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SPECTRAL, SCATTERING, AND REGULARITY PROPERTIES RELATED TO VARIOUS FUNCTIONAL AND DIFFERENTIAL EQUATIONS

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Spectral, scattering, and regularity properties related to various functional and differential equations

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Abstract

Waves occur in many real world phenomena and much of the scientific literature concerns the study of waves. Different types of waves are described by different wave equations, for example Schrödinger's equation describing the time evolution of the wave function in a quantum mechanical system, or the Euler-Bernoulli equation describing bending waves in beams. This dissertation mainly considers these two types of waves and related topics. However, the dissertation also has a secondary focus unrelated to the topic of wave equations, which is the study of random variables with stationary digits. The dissertation consists of two parts. The first gives an introduction to the subjects listed above and an overview of the papers included in the second part. The second part consists of 6 papers, labeled A–F, concerning the aforementioned subjects.

Paper A and B deal with topics related Schrödinger's equation. Specifically, in Paper A spectral results for the magnetic Weyl quantization of $S^0_{0,0}$ symbols are obtained. A decomposition for the magnetic Weyl quantization is established, and from this the $\frac{1}{2}$-Hölder continuity of the spectrum with respect to the magnetic field strength is shown, when the operator is self-adjoint. If additionally the magnetic field is constant, then spectral gaps are shown to be Lipschitz continuous in the magnetic field strength. Paper B presents results on the regularity of powers of the resolvent of magnetic Schrödinger operators in the half plane. In particular, it is shown that the regularity of the range of powers of the resolvent increases with the exponent. These results are obtained using ideas from the magnetic perturbation theory. Using the regularity results for the resolvent, it is shown that Schwartz functions of the magnetic Schrödinger operator have smooth integral kernels. Furthermore, the asymptotic behavior of the particle current density corresponding to the magnetic Schrödinger operator is analyzed using geometric perturbation theory.

Paper C and D concern the acoustic black hole effect related to bending waves. Both papers concern the same results, as Paper D is a conference paper, which was extended to the journal paper, Paper C. In these papers, an optimal height profile h for the edge of a plate is sought in order to minimize the reflection of bending waves from the edge. This is done using the usual first order WKB approximation to Euler-Bernoulli's equation. Explicit

optimal solutions are found using methods from variational calculus. These solutions generalize the commonly considered profile $h(x) = \varepsilon x^2$.

Paper E and F concern the topic of random variables $X \in [0, 1]$ with stationary digits $\{X_n\}_{n \geq 1}$. Note that this topic is not related to the overarching theme of wave equations in the papers A–D. In Paper E a functional equation is established for the cumulative distribution function F of X. From this functional equation further characterizations of stationarity of the digits $\{X_n\}_{n \geq 1}$ are given in terms of the CDF F. In Paper F specific models for stationary digits are considered, specifically stationary Markov chains and stationary renewal processes. A law of pure type is established for F in these cases; either $F(x) = x$ for $x \in [0, 1]$ or F is singular, i.e. $F'(x) = 0$ for almost all $x \in [0, 1]$. Mixtures of stationary Markov chains, or stationary renewal processes are also treated in Paper F, where the normal number theorem plays a crucial role in the analysis.

Resumé

Bølgefænomener fremkommer inden for mange grene af videnskaben, og en stor del af den videnskabelige litteratur omhandler netop bølgefænomener. Forskellige typer af bølger bliver beskrevet af forskellige bølgeligninger, for eksempel Schrödingers ligning, der beskriver tidsudviklingen af bølgefunktionen for et kvantemekanisk system, eller Euler-Bernoullis ligning der beskriver bøjningsbølger i bjælker. Denne afhandling beskæftiger sig hovedsageligt med emner relateret til disse to bølgeligninger. Dog har afhandlingen også et sekundært fokus på stokastiske variable, hvis cifre er givet ved en stationær stokastisk proces. Afhandlingen består af to dele. Første del består af en introduktion til de ovenstående emner og en sammenfatning af indholdet af de artikler, som er inkluderet i afhandlingens anden del. I anden del af denne afhandling findes seks artikler nummereret A til F, som omhandler ovenstående emner.

Artikel A og B vedrører begge emner relateret til Schrödingers ligning. Artikel A omhandler spektralanalyse af den magnetiske Weyl-kvantisering af symboler af klassen $S^0_{0,0}$. Et dekompositionsresultat bevises for den magnetiske Weyl-kvantisering og dette resultat bruges efterfølgende til at vise $\frac{1}{2}$-Hölder-kontinuitet af spektret med hensyn til styrken af det magnetiske felt, når operatoren er selvadjungeret. Når det yderligere antages at det magnetiske felt er konstant, så er huller i spektret Lipschitz-kontinuerte med hensyn til styrken af det magnetiske felt. I Artikel B vises resultater angående regularitet af resolventen for magnetiske Schrödinger-operatorer i halvplanen. Et vigtigt værktøj for at opnå disse resultater er den magnetiske perturbationsteori. Ved at anvende de førnævnte regularitetsresultater vises det også, at Schwartz-funktioner af den magnetiske Schrödinger-operator har glatte integralkerner. Ved anvendelse af geometrisk perturbationsteori vises endvidere asymptotiske resultater for strømtætheden defineret ud fra den magnetiske Schrödinger-operator.

Artikel C og D vedrører den akustiske sorthuls-effekt i relation til bøjningsbølger. Begge artikler omhandler samme resultater, da Artikel D er en konferenceartikel, der er blevet videreudviklet til tidsskriftsartiklen Artikel C. I disse artikler bestemmes en optimal højdeprofil h nær kanten af en plade, således at reflektion af bøjningsbølger minimeres. Dette gøres for førsteordens WKB approksimationen til Euler-Bernoullis ligning. Den opti-

male profil h bestemmes ved at anvende metoder fra variationsregningen, og det vises, at det optimale højdeprofil h er en generalisering af højdeprofilen $h(x) = \varepsilon x^2$, der har fået meget opmærksomhed i litteraturen.

Artikel E og F omhandler emner relateret til stokastiske variabler $X \in [0,1]$, hvis cifre $\{X_n\}_{n\geq 1}$ udgør en stationær stokastisk proces. Bemærk at dette emne ikke har nogen direkte sammenhæng med de foregående emner relateret til bølgefænomener. I Artikel E bevises en funktionalligning for den kumulative fordelingsfunktion F tilhørende X. Fra funktionalligningen bevises yderligere karakteriseringer af stationaritet af cifrene $\{X_n\}_{n\geq 1}$ ud fra egenskaberne ved funktionen F. I Artikel F betragtes bestemte stationære stokastiske modeller for cifrene. Konkret betragtes stationære Markovkæder og stationære fornyelsesprocesser. Det vises, at for sådanne processer vil F enten være givet ved $F(x) = x$ for $x \in [0,1]$, eller være singulær, dvs. $F'(x) = 0$ for næsten alle $x \in [0,1]$. Miksturer af stationære Markovkæder eller stationære fornyelsesprocesser behandles ligeledes i Artikel F, og helt fundamentalt for denne analyse er egenskaberne ved normale tal.

Contents

Preface

This dissertation is the final product of my studies as a PhD student at Aalborg University. My time as a PhD student has been shared between the Department of Mathematical Sciences and the Department of Materials and Production both at Aalborg University.

The initial objective of this dissertation was to study various aspects of certain wave equations from both a theoretical and an applied perspective. From a mathematical point of view, the aim was to study properties of magnetic Schrödinger operators. From an engineering point of view, the focus was on dampening of bending waves through the so-called "acoustic black hole effect".

Furthermore, during my time as a PhD student I got involved in a collaboration with Professor Jesper Møller, who was supported by the "Centre for Stochastic Geometry and Advanced Bioimaging". This collaboration concerns characterization of random variables with digits given by stationary stochastic processes. This work, although unrelated to aforementioned topics, is interesting in its own right and of an extent which calls for inclusion in this dissertation.

This dissertation is a collection of papers and is comprised of two parts. Part II contains the scientific papers produced by me and my co-authors during my time as a PhD student, whereas Part I presents the necessary background for the papers in Part II and briefly discusses the main result of these papers. The papers included in Part II are:

Paper A : H. D. Cornean, H. Garde, B. B. Støttrup, and K. S. Sørensen, "Magnetic pseudodifferential operators represented as generalized Hofstadter-like matrices," *J. Pseudodiffer. Oper. Appl.*, vol. 10, pp. 307–336, 2019.

Paper B : M. Moscolari and B. B. Støttrup, "Regularity properties of bulk and edge current densities at positive temperature," 2022, [arXiv preprint arXiv:2201.08803]

Paper C : B. B. Støttrup, S. Sorokin, and H. D. Cornean, "A rigorous approach to optimal profile design for acoustic black holes," *J. Acoust. Soc. Am.*, vol. 149, no. 1, pp. 447–456, 2021.

Paper D : H. D. Cornean, S. Sorokin, and B. B. Støttrup, "Acoustic black hole profile optimization," in EURODYN 2020 - 11th International Conference on Structural Dynamics, Proceedings, M. Papadrakakis, M. Fragiadakis, and C. Papadimitriou, Eds., vol. 2. European Association for Structural Dynamics (EASD), 2020, pp. 2482–2488.

Paper E : H. D. Cornean, I. W. Herbst, J. Møller, B. B. Støttrup, and K. S. Sørensen, "Characterization of random variables with stationary digits," 2021, [arXiv preprint arXiv:2001.08492], *Accepted for publication in J. Appl. Probab.*

Paper F : H. D. Cornean, I. W. Herbst, J. Møller, B. B. Støttrup, and K. S. Sørensen, "Singular distribution functions for random variables with stationary digits," 2022, [arXiv preprint arXiv:2201.01521].

In order to adhere to copyright law and avoid cross-publication of submitted but not yet published results, the above papers are not available in full text in this dissertation. In Part II a title page is presented for each paper with links to either a preprint or full paper.

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Benjamin Buus Støttrup
Aalborg University, January 28, 2022

Part I

Background

Background

1 Introduction

The papers included in this dissertation can be divided into two groups. The papers A–D deal with topics related to wave propagation in various ways which will be clarified in Sections 2–4 below. Paper E and F concern characterizations of random variables with digits given by stationary stochastic processes. This topic is "orthogonal" to those treated in papers A–D, but still interesting in its own right. A brief introduction to this subject is given in Section 5. Lastly, in Section 6 the main results of each paper in Part II will be briefly discussed.

2 Some wave equations

Waves occur in many different areas within science, engineering, and mathematics. For example, in quantum mechanics the state of a quantum system is represented by a unit vector ψ in some Hilbert space and the time-evolution of the system is governed by the Schrödinger equation given (in appropriate units) by

$$\mathrm{i}\frac{\partial}{\partial t}\psi = H\psi, \tag{2.1}$$

where H is the Hamiltonian of the system. Another example, which is relevant for this dissertation, comes from the theory of thin plates. Bending waves (often also referred to as flexural waves) in a thin plate of variable height h extending infinitely in both the y direction and the positive x direction are described by the differential equation

$$\frac{\partial^2}{\partial x^2}\left(\frac{\rho c^2 h^3(x)}{12}\frac{\partial^2}{\partial x^2}w(x,\omega)\right) - \omega^2\rho h(x)w(x,\omega) = 0, \tag{2.2}$$

where w denotes the displacement of the midsurface, ω denotes the angular frequency, ρ denotes the density of the material, and c is the speed of quasi-longitudinal waves in thin plates.

Although these two equations look different, their solutions exhibit wave-like behavior and can, in some sense, be treated with similar methods. In this dissertation various aspects relating to these two of wave equations are considered.

3 Quantum mechanics

The following section presents a brief introduction to the theory of quantum mechanics necessary to motivate and understand the content of Paper A and B in Part II. It is assumed that the reader is somewhat familiar with the subjects covered, and therefore many arguments are done formally to not obfuscate the exposition. The unfamiliar reader is urged to consult [37] or [76] for a more detailed introduction, suitable for mathematicians with minimal physics background. In general, units will be chosen in a way which simplifies expressions as much as possible.

In quantum mechanics, the state of a particle moving in $\mathbb{R}^n$ can be described by unit vectors in the Hilbert space $L^2(\mathbb{R}^n)$. As briefly mentioned in Section 2, the time-evolution of the system is governed by Schrödinger's equation (2.1), and the Hamiltonian H (often called a *Schrödinger operator*) is given by

$$H = -\Delta + V$$

where $\Delta = \sum_{j=1}^n \partial_j^2$ is the *Laplace operator* (which is essentially self-adjoint on $C_c^\infty(\mathbb{R}^n)$) and V is called a *potential*. As described in Subsection 3.1 below, H is a quantum observable which corresponds to the energy function of a classical particle and therefore should be self-adjoint. Then, by the functional calculus for self-adjoint operators, the Schrödinger equation (2.1) is solved by setting $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi$. Thus it is of interest to know for which potentials V the operator H is self-adjoint. The key condition is that V should be symmetric and relatively bounded with respect to Δ with bound strictly less than 1, as established by the well-known Kato-Rellich theorem [50, 74]. However, in the following sections we are interested in particles moving in magnetic fields. Then question of self-adjointness becomes more intricate, cf. Subsection 3.2.

A general question concerning Schrödinger operators is how the spectrum of the operator $-\Delta$ is affected when it is perturbed by a potential V. More precisely, how does the spectrum of $-\Delta + \lambda V$ vary with $\lambda \in \mathbb{R}$? For the aforementioned relatively bounded potential, analytic perturbation theory applies [50, 75]. For example it is not hard to establish that the Hausdorff distance between $\sigma(-\Delta)$ and $\sigma(-\Delta + \lambda V)$ goes like $|\lambda|$. In Paper A similar results are addressed, but in the more advanced context of the magnetic Weyl quantization cf. Subsection 3.4. Perturbation theoretical results are also obtained in Paper B for magnetic Schrödinger operators, cf. Subsection 3.3. Therefore, this dissertation does not go further into the analytic perturbation theory as it will mostly concern what happens when a magnetic field is turned on, cf. Subsection 3.2.

3.1 Weyl Quantization

The correspondence principle of quantum mechanics states that a classical observable a (i.e. a function $a(x, \xi)$ defined on the phase space $\mathbb{R}^{2n}$ with $x \in \mathbb{R}^n$ being the position of the particle and $\xi \in \mathbb{R}^n$ being the momentum) should correspond to a self-adjoint operator $\mathrm{Op}(a)$ on $L^2(\mathbb{R}^n)$ [37, 76]. The map $a \mapsto \mathrm{Op}(a)$ is often called the *quantization* of a. This correspondence should satisfy that the classical observables for position, i.e. $a(x, \xi) = x_j$, correspond to the position operator X_j (i.e. multiplication by x_j on $L^2(\mathbb{R}^n)$). Likewise, the classical observable for momentum, i.e. $a(x, \xi) = \xi_j$, should correspond to the momentum operator $P_j = -\mathrm{i}\partial_j$. The classical Hamiltonian (energy function) is given by the function $h(x, \xi) = |\xi|^2 + V(x)$ with V being the potential energy of the particle. The obvious quantization of h is to replace x_j with X_j and ξ_j with P_j leading to the (quantum) Hamiltonian (cf. (2.1))

$$H = \mathrm{Op}(h) = -\Delta + V,$$

where V is the operator defined by multiplication by $V(x)$ (defined on a suitable dense subset of $L^2(\mathbb{R}^n)$). For more complicated observables the choice of quantization is not obvious due to the *canonical commutator relations*

$$[X_j, P_k] = -\mathrm{i}\delta_{jk},$$

where $\delta_{jk} = 0$ for $j \neq k$ and $\delta_{jk} = I$ for $j = k$. This suggests that the straightforward strategy of replacing x_j with X_j and ξ_j with P_j might not be the "best" choice of quantization in general. For this reason, finding a reasonable map Op which quantizes classical observables is often also referred to as the *quantization problem* [31].

Although there exist a number of different quantization schemes, the most commonly used is known as the *Weyl quantization* (often also referred to as the Weyl correspondence or Weyl calculus) [31, 37]. The Weyl quantization is established by defining the quantization of $a(x, \xi) = \mathrm{e}^{\mathrm{i}(px+q\xi)}$ as $\mathrm{Op}(a) = \mathrm{e}^{\mathrm{i}(pX+qP)}$, for any $p, q \in \mathbb{R}^n$. Note that we here write the dot product of $x, y \in \mathbb{R}^n$ simply as xy. Other "functions" $a(x, \xi)$ are quantized through the Fourier transform by making suitable sense of the expression

$$\mathrm{Op}(a) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \hat{a}(p, q) \mathrm{e}^{\mathrm{i}(pX+qP)} \, \mathrm{d}p \, \mathrm{d}q. \tag{3.1}$$

Often the notation $a(X, P)$ is used for $\mathrm{Op}(a)$. This can be done in general for temperate distributions $a \in \mathscr{S}'(\mathbb{R}^{2n})$ [31] by constructing an explicit integral kernel for sufficiently nice a and then extending to a Schwartz kernel in the general case. Let us formally manipulate (3.1) to obtain a formula for this integral kernel. By the functional calculus for self-adjoint operators

$$[\mathrm{e}^{\mathrm{i}(pX+qP)}\psi](x) = \mathrm{e}^{\mathrm{i}pq/2}\mathrm{e}^{\mathrm{i}px}\psi(x+q), \tag{3.2}$$

and since (at least formally)

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}p(x-w+\frac{q}{2})}\,\mathrm{d}p = [\mathscr{F}^{-1}(1)]\Big(x-w+\frac{q}{2}\Big) = \delta\Big(x-w+\frac{q}{2}\Big),$$

where $\mathscr{F}$ denotes the Fourier transform and $x, w, p, q \in \mathbb{R}^n$, it follows that

$$\begin{aligned}
[\mathrm{Op}(a)f](x) &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{4n}} a(w,\xi)\mathrm{e}^{-\mathrm{i}(pw+q\xi)}\mathrm{e}^{\mathrm{i}pq/2}\mathrm{e}^{\mathrm{i}px}f(x+q)\,\mathrm{d}w\,\mathrm{d}\xi\,\mathrm{d}p\,\mathrm{d}q \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} a\Big(x+\frac{q}{2},\xi\Big)\mathrm{e}^{-\mathrm{i}q\xi}f(x+q)\,\mathrm{d}\xi\,\mathrm{d}q \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} a\Big(\frac{x+y}{2},\xi\Big)\mathrm{e}^{\mathrm{i}\xi(x-y)}f(y)\,\mathrm{d}\xi\,\mathrm{d}y.
\end{aligned}$$

The integral kernel above extends to all temperate distributions $a \in \mathscr{S}'(\mathbb{R}^{2n})$ [31]. However, when a particle is moving in $\mathbb{R}^n$ with a magnetic field turned on, then the above quantization is no longer the right choice. We consider the quantization problem in the presence of magnetic fields in Subsection 3.4.

3.2 Magnetic fields

In this subsection, we will consider a magnetic field in $\mathbb{R}^n$ described by a closed 2-form $B(x) = \sum_{i,j=1}^n B_{ij}(x)\mathrm{d}x_i \wedge \mathrm{d}x_j$ with $B_{ij} = -B_{ji}$ and $B_{ij} \in BC^\infty(\mathbb{R}^n)$. Then, $B = \mathrm{d}\mathcal{A}$ for some 1-form $\mathcal{A}$, called a *magnetic potential* or *gauge*. A particular choice of $\mathcal{A}$ is the *transverse gauge*, defined for any $x' \in \mathbb{R}^n$ as (cf. Paper A)

$$\mathcal{A}_j(x,x') := -\sum_{k=1}^n \int_0^1 s(x_k - x'_k)B_{jk}(x' + s(x-x'))\,\mathrm{d}s. \tag{3.3}$$

For any $x' \in \mathbb{R}^n$, explicit calculations show that $B = \mathrm{d}\mathcal{A}(\cdot, x')$ and that $\mathcal{A}_j \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with all derivatives bounded linearly in $\langle x - x'\rangle$. Here $\langle x\rangle := (1+|x|^2)^{1/2}$. Note that even a constant magnetic field has a magnetic potential which grows towards infinity.

To describe observables of a quantum system in the presence of a magnetic field, the momentum operators P_j are replaced by the *magnetic momentum operators* $\Pi_j^{\mathcal{A}} = P_j - \mathcal{A}_j$ [64]. Here $\mathcal{A}_j$ is the operator in $L^2(\mathbb{R}^n)$ given by multiplication with the j'th component of the magnetic potential $\mathcal{A}$. Hence, quantizing the Hamiltonian of the system using the same principles as in Subsection 3.1 gives

$$H = (-\mathrm{i}\nabla - \mathcal{A})^2 + V. \tag{3.4}$$

Operators of the form in (3.4) are often referred to as *magnetic Schrödinger operators* and have been the focus of much study in the literature (see e.g. the general references [2, 20, 32] and references therein).

Two immediate issues should be addressed. The first issue is that the choice of magnetic potential is not unique. Specifically, $\mathrm{d}(\mathrm{d}\varphi) = 0$ for any smooth function φ, and thus $B = \mathrm{d}(\mathcal{A} + \mathrm{d}\varphi)$. However, from a physical standpoint, choosing another magnetic potential A satisfying $B = \mathrm{d}A$ should not change the quantum system, as B is unchanged. In particular, we should choose a quantization scheme $\mathrm{Op}^{\mathcal{A}}$, depending on $\mathcal{A}$, that is *gauge covariant*, i.e. if A satisfies $\mathrm{d}A = \mathrm{d}\mathcal{A}$ then $\mathrm{Op}^{A}(a)$ and $\mathrm{Op}^{\mathcal{A}}(a)$ are unitarily equivalent. For certain operators such as $\Pi^{\mathcal{A}}$ (and hence also H) this is indeed the case. For example, if $\mathrm{d}\varphi = A - \mathcal{A}$, then a simple (formal) calculation shows $\Pi_j^{\mathcal{A}}$ and Π_j^{A} are unitary equivalent, specifically

$$\mathrm{e}^{\mathrm{i}\varphi}\Pi_j^{\mathcal{A}}\mathrm{e}^{-\mathrm{i}\varphi} = \Pi_j^{A}.$$

In the presence of a magnetic field the quantization problem becomes more complicated as the quantization should be gauge covariant. As for the usual Weyl quantization in Subsection 3.1, simply replacing the classical position x_j and momentum ξ_j with their (magnetic) quantum counterparts X_j and $\Pi_j^{\mathcal{A}}$ is not the "right" approach. The quantization problem when a magnetic field is present will be considered in Subsection 3.4 and is of central interest in Paper A.

The second issue is that, $\mathcal{A}$ can grow towards infinity. Thus, H is not necessarily a relatively bounded perturbation of $-\Delta$. However, there exist general conditions on V which ensure essentially self-adjointness of H [43, 74], but one can no longer apply the analytic perturbation theory to analyze "magnetic perturbations" of the form

$$H_b = (-\mathrm{i}\nabla - bA - \mathcal{A})^2 + V, \tag{3.5}$$

where $b \in \mathbb{R}$ and A is some magnetic potential.

3.3 Magnetic Perturbation Theory

The following section presents various results related to the magnetic perturbation theory, i.e. the study of how the spectrum of H_b in (3.5) depends on b. For convenience, assume that A and $\mathcal{A}$ are smooth and polynomially bounded. The papers A and B in Part II both concern topics related to magnetic perturbation theory. In addition to the continuous models considered in both Paper A and B, it is also relevant to look at some related discrete models. Note that some of the results described below hold for more general magnetic fields than considered in this dissertation. However, to be consistent with Paper A and B, we stick to the assumptions on A and $\mathcal{A}$ above.

Define the function

$$\varphi(x,x') = \int_{[x',x]} A,$$

where $[x',x]$ is the oriented line segment from x' to x. It is not hard to see that φ is anti-symmetric and that for all $\alpha,\beta \in \mathbb{N}_0^n$

$$|\partial_x^\alpha \partial_x'^\beta \varphi(x,x')| \leq C_{\alpha\beta}|x||x'|. \tag{3.6}$$

Next we consider discrete models. The *generalized Harper operators* (see [67]) are bounded operators h_b on $\ell^2(\mathbb{Z}^2)$ defined by

$$[h_b\psi](\gamma) = \sum_{\gamma' \in \mathbb{Z}} \mathrm{e}^{ib\varphi(\gamma,\gamma')} h(\gamma,\gamma')\psi(\gamma'), \quad \gamma \in \mathbb{Z} \tag{3.7}$$

where the infinite matrix h decays sufficiently fast in $\langle \gamma - \gamma' \rangle$ and is Hermitian, i.e. $h(\gamma,\gamma') = \overline{h(\gamma',\gamma)}$. Such operators have been treated numerous times in the literature, see e.g. [11, 67] and references therein. For instance when $h(\gamma,\gamma')$ depends only on $\gamma - \gamma'$ and the magnetic field is constant, Bellissard [5] proved Lipschitz continuity of spectral gap edges. For a general h_b in (3.7) and a smooth and bounded magnetic field it has been shown that the gap edges goes like $|b - b_0|\, |\ln(|b - b_0|)|$ [67]. Furthermore, in [67] it also established that $\sigma(h_b)$ is $\frac{1}{2}$-Hölder continuous in the Hausdorff distance with respect to b. Later regularity of spectral gap edges was show to be Lipschitz, but only for constant magnetic fields [11]. It should be noted that various authors have gradually pushed the Hölder exponent up to $\frac{1}{2}$ (see the discussion in [11]).

A reason to consider the discrete case is that the spectral regularity of the magnetic Schrödinger operator H_b can be reduced to "continuous generalized Harper operators" (i.e. where the sum is replaced with an integral and $\ell^2(\mathbb{Z}^2)$ is replaced by $L^2(\mathbb{R}^n)$) [11, 16, 17]. Therefore many of the results obtained in the discrete setting transfer directly to the continuous setting using similar proofs.

This paragraph will discuss the development of the spectral regularity for the continuous model, that is H_b. The stability of spectral gaps for H_b was first established independently in [65] and [3], in the sense that gaps in the spectrum of H_0 persist for sufficiently small b. In fact, the result of [65] implicitly gives $\frac{1}{2}$-Hölder continuity $\sigma(H_b)$ in the Hausdorff distance with respect to b [11]. In [66] Nenciu proved that the phase $\mathrm{e}^{ib\varphi(x,x')}$ could be factored out of the integral kernel of the resolvent $(H_b - z)^{-1}$ for $z \in \rho(H_b)$, leaving an integral operator $K_b(z)$ which has a norm convergent expansion in b for $b \in [0,\varepsilon]$ where $\varepsilon > 0$ is sufficiently small, i.e.

$$(H_b - z)^{-1}(x,x') = \mathrm{e}^{ib\varphi(x,x')} K_b(z)(x,x'). \tag{3.8}$$

Specifically, expanding $K_b(z)$ in a power series gives a norm convergent perturbation formula

$$(H_b - z)^{-1} = S_b(z) \sum_{n=0}^{\infty} b^n T_b(z)^n = S_b(z)(1 - bT_b(z))^{-1} \tag{3.9}$$

for $z \in \rho(H_b)$ and b sufficiently small. Here $S_b(z)$ is an integral operator with kernel given by

$$S_b(z)(x,y) = \mathrm{e}^{ib\varphi(x,y)}(H_0 - z)^{-1}(x,y),$$

and $T_b(z)$ is an operator which is bounded uniformly for $b \in [0, \varepsilon]$.

From this, one can proceed as in the analytic perturbation theory [66]. By constructing an explicit pseudo-inverse $S_b(z)$ for $(H_b - z)^{-1}$, (similar to (3.9)) it has been shown that spectral gap edges are $\frac{2}{3}$-Hölder continuous in the Hausdorff distance with respect to b for smooth magnetic potentials [9]. In two dimension and for a constant magnetic field, the spectral gap edges has been shown to be Lipschitz continuous in b [11]. This replicates the result of Bellissard for the discrete setting [5]. The regularity results for spectral gap edges of the generalized Harper operator obtained in [67] (cf. the discussion above) have also been extended in the continuous case [17].

3.4 Magnetic Weyl quantization

As mentioned in Subsection 3.2, a natural question is how to set up a gauge covariant quantization scheme Op^A when a magnetic field $B = \mathrm{d}A$ is present. Interestingly, the naive quantization of a given by $\mathrm{Op}(a^A)$ where $a^A(x,\xi) = a(x, \xi - A(x))$ is not necessarily gauge covariant. In particular, if $\mathcal{A} = A + \mathrm{d}\varphi$ for some smooth φ, then for $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \leq 2$, the polynomial $a(x,\xi) = \xi^\alpha$ satisfies $\mathrm{e}^{\mathrm{i}\varphi}\mathrm{Op}(a^A)\mathrm{e}^{-\mathrm{i}\varphi} = \mathrm{Op}(a^{\mathcal{A}})$, but there exists some α of order 3 for which $\mathrm{Op}(a^A)$ and $\mathrm{Op}(a^{\mathcal{A}})$ are not unitarily equivalent [41].

A gauge covariant quantization, now known as the *magnetic Weyl quantization*, has been proposed independently in [47] and [64] although with different motives. It is obtained similar to the Weyl quantization in Subsection 3.1 by setting $\mathrm{Op}^A(a) = \mathrm{e}^{\mathrm{i}(pX + q\Pi^A)}$ for $a(x,\xi) = \mathrm{e}^{\mathrm{i}(px+q\xi)}$ with $p, q \in \mathbb{R}^n$ and extending to other "functions" $a(x,\xi)$ by making suitable sense of the expression

$$\mathrm{Op}^A(a) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \hat{a}(p,q)\mathrm{e}^{\mathrm{i}(pX+q\Pi)}\,\mathrm{d}p\,\mathrm{d}q. \tag{3.10}$$

Note that in [64] a symplectic Fourier transform was used in the definition, but it leads to the same operator (3.10) (see also the remark in [31, p. 80]).

Arguing formally as in Subsection 3.1 and using an explicit expression for $\mathrm{e}^{\mathrm{i}(pX+q\Pi)}$ (cf. [64, Equation (2.17)]) it is not hard to obtain the formula

$$[\mathrm{Op}^A(a)f](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \mathrm{e}^{\mathrm{i}\xi(x-x')}\mathrm{e}^{\mathrm{i}\varphi(x,x')} a\Big(\frac{x+x'}{2}, \xi\Big) f(x')\,\mathrm{d}x'\,\mathrm{d}\xi.$$

Since A is assumed to be smooth with polynomially bounded derivatives of all orders, the above expression can be defined for all $a \in \mathscr{S}'(\mathbb{R}^{2n})$ and it can be shown that $\mathrm{Op}^A(a)$ is gauge covariant [64, Proposition 3.6]. A key point here is that under these assumptions, multiplication by the components of A map $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$. It should be noted that when B is constant and A is linear then $\mathrm{Op}^A(a) = \mathrm{Op}(a^A)$ [64, Lemma 3.11]. When B is not constant the same result holds for any A when a is a polynomial in ξ of order at most 2 [64, Proposition 3.12]

Going forward only functions a in the symbol classes $S^m_{\rho,\delta}(\mathbb{R}^{2n})$ of Hörmander [39, 40] are considered, i.e. $a \in C^\infty(\mathbb{R}^{2n})$ such that for all $\alpha, \beta \in \mathbb{N}_0^n$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}, \tag{3.11}$$

where m, ρ, δ are real numbers. Of special interest is the class $S^0_{0,0}(\mathbb{R}^{2n})$ which is (with minor adjustments) considered in Paper A. For $a \in S^m_{\rho,\delta}(\mathbb{R}^{2n})$ with $\delta < 1$ we can write $\mathrm{Op}^A(a)$ explicitly as a map from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ by the formula

$$\langle \mathrm{Op}^A(a) f, g \rangle := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{3n}} \mathrm{e}^{\mathrm{i}\xi(x-x')} \mathrm{e}^{\mathrm{i}\varphi(x,x')} a\Big(\frac{x+x'}{2}, \xi\Big) f(x') \overline{g(x)} \, \mathrm{d}x' \, \mathrm{d}x \, \mathrm{d}\xi, \tag{3.12}$$

where $f, g \in \mathscr{S}(\mathbb{R}^n)$. To see that this formula is indeed well-defined one uses Green's formulas to introduce $\langle \xi \rangle^{-k}$ in the integral for some large $k \in \mathbb{N}$. This happens at the expense of also introducing polynomially increasing functions in x and x' but since both f and g are Schwartz functions $\mathrm{Op}^A(a)$ is well-defined. Note that when the magnetic field B is 0, then Op^A reduces to the usual Weyl quantization discussed in Subsection 3.1. Note also that taking $a(x,\xi) = |\xi|^2 + V(x)$ for a smooth and bounded function V gives $\mathrm{Op}^A(a) = (-\mathrm{i}\nabla - A)^2 + V = H$, cf. (3.4).

Extending the known results for the usual Weyl quantization of $S^0_{0,0}(\mathbb{R}^n)$ symbols to the magnetic setting is not trivial, since by (3.6) the function $\mathrm{e}^{\mathrm{i}\varphi(x,x')} a((x+x')/2, \xi)$ does not have bounded derivatives of all orders with respect to x and x'. The first efforts to consider the magnetic Weyl quantization for Hörmander symbols goes back to [41] where many of the "non-magnetic" properties of the Weyl quantization is extended to the magnetic setting.

The following paragraph highlights some of the developments in the study of the magnetic Weyl quantization $\mathrm{Op}^A(a)$ for $a \in S^m_{\rho,\delta}(\mathbb{R}^{2n})$. It has been shown [41, Proposition 6.7, Proposition 6.9] that there exist unique functions $a_1, a_2 \in S^m_{\rho,\delta}(\mathbb{R}^{2n})$ such that

$$\mathrm{Op}^A(a) = \mathrm{Op}(a_1^A), \quad \text{and} \quad \mathrm{Op}^A(a_2) = \mathrm{Op}(a^A).$$

Hence, the magnetic Weyl calculus can be obtained from the usual Weyl calculus. In fact, if B is constant and A is linear then $a = a_1 = a_2$ [64, Lemma 3.11]. Furthermore, in the paper [41] a Calderón-Vaillancourt type result was obtained for $a \in S^0_{\rho,\rho}(\mathbb{R}^{2n})$ with $\rho \in [0,1)$, i.e. for such symbols the corresponding operator $\mathrm{Op}^A(a)$ extends to a bounded operator on $L^2(\mathbb{R}^n)$. The Beals criterion [4] has been generalized to the magnetic Weyl quantization [13, 42] for symbols of type $S^0_{0,0}(\mathbb{R}^{2n})$.

Next, spectral results related to $\mathrm{Op}^A(a)$ for $a \in S^m_{\rho,\delta}(\mathbb{R}^{2n})$ are summarized. Recall that such symbols are called *elliptic* if $|a(x,\xi)| \geq C\langle\xi\rangle^m$ for some $C > 0$ and all ξ with $|\xi|$ sufficiently large. In [59], the essential spectrum of $\mathrm{Op}^A(a)$ was characterized for elliptic symbols $a \in S^m_{\rho,0}(\mathbb{R}^{2n})$ with $m > 0$ and $\rho \in [0,1]$ using C^* algebra methods. When considering elliptic symbols in $S^m_{1,0}(\mathbb{R}^{2n})$ with $m > 0$, stability of spectral gaps and spectral islands has been established in [1]. In [17] the results of [67] (cf. the discussion in Subsection 3.3) was extended from the discrete case to $\mathrm{Op}^A(a)$ where $a \in S^m_{1,0}(\mathbb{R}^{2n})$ for $m < 0$. The magnetic Weyl quantization has also played a role in further spectral analysis of magnetic Schrödinger operators [12, 14].

4 Acoustic black holes

This section considers the equation of bending waves in (2.2) and the behavior of its (approximate) solutions in relation to the theory of acoustic black holes. This subject is the focus of Paper C and D in Part II of this dissertation.

Consider a thin plate of variable height h with its midsurface contained in the xy-plane. Furthermore, suppose that the plate is extending infinitely in the y direction and in the positive x direction starting at $x = x_0$, see also Fig. 1 in Paper C. Note that under these assumptions, the plate is essentially one-dimensional. The plate is considered in an Euler-Bernoulli setting, meaning that the cross sections of the plate are assumed to be perpendicular to the midsurface, even when the plate is bending. Hence, if w denotes the displacement of the midsurface, then a bending wave in the plate is described by (2.2), which is recalled here (see also [48, 52, 57]):

$$\frac{\partial^2}{\partial x^2}\left(\frac{\rho c^2 h^3(x)}{12}\frac{\partial^2}{\partial x^2}w(x,\omega)\right) - \omega^2\rho h(x)w(x,\omega) = 0. \tag{4.1}$$

Here ω denotes the angular frequency, ρ denotes the density of the material, and c is the speed of quasi-longitudinal waves in thin plates.

A key objective in mechanical engineering is to dampen or absorb bending waves [18, 51] and different ways of doing so exists in the literature [51, 54, 87]. In this section a method utilizing the so-called *acoustic black hole effect* is discussed. Put loosely, one manipulates the plate (its height and loss factor) in an interval $[x_0, x_1]$ near the edge to achieve less reflection of the bending waves from the edge. In theory it is possible to choose h such that no reflection occurs from the edge [63]. During the last two decades, this method of dampening bending waves in plates has received much attention in the literature [19, 23, 29, 30, 34, 35, 48, 54, 54, 55, 71, 81].

The theory behind the acoustic black hole effect is often based on finding approximate solutions of (4.1) using the WKB approximation [48, 52]. The use of the WKB approximation in the context of the acoustic black hole effect is very well explained in the paper [48]. For completeness, a short summary is given in the following paragraphs. It should be noted that other approaches have been used [23, 34]. The WKB method [6, 37] is in general used to obtain approximate solutions to differential equations of the form

$$\varepsilon \frac{\mathrm{d}^n}{\mathrm{d}x^n} y(x) + a_{n-1}(x) \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} y(x) + \cdots + a_1(x) \frac{\mathrm{d}}{\mathrm{d}x} y(x) + a_0(x) y(x) = 0, \quad (4.2)$$

when ε is small. In the WKB method, one looks for a solution which is of the asymptotic form

$$y(x) \sim A(x) \exp\Big(\delta^{-1} \sum_{n=0}^{\infty} \delta^n S_n(x)\Big), \quad (4.3)$$

when δ goes to 0. This method originates from quantum mechanics where it is used to find solve the eigenvalue problem for one-dimensional Schrödinger operators (cf. Section 3) [37, 49]. In practice the right hand side of (4.3) is inserted in (4.2). Then by the method of dominant balance, the asymptotic behavior of δ with respect to ε is determined, and equations for S_n are found by comparing terms of the same order. Often the series in (4.3) is truncated at some $N \in \mathbb{N}$, and the function obtained is referred to as the N'th order WKB approximation.

When applying the WKB approximation to "solve" (4.1) one uses $\varepsilon = \omega^{-2}$. In the literature concerning the acoustic black hole effect, often only the first order WKB approximation (also called the geometrical acoustics approximation) is used, see e.g. [29, 30, 52, 54, 63, 81]. This approximation is of the form [48]

$$w(x, \omega) = A_0 \Big(\frac{h_0}{h(x)}\Big)^{3/4} \exp\Big(\int_{x_0}^{x} k(s)\,\mathrm{d}s\Big), \quad (4.4)$$

where $h(x_0) = h_0$, $A_0 \in \mathbb{R}$ and k is the wave number defined as a solution to

$$(k(x))^4 = \frac{12\omega^2}{c^2 h^2(x)}. \quad (4.5)$$

In this dissertation the higher order expansions will not be described further as Paper C and D in Part II only considers the first order WKB approximation. The reflection coefficient is then defined as the ratio of amplitudes of the bending wave travelling from x_1 towards the edge at x_0 and the wave which is reflected from the edge. When using the first order WKB approximation, the reflection coefficient is given by [52]

$$R = \exp\Big(-2 \int_{x_0}^{x_1} \operatorname{Im}(k(x))\, \mathrm{d}x \Big). \tag{4.6}$$

The quantity R is used as measure for the effectiveness of an acoustic black hole. Theoretically R can equal 0 but in practice this is not possible, cf. the discussion in Subsection 4.1

Of course, since the expansion in (4.3) is asymptotic one needs to ensure that it is valid (see [6] for a discussion of general conditions). In the theory of the acoustic black hole effect, the condition that

$$\left| \frac{1}{k^2} \frac{\mathrm{d}k}{\mathrm{d}x} \right| \ll 1, \tag{4.7}$$

is generally the only condition imposed to ensure the validity of the approximation in (4.4) [30, 52, 63].

4.1 The acoustic black hole effect

In this subsection, a short review of the history of the acoustic black hole effect is given. The acoustic black hole effect was first demonstrated in the paper [63] by Mironov where it was realized that if $h(x) = \varepsilon x^m$ for $x \in [0, x_1]$, $m \geq 2$, and $\varepsilon > 0$ is sufficiently small, then no bending wave is reflected from the edge at $x_0 = 0$, i.e. the reflection coefficient is 0. This vanishing of the reflected wave means that the incoming wave is simply absorbed. This behavior is the motivation behind the name "acoustic black hole effect". An immediate conclusion is that the lack of reflection from the edge leads to vibration dampening in the plate. However, in the same paper [63] Mironov argued that in a practical setting the edge will always have a certain minimal height, i.e. $h(x_0) > 0$ (see Fig. 1 (b) in Paper C) and further showed that for such truncated edges the reflection coefficient could be too large for practical applications. In the papers [52, 54] it has been proposed to add additional dampening material to the wedge shaped edges of [63]. This increases the loss factor of the material and hence leads to a lower reflection coefficient. Note that this method is easy to apply in practice. Furthermore, the paper [52] also considered other functions h than the power-law profile $h(x) = \varepsilon x^m$ in [63]. Further study of the height profile h was conducted in [81] where an optimal h of the form $h(x) = \varepsilon x^m + h_0$ was sought by means of numerical multiobjective methods. The objectives of the optimization was to keep R low while not violating the condition (4.7).

It is worth mentioning that the acoustic black hole effect has also been investigated in a two-dimensional setting [34, 35, 53].

5 Numbers with random digits

The exposition in this section deviates from Sections 2–4 in the sense that the focus is no longer on subjects motivated by waves. However, the mathematics covered in this section is interesting in its own right.

This section treats real stochastic variables $X \in [0,1]$ where the digits are given by a stochastic process $\{X_n\}_{n\geq 1}$ taking values in $\{0,1,\ldots,q-1\}$ for some integer $q \geq 2$. Hence, X is defined by its *base-q expansion*

$$X = \sum_{n=1}^{\infty} X_n q^{-n}, \tag{5.1}$$

and $F(x) = \mathrm{P}(X \leq x)$ denotes the *cumulative distribution function* (CDF) of X.

In Paper E and F this setting is considered, under the additional assumption that $\{X_n\}_{n\geq 1}$ is *stationary*, i.e. $\{X_n\}_{n\geq 1}$ and $\{X_n\}_{n\geq 2}$ are identically distributed. The motivation behind Paper E and F was to investigate if any relation between F and $\{X_n\}_{n\geq 1}$ could be established and exploited to construct new point processes with interesting properties from known CDFs.

The idea of considering a stochastic variable X as in (5.1) goes back to at least the paper [61] by Borel. In this paper, Borel established the strong law of large numbers by noting that X is uniformly distributed on $[0,1]$ if and only if the digits X_n are independent and uniformly distributed on $\{0,1,\ldots,q-1\}$. In this case, the CDF of X is just the *uniform CDF* on $[0,1]$ given by $F(x) = x$ for $x \in [0,1]$. From the strong law of large numbers Borel proved his famous normal number theorem [8, 61] which states loosely that for (Lebesgue) almost all numbers in $\mathbb{R}$ and any integer $q \geq 2$, every finite string of numbers $x_1,\ldots,x_n \in \{0,1,\ldots,q-1\}$ occurs in the base-q expansion of x with asymptotic frequency q^{-n}.

A natural question is then what happens when $\{X_n\}_{n\geq 1}$ is not a sequence of independent and uniformly distributed random variables on $\{0,1,\ldots,q-1\}$. The simplest generalization is to assume that the X_n's are *independent and identically distributed* (IID) but with a distribution $\pi = (\pi_0,\ldots,\pi_{q-1})$ which is not uniform on $\{0,1,\ldots,q-1\}$. When this is the case it is well known that F is a *singular function*, i.e. $F'(x)$ exists and equals 0 for almost all $x \in [0,1]$ [8]. Note that since F is non-decreasing, it is a classic result of Lebesgue that F is differentiable almost everywhere [58]. The term "singular" is used to describe such functions as the corresponding measure $\mathrm{d}F$ (uniquely determined by $\mathrm{d}F((a,b]) = F(b) - F(a)$ for any real numbers $a < b$) is singular with respect to the Lebesgue measure. The first paper to loosen the assumption of independence seems to be [38]. In this paper, Harris shows that when $\{X_n\}_{n\geq 1}$ is stationary and satisfies a mixing condition then F is one of three distinct types: 1) either F is the uniform CDF on $[0,1]$. 2) F is a discrete CDF (i.e. step function) with a unit step at $\frac{k}{q-1}$ for some $k \in \{0,1,\ldots,q-1\}$. 3) F is singular continuous. A similar result is obtained by Dym [28] under the assumption that $\{X_n\}_{n\geq 1}$ is stationary and ergodic. Under these assumptions, the only difference from Harris's result is that when F is a discrete CDF, it can have m jumps of height $1/m$. The jumps occurs at the "purely repeating base-q fractions" described in detail in Paper E.

For the sake of illustration, consider F in the IID case with a distribution $\pi = (\pi_0, \ldots, \pi_{q-1})$ where $\pi_k < 1$ for all $k \in \{0, 1, \ldots, q-1\}$. Let $\{X_n\}_{n \geq 1}$ be a stochastic process of IID random variables such that $\mathrm{P}(X_n = k) = \pi_k$. Since no π_k equals 1, it follows that $\mathrm{P}(X_1 = x_1, X_2 = x_2, \ldots) = 0$ for all $\{x_n\}_{n \geq 1} \subset \{0, 1, \ldots, q-1\}^{\mathbb{N}}$. Hence $\mathrm{P}(X = x) = 0$ for all $x \in [0, 1]$, and thus it does not matter which representation of x is chosen in the case that x has two different base-q representations. The numbers $x \in [0, 1]$ which have a non-unique base-q expansion are referred to as *base-q fractions*.

In the following, the notation $x = (0.x_1 \ldots x_n)_q := \sum_{k=1}^{n} x_k q^{-k}$ will be used. A straightforward calculation using (2.2) in Paper F and the independence of the X_n's shows that for any $x = (0.x_1 \ldots x_n)_q$, where $x_k \in \{0, 1, \ldots, q-1\}$, and any $j \in \{0, 1, \ldots, q-1\}$, we have

$$F((0.x_1 \ldots x_n j)_q) = F(x) + (F(x + 1/q^n) - F(x)) \sum_{k=0}^{j-1} \mathrm{P}(X_1 = k), \tag{5.2}$$

where the sum is defined as 0 when $j = 0$. This formula implies that F is obtained from a simple generalization of the geometric construction appearing in [78, 80] which is presented in the following. First, let F_0 be the uniform CDF on $[0, 1]$. Then, define $F_1(0) = F_0(0)$, $F_1(1) = F_0(1)$ and $F_1(j/q) = \sum_{k=0}^{j-1} \mathrm{P}(X_1 = k)$ for $j \in \{1, 2, \ldots, q-1\}$. Between these points F_1 is defined by linear interpolation. Next, continuing in the same way, define $F_2(j/q) = F_1(j/q)$ for all $j \in \{0, 1, \ldots, q-1\}$ and set

$$F_2((0.x_1 j)_q) = F_1((0.x_1)_q) + [F_1((0.x_1)_q + 1/q) - F_1((0.x_1)_q)] \sum_{k=0}^{j-1} \mathrm{P}(X_1 = k),$$

for all $x_1, j \in \{0, 1, \ldots, q-1\}$. For all other points in $[0, 1]$, F_2 is defined by linear interpolation. Continuing this recursive construction gives a sequence F_n of continuous non-decreasing functions, which converge uniformly to some continuous function F_∞. See Fig. 1 below for an illustration of this construction. By construction $F_\infty(x) = F(x)$ on a dense set and hence by (right) continuity $F_\infty = F$. As previously noted this construction can not be considered novel. However, it is still worth mentioning as recent papers have presented special cases of this construction without realizing the simple probabilistic nature behind it [56, 68, 84].

5.1 Singular functions

Singular functions are by no means hard to come by. For example [89] showed that "most" monotone functions are singular. In this context the word "most" means that all continuous monotone functions on $[0, 1]$ are singular, except

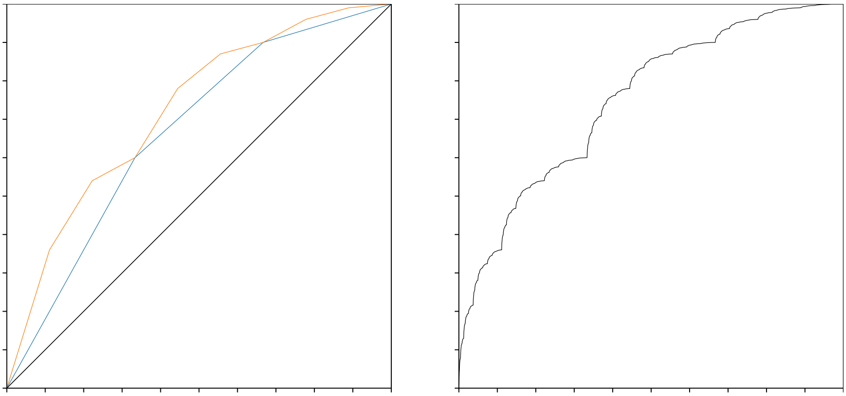

Fig. 1: Left: From bottom to top: $F_0, \dots, F_2$ for $q = 3$ and $\pi = (0.6, 0.3, 0.1)$ Right: F_{10} for the same parameters.

for a set of the first Baire category. In a similar Baire category sense Simon's "Wonderland theorem" proved that most Schrödinger operators (cf. Section 3) have singular continuous spectra [82]. For any singular continuous measure μ the function $x \mapsto \mu((-\infty, x])$ is a singular continuous function [79]. Hence, the construction of singular functions are in principle easy, but obtaining explicit formulas for such functions is often harder.

The first example of a singular continuous non-decreasing function goes back to the well-known *Cantor function* [10]. The Cantor function can be obtained from the construction in Section 5 by choosing $q = 3$ and $\pi_0 = \pi_2 = \frac{1}{2}$ [8, 26]. In [62], Minkowski defined his *question mark function* denoted $?(x)$ (see Paper E for a definition) which was proven to be singular by Denjoy [24, 25]. In the paper [80] Salem gave a elegant geometric construction of a class of strictly increasing singular functions. This is the construction presented in Section 5 when $q = 2$. In the same paper, Salem showed that unless F is the uniform CDF on $[0, 1]$, the Fourier transform of $\mathrm{d}F$ does not go to 0 at infinity, i.e. $\mathrm{d}F$ is not a so-called Rajchmann measure [60]. In the textbook [78] by Riesz and Nágy the construction of Salem appeared again, which has given the resulting functions the name "Riesz-Nágy functions" in the literature [69]. In fact these functions have been treated multiple times in the literature [7, 8, 77, 86].

The above mentioned singular functions are the most well-known and studied in the literature. Interestingly, these functions can also be described as solutions to various functional equations. The paper [46] gives an excellent exposition of this subject. For example in [46] it is shown that the Cantor function is the unique bounded function on $[0, 1]$ which solves the functional equations

$$f(x/3) = \frac{1}{2} f(x), \quad f((x+1)/3) = 1/2, \quad f((x+2)/3) = \frac{1}{2} f(x) + \frac{1}{2}.$$

Likewise it is shown in [22, 46] that each Riesz-Nágy function solves

$$f(x/2) = \alpha f(x), \quad f((x+1)/2) = (1-\alpha)f(x) + \alpha$$

for some $\alpha \in (0,1)$.

More recently numerous constructions of singular functions and generalizations of the previously discussed functions have appeared in the literature [21, 56, 68–70, 83–85, 88].

6 Overview

This section briefly discusses the papers included in Part II of this dissertation in relation to the background presented in the previous sections. The notation is kept consistent with the notation used in Section 2–5 and not necessarily the papers in Part II.

6.1 Paper A

In Paper A, a variation of the magnetic Weyl quantization (cf. Subsection 3.4) is considered with a magnetic field of class $BC^\infty(\mathbb{R}^d;\mathbb{R}^d)$. The transverse gauge (3.3) is used to define the magnetic potential A and the magnetic field strength is represented by the parameter $b \in \mathbb{R}$. The aforementioned variation consists of writing the magnetic field strength b explicitly in front of φ and then replacing $e^{ib\varphi(x,x')}a((x+x')/2,\xi)$ in (3.12) with a so-called *magnetic symbol* $a_b(x,x',\xi)$. A magnetic symbol is defined as a function $a_b(x,x',\xi) = e^{ib\varphi(x,x')}a(x,x',\xi)$ where $a \in C^\infty(\mathbb{R}^{3n})$ such that there exists a constant M_a with the property that for all $\alpha,\alpha',\beta \in \mathbb{N}_0^n$:

$$\left|\partial_x^\alpha \partial_{x'}^{\alpha'} \partial_\xi^\beta a(x,x',\xi)\right| \le C_{\alpha,\alpha',\beta}\langle x-x'\rangle^{M_a}.$$

The set of all such symbols is denoted by $M_\varphi(\mathbb{R}^{3n})$. Note that since the phase factor $e^{ib\varphi(x,x')}$ is included in the symbol a_b, the growth in $x-x'$ on the right hand side above does not complicate the calculations much.

To summarize, Paper A concerns operators defined weakly by

$$\langle \mathrm{Op}(a_b)f,g\rangle := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{3n}} e^{i\xi(x-x')}a_b(x,x',\xi)f(x')\overline{g(x)}\,dx'\,dx\,d\xi, \qquad (6.1)$$

where $f,g \in \mathscr{S}(\mathbb{R}^n)$ and $a_b \in M_\varphi(\mathbb{R}^{3n})$. A priori (6.1) might seem like a generalization of (3.12) for $a \in S^0_{0,0}(\mathbb{R}^{2n})$. However this is not the case, since an application of the magnetic Beals criterion [13, 42] shows that for every symbol in $M_\varphi(\mathbb{R}^{3n})$ there is a corresponding symbol $\tilde{a} \in S^0_{0,0}(\mathbb{R}^{2n})$ such that $\mathrm{Op}(a_b) = \mathrm{Op}(\tilde{a})$ (cf. Remark 2.3 in Paper A).

The main result of Paper A can be divided into thee parts. Firstly, by discretizing $L^2(\mathbb{R}^n)$ into to the space $\mathscr{H} := \bigoplus_{\gamma\in\mathbb{Z}^n} L^2(\Omega)$, where $\Omega = (-\frac{1}{2}, \frac{1}{2})^n$, it is established that $\mathrm{Op}(a_b)$ in (6.1) is unitary equivalent to an infinite operator valued matrix of the form

$$\left\{ \mathrm{e}^{ib\varphi(x,x')} \mathcal{A}_{\gamma,\gamma',b} \right\}_{\gamma,\gamma'\in\mathbb{Z}^n} \in B(\mathscr{H}), \tag{6.2}$$

where $\mathcal{A}_{\gamma,\gamma',b} \in B(L^2(\Omega))$ and $\|A_{\gamma,\gamma',b}\|$ decay faster than any inverse polynomial in $\langle \gamma - \gamma' \rangle$. Essentially this result calculates the operator kernel of $\mathrm{Op}(a_b)$, see also [36]. Then, an obvious modification to Schur's test [40] immediately gives a Calderón-Vaillancourt type result, namely that $\mathrm{Op}(a_b) \in B(L^2(\mathbb{R}^n))$. This is not a novel result for the magnetic Weyl quantization [41]. However, the formula (6.2) is crucial when extending some of the known spectral results for Harper operators and magnetic Schrödinger operators listed in Subsection 3.3. For this, an important point is that the operator in (6.2) is a generalization of (3.7). In light of this it should maybe not be surprising that some of the results detailed in Subsection 3.3 generalizes to the setting discussed here. Indeed, the second result of Paper A establishes that when $\mathrm{Op}(a_b)$ is self-adjoint, i.e. when $a(x, x', \xi) = \overline{a(x', x, \xi)}$, then $\sigma(\mathrm{Op}(a_b))$ is $\frac{1}{2}$-Hölder continuous with respect to b in the Hausdorff distance. Due to the similarity between (6.2) and (3.7) this is shown by adapting the methods used in [16]. The third and final main result of Paper A establishes Lipschitz continuity of spectral gap edges, when $\mathrm{Op}(a_b)$ is self-adjoint and the magnetic field is constant. This is also an extension of well-known results for Harper operators and magnetic Schrödinger operators, as discussed in Subsection 3.3. As in [11], an important technique is to use the heat kernel to rewrite the phase factor $\mathrm{e}^{ib\varphi(x,x')}$.

6.2 Paper B

Paper B concerns two related magnetic Schrödinger operators. The first operator is of the form in (3.5), i.e. $H_b = (-\mathrm{i}\nabla - bA - \mathcal{A})^2 + V$, acting on $L^2(\mathbb{R}^2)$. Here A is a magnetic potential given in the Landau gauge, i.e. $A(x) = (-x_2, 0)$, $\mathcal{A} \in BC^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is some other magnetic potential, and $V \in BC^\infty(\mathbb{R}^2; \mathbb{R})$ is a scalar potential. The parameter b represents the strength of the constant magnetic field $b\mathrm{d}A$. The second operator is the Dirichlet realization of H_b in the half-space $E = \{x \in \mathbb{R}^2 \mid x_2 > 0\}$ and is denoted by H_b^E. This setting is motivated by the study of the bulk-edge correspondence in the field of topological insulators, and in particular the recent paper [15] giving a very general bulk-edge correspondence. For this reason H_b is referred to as the *bulk Hamiltonian* and H_b^E is referred to as the *edge Hamiltonian*. The aim of Paper B is to extend certain regularity properties of the edge Hamiltonian H_b^E established in [15]. In particular, it is shown in [15] that if $F\colon \sigma(H_b^E) \to \mathbb{R}$ is a function which can be extended to a function in $\mathscr{S}(\mathbb{R})$, then for any

$g_1, g_2 \in C_c^\infty(E)$ the operators $g_1 F(H_b^E) g_2$ and $g_1 \mathrm{i}[H_b^E, X_j] F(H_b^E) g_2$ have continuous integral kernels. In Paper B we establish that these kernels are not merely continuous but in fact smooth. For the bulk Hamiltonian H_b this can be done using the Beals criterion for the magnetic Weyl quantization [15], but for the edge Hamiltonian this type of argument no longer works. From this regularity result, it is then established that the bulk and edge particle current densities $\mathcal{J}_{1,b}$ and $\mathcal{J}_{1,b}^E$ defined as

$$\mathcal{J}_{1,b}{}^{/E}(x_1, x_2) := \left(\mathrm{i}[H_b{}^{/E}, X_1] F(H_b{}^{/E})\right)((x_1, x_2), (x_1, x_2))$$

are smooth and that

$$\sup_{x_1 \in \mathbb{R}} \left| \mathcal{J}_{1,b}^E(x_1, x_2) - \mathcal{J}_{1,b}(x_1, x_2) \right| = \mathcal{O}(x_2^{-\infty}) \text{ for } x_2 \to +\infty.$$

The basic idea for obtaining these results is to first define $\mathfrak{H}_b^E := (-\mathrm{i}\nabla - bA)^2$ on the set of functions $f \in C^\infty(E)$ for which all partial derivatives extend continuously to ∂E and which satisfy $f(x) = 0$ for $x \in \partial E$. Following the magnetic perturbation theory, a norm convergent resolvent expansion is established for $h_b^E := \overline{\mathfrak{H}_b^E}$,

$$(h_b^E + \lambda)^{-1} = S_b(-\lambda) \sum_{n=0}^{\infty} (-T_b(-\lambda))^n, \tag{6.3}$$

whenever $\lambda > 0$ is sufficiently large (compare with (3.9)). Here $S_b(-\lambda)$ is an integral operator with kernel

$$S_b(-\lambda)(x, y) = \mathrm{e}^{\mathrm{i}b\varphi(x,y)}[(-\Delta + \lambda)^{-1}(x, y) - (-\Delta + \lambda)^{-1}(x, y^*)],$$

where $(-\Delta + \lambda)^{-1}(x, y)$ is the integral kernel of the resolvent for the usual Laplacian defined on $L^2(\mathbb{R}^2)$ and $y^* = (y_1, -y_2)$. Furthermore, $T_b(-\lambda) = (h_b^E + \lambda) S_b(-\lambda) - I$ is also an integral operator with a kernel that can be determined explicitly. Then, using the Kato-Rellich theorem, H_b^E is defined as a relatively bounded perturbation of h_b^E. This definition is shown to give the same operator as the usual definition using the Friedrichs extension.

The resolvent expansion (6.3) and the properties of the kernels of $S_b(-\lambda)$ and $T_b(-\lambda)$ are crucial for results obtained in Paper B. Combining these with the second resolvent identity it is possible to show that for b in some compact set $\Omega \subset \mathbb{R}$ and $\lambda > 0$ sufficiently large we have that

$$(H_b^E + \lambda)^{-1}(x, y) = \mathrm{e}^{\mathrm{i}b\varphi(x,y)} K_b(-\lambda)(x, y),$$

where the operator corresponding to the kernel $K_b(-\lambda)(x, y)$ is smooth with respect to $b \in \Omega$ in the operator norm topology. Note that this is similar to (3.8) but for the edge Hamiltonian. A central technical result, which is used to establish the smoothness of $g_1 F(H_b^E) g_2$ and $g_1 \mathrm{i}[H_b^E, X_j] F(H_b^E) g_2$, is that when λ is sufficiently large, then for all $m \in \mathbb{N}$ one can find $N \in \mathbb{N}$ and $\varepsilon > 0$ such that $(\mathrm{i}[H_b^E, X_j])^k (H_b^E + \lambda)^{-N} \mathrm{e}^{\varepsilon|X|}$ map $L^2(E)$ continuously into $C^m(E)$ for $j \in \{1, 2\}$ and $k \in \{0, 1\}$. This result is established from (6.3) and a detailed analysis of the integral kernels of $S_b(-\lambda)$ and $T_b(-\lambda)$.

6.3 Paper C and D

Paper C and D both concern the theory of acoustic black holes, cf. Section 4. Paper D is a conference paper made to present our results prior to the creation of the full journal paper, Paper C. As such the two papers present essentially the same results, but both are included in this dissertation for completeness.

The main idea behind these papers is inspired by the paper [81] where a plate with a height profile of the form $h(x) = \varepsilon x^m + h_0$ is considered in an interval $[x_0, x_1]$. Numeric optimization methods are used determine "optimal" values for the power m, length $x_1 - x_0$, initial height $h(x_0)$, and final height $h(x_1)$. In this paper optimality was defined as minimizing the reflection coefficient R in (4.6) subject to the constraint that $|k'/k^2| < 0.4$, cf. (4.7). The specific number 0.4 was established numerically in [30] as a reasonable upper bound for the validity condition (4.7). However in the literature other functions h have been considered as the height profile, e.g. $h(x) = \varepsilon \sin^m(x)$ [52] which is not much different from εx^m for small x. Yet, this choice of h leads to closed form expressions for R. A natural question is then if it is possible to minimize R subject to the constraint (4.7) over all C^1 functions. Of course some suitable interpretation of the constraint (4.7) is required. This vaguely formulated optimization problem is what Paper C and D try to formulate precisely and then solve. Note also that by (4.5) it is clear that (4.7) is not satisfied for all frequencies ω. The starting point for Paper C and D (as well as the other papers cited in this paragraph) is to use the first order WKB approximation outlined in Section 4. Thus the word "optimal" should be interpreted in this context and the work should be seen as a step towards maturing the theory of the acoustic black hole effect and pushing it towards more advanced models. In this regard an obvious extension of the theory would be to consider more general bending wave equations (e.g. Timoshenko theory [27]).

The basic assumption of Paper C and D is to consider a plate as described in Section 4 with a height profile h which varies in an interval $[x_0, x_1]$ and satisfies the boundary conditions $h(x_0) = h_0 > 0$ and $h(x_1) = h_1 > h_0$. A key point of our approach is the observation that the integral in (4.6) can be considered as a functional of h. Then, by approximating the pointwise bound in (4.7) by an L^p norm for some large p we arrive at a variational problem where R is minimized with respect to h subject to the boundary conditions above and the additional constraint

$$\int_{x_0}^{x_1} \left| \frac{1}{k^2} \frac{\mathrm{d}k}{\mathrm{d}x} \right|^p \mathrm{d}x = L, \tag{6.4}$$

where L is some number which is chosen small. This problem is similar to the well-known isoperimetric problem [33] and can be solved by the method of Lagrange multipliers. Solving this problem shows that if the additional constraint (6.4) is not taken into account, then h is given by a quadratic polynomial, generalizing the much studied choice $h(x) = \varepsilon x^2$. Furthermore, it is shown that when the constraint (6.4) is taken into account the optimal profile is of the form

$$h(x) = \begin{cases} h_0, & x \in [x_0, \tilde{x}], \\ \tilde{h}(x), & x \in (\tilde{x}, x_1], \end{cases}$$

where $\tilde{h}$ is a quadratic polynomial satisfying $\tilde{h}(\tilde{x}) = h_0$ and $\tilde{x}$ is some number in $[x_0, x_1)$.

6.4 Paper E and F

The aim of Paper E is to completely characterize stationarity of the digits $\{X_n\}_{n\geq 1}$ in terms of properties of the CDF F (cf. Section 5). Three different equivalences are given. All three generalize some known results in the literature.

A basic result is that when assuming stationarity of $\{X_n\}_{n\geq 1}$, then the ambiguity of the base-q expansion plays no role as $P(X = x) = 0$ for all base-q fractions $x \in (0, 1)$. From this the most fundamental result of the paper is obtained, namely that stationarity of $\{X_n\}_{n\geq 1}$ is equivalent to F being a solution to the functional equation

$$F(x) = F(0) + \sum_{j=0}^{q-1} \left[F\left(\frac{x+j}{q}\right) - F\left(\frac{j}{q}\right) \right], \quad x \in [0, 1]. \tag{6.5}$$

This functional equation immediately gives an equation for the measure dF, which is used to show the second characterization of stationarity of $\{X_n\}_{n\geq 1}$. Specifically, it is established that $\{X_n\}_{n\geq 1}$ is stationary if and only if the characteristic function f of X (i.e. the Fourier transform of the measure dF) satisfies $f(2\pi k q) = f(2\pi k)$ for all $k \in \mathbb{Z}$. Interestingly, this implies that $\lim_{t\to\infty} f(t) = 0$ if and only if F is the uniform CDF on $[0, 1]$. Note that this is an extension of the results obtained in [80] for the Riesz-Nágy functions. Interestingly, as the Fourier transform of d? goes to 0 at infinity [45, 73], it follows that ? is not the CDF of a random variable X given by (5.1) with stationary digits $\{X_n\}_{n\geq 1}$. The functional equation in (6.5) also leads directly to a generalization of the decomposition results of Harris and Dym, [28, 38]. Specifically it is shown that the Lebesgue decomposition of F is a mixture (convex combination) of the uniform CDF on $[0,1]$, step functions with $m \in \mathbb{N}$ jumps of height m^{-1} and a singular continuous CDF satisfying (6.5). Note that under the assumptions in [28, 38] F could only be either one of the aforementioned cases. The only case not completely described in Paper E is when F is singular continuous and satisfies (6.5).

In an effort to gain an understanding of the singular continuous part of F, Paper F considers specific stationary stochastic processes $\{X_n\}_{n\geq 1}$. In particular, stationary Markov chains of arbitrary orders, stationary renewal processes, and mixtures of such models are considered. The well-known case when the X_n's are IID (cf. Section 5) is considered as a 0'th order Markov chain for completeness. When $\{X_n\}_{n\geq 1}$ is a stationary Markov chain, it is shown that F is either the uniform CDF on $[0,1]$ or a singular (not necessarily continuous) function. Borrowing the terminology of the field of Bernoulli convolutions we say that F is of *pure type* [44, 72]. The same law of pure type is established when $\{X_n\}_{n\geq 1}$ corresponds to a stationary renewal process. In fact, more is shown in the case when F is not the uniform CDF on $[0,1]$, namely that $F'(x) = 0$ for all $x \in [0,1]$ where the derivative exists. This is stronger than F being merely singular since there exist examples of singular functions with non-zero derivatives [83–85]. A key ingredient in the proof of these "pure type" results is the observation that if $x = (0.x_1x_2\dots)_q \in [0,1]$ is a non-base-q fraction where $F'(x)$ exists, then for any $m \in \mathbb{N}$ and $\xi_1,\dots,\xi_m \in \{0,1,\dots,q-1\}$,

$$F'(x) = \lim_{n\to\infty} q^{n+m}\mathrm{P}(X_1 = x_1,\dots,X_n = x_n, X_{n+1} = \xi_1,\dots,X_{n+m} = \xi_m). \tag{6.6}$$

When considering mixtures of either stationary Markov chains or stationary renewal processes, it is assumed that Π is a random variable that determines the distribution of $\{X_n\}_{n\geq 1}$. Thus the conditional CDF is given by $F_\Pi(x) = \mathrm{P}(X \leq x \mid \Pi)$ and the CDF of X is then obtained by averaging, i.e. $F = \mathrm{E}F_\Pi$. It is then shown that for almost all $x \in [0,1]$, $F'(x) = \mathrm{P}(F_\Pi = F_1)$ where F_1 denotes the uniform CDF on $[0,1]$. The proof of this requires that x is a normal number and detailed analysis of the probabilities occurring on the right hand side of (6.6) using specific formulas for Markov chains and renewal processes. Note that in this case F is of pure type if and only if $\mathrm{P}(F_\Pi = F_1) = 0$.

References

[1] N. Athmouni, M. Măntoiu, and R. Purice, "On the continuity of spectra for families of magnetic pseudodifferential operators," *J. Math. Phys.*, vol. 51, no. 8, p. 083517, 2010.

[2] J. Avron, I. Herbst, and B. Simon, "Schrödinger operators with magnetic fields. I. general interactions," *Duke Math. J.*, vol. 45, no. 4, pp. 847 – 883, 1978.

[3] J. Avron and B. Simon, "Stability of gaps for periodic potentials under variation of a magnetic field," *J. Phys. A: Math. Gen.*, vol. 18, pp. 2199–2205, 1985.

[4] R. Beals, "Characterization of pseudodifferential operators and applications," *Duke Math. J.*, vol. 44, no. 1, pp. 45 – 57, 1977.

[5] J. Bellissard, "Lipshitz continuity of gap boundaries for Hofstadter-like spectra," *Commun. Math. Phys.*, vol. 160, no. 3, pp. 599 – 613, 1994.

[6] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill, 1978.

[7] P. Billingsley, *Ergodic Theory and Information*. Wiley, 1965.

[8] ——, *Probability and Measure*, ser. Wiley Series in Probability and Statistics. Wiley, 1995.

[9] P. Briet and H. D. Cornean, "Locating the spectrum for magnetic Schrödinger and Dirac operators," *Commun. Partial Differ. Equ.*, vol. 27, no. 5-6, pp. 1079–1101, 2002.

[10] G. Cantor, "De la puissance des ensembles parfaits de points," *Acta Math.*, vol. 4, pp. 381–392, 1884.

[11] H. D. Cornean, "On the Lipschitz continuity of spectral bands of Harper-like and magnetic Schrödinger operators," *Ann. Henri Poincaré*, vol. 11, no. 5, pp. 973–990, 2010.

[12] H. D. Cornean, B. Helffer, and R. Purice, "Low lying spectral gaps induced by slowly varying magnetic fields," *J. Funct. Anal.*, vol. 273, no. 1, pp. 206–282, 2017.

[13] ——, "A Beals criterion for magnetic pseudo-differential operators proved with magnetic Gabor frames," *Commun. Partial Differ. Equ.*, vol. 43, no. 8, pp. 1196–1204, 2018.

[14] ——, "Peierls' substitution for low lying spectral energy windows," *J. Spectr. Theory*, vol. 9, no. 4, pp. 1179–1222, 2019.

[15] H. D. Cornean, M. Moscolari, and S. Teufel, "General bulk-edge correspondence at positive temperature," 2021, [arXiv preprint arXiv:2107.13456].

[16] H. D. Cornean and R. Purice, "On the regularity of the Hausdorff distance between spectra of perturbed magnetic Hamiltonians," in *Spectral Analysis of Quantum Hamiltonians*, R. Benguria, E. Friedman, and M. Mantoiu, Eds., vol. 244. Basel: Springer Basel, 2012, pp. 55–66.

[17] ——, "Spectral edge regularity of magnetic Hamiltonians," *J. London Math. Soc.*, vol. 92, no. 1, pp. 89–104, 06 2015.

[18] L. Cremer, M. Heckl, and B. Petersson, *Structure-Borne Sound: Structural Vibrations and Sound Radiation at Audio Frequencies*. Springer Berlin Heidelberg, 2005.

[19] J. Cuenca, "Wave models for the flexural vibrations of thin plates - model of the vibrations of polygonal plates by the image source method - vibration damping using the acoustic black hole effect," Ph.D. dissertation, Université du Maine, 2009.

[20] H. Cycon, B. Simon, R. Froese, and W. Kirsch, *Schrödinger Operators: With Applications to Quantum Mechanics and Global Geometry*, ser. Springer study edition. Springer, 1987.

[21] E. de Amo, M. D. Carrillo, and J. Fernández-Sánchez, "Singular functions with applications to fractal dimensions and generalized Takagi functions," *Acta Appl. Math.*, vol. 119, pp. 129–148, 2012.

[22] G. de Rham, "Sur quelques courbes définies par des équations fonctionnelles," *Rend. Sere. Math. Torino*, vol. 16, pp. 101–113, 1956.

[23] V. Denis, F. Gautier, A. Pelat, and J. Poittevin, "Measurement and modelling of the reflection coefficient of an acoustic black hole termination," *J. Sound Vib.*, vol. 349, pp. 67 – 79, 2015.

[24] A. Denjoy, "Sur quelques points de la théorie des fonctions," *C. R. Acad. Sci. Paris Sér. I Math.*, vol. 194, pp. 44–46, 1932.

[25] ——, "Sur une fonction de Minkowski," *C. R. Acad. Sci. Paris Sér. I Math.*, vol. 198, pp. 44–47, 1934.

[26] O. Dovgoshey, O. Martio, V. Ryazanov, and M. Vuorinen, "The Cantor function," *Expo. Math.*, vol. 24, pp. 1–37, 2006.

[27] C. L. Dym and I. H. Shames, *Solid Mechanics : A Variational Approach, Augmented Edition*. New York, NY: Springer New York, 2013.

[28] H. Dym, "On a class of monotone functions generated by ergodic sequences," *Am. Math. Mon.*, vol. 75, pp. 594–601, 1968.

[29] P. A. Feurtado, S. C. Conlon, and F. Semperlotti, "A normalized wave number variation parameter for acoustic black hole design," *J. Acoust. Soc. Am.*, vol. 136, no. 2, pp. EL148–EL152, 2014.

[30] P. A. Feurtado and S. C. Conlon, "Investigation of boundary-taper reflection for acoustic black hole design," *J. Noise Cont. Eng.*, vol. 63, no. 5, pp. 460–466, 2015.

[31] G. B. Folland, *Harmonic Analysis in Phase Space. (AM-122)*. Princeton University Press, 1989.

[32] S. Fournais and B. Helffer, *Spectral Methods in Surface Superconductivity*, ser. Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Verlag, 2009.

[33] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, ser. Dover Books on Mathematics. Mineola, N.Y: Dover Publications, 2012.

[34] V. B. Georgiev, J. Cuenca, F. Gautier, L. Simon, and V. V. Krylov, "Damping of structural vibrations in beams and elliptical plates using the acoustic black hole effect," *J. Sound Vib.*, vol. 330, no. 11, pp. 2497 – 2508, 2011.

[35] V. B. Georgiev, J. Cuenca, M. A. Moleron Bermudez, F. Gautier, and L. Simon, "Recent progress in vibration reduction using Acoustic Black Hole effect," in *10ème Congrès Français d'Acoustique*, Lyon, France, Apr 2010, pp. 1–6.

[36] F. Germinet and A. Klein, "Operator kernel estimates for functions of generalized Schrödinger operators," *Proc. Am. Math. Soc*, vol. 131, no. 3, pp. 911–920, 2003.

[37] B. C. Hall, *Quantum Theory for Mathematicians*, ser. Graduate Texts in Mathematics. Springer New York, 2013, no. 267.

[38] T. E. Harris, "On chains of infinite order," *Pacific J. Math.*, vol. 5, pp. 707–724, 1955.

[39] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, ser. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1990, vol. 1.

[40] ——, *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*, ser. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 1994, vol. 3.

[41] V. Iftimie, M. Măntoiu, and R. Purice, "Magnetic pseudodifferential operators," *Publ. RIMS.*, vol. 43, no. 3, pp. 585–623, 2007.

[42] ——, "Commutator criteria for magnetic pseudodifferential operators," *Commun. Partial Differ. Equ.*, vol. 35, no. 6, pp. 1058–1094, 2010.

[43] T. Ikebe and T. Kato, "Uniqueness of the self-adjoint extension of singular elliptic differential operators," *Arch. Ration. Mech. Anal.*, vol. 9, no. 1, pp. 77–92, 1962.

[44] B. Jessen and A. Wintner, "Distribution functions and the Riemann zeta function," *Trans. Am. Math. Soc.*, vol. 38, pp. 48–88, 1935.

[45] T. Jordan and T. Sahlsten, "Fourier transforms of Gibbs measures for the Gauss map." *Math. Ann.*, vol. 364, pp. 983–1023, 2016.

[46] H. Kairies, "Functional equations for peculiar functions," *Aequ. Math.*, vol. 53, pp. 207–241, 1997.

[47] M. V. Karasev and T. A. Osborn, "Symplectic areas, quantization, and dynamics in electromagnetic fields," *J. Math. Phys.*, vol. 43, no. 2, pp. 756–788, 2002.

[48] A. Karlos, S. J. Elliott, and J. Cheer, "Higher-order WKB analysis of reflection from tapered elastic wedges," *J. Sound Vib.*, vol. 449, pp. 368 – 388, 2019.

[49] B. M. Karnakov and V. P. Krainov, *WKB Approximation in Atomic Physics*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013.

[50] T. Kato, *Perturbation Theory for Linear Operators*, ser. Classics in Mathematics. Springer Berlin Heidelberg, 1995.

[51] E. M. Kerwin, "Damping of Flexural Waves by a Constrained Viscoelastic Layer," *J. Acoust. Soc. Am.*, vol. 31, pp. 952–962, 1959.

[52] V. V. Krylov and F. J. B. S. Tilman, "Acoustic 'black holes' for flexural waves as effective vibration dampers," *J. Sound Vib.*, vol. 274, no. 3, pp. 605 – 619, 2004.

[53] V. V. Krylov and R. E. T. B. Winward, "Experimental investigation of the acoustic black hole effect for flexural waves in tapered plates," *J. Sound Vib.*, vol. 300, no. 1, pp. 43 – 49, 2007.

[54] V. V. Krylov, "New type of vibration dampers utilising the effect of acoustic 'black holes'," *Acta Acust. united Acust.*, vol. 90, pp. 830–837, 08 2004.

[55] ——, "Acoustic black holes: Recent developments in the theory and applications," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control*, vol. 61, no. 8, pp. 1296–1306, 2014.

[56] J. Kyeonghee, "A construction of strictly increasing continuous singular functions," *J. Korean Soc. Math. Educ. Ser. B*, vol. 23, pp. 21–34, 2016.

[57] L. Landau and E. Lifshitz, *Theory of Elasticity*, ser. Course of Theoretical Physics Series. Pergamon Press, 1970.

[58] H. Lebesgue, *Leçons sur l'intégration et la recherche des fonctions primitives: professées au Collège de France*, ser. Collection de monographies sur la théorie des fonctions. Gauthier-Villars, 1904.

[59] M. Lein, M. Măntoiu, and S. Richard, “Magnetic pseudodifferential operators with coefficients in C^*-algebras,” *Publ. RIMS.*, vol. 46, no. 4, pp. 755–788, 2010.

[60] R. Lyons, “Seventy years of Rajchman measures. Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993),” *J. Fourier Anal. Appl.*, pp. 363–377, 1995.

[61] M. Émile Borel, “Les probabilités dénombrables et leurs applications arithmétiques,” *Rend. Circ. Matem. Palermo*, vol. 27, pp. 247–271, 1909.

[62] H. Minkowski, *Zur Geometrie der Zahlen, Verhandlungen des III. Internationalen Mathematiker-Kongresses in Heidelberg, 1904, pp. 164–173. (Gesammelte Abhandlungen von Hermann Minkowski. Bd. II. B. G. Teubner, Leipzig, 1911, pp. 43–52).* Reprinted by Chelsea, New York, 1967, 1904.

[63] M. A. Mironov, “Propagation of a flexural wave in a plate whose thickness decreases smoothly to zero in a finite interval,” *Sov. Phys. Acoust.*, vol. 34, pp. 318–319, 1988.

[64] M. Măntoiu and R. Purice, “The magnetic Weyl calculus,” *J. Math. Phys.*, vol. 45, no. 4, pp. 1394–1417, 2004.

[65] G. Nenciu, “Stability of energy gaps under variations of the magnetic field,” *Lett. Math. Phys.*, vol. 11, no. 2, pp. 127–132, 1986.

[66] ——, “On asymptotic perturbation theory for quantum mechanics: Almost invariant subspaces and gauge invariant magnetic perturbation theory,” *J. Math. Phys.*, vol. 43, no. 3, pp. 1273–1298, 2002.

[67] ——, “On the smoothness of gap boundaries for generalized Harper operators,” in *Advances in operator algebras and mathematical physics*, ser. Theta Ser. Adv. Math., 5,. Bucharest: Theta, 2005, vol. 2, pp. 173–182.

[68] H. Okamoto and M. Wunsch, “A geometric construction of continuous, strictly increasing singular functions,” *Proc. Jpn. Acad. A: Math. Sci.*, vol. 83, no. 7, pp. 114–118, 07 2007.

[69] J. Paradís, P. Viader, and L. Bibiloni, “Riesz-Nágy singular functions revisited,” *J. Math. Anal. Appl.*, vol. 329, pp. 592–602, 2007.

[70] ——, “A new singular function,” *Am. Math. Mon.*, vol. 118, no. 4, pp. 344–354, 2011.

[71] A. Pelat, F. Gautier, S. C. Conlon, and F. Semperlotti, “The acoustic black hole: A review of theory and applications,” *J. Sound Vib.*, vol. 476, p. 115316, 2020.

[72] Y. Peres, W. Schlag, and B. Solomyak, “Sixty years of Bernoulli convolutions,” in *Fractal Geometry and Stochastics II*, C. Brandt, S. Graf, and M. Zähle, Eds., vol. 46. Birkhäuser Basel, 03 2000, pp. 39–65.

[73] T. Persson, “On a problem by R. Salem concerning Minkowski’s question mark function,” 2015, [arXiv preprint arXiv:1501.00876].

[74] M. Reed and B. Simon, *II: Fourier Analysis, Self-Adjointness*, ser. Methods of Modern Mathematical Physics. Elsevier Science, 1975.

[75] ——, *Methods of Modern Mathematical Physics. IV Analysis of Operators*. New York: Academic Press, 1978.

[76] ——, *I: Functional Analysis*, 2nd ed., ser. Methods of Modern Mathematical Physics. Academic Press, 1980.

[77] S. Reese, "Some Fourier-Stieltjes coefficients revisited," *Proc. Am. Math. Soc*, vol. 105, no. 2, pp. 384–386, 1989.

[78] F. Riesz and B. Sz.-Nagy, *Functional Analysis*. Dover Publications, 1955.

[79] W. Rudin, *Real and Complex Analysis*, ser. Higher Mathematics Series. McGraw-Hill Education, 1987.

[80] R. Salem, "Some singular monotonic functions which are strictly increasing," *Trans. Am. Math. Soc.*, vol. 53, pp. 427–439, 1943.

[81] M. R. Shepherd, P. A. Feurtado, and S. C. Conlon, "Multi-objective optimization of acoustic black hole vibration absorbers," *J. Acoust. Soc. Am.*, vol. 140, no. 3, pp. EL227–EL230, 2016.

[82] B. Simon, "Operators with singular continuous spectrum: I. General operators," *Ann. Math.*, vol. 141, pp. 131–145, 1995.

[83] J. F. Sánchez, P. Viader, J. Paradís, and M. D. Carrillo, "A singular function with a non-zero finite derivative," *Nonlinear Anal. Theory Methods Appl.*, vol. 75, no. 13, pp. 5010 – 5014, 2012.

[84] ——, "A singular function with a non-zero finite derivative on a dense set," *Nonlinear Anal. Theory Methods Appl.*, vol. 95, pp. 703 – 713, 2014.

[85] ——, "A singular function with a non-zero finite derivative on a dense set with Hausdorff dimension one," *J. Math. Anal. Appl.*, vol. 434, no. 1, pp. 713 – 728, 2016.

[86] L. Takács, "An increasing continuous singular function," *Am. Math. Mon.*, vol. 85, pp. 35–37, 1978.

[87] C. Vemula and A. N. Norris, "Attenuation of waves in plates and bars using a graded impedance interface at edges," *J. Sound Vib.*, vol. 196, no. 1, pp. 107 – 127, 1996.

[88] L. Wen, "An approach to construct the singular monotone functions by using Markov chains," *Taiwan. J. Math.*, vol. 2, pp. 361–368, 1998.

[89] T. Zamfirescu, "Most monotone functions are singular," *Am. Math. Mon.*, vol. 88, no. 1, pp. 47–49, 1981.

Part II

Papers

Paper A

Magnetic pseudodifferential operators represented as generalized Hofstadter-like matrices

Horia D. Cornean, Henrik Garde, Benjamin B. Støttrup, Kasper S. Sørensen

The paper has been published in the
Journal of Pseudo-Differential Operators and Applications Vol. 10, pp. 307–336, 2019
Paper available at DOI:10.1007/s11868-018-0271-y

Paper B

Regularity properties of bulk and edge current densities at positive temperature

Massimo Moscolari and Benjamin B. Støttrup

The paper has been submitted to the
Annales Henri Poincaré
Preprint available at arXiv:2201.08803

Paper C

A rigorous approach to optimal profile design for acoustic black holes

Benjamin B. Støttrup, Sergey Sorokin, Horia D. Cornean

The paper has been published in the
The Journal of the Acoustical Society of America Vol. 149, pp. 447–456, 2021
Paper available at DOI:10.1121/10.0003337

Paper D

Acoustic Black Hole Profile Optimization

Horia D. Cornean, Sergey Sorokin, and Benjamin B. Støttrup

The paper has been published in the
Proceedings of the International Conference on Structural Dynamic , EURODYN
Vol. 2, pp. 2482–2488, 2020
Paper available at EURODYN2020.org

Paper E

Characterization of random variables with stationary digits

Horia D. Cornean, Ira W. Herbst, Jesper Møller, Benjamin B. Støttrup, Kasper S. Sørensen

The paper has been accepted for publication in
Journal of Applied Probability
Preprint available at arXiv:2001.08492

Paper F

Singular distribution functions for random variables with stationary digits

Horia D. Cornean, Ira W. Herbst, Jesper Møller, Benjamin B. Støttrup, Kasper S. Sørensen

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