1+U)^\lambda (1+|x|_B)^N} \leq \frac{Cm^{-\lambda^2/2N}}{(1+|x|_B)^N}. \end{aligned} \tag{5.8}$$

Integrating over Ω^c with $|x|_B > U$ gives

$$\begin{aligned} &\int_{\Omega^c} |\mu_k(x) - \mu_k(x-y)||g_m(y)|\, \mathrm{d}y \\ &\qquad \leq \frac{C}{(1+|x|_B)^{N'}} + \int_{\Omega^c} \frac{Cm^\nu}{(1+|x-y|_B)^{N'}(1+m|y|_B)^{N+\alpha_1}}\, \mathrm{d}y \\ &\qquad \leq \frac{C}{(1+|x|_B)^{N'}} + \frac{Cm^{-\lambda}}{(1+|x|_B)^N} \leq \frac{C(m^{-\lambda^2/2N} + m^{-\lambda})}{(1+|x|_B)^N}. \end{aligned} \tag{5.9}$$

So by choosing m sufficiently large in (5.7)-(5.9), we get

$$|\mu_k(x) - \omega_m(x)| \leq \frac{\varepsilon}{3}(1+|x|_B)^{-N}. \tag{5.10}$$

For the next step we fix m and choose $q \in \mathbb{N}$. Let $H_{l,q}$ denote the smallest set of dyadic cubes aligned with the coordinate axes and sidelength 2^{-l}, $l \in \mathbb{N}$, that covers $\mathcal{B}_B(0, 2^q)$. We then approximate ω_m with $\theta_{q,m}$ defined as

$$\theta_{q,m}(\cdot) = \int_Q \mu_k(y) g_m(\cdot - y)\, \mathrm{d}y,$$

where $Q = \cup_{I \in H_{l,q}} I$. In which case we have

$$\omega_m(x) - \theta_{q,m}(x) = \int_{Q^c} \mu_k(y) g_m(x-y)\, \mathrm{d}y,$$

and it follows that

$$|\omega_m(x) - \theta_{q,m}(x)| \le \int_{|y|_B \ge 2^q} \frac{Cm^\nu}{(1+|y|_B)^{N'}(1+m|x-y|_B)^{N+\alpha_1}}\, \mathrm{d}y := L.$$

We first estimate the integral for $|x|_B \le 2^{q-1}/C_B$ which gives $|y|_B \ge |x|_B$ and $|x-y|_B \ge 2^{q-1}/C_B$. Hence we obtain

$$L \le \frac{C}{(1+|x|_B)^{N'}} \int_{|u| \ge \frac{2^{q-1}}{C_B}} \frac{m^\nu}{(1+m|u|_B)^{N+\alpha_1}}\, \mathrm{d}u \le \frac{Cm^{-\lambda}2^{-\lambda q}}{(1+|x|_B)^{N'}}. \tag{5.11}$$

For $|x|_B > 2^{q-1}/C_B$, we split the integral over $\Omega := \{y : |y|_B \ge 2^q\} \cap \{y : |y|_B \le |x|_B/2C_B\}$ and $\Omega' := \{y : |y|_B \ge 2^q\} \cap \{y : |y|_B > |x|_B/2C_B\}$. If $y \in \Omega$, then $|x-y|_B \ge |x|_B/2C_B$, and we get

$$\begin{aligned}\int_\Omega \frac{m^\nu}{(1+|y|_B)^{N'}(1+m|x-y|_B)^{N+\alpha_1}}\, \mathrm{d}y &\le \frac{Cm^\nu}{(1+m|x|_B)^{N+\alpha_1}} \int_{|y|_B \ge 2^q} \frac{1}{(1+|y|_B)^{N'}}\, \mathrm{d}y \\ &\le \frac{Cm^{-\lambda}2^{-\lambda q}}{(1+|x|_B)^{N}}. \end{aligned} \tag{5.12}$$

Similar for Ω' we have

$$\begin{aligned}\int_{\Omega'} \frac{m^\nu}{(1+|y|_B)^{N'}(1+m|x-y|_B)^{N+\alpha_1}}\, \mathrm{d}y &\le \frac{C}{(1+|x|_B)^{N'}} \int_{\mathbb{R}^d} \frac{m^\nu}{(1+m|x-y|_B)^{N+\alpha_1}}\, \mathrm{d}y \\ &\le \frac{C}{(1+|x|_B)^{N'}} \le \frac{C2^{-\lambda q}}{(1+|x|_B)^{N}}. \end{aligned} \tag{5.13}$$

By choosing q sufficiently large in (5.11)-(5.13), we obtain

$$|\omega_m(x) - \theta_{q,m}(x)| \le \frac{\varepsilon}{3}(1+|x|_B)^{-N}. \tag{5.14}$$

For the final step, we fix q and approximate $\theta_{q,m}$ by a discretization $\tau_{l,q,m}$,

$$\tau_{l,q,m}(\cdot) = \sum_{I \in H_{l,q}} |I| \mu_k(x_I) g_m(\cdot - x_I),$$

where x_I is the center of the dyadic cube I. Now choose $q' > q$ such that $Q \subset \mathcal{B}_B(0, 2^{q'})$, and note that $\tau_{l,q,m} \in \Theta_{K,m}$, $K < 2^{dl+\nu q'}$. We introduce $F(\cdot) := \mu_k(\cdot) g_m(x - \cdot)$ which gives

$$\begin{aligned} |\theta_{q,m}(x) - \tau_{l,q,m}(x)| &\le \sum_{I \in H_{l,q}} \int_I |\mu_k(y) g_m(x-y) - \mu_k(x_I) g_m(x - x_I)| \, \mathrm{d}y \\ &\le \sum_{I \in H_{l,q}} \int_I |F(y) - F(x_I)| \, \mathrm{d}y. \end{aligned}$$

By using the mean value theorem, we then get

$$\begin{aligned} |\theta_{q,m}(x) - \tau_{l,q,m}(x)| &\le \sum_{I \in H_{l,q}} \int_I |y - x_I| \max_{\substack{z \in l(x_I, y) \\ |\kappa| \le 1}} |F^{(\kappa)}(z)| \, \mathrm{d}y \\ &\le C 2^{\nu q' - l} \max_{\substack{z \in \mathcal{B}_B(0, 2^{q'}) \\ |\kappa| \le 1}} |g_m^{(\kappa)}(x - z)|, \end{aligned} \tag{5.15}$$

where $l(x_I, y)$ is the line-segment between x_I and y. If $|x|_B \le 2^{q'+1} C_B$ and $|\kappa| \le 1$, then we have

$$|g_m^{(\kappa)}(x - z)| \le C m^{\nu + \alpha_2} \le \frac{C m^{\nu + \alpha_2} 2^{q' N}}{(1 + |x|_B)^N}. \tag{5.16}$$

For $|x|_B > 2^{q'+1} C_B$ and $z \in \mathcal{B}_B(0, 2^{q'})$, we have $|x - z|_B \ge |x|_B / 2C_B$, and hence for $|\kappa| \le 1$, it follows that

$$|g_m^{(\kappa)}(x - z)| \le \frac{C m^{\nu + \alpha_2}}{(1 + m|x|_B)^{N + \alpha_1}} \le \frac{C m^{\alpha_2 - \alpha_1}}{(1 + |x|_B)^{N + \alpha_1}}. \tag{5.17}$$

By choosing l sufficiently large, and combining (5.15)-(5.17), we have

$$|\theta_{q,m}(x) - \tau_{l,q,m}(x)| \le \frac{\varepsilon}{3} (1 + |x|_B)^{-N}. \tag{5.18}$$

Finally by combining (5.10), (5.14) and (5.18), we get

$$|\mu_k(x) - \tau_{l,q,m}(x)| \le \varepsilon (1 + |x|_B)^{-N}. \tag{5.19}$$

To approximate μ_k in frequency space we use three steps similar to the approximation in direct space. Note that $\tau_{l,q,m}$ still fulfills (5.19) if we choose l, q, m even larger. First, we use $\hat{\omega}_m$ to approximate $\hat{\mu}_k$ in which case we have

$$\begin{aligned} |\hat{\mu}_k(\xi) - \hat{\omega}_m(\xi)| &= |\hat{\mu}_k(\xi)(1 - C_g \hat{g}(\delta_{\frac{1}{m}} \xi))| \\ &\le C(1 + |\xi|_A)^{-M} (1 + |\xi|_A)^{M - M'} |1 - C_g \hat{g}(\delta_{\frac{1}{m}} \xi)|. \end{aligned}$$

By choosing $a > 0$ such that $C(1+a)^{M-M'}|1 - C_g\hat{g}(\delta_{\frac{1}{m}}\xi)| \leq \varepsilon/3$ and m such that $C|1 - C_g\hat{g}(\delta_{\frac{1}{m}}\xi)| \leq \varepsilon/3$ for $|\xi|_A < a$, we get

$$|\hat{\mu}_k(\xi) - \hat{\omega}_m(\xi)| \leq \frac{\varepsilon}{3}(1+|\xi|_A)^{-M}. \tag{5.20}$$

Next, we fix m, choose q and limit the Fourier integral of μ_k to Q from the approximation in direct space,

$$\theta'_{q,m}(\xi) = \hat{g}_m(\xi)\int_Q \mu_k(x)e^{ix\cdot\xi}\,\mathrm{d}x.$$

This gives

$$|\hat{\omega}_m(\xi) - \theta'_{q,m}(\xi)| \leq |\hat{g}_m(\xi)|\int_{|x|_B \geq 2^q} |\mu_k(x)e^{ix\cdot\xi}|\,\mathrm{d}x \leq \frac{Cm^M 2^{-\lambda q}}{(1+|\xi|_A)^M}. \tag{5.21}$$

In the last step, we fix q and approximate $\theta'_{q,m}$ by $\hat{\tau}_{l,q,m}$. We introduce $G(x) := \mu_k(x)e^{ix\cdot\xi}$ and reuse q' from the approximation in direct space to get

$$\begin{aligned}|\theta'_{q,m}(\xi) - \hat{\tau}_{l,q,m}(\xi)| &\leq |\hat{g}_m(\xi)|\Big|\int_Q \mu_k(x)e^{ix\cdot\xi}\,\mathrm{d}x - \sum_{I\in H_{l,q}} |I|\mu_k(x_I)e^{ix_I\cdot\xi}\Big| \\ &\leq |\hat{g}_m(\xi)|\sum_{I\in H_{l,q}}\int_I |G(x) - G(x_I)|\,\mathrm{d}x \\ &\leq \frac{Cm^{M+\alpha_2}2^{\nu q'-l}}{(1+|\xi|_A)^{M+\alpha_2}}\max_{\substack{x\in\mathbb{R}^d\\|\kappa|\leq 1}}|G^{(\kappa)}(x)| \leq \frac{Cm^{M+\alpha_2}2^{\nu q'-l}}{(1+|\xi|_A)^M}.\end{aligned} \tag{5.22}$$

By combining (5.20)-(5.22) with sufficiently large l, q, m, we get

$$|\hat{\mu}_k(\xi) - \hat{\tau}_{l,q,m}(\xi)| \leq \varepsilon(1+|\xi|_A)^{-M}.$$

It follows that by choosing l, q, m large enough, $\tau_{l,q,m}$ fulfills both (5.4) and (5.5). Furthermore, we have $\tau_{l,q,m} \in \Theta_{K,m}$, $K < 2^{dl+\nu q'}$.

■

6. Discussion and further examples

In this paper we studied a flexible method of generating frames for T-L type spaces and the associated modulation spaces. With Proposition 4.2, Lemma 4.3 and Theorem 4.4, we proved that a system, which is sufficiently close to a frame for certain types of T-L type spaces and the associated modulation spaces, also constitutes a frame for these spaces. Furthermore, with Proposition 5.1 we construct such a system from finite linear combinations of shifts and dilates of a single function with sufficient decay in direct and frequency space.

Examples of functions with sufficient decay in direct and frequency space are $e^{-|\cdot|_B}$ and $(1+|\cdot|_B)^{-N}$ with N sufficiently large. By using (2.2), we can simplify this even further and use the exponential function $e^{-|\cdot|^2}$ or the rational functions $(1+|\cdot|^2)^{-N/2\alpha_1}$. An example with compact support can be constructed by using a spline with compact support. Furthermore, as the system is constructed using finite linear combinations of splines, we get a system consisting of compactly supported splines.

As a last remark, we draw attention to the fact that the methods used in Section 4 and 5 do not depend on the assumptions made on the function h in the beginning of Section 3. These assumptions are only needed to prove that the "change of frame coefficient" matrices are bounded and closed under compositions. For the anisotropic α-modulation spaces, spaces with $0 \le \alpha < 1$ satisfy the assumptions in Section 3, but the case $\alpha = 1$ does not. To deal with the case $\alpha = 1$, we mention that one can use a definition of almost diagonal matrices closer to the one for the classical Triebel-Lizorkin and Besov spaces which does not require these assumptions. These almost diagonal matrices were introduced in [3] and proven to be bounded. Furthermore, they can be used to show that the "change of frame coefficient" matrices are also bounded and closed under compositions in the case $\alpha = 1$. It follows that the methods in Section 4 and 5 can be used to construct frames in a variety of decomposition spaces given the right definition of almost diagonal matrices.

Appendix

In this appendix we prove five technical lemmas which we used in Section 3. We use the same notation as in Sections 2 and 3. First, we used the following lemma to prove Lemma 3.1.

Lemma A.1.
Let $N > \nu$ and suppose $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (3.2), and $\{\psi_{k,n}\}_{k,n\mathbb{Z}^d}$ satisfies (3.5). We then have

$$|\langle \eta_{k,n}, \psi_{j,m}\rangle| \le C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}} (1+\min(t_k,t_j)|x_{k,n}-x_{j,m}|_B)^{-N}, \tag{A.1}$$

with t_k defined in (2.8) and $x_{k,n}$ in (3.4).

Proof:
Without loss of generality assume that $t_k \le t_j$. First we consider the case $t_k|x_{k,m}-x_{j,m}|_B \le 1$. It then follows that

$$\frac{t_k^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x|_B)^N} \le t_k^{\frac{\nu}{2}} \le \frac{2^N t_k^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N}, \tag{A.2}$$

and we have

$$|\langle \eta_{k,n}, \psi_{j,m}\rangle| \leq \frac{C t_k^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N} \int_{\mathbb{R}^d} \frac{t_j^{\frac{\nu}{2}}}{(1+t_j|x_{j,m}-x|_B)^N}\,dx$$

$$= \frac{C t_k^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N} \int_{\mathbb{R}^d} \frac{t_j^{-\frac{\nu}{2}}}{(1+|x|_B)^N}\,dx$$

$$\leq C\left(\frac{t_k}{t_j}\right)^{\frac{\nu}{2}} (1+t_k|x_{k,n}-x_{j,m}|_B)^{-N} \tag{A.3}$$

since the space associated with $|\cdot|_B$ has homogeneous dimension ν. For the other case, $t_k|x_{k,m}-x_{j,m}|_B > 1$, we consider two additional cases. In the first case, we assume that $|x_{k,n}-x|_B \geq \frac{1}{2C_B}|x_{k,n}-x_{j,m}|_B$. Similar to above we then get (A.2) which leads to (A.3). In the last case, we have $|x_{k,n}-x|_B < \frac{1}{2C_B}|x_{k,n}-x_{j,m}|_B$ which gives $|x_{j,m}-x|_B > \frac{1}{2C_B}|x_{k,n}-x_{j,m}|_B$. It then follows that

$$\frac{1}{(1+t_j|x_{j,m}-x|_B)^N} \leq \frac{C_1}{(1+t_j|x_{k,n}-x_{j,m}|_B)^N} \leq \frac{C_2(t_k/t_j)^N}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N},$$

and we have

$$|\langle \eta_{k,n}, \psi_{j,m}\rangle| \leq \frac{C (t_k/t_j)^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N} \int_{\mathbb{R}^d} \frac{t_k^{\nu}}{(1+t_k|x_{k,n}-x|_B)^N}\,dx$$

$$\leq C\left(\frac{t_k}{t_j}\right)^{\frac{\nu}{2}} (1+t_k|x_{k,n}-x_{j,m}|_B)^{-N}.$$

■

The following estimate in direct space was used to prove Proposition 3.3.

Lemma A.2.
Suppose that $0 < r \leq 1$ and $N > \nu/r$. Then for any sequence $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset \mathbb{C}$, and for $x \in Q(j,m)$, we have

$$\sum_{n\in\mathbb{Z}^d} \frac{|s_{k,n}|}{(1+\min(t_k,t_j)|x_{k,n}-x_{j,m}|_B)^N} \leq C \max\left(\frac{t_k}{t_j}, 1\right)^{\frac{\nu}{r}} \times M_r^B\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)(x), \tag{A.4}$$

with t_k defined in (2.8), $x_{k,n}$ in (3.4), $Q(j,m)$ in (2.12) and M_r^B in (2.3).

Proof:
Without loss of generality we may assume $x_{j,m} = 0$ and begin by considering

the case $t_k \le t_j$. We define the sets,

$$A_0 = \{n \in \mathbb{Z}^d : t_k|x_{k,n}|_B \le 1\},$$
$$A_i = \{n \in \mathbb{Z}^d : 2^{i-1} < t_k|x_{k,n}|_B \le 2^i\}, \ \ i \ge 1.$$

Choose $x \in Q(j,m)$. There exists $C_1 > 0$ such that $\cup_{n\in A_i} Q(k,n) \subset \mathcal{B}_B(x, C_1 2^i t_k^{-1})$, and by using $\int \chi_{Q(k,n)} = \omega_d^B t_k^{-\nu}$, we get

$$\sum_{n\in A_i} \frac{|s_{k,n}|}{(1+t_k|x_{k,n}|)^N} \le C2^{-iN} \sum_{n\in A_i} |s_{k,n}| \le C2^{-iN}\Big(\sum_{n\in A_i} |s_{k,n}|^r\Big)^{\frac{1}{r}}$$
$$\le C2^{-iN}\left(\frac{t_k^\nu}{\omega_d^B}\int_{\mathcal{B}_B(x,C_1 2^i t_k^{-1})} \sum_{n\in A_i} |s_{k,n}|^r \chi_{Q(k,n)}\right)^{\frac{1}{r}}.$$

Hence by the definition of the maximal operator (2.3) we have

$$\sum_{n\in A_i} \frac{|s_{k,n}|}{(1+t_k|x_{k,n}|)^N} \le C2^{i(\frac{\nu}{r}-N)}\left(\frac{t_k^\nu}{2^{i\nu}\omega_d^B}\int_{\mathcal{B}_B(x,C_1 2^i t_k^{-1})} \sum_{n\in A_i} |s_{k,n}|^r \chi_{Q(k,n)}\right)^{\frac{1}{r}}$$
$$\le C2^{i(\frac{\nu}{r}-N)} M_r^B\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)(x)$$

by using $\sum_{n\in\mathbb{Z}^d} \chi_{Q(k,n)} \le n_0$. Summing over $i \ge 0$ and using $N > \nu/r$ gives (A.4). For the second case, $t_k > t_j$, we redefine the sets,

$$A_0 = \{n \in \mathbb{Z}^d : t_j|x_{k,n}|_B \le 1\}$$
$$A_i = \{n \in \mathbb{Z}^d : 2^{i-1} < t_j|x_{k,n}|_B \le 2^i\}, i \ge 1.$$

As before we have

$$\sum_{n\in A_i} \frac{|s_{k,n}|}{(1+t_j|x_{k,n}|)^M} \le C2^{-iN}\left(\frac{t_k^\nu}{\omega_d^B}\int_{\mathcal{B}_B(x,C_1 2^i t_j^{-1})} \sum_{n\in A_i} |s_{k,n}|^r \chi_{Q(k,n)}\right)^{\frac{1}{r}}$$
$$\le C2^{i(\frac{\nu}{r}-N)}\left(\frac{t_k}{t_j}\right)^{\frac{\nu}{r}} M_r^B\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)(x).$$

Summing over $i \ge 0$ again gives (A.4).

■

To prove Proposition 3.3 we also used the following estimate in frequency space.

Lemma A.3.
Let $\delta > 0$. There exists $C > 0$ independent of k such that

$$\sum_{j\in\mathbb{Z}^d} \min\left(\left(\frac{t_j}{t_k}\right)^{\nu}, \left(\frac{t_k}{t_j}\right)^{\delta}\right)(1+\max(t_k,t_j)^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta} \leq C,$$

with t_k defined in (2.8) and ξ_k in Lemma 2.6.

Proof:
We begin by dividing the indices into sets,

$$A_0 = \{j \in \mathbb{Z}^d : |\xi_j - \xi_k|_A \leq \rho_1 t_k\}$$
$$A_i = \{j \in \mathbb{Z}^d : 2^{i-1}\rho_1 t_k < |\xi_j - \xi_k|_A \leq 2^i \rho_1 t_k\}, \ \ i \geq 1,$$

with ρ_1 defined in (3.1). For $j \in A_i$, we have $\mathcal{B}_A(\xi_j, t_j) \subset \mathcal{B}_A(\xi_k, C_1 2^i t_k)$ which follows from using (3.1):

$$\begin{aligned}|\xi_k - \xi|_A \leq C_A(|\xi_k - \xi_j|_A + |\xi_j - \xi|_A) &\leq C_A(2^i\rho_1 t_k + t_j)\\ &\leq C_A(2^i \rho_1 t_k + R_1 2^i t_k)\\ &= C_1 2^i t_k,\end{aligned}$$

for $\xi \in \mathcal{B}_A(\xi_j, t_j)$. Next, we divide the sum even further by first looking at $t_k \geq t_j$, and by using that the covering $\{\mathcal{B}_A(\xi_j, t_j)\}_j$ is admissible, we get

$$\begin{aligned}&\sum_{\substack{j\in A_i\\ j:t_j\leq t_k}} \left(\frac{t_j}{t_k}\right)^{\nu} (1+t_k^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta}\\ &\qquad\leq C2^{-i(\nu+\delta)} \sum_{\substack{j\in A_i\\ j:t_j\leq t_k}} \left(\frac{t_j}{t_k}\right)^{\nu} \frac{1}{\omega_d^A t_j^{\nu}} \int_{\mathcal{B}_A(\xi_j,t_j)} \chi_{\mathcal{B}_A(\xi_j,t_j)}(\xi)\,\mathrm{d}\xi\\ &\qquad\leq C2^{-i(\nu+\delta)} \frac{1}{\omega_d^A t_k^{\nu}} \int_{\mathcal{B}_A(\xi_k, C_1 2^i t_k)} \sum_{\substack{j\in A_i\\ j:t_j\leq t_k}} \chi_{\mathcal{B}_A(\xi_j,t_j)}(\xi)\,\mathrm{d}\xi\\ &\qquad\leq C2^{-i\delta}.\end{aligned}$$

Summing over i gives the lemma for the $t_k \geq t_j$ part of the sum. In a similar way, the result for $t_k < t_j$ follows by using

$$\sum_{\substack{j\in A_i\\ j:t_j>t_k}} \left(\frac{t_k}{t_j}\right)^{\delta} (1+t_j^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta} \leq \sum_{\substack{j\in A_i\\ j:t_j>t_k}} \left(\frac{t_j}{t_k}\right)^{\nu} (1+t_k^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta}.$$

■

The following estimate in direct space was used to prove Proposition 3.4.

Lemma A.4.
Assume that $t_j \le t_k$, $N > \nu$ and

$$g := \sum_{l\in\mathbb{Z}^d} \frac{1}{(1+\min(t_j,t_i)|x_{j,m}-x_{i,l}|_B)^N} \frac{1}{(1+\min(t_k,t_i)|x_{k,n}-x_{i,l}|_B)^N},$$

with t_k defined in (2.8) and $x_{k,n}$ in (3.4). We then have

$$g \le \frac{C}{(1+\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B)^N} \max\left(\frac{t_i}{t_k},1\right)^\nu.$$

Proof:
Note that from lemma A.2 with $r = 1$ and $s_{k,n} = 1$, it follows that

$$\sum_{l\in\mathbb{Z}^d} \frac{1}{(1+\min(t_k,t_i)|x_{k,n}-x_{i,l}|_B)^N} \le C \max\left(\frac{t_i}{t_k},1\right)^\nu. \tag{A.5}$$

We first consider the case $\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B \le 1$ which gives

$$\begin{aligned} g &\le \sum_{l\in\mathbb{Z}^d} \frac{1}{(1+\min(t_k,t_i)|x_{k,n}-x_{i,l}|_B)^N} \\ &\le C \max\left(\frac{t_i}{t_k},1\right)^\nu \\ &\le \frac{C}{(1+\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B)^N} \max\left(\frac{t_i}{t_k},1\right)^\nu. \end{aligned}$$

For the case $\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B > 1$ we split the sum into

$$A = \left\{l \in \mathbb{Z}^d : |x_{j,m}-x_{i,l}|_B < \tfrac{1}{2C_B}|x_{j,m}-x_{k,n}|_B\right\}$$

and its complement. For A^c we have

$$\frac{1}{(1+\min(t_j,t_i)|x_{j,m}-x_{i,l}|_B)^N} \le \frac{(2C_B)^N}{(1+\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B)^N},$$

and by using (A.5), the desired estimate follows. For $l \in A$, we notice that $|x_{k,n}-x_{i,l}|_B > \frac{1}{2C_1}|x_{j,m}-x_{k,n}|_B$ and get

$$\begin{aligned} &(1+\min(t_k,t_i)|x_{k,n}-x_{i,l}|_B)^{-N} \\ &\qquad \le \left(1+\tfrac{1}{2C_B}\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B \frac{\min(t_k,t_i)}{\min(t_j,t_i)}\right)^{-N} \\ &\qquad \le \frac{C}{(1+\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B)^N} \left(\frac{\min(t_j,t_i)}{\min(t_k,t_i)}\right)^\nu. \end{aligned} \tag{A.6}$$

Next, by using (A.5) with j instead of k we get

$$\sum_{l\in\mathbb{Z}^d} \frac{1}{(1+\min(t_j,t_i)|x_{j,m}-x_{i,l}|_B)^N} \le C\max\left(\frac{t_i}{t_j},1\right)^{\nu}. \tag{A.7}$$

The lemma follows by combining (A.6) and (A.7).

■

Finally, we also used the following estimate in frequency space to prove Proposition 3.4.

Lemma A.5.
Let $\delta > 0$ and $0 < r \le 1$. We then have

$$h := \sum_{i\in\mathbb{Z}^d} c_{ji}^{\delta} c_{ik}^{\delta} \le C c_{jk}^{\delta/2},$$

where

$$c_{jk}^{\delta} := \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\delta}, \left(\frac{t_k}{t_j}\right)^{\delta}\right)(1+\max(t_k,t_j)^{-1}|\xi_k-\xi_j|_A)^{-\frac{\nu}{r}-\delta},$$

with t_k defined in (2.8) and ξ_k in Lemma 2.6.

Proof:
Without loss of generality assume that $r = 1$. We will begin with assuming that $t_j \le t_k$. Furthermore, if $t_k^{-1}|\xi_j-\xi_k|_A \le \rho_0$ we have $t_k/t_j \le R_0$ by using that h is moderate (see Definition 2.4). Combining this with Lemma A.3 gives

$$h \le \sum_{i\in\mathbb{Z}^d} c_{ik}^{\delta} \le C_1 \le C_2 c_{jk}^{\delta}.$$

In the other case, $t_k^{-1}|\xi_j-\xi_k|_A > \rho_0$, we split the sum into

$$A = \{i : |\xi_j-\xi_i|_A < \tfrac{1}{2C_A}|\xi_j-\xi_k|_A\}$$

and its complement. For $i \in A^c$ and $t_i \ge t_k \ge t_j$ we have

$$\begin{aligned} h \le& C\sum_{\substack{i\in A^c\\ i:t_i\ge t_k}} \left(\frac{t_j}{t_i}\right)^{\nu+\delta}(1+t_i^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta}c_{ik}^{\delta} \\ \le& C\left(\frac{t_j}{t_k}\right)^{\nu+\delta}(1+t_k^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta}\sum_{\substack{i\in A^c\\ i:t_i\ge t_k}} c_{ik}^{\delta} \\ \le& Cc_{jk}^{\delta} \end{aligned}$$

and similarly for $t_k > t_i \geq t_j$. For $t_k \geq t_j > t_i$ we get

$$\begin{aligned} h \leq & C \sum_{\substack{i \in A^c \\ i: t_i < t_j}} \left(\frac{t_i}{t_j}\right)^{\delta} (1 + t_j^{-1} |\xi_j - \xi_k|_A)^{-\nu-\delta} c_{ik}^{\delta} \\ \leq & C \left(\frac{t_j}{t_k}\right)^{\nu+\delta} (1 + t_k^{-1} |\xi_j - \xi_k|_A)^{-\nu-\delta} \sum_{\substack{i \in A^c \\ i: t_i < t_j}} c_{ik}^{\delta} \\ \leq & C c_{jk}^{\delta}. \end{aligned}$$

Finally, when $i \in A$ we have $|\xi_i - \xi_k|_A > \frac{1}{2C_A} |\xi_j - \xi_k|_A$ which for $t_i \geq t_k \geq t_j$ gives

$$\begin{aligned} h \leq & C \sum_{\substack{i \in A \\ i: t_i \geq t_k}} \left(\frac{t_k}{t_i}\right)^{\delta} \left(\frac{t_j}{t_i}\right)^{\nu+\delta} (1 + t_i^{-1} |\xi_j - \xi_i|_A)^{-\nu-\frac{\delta}{2}} (1 + t_i^{-1} |\xi_j - \xi_k|_A)^{-\nu-\frac{\delta}{2}} \\ \leq & C \left(\frac{t_j}{t_k}\right)^{\nu+\frac{\delta}{2}} (1 + t_k^{-1} |\xi_j - \xi_k|_A)^{-\nu-\frac{\delta}{2}} \sum_{\substack{i \in A \\ i: t_i \geq t_k}} \left(\frac{t_j}{t_i}\right)^{\frac{\delta}{2}} (1 + t_i^{-1} |\xi_j - \xi_i|_A)^{-\nu-\frac{\delta}{2}} \\ \leq & C c_{jk}^{\delta/2}. \end{aligned}$$

For $t_k > t_i \geq t_j$ and $t_k \geq t_j > t_i$ the argument can be repeated in a similar way which proves the lemma when $t_k \geq t_j$. For $t_k < t_j$, it suffices to use that $c_{jk}^{\delta} = (t_j/t_k)^{\nu} c_{kj}^{\delta}$, and we get

$$h = \sum_{i \in \mathbb{Z}^d} \left(\frac{t_j}{t_i}\right)^{\nu} c_{ij}^{\delta} \left(\frac{t_i}{t_k}\right)^{\nu} c_{ki}^{\delta} \leq C \left(\frac{t_j}{t_k}\right)^{\nu} c_{kj}^{\delta/2} = c_{jk}^{\delta/2}.$$

■

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