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ON SOME DIFFERENTIAL EQUATIONS IN MATHEMATICAL-PHYSICS AND SINGULAR FUNCTIONS IN PROBABILITY THEORY

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On some differential equations in mathematical-physics and singular functions in probability theory

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Abstract

This dissertation deals with three different mathematical subjects: mathematical physics, acoustics and probability theory. The manuscript starts with a brief description of the aforementioned fields, and introduces the notation and terminology needed to understand the content of the published/submitted papers.

Paper A deals with the theory of pseudodifferential operators with magnetic fields. Here a generalization of the magnetic Weyl quantization is proposed and then shown that this choice makes it possible to represent a magnetic pseudodifferential operator as a generalized Hofstadter matrix. This generalized matrix structure is then used to show spectral stability when the generating symbol is real, i.e. the corresponding operator is self-adjoint. Specifically, it is shown that the spectrum of the operator is locally Hölder continuous in the strength of the magnetic field, with a Hölder exponent 1/2. Furthermore, if the magnetic field is constant, it is shown that the extreme spectral values, as well as the gap edges (if such gaps exist and they stay open when the strength of the magnetic field is varied) are Lipschitz continuous.

Paper B is concerned with the bulk-boundary correspondence for unbounded Dirac operators. First, one shows essential self-adjointness of the edge magnetic Dirac operator defined with infinite mass boundary conditions, along with a detailed analysis of the integral kernel of its resolvent. This is then used to formulate a relativistic bulk-boundary correspondence and a gap labelling theorem, which extends some known results from the Schrödinger setting.

Paper C investigates the possibility of constructing acoustic black holes within the Timoshenko beam theory setting. More specifically, the paper deals with determining an optimal height profile of a wedge at the end of thin plate, which minimizes the reflection of waves at the boundary. This is done by considering the partial differential equations describing the beam motion and then derive the Timoshenko dispersion relation. A functional depending on the height profile is then derived, and the associated Euler-Lagrange equation is determined. By solving this equation numerically an optimal profile is determined and compared with the corresponding profile obtained by considering the waves using the Euler-Bernoulli beam theory

instead.

Papers D and E deals with stochastic variables on the unit interval given by a base-q expansion with digits coming from a stationary stochastic process. In Paper D a functional equation for the CDF corresponding to the stationary stochastic process is derived. By applying this functional equation, a characterization of stationarity of the stochastic process in terms of the corresponding CDF is established. In Paper E specific stationery models are characterized, hereunder stationary Markov chains and stationary renewal processes.

Resumé

I denne afhandling behandles tre forskellige matematiske områder: matematisk fysik, akustik og sandsynlighedsregning. Afhandlingen er delt i to dele, hvor formålet med den første er at introducere de tre områder samt at forberede læseren på notationen of terminologien anvendt i artiklerne i den anden del af afhandlingen.

Artikel A omhandler teorien om pseudodifferentiale operatorer i magnetiske felter. Der præsenteres en generalisering af den magnetiske Weyl kvantisering og det vises så hvordan denne kvantisering gør det muligt at betragte pseudodifferential operatoren som en generaliseret matrix. Denne generaliseret matrix struktur bliver dernæst anvendt til at vise stabilitets resultater af spektrumet, når det betragtede symbol er reelt (hvilket medfører at den betragtede operator er selvadjungeret). Mere præcist, bliver det vist at spektrummet er Hölder kontinuert, i styrken af det magnetiske felt, med eksponent 1/2. Endvidere, vises det at hvis det magnetiske felt er konstant, så er den maksimale og den minimale værdi af spektrummet samt endepunkterne i et hul i spektrummet (hvis et sådan eksisterer og hulet forbliver åbent når styrken af det magnetisk felt varierer) Lipschitz kontinuerte.

Artikel B omhandler bulk-boundary korrespondancen for ubegrænsede Dirac operatorer. Først bestemmes integral kernen af den frie bulk Dirac operator og så anvendes den til at vise essentielt selvadjungerethed af den magnetiske edge Dirac operator samt til at bestemme et eksplicit udtryk for integral kernen af resolventen til den magnetiske edge operator. Dette bliver så brugt til at formulerer en relativistisk bulk-boundary korrespondance og en gap labeling sætning, som udvider nogle velkendte resultater fra Schrödinger setuppet.

Artikel C omhandler teorien om akustiske sorte huller i et Timoshenko setup. Mere præcist, bestemmes der i artiklen en optimal højdeprofil af en kile for enden af en tynd plade, som minimere reflektionen af bølger der rammer enden af pladen. Dette bliver gjort ved at betragte differential ligningerne der beskriver bevægelsen af en bjælke i Timoshenkos teori og så ved hjælp af disse udlede Timoshenkos spredningsligning, der beskriver bølgenumrene. Dernæst udledes et funktionalle der er afhængig af højdeprofilen og den tilhørende Euler-Lagrange ligning bestemmes. Denne ligning løses dernæst numerisk for at bestemme den optimale højdeprofil, som

sammenlignes med højdeprofilen fundet ved at betragte bølger med Euler-Bernoullis bjælke teori.

Artikel D og E omhandler stokastiske variable på enhedsintervallet der er givet ved en base-q udvikling hvor cifrene kommer fra en stationær stochastisk proces. I Artikel D udledes en funktional ligning for CDFen der hører til den den stationære stokastiske process. Ved at benytte denne funktional ligning karakteriseres stationaritet af den stochastiske proces med hensyn til den tilhørende CDF. I Artikel E bliver specifikke stationære modeller karakteriseret, herunder stationære Markov kæder og stationære fornyelsesprocesser.

Contents

Preface

This dissertation is the culmination of my studies as a PhD student at Aalborg University. My PhD studies has been a joint project between the Department of Mathematical sciences and Department of Materials and Production, both at Aalborg University.

When my PhD studies began, the plan was to pursuit two, more or less, independently research directions, one in the field of mathematical physics with Professor Horia Cornean and another in acoustics with Professor Sergey Sorokin. This has resulted in two papers in mathematical physics regarding pseudodifferential operators in magnetic fields and bulk-boundary correspondence for Dirac operators along with one paper in acoustics regarding acoustic black holes. Almost halfway through my PhD, I also got involved in a research project regarding characterization of random variables on the unit interval, which was supported by the research centre CSGB and run by Professor Jesper Møller. This work led to the last two papers included in this dissertation.

Acknowledgements

By writing this dissertation I now have the opportunity to express my gratitude to the many people who has helped me, more than they know, during the last four years.

First, I would like to express my sincere gratitude to my advisor Professor Horia Cornean, for all our interesting discussions and all the time you have spend answering my countless stupid questions, whether it was in front of a blackboard at the department or over a beer at some bar. I am also very thankful to him for introducing me to the amazing world of mathematical physics and for including me in the mathematical physics community.

I would also like to thank my two co-supervisors Professor Sergey Sorokin and Professor Jesper Møller, for the very stimulating collaborations we have had doing my PhD-studies.

Then I would like to thank all the employees I have interacted with at the Department of Mathematical Sciences at Aalborg University during both my time as a student and as your colleague. That I have enjoyed the last nine years here is especially due to all of you.

Moreover, I wish to thank Professor Stefan Teufel and Postdoc Massimo Moscolari for inviting me to Tübingen and making my stay there extremely joyful. A big thanks here also to the rest of the mathematical physics group at the Eberhard Karls Universität Tübingen, I hope it is not the last time I will visit Tübingen.

A special thanks also to Benjamin Buus Støttrup. For all our interesting discussions during the last eight years, both as fellow students, colleagues and most importantly friends.

Last, but definitely not least, my deepest thanks goes to my family, who has supported me through thick and thin. I am more grateful to you than you will ever know.

Kasper Studsgaard Sørensen
Aalborg University, August 11, 2022

Part I

Background

Background

1 Introduction

The scope of my PhD-studies ended up being threefold. The Papers A and B belongs in the subject of mathematical physics, the Paper C in acoustics and the Papers D and E in probability theory. Therefore, this chapter will also be split into three, more or less, independent sections. Each section will contain some preliminary material, intended to prepare the reader to dive into the papers, a summary of the main results of each paper and some references to other similar works, for the interested reader. This introduction will be quite brief and superficial. This means that proofs will be omitted and we are not always strict about domains of operators and such. We instead give several references which covers the material in great depth.

2 Quantum mechanics

We begin with a very short historical overview. It was believed in the 19th century, that the deterministic theory classical mechanics was the correct theory to predict the motion of bodies [20]. In classical mechanics one is interested in classical observables, which are real functions $a(q,p)$ defined on the phase space $\mathbb{R}^{6N} = \mathbb{R}^{3N} \times \mathbb{R}^{3N}$, where N is the number of particles, $q \in \mathbb{R}^{3N}$ is the position of the particles and $p \in \mathbb{R}^{3N}$ is the momentum of the particles. This theory is deterministic in the following sense: if you know the exact state of the system at some type (i.e. the position and momentum of a system at some time), then by solving the laws of motion, you can predict the future states of that system at any time. In physics one is interested in considering the total energy of a system. In classical mechanics, this is done using the Hamiltonian

$$H^{\mathrm{CL}}(q,p) = \frac{p^2}{2m} + V(q),$$

where m is the mass, $p^2/2m$ describes kinetic energy and V describes potential energy. Classical mechanics is still considered to be quite accurate when considering "large" objects not moving with a speed comparable to the speed of light.

In the beginning of the 20th century experiments starting showing that this classical theory did not approximate the behaviour at an atomic level very well. The outcome of these observation led to a new theory, namely quantum mechanics. In quantum mechanics one is interested in quantum observables which are given by self-adjoint operators in the Hilbert space $L^2(\mathbb{R}^n)$. Unlike the classical observables, these can not be simultaneously measured exactly, due to the uncertainty principle. In quantum mechanics the state of the system is given by the wave function, which does not give the exact state but instead describes the probability of finding the system in some state. This means that quantum mechanics is not a deterministic theory.

If we consider one particle in $\mathbb{R}$, described by the wave function $\psi \in L^2(\mathbb{R}^n)$ then the future states of this particle are determined by the time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t}(t) = H\psi(t), \tag{2.1}$$

with $H = -\Delta + V$ where Δ is the Laplace operator, V is a multiplication operator and we have applied suitable units, such that Planck's constant $\hbar = 1$ and the mass $m = 1$. The operator H is called the quantum mechanical Hamiltonian and as in classical mechanics it describes the total energy of a system (the kinetic energy is here given by $-\Delta$ and the potential energy by V). The Hamiltonian is an example of an unbounded operator and thus the theory of unbounded operators on Hilbert spaces is fundamental for the mathematical study of quantum mechanics. We refer the reader to the monographs [66], [64] and [76] for the basic mathematics needed to study Hamiltonians.

When H is self-adjoint, the solution of (2.1) is given by $\psi(t) = e^{-itH}\psi_0$, where ψ_0 is the initial condition. Hence, the time evolution of a quantum system is generated by a self-adjoint operator [76]. One interesting question in this regard, is whether the spectrum is stable under perturbations of the Hamiltonian. A common example regarding this is if $H_k = H_0 + kV$ is a family of Hamiltonians where V is relatively bounded by H_0, then the Hausdorff distance between the spectrum of H_k and H_0 varies proportional to $|k|$ [19]. The interested reader can consider [47], [65] or [36] for the general theory of perturbation theory.

2.1 Magnetic Schrödinger operators

When considering quantum particle as above, another interesting question arises: What happens if one impose a magnetic field on the system? A magnetic field is a closed 2-form B given as $B = dA$, where A is a 1-form, called the magnetic potential [28]. An important thing to note here, is that the magnetic potential A is not unique. Due to this, we are interested in a theory that is gauge covariant, i.e. if we choose another magnetic potential, say $\tilde{A}$, then the operators corresponding to A and those corresponding to $\tilde{A}$ should be unitarily equivalent. To go from the free Schrödinger operator (i.e. no magnetic field) to the magnetic Schrödinger operator, one replaces the momentum operator $p = -i\nabla$ with the magnetic momentum operator $\Pi^A = -i\nabla + bA$, where b is the strength of the magnetic potential A. Note that this operator is gauge covariant, since $d(bA - b\tilde{A}) = 0$, implies by Poincaré's lemma [50] that $bA - b\tilde{A} = d\varphi$, for some sufficiently regular φ, and thus $e^{i\varphi}\Pi^A e^{-i\varphi} = \Pi^{\tilde{A}}$. This leads to the the magnetic Schrödinger operator

$$H_b := (-i\nabla - bA)^2 + V,$$

which clearly also is gauge covariant. As with the free Schrödinger operator, we are interested in properties of the spectrum. Before we can start considering the spectrum, we need to consider the question of self-adjointness (or essentially self-adjointness) of such operators. This is a more difficult problem then in the free case, since the behavior at infinity of some of the most interesting magnetic fields, is not necessarily decaying (eg. the constant magnetic field) [3]. General results in this direction can be found in [44] and [51]. After the problem of self-adjointness has been considered, one would be interested in the same questions as for the free Schrödinger operator, eg. the stability of the spectrum. We will return to questions of this kind later, in Paper A. Here we refer the reader to [2], [9], [14] and [19] for a small selection of results in this direction.

2.2 Quantization

As stated in the previous section, classical mechanics give quite well predictions for certain systems. This lead Niels Bohr to formulate the correspondence principle, which says that when systems become "large" enough to be described using classical mechanics, then the description should agree with the one by quantum mechanics [60]. Thus we are interested in finding a correspondence between the observables $a(q, p)$ from classical mechanics and the self-adjoint operators which acts as observables in quantum mechanics. There has not yet been found a final answer to this question which it known as the the quantization problem [27] (if it is even possible to find a final answer). A simple suggestion is to just replace the position and momentum functions $a(q, p) = q$ and $a(q, p) = p$ from classical mechanics with the self-adjoint operators X and P given by $X\psi(x) = x\psi(x)$ and $P\psi(x) = -i\nabla\psi(x)$, which represents position and momentum in quantum mechanics. If we use this naive quantization then we arrive at a problem regarding the quantization of products. In classical mechanics qp and pq are equal since they commute. This is not the case for the quantum analogs, due to the canonical commutator relation

$$[X, P] = i\delta_{jk},$$

where δ_{jk} is Kronecker's delta [32]. One often used quantization is the Weyl quantization, which associates to a self-adjoint operator its classical counterpart, using the theory of pseudodifferential operators. For a general introduction of pseudodifferential operators we refer the reader to [37], [38] and [31]. If $\tilde{a} \in S^0_{0,0}(\mathbb{R}^{2n})$ is a Hörmander symbol on the phase space, then the corresponding operator $\mathrm{Op}^W(\tilde{a})$ on the Hilbert space $L^2(\mathbb{R}^n)$ (using the Weyl quantization) is given by (in weak sense)

$$\langle Op^W(\tilde{a})f, g\rangle := \frac{1}{(2\pi)^n}\int_{\mathbf{R}^{3n}} e^{i(x-x')\cdot\xi}\tilde{a}\Big(\frac{x+x'}{2},\xi\Big)f(x')\overline{g(x)}\,\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}\xi,$$

where $f, g \in \mathscr{S}(\mathbb{R}^n)$ are Schwartz functions [57]. By a straight forward calculation, it then follows that for the classical position $\tilde{a}_1(q,p) = q$ and momentum $\tilde{a}_2(q,p) = p$ we get

$$(\mathrm{Op}^W(\tilde{a}_1)f)(x) = xf(x),$$
$$(\mathrm{Op}^W(\tilde{a}_2)g)(x) = (-i\nabla f)(x),$$

which are the quantum position and momentum operators.

In classical mechanics, the corresponding notation of the commutator relation in quantum mechanics, is the Poisson bracket. It can be shown (see [32]) that there is a correspondence between these two notations, using Weyl quantization, of polynomials in position and momentum of degree at most two. Since this relation is not generally satisfied, one could think that maybe there exists another quantization for which this is satisfied generally. This it not the case, as follows from Groenewold's "no go" theorem [32], which states that there does not exist a correspondence between the commutator and the Poisson bracket for polynomials of degree at most four.

Another interesting question is the quantization when a magnetic field is present, since this adds the extra condition of gauge covariance. This problem has been considered in [40], [41], [42], [43] and [45], where they suggested that for a symbol $\tilde{a} \in S^0_{0,0}(\mathbb{R}^{2n})$, the Weyl quantization in the presence of a magnetic field should be given as (in weak sense)

$$\langle Op^W_b(\tilde{a})f, g\rangle := \frac{1}{(2\pi)^n}\int_{\mathbf{R}^{3n}} e^{i(x-x')\cdot\xi}e^{ib\varphi(x,x')}\tilde{a}\Big(\frac{x+x'}{2},\xi\Big)f(x')\overline{g(x)}\,\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}\xi,$$

where φ describes the magnetic flux through the oriented triangle with $0, x, x'$ as vertices. In Paper A we give another quantization, which are closely related to the magnetic Weyl quantization. Using this quantization we then prove two results regarding the stability of the spectrum.

2.3 Overview of Paper A

Paper A is concerned with the study of a pseudo-differential operator quantization which is closely related to the magnetic Weyl quantization presented in Section 2.2. We consider a magnetic field which is smooth, with all derivatives bounded and is given by a 2-form $B = \mathrm{d}A$, for some 1-form A, which is not unique. We choose to work with the transverse gauge. If we denote by $\Gamma_{x,x'}$, the oriented line from x' to x, we can define the anti symmetric function φ, which describes the magnetic flux in the oriented triangle with vertices $0, x, x'$, by

$$\varphi(x,x') = \int_{\Gamma_{x,x'}} A(\cdot,0),$$

which satisfies $\partial_{x_j}\varphi(x,x') = A_j(x,0) - A_j(x,x')$. An important property of φ is the following estimate

$$|\partial_x^{\alpha}\partial_{x'}^{\alpha'}\varphi(x,x')| \leq C_{\alpha,\alpha'}|x||x'|,$$

for every $x, x' \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$. Using this magnetic flux, we introduce a new class of symbols which we call magnetic symbols. These magnetic symbols are given by

$$a_b(x,x',\xi) = e^{ib\varphi(x,x')}a(x,x',\xi),$$

for $b \in \mathbb{R}$, the strength of the magnetic field, and $a \in C^\infty(\mathbb{R}^n)$, satisfying that for some $M \geq 0$ we have

$$|\partial_x^{\alpha}\partial_{x'}^{\alpha'}\partial_{\xi}^{\beta}a(x,x',\xi)| \leq C_{\alpha,\alpha',\beta}\langle x - x'\rangle^M,$$

for $\alpha, \alpha'\beta \in \mathbb{N}_0^n$. Given a magnetic symbol we define the magnetic pseudodifferential operator $\mathrm{Op}(a_b)\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, given by (in weak sense)

$$\langle Op(a_b)f,g\rangle := \frac{1}{(2\pi)^n}\int_{\mathbf{R}^{3n}} e^{i(x-x')\cdot\xi}e^{ib\varphi(x,x')}a_b\Big(x,x',\xi\Big)f(x')\overline{g(x)}\,\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}\xi,$$

where $f, g \in \mathscr{S}(\mathbb{R}^n)$. At a first glance, this seems to be a larger operator class than the magnetic Weyl operators introduced in Section 2.2, but by applying the magnetic Beals criterion [15] it follows that the two operator classes actually are identical.

We then show that the magnetic pseudodifferential operator is unitarily equivalent to, what we call a generalized matrix on the Hilbert space $\bigoplus_{\gamma\in\mathbb{Z}^n} L^2((-1/2,1/2)^n)$. More precisely

$$U_b\mathrm{Op}(a_b)U_b^* = \{e^{ib\varphi(x,x')}\mathcal{A}_{\gamma\gamma',b}\}_{\gamma,\gamma'\in\mathbb{Z}^n},$$

where U_b is a unitary operator and $\mathcal{A}_{\gamma\gamma'}$ is a bounded operator on the Hilbert space $L^2((-1/2,1/2)^n)$. We then proceed with showing that the norm of the matrix elements $\|\mathcal{A}_{\gamma\gamma',b}\|$ decay as $\langle \gamma - \gamma' \rangle^{-N}$, for arbitrarily N and use this together with a Schur-Holmgren type result to show that the magnetic pseudodifferential operator $\mathrm{Op}(a_b)$ is bounded in $L^2(\mathbb{R}^n)$. By applying the generalized matrix structure we then prove two results regarding the stability of the spectrum when the symbol a_b is real and thus $\mathrm{Op}(a_b)$ is self-adjoint. First we show that for b in some small compact interval, the Hausdorff distance of the spectrum varies Hölder continuously in b with exponent $1/2$. Secondly, we prove that if B is a constant magnetic field, then the extreme spectral values, along with gap edges (if there exists a gap that does not close when b varies) varies Lipschitz continuously in b on some small compact interval.

3 Condensed matter physics

We now turn our attention to the field of condensed matter physics, more particularly the bulk-boundary correspondence of topological insulators, which establishes a relation between a properties in the bulk of the insulator with properties at the boundary. The most classical example, and the most studied one is the two-dimensional quantum Hall system, see etc. [78]. The interest in the bulk-boundary correspondence originates from the seminal paper [33] and a little later [35] and [29]. This has lead to a large mathematical literature on bulk-boundary correspondence, here we mention the papers [23], [24], [48], [69] on zero-temperature, [17] at positive temperatures and the monograph [63].

3.1 Bulk-boundary correspondence

Since the bulk-boundary correspondence is about a relation between properties in the bulk and at the boundary, we begin by introducing two different operators: one to describe the bulk properties and one to describe the boundary properties. The bulk dynamics in $L^2(\mathbb{R}^2)$ is described by

$$H_b = (-i\nabla - bA)^2 + V,$$

where A is the magnetic potential given by $A(x) = (-x_2, 0)$, b is the strength of the magnetic field and V is the potential. By defining $E := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$, the edge dynamics in $L^2(E)$ is described by

$$H_b^E := \overline{H_b|_{C_c^\infty(\bar{E})}},$$

where $H_b|_{C_c^\infty(\bar{E})}$ is considered with a Dirichlet boundary condition at $x_2 = 0$. Under suitable conditions on the potential, it can be shown that H_b is essentially self-adjoint on $C_c^\infty(\mathbb{R}^2)$ [28] and H_b^E is essentially self-adjoint on $C_c^\infty(\bar{E})$ [56]. For some results regarding the bulk operator we refer the reader to [1], [4], [7] and [8]. For the edge operator, see [25] and [30].

Denote by χ_Ω the indicator function on $\Omega := [0,1] \times [0,1]$ and χ_L the indicator function on the finite strip $S_L := [0,1] \times [0,L]$ for $L \geq 1$. In [17] it is shown that if F is a real-valued Schwartz function on $\mathbb{R}$, then $\chi_\Omega F(H_b)$ and $\chi_L F(H_b^E)$ are both trace-class and one can define the physical quantities

$$\rho_L(b) := \frac{1}{L}\mathrm{Tr}(\chi_L F(H_b^E)), \qquad B_F(b) := \mathrm{Tr}(\chi_\Omega F(H_b)),$$

where B_F is a generalization of the integrated density of states of the bulk operator (i.e. it measures the number of energy levels pr. unit volume below some energy) and ρ_L is a generalization of the integrated density of states of the edge operator on the strip S_L. The bulk-boundary correspondence is then

$$\lim_{L\to\infty} \rho_L(b) = B_F(b), \qquad \lim_{L\to\infty} \frac{\mathrm{d}\rho_L}{\mathrm{d}b}(b) = \frac{\mathrm{d}B_F}{\mathrm{d}b}(b),$$

i.e. the number of energy levels pr. unit volume below some energy for the edge operator restricted to the strip S_L convergences to the number of energy level pr. unit volume below some energy for the bulk operator. The setting in [17] is more general than presented here, since they prove a bulk-boundary correspondence for positive temperatures. If $g \in C^1([0,1])$ is a function which satisfies $g(0) = 1$ and $g(1) = 0$, then one can prove the following explicit formula for B_F' [17]

$$\frac{\mathrm{d}B_F}{\mathrm{d}b}(b) = -\lim_{L\to\infty} \mathrm{Tr}(\tilde{\chi}_L i[H_b^E, X_1]F'(H_b^E)),$$

where $\tilde{\chi}_L = \chi_L(x)g(x_2/L)$.

If the bulk operator H_b has an isolated spectral island (i.e. a part of the spectrum isolated from the rest of the spectrum by two gaps), then by [58] the gap varies continuously in b and we can define the Riesz projection by

$$\Pi_b = \frac{i}{2\pi}\oint_{\mathcal{C}} (H_b - z)^{-1}\,\mathrm{d}z,$$

where $\mathcal{C}$ is a positively oriented simple contour which does not close for $b \in [b_1, b_2]$, for some $b_1, b_2 \in \mathbb{R}$. It is shown in [16] that if we define the integrated density of states of Π_b by

$$\mathcal{I}(\Pi_b) := \lim_{L\to\infty} \frac{1}{L}\mathrm{Tr}(\chi_L \Pi_b),$$

then

$$\frac{d\mathcal{I}(\Pi_b)}{\mathrm{d}b} = \frac{1}{2\pi}\mathrm{Ch}(\Pi_b), \tag{3.1}$$

where

$$\mathrm{Ch}(\Pi_b) := 2\pi \int_\Omega (i\Pi_b[[X_1, \Pi_b], [X_2, \Pi_b]])(x,x)\,\mathrm{d}x \in \mathbb{Z}$$

is a constant. The equality (3.1) is known as the Stręda formula. If one chooses a function $F_0 \in C^\infty(\mathbb{R})$ which satisfies that $0 \leq F_0 \leq 1$, $F_0 = 1$ on the isolated spectral island, $F_0 = 0$ on the rest of the spectrum and that F_0' has support only in the two gaps surrounding the isolated spectral island, then it can be shown [17] that $B_{F_0}(b) = \mathcal{I}(\Pi_b)$.

The proofs of all these statements relies heavily on the geometric perturbation theory introduced in [59] and [18]. We give here a short introduction to this theory, based on [17]. Define four functions, $0 \leq \eta_0, \eta_L \leq 1$ and $0 \leq \tilde{\eta}_0, \tilde{\eta}_L \leq 1$ which should all be smooth, only depend on x_2 and η_0 and η_L should satisfy that $\eta_0(x) + \eta_L(x) = 1$. All four functions should also satisfy some further conditions regarding their support and derivatives. These conditions can be found explicitly in [17]. We omit the details here to keep the presentation more simple and just mention that $\tilde{\eta}_0$ and $\tilde{\eta}_L$ can be thought of as strecthed out versions of η_0 and η_L. One can then show the following identity

$$(H_b^E - z)\tilde{\eta}_L = (H_b - z)\tilde{\eta}_L,$$

which shows that away from the boundary, the bulk and the edge operator behaves similarly. By defining the operator

$$U_L(z) := \tilde{\eta}_L(H_b - z)^{-1}\eta_L + \tilde{\eta}_0(H_b^E - z)^{-1}\eta_0$$

which is bounded in $L^2(E)$, we get that

$$(H_b^E - z)U_L(z) = 1 + W_L(z),$$

where W_L is a bounded operator, which one can get an explicit expression for. We say that U_L is an almost resolvent of $(H_b^E - z)$, in the sense that it satisfies the following resolvent-like identity, which will play a central role in the proofs

$$(H_b^E - z)^{-1} = U_L(z) - (H_b^E - z)^{-1}W_L(z).$$

3.2 Dirac operators

Historically in mathematical physics, the Schrödinger operator has been, by far, the most considered operator in the study of quantum mechanics. In the recent years, especially in the study of graphene related areas, people has begun considering the Dirac operators more thoroughly, see etc. [5], [6], [10], [11] and [73]. For a general introduction the Dirac operator, we refer the interested reader to the monograph [77].

We denote by $\{\sigma_1, \sigma_2, \sigma_3\}$ the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and by $\sigma := (\sigma_1, \sigma_2)$. By a straight forward calculation, it follows that the Pauli matrices satisfies the following commutation ($[A, B] = AB - BA$) and anticommutation ($\{A, B\} = AB + BA$) relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k, \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij},$$

for $i, j, k \in \{1, 2, 3\}$, where ε_{ijk} is the Levi-Civita symbol and δ_{ij} is the Kronecker delta. The free massless Dirac operator is considered on $L^2(\mathbb{R}^2, \mathbb{C}^2) \equiv L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ and is given by

$$H_0 = -i\nabla \cdot \sigma = p \cdot \sigma = -i\frac{\partial}{\partial x_1}\sigma_1 - i\frac{\partial}{\partial x_2}\sigma_2 = \begin{pmatrix} 0 & -i\partial_{x_1} - \partial_{x_2} \\ -i\partial_{x_1} + \partial_{x_2} & 0 \end{pmatrix}.$$

It can be shown that H_0 is an elliptic operator, which is essentially self-adjoint on $C_c^\infty(\mathbb{R}^2) \times C_c^\infty(\mathbb{R}^2)$, self-adjoint on the Sobolev space $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ and has purely absolutely continous spectrum $\sigma(H) = (-\infty, \infty)$ [77]. It is often also of interest, to consider the Dirac operator with mass. For $m > 0$, the free Dirac operator with mass, also considered on $L^2(\mathbb{R}^2, \mathbb{C}^2)$, is given by

$$H_{0,m} := H_0 + m\sigma_3.$$

A well-known fact is that the addition of mass creates a symmetric gap around 0 in the spectrum of the Dirac operator, i.e. $\sigma(H_{0,m}) = (-\infty, -m) \cup (m, \infty)$ [77]. Finally, as in the Schrödinger case, we are interested in considering systems, described by the Dirac operator, which are imposed with a magnetic field. The magnetic massless Dirac operator is given by

$$H_b := (-i\nabla - bA(x)) \cdot \sigma,$$

where A is the magnetic potential and b is the strength of the magnetic field. As in the Schrödinger case, we are also interested in the Dirac operator defined on the half-space E. Unlike the Schrödinger case, where we considered the Dirichlet boundary condition, we here consider the infinite mass boundary condition, i.e. if ψ is the restriction of a Schwartz function on $\mathbb{R}^2$ to E, then it must satisfy $\psi_1(x_1, 0) = \psi_2(x_1, 0)$, for every $x_1 \in \mathbb{R}$. The bulk and edge magnetic Dirac operators are the main focus of Paper B, where a Bulk-boundary correspondence is proved for the magnetic Dirac operator.

3.3 Overview of Paper B

Paper B is concerned with proving a bulk-boundary correspondence (as described in Section 3.1) for magnetic Dirac operators. We consider the bulk, purely magnetic and massless Dirac-Landau operator defined on the Hilbert space $L^2(\mathbb{R}^2, \mathbb{C}^2)$, given by

$$H_b := (-i\nabla - b\tilde{A}(x)) \cdot \sigma,$$

where σ is given as in Section 3.2 and $\tilde{A} = (-x_2, 0)$ is the magnetic potential. It is well-known that this function is essentially self-adjoint on $C_c^\infty(\mathbb{R}^2) \times C_c^\infty(\mathbb{R}^2)$.

In order to introduce the magnetic edge operator, we denote the set consisting of the restriction of Schwartz functions on $\mathbb{R}^2$ to E by $\mathscr{S}_+$ and define the two spaces

$$\mathscr{E} := \{\psi = (\psi_1, \psi_2) \in \mathscr{S}_+ \oplus \mathscr{S}_+ \mid \psi_1(x_1, 0) = \psi_2(x_1, 0), \forall\, x_1 \in \mathbb{R}\},$$
$$\mathscr{M} := \{\psi \in \mathscr{E} \mid \forall \alpha \in \mathbb{N}_0^2,\ \exists c, C > 0 \text{ such that } |\partial^\alpha \psi| \le Ce^{-c|x|}\}.$$

Then the edge magnetic Dirac operator is given by

$$H_b^E := \overline{\tilde{H}_b|_{\mathscr{M}}}$$

and we prove that the restriction of this operator to $\mathscr{M}$ is essentially self-adjoint.

By en explicit calculation, we then determine an expression for the integral kernel of the free edge Dirac operator

$$\mathcal{K}_0^E(x, x', \sqrt{\lambda}) := \mathcal{K}_0(x; x_1', -x_2', \sqrt{\lambda})\sigma_1 + \mathcal{K}_0(x, x', \sqrt{\lambda}),$$

where $\mathcal{K}_0$ is given by

$$\begin{aligned}\mathcal{K}_0(x, x', \sqrt{\lambda}) = (2\pi)^{-1} i\sqrt{\lambda} K_0(\sqrt{\lambda}|x - x'|) I_2 \\ - i\sqrt{\lambda}(2\pi)^{-1}\sigma \cdot \frac{x - x'}{|x - x'|} K_0'(\sqrt{\lambda}|x - x'|),\end{aligned}$$

and K_0 is the Macdonald function (see [61] for details of the Macdonald function).

Using this integral kernel of the free edge Dirac operator, we can define two operators $S_b(i\sqrt{\lambda})$ and $T_b(i\sqrt{\lambda})$ on $L^2(E, \mathbb{C}^2)$ which have integral kernels

$$\begin{aligned}S_b(x, x', \sqrt{\lambda}) &:= e^{ib\varphi_2(x,x')}\mathcal{K}_0^E(x, x', \sqrt{\lambda}) \\ T_b(x, x', \sqrt{\lambda}) &:= e^{ib\varphi_2(x,x')}(-b\tilde{A}(x - x') \cdot \sigma)\mathcal{K}_0^E(x, x'; \sqrt{\lambda}).\end{aligned}$$

These two operators play a central role in our analysis and we show several properties of them, eg. that both $S_b(i\sqrt{\lambda})$ and $T_b(i\sqrt{\lambda})$ are bounded on $L^2(E, \mathbb{C}^2)$ and that $\|T_b(i\sqrt{\lambda})\| \le Cb\lambda^{-1}$. This last property is particularly important, since it gives that for λ large enough, then $(1 + T_b(i\sqrt{\lambda}))^{-1}$ exists as a Neumann series.

If we consider the magnetic phase $\varphi_2(x, x') := (x_1' - x_1)x_2'$, then the resolvent $(H_b^E - i\sqrt{\lambda})^{-1}$, which we show can be written in terms of $S_b(i\sqrt{\lambda})$ and $T_b(i\sqrt{\lambda})$, has an integral kernel given by

$$(H_b^E - i\sqrt{\lambda})^{-1}(x, x') = e^{ib\varphi(x,x')} K_b(i\sqrt{\lambda})(x, x'),$$

where $K_b(i\sqrt{\lambda})(x, x')$ is smooth in the operator norm topology, for b in some small compact interval. This shows the only obstruction for the resolvent to be smooth in the norm topology comes from the phase.

We then show that the geometric perturbation theory introduced in [59] and [18] can be extended to the Dirac setting. Using this geometric perturbation theory then leads, with proofs very similar to those given in [17], us to extend both the bulk-boundary correspondence and the Středa formula considered in Section 3.1 to the Dirac setting.

4 Acoustic black holes

We now move away from quantum mechanics, which constituted the first part of this dissertation and move into the second part which is concerned with acoustics.

In acoustics one is interested in the behaviour of waves in a medium, say a solid. The most simple geometry of a solid is the thin plate, which can be considered one dimensional. Here we consider a semi-infinite, thin plate of height $h(x)$, i.e. it has a boundary at the one end, but not the other. Furthermore, we suppose that the plate has constant height except near the end with boundary, where the height can vary. We are then interested in the reflection of waves at this boundary. Historically, the Euler-Bernoulli beam theory has been widely used in mechanical engineering, since it is a very simple approximation of the linear theory of elasticity, which still provides quite good results. In Euler-Bernoulli beam theory, the wave in a one dimensional thin plate of thickness $h(x)$ is described using the differential equation [46]

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{Eh^3(x)}{12(1-\nu^2)}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(x,\omega)\right) - \omega^2\rho h(x) w(x,\omega) = 0, \tag{4.1}$$

where $w(x, \omega)$ is the transverse displacement of the midsurface, $E = E_0(1 - i\eta)$ is Young's modulus with loss, ν is Poisson's ratio, ω is the angular frequency and ρ is the material density of the plate.

In order to determine a solution (or more precisely to approximate a solution) to (4.1), it is customary in acoustics to use the WKB approximation, which was developed as a method to approximate solutions to differential equations in quantum mechanics (see [46] for details of the use of the WKB method in acoustics). By applying a first order WKB approximation it is possible to derive a measure of how much of a wave is reflected at the end of the plate, called the reflection coefficient. This reflection coefficient is given by

$$R = \exp\Big(\int_{x_0}^{x_1} \mathrm{Im}(k(x))\,\mathrm{d}x\Big),$$

where x_0 is where the boundary of the semi-infinite plate is located and k is the wave number, which in Euler-Bernouli theory is given by

$$k(x) = \sqrt[4]{\frac{12\omega^2}{c^2 h(x)^2}} \approx \frac{\sqrt[4]{12}}{\sqrt{h(x)}}\sqrt{\Omega},$$

where $c^2 = \frac{E}{\rho}$ and $\Omega = \frac{\omega h_1}{c}$, where h_1 is the height of the constant part of the plate. The application of the WKB approximation used here is only valid under the condition

$$\left| \frac{1}{k^2} \frac{\mathrm{d}k}{\mathrm{d}x} \right| \ll 1.$$

In applications this is generally considered to be satisfied when it is less than 0.3, see eg. [26].

The aim with the study of acoustic black holes is to minimize the reflection coefficient, so as much as possible of the wave is being absorbed at the boundary of plate. The field of acoustic black holes originates from the seminal work of Mironov in [55], where he showed that theoretically it is possible to get a reflection coefficient equal to zero. Indeed, by letting the height of the plate go to zero when approaching x_0, the speed of the wave will go to zero and thus never reach the end. In practice it is not possible to make the height go to zero and thus one needs other ideas to minimize the reflection coefficient. Two often considered methods are to make an optimal height profile $h(x)$ of the plate near the boundary or by attaching a dampening layer to the height profile which can absorb a part of the wave. These methods has been thoroughly studied in the Euler-Bernoulli setting, where we refer the reader to [26], [49], [74] and the review paper [62] and the references therein.

It is well known, that the Euler-Bernoulli beam theory is only valid for low frequencies, up to around $\Omega \approx 0.3$. Other beam theories are expected to hold for much higher frequencies. One of these is the Timoshenko beam theory (see [53] for an introduction to Timoshenko theory) which agrees with the Euler-Bernoulli theory for low frequencies [71], but unlike the Euler-Bernoulli, it is expected to be accurate up to $\Omega \approx 3.5$ [72]. A reason for this is that the Timoshenko theory relies on the shear deformation and rotational bending, while these are neglected in the Euler-Bernoulli theory. To the best of our knowledge, the research of acoustic black holes has only (up until now) been done in the Euler-Bernoulli framework.

4.1 Overview of Paper C

In Paper C we minimize the reflection coefficient of an acoustic black hole in a one-dimensional, semi-infinite, thin plate, where we consider the waves using the Timoshenko beam theory. To do so we apply the calculus of variations to determine the optimal height profile which is covered by a dampening layer.

The motion of a Timoshenko beam is described using the differential equations

$$-\rho A\frac{\partial^2 w}{\partial \tau^2}(x,\tau)+\kappa GA\Big(\frac{\partial^2 w}{\partial x^2}(x,\tau)-\frac{\partial \psi}{\partial x}(x,\tau)\Big)+q(x,\tau)=0,$$
$$-\rho I\frac{\partial^2 \psi}{\partial \tau^2}(x,\tau)+EI\frac{\partial^2 \psi}{\partial x^2}(x,\tau)+\kappa GA\Big(\frac{\partial w}{\partial x}(x,\tau)-\psi(x,\tau)\Big)=0,$$

where ρ is the density of the material, $A = Bh_d(x)$ is the cross section area where B is the constant width, while $h_d(x)$ is the height, $E = E_0(1 - i\eta)$ is the complex elastic modulus with loss $\eta > 0$, $G = E(1+\nu)^{-1}/2$ is the shear modulus with ν its Poisson ratio, $I = B(h_d(x))^3/12$ is the second moment area, κ is the Timoshenko shear coefficient and q is the distributed load.

By introducing dimensionless variables and making an WKB-like approximation of this system of differential equations we derive the Timoshenko dispersion equation which we solve to get an expression for the wave numbers. From this expression, we then derive the well-known fact that for low frequencies the Timoshenko theory reduces to the Euler-Bernoulli theory. Furthermore, we derive that for higher frequencies the Timoshenko theory predicts the appearence of two waves instead of just the one predicted by the Euler-Bernoulli theory. The first wave, which exists for all frequencies ends up becoming a Rayleigh wave and thus independent of the profile of the plate for the high frequencies. Then we show, as expected, that the second wave only appears for very high frequencies.

In order to optimize the reflection coefficient under the constraint explained in Section 4, we then introduce a Lagrange multiplier inspired functional, where we consider the constraint as an integral condition which puts a penalty on the functional when we violate the constraint. We then derive the Euler-Lagrange equation associated to this function analytically. This gives a first order, separable, autonomous differential equation which we have not been able to solve analytically. We therefore instead consider the differential equation numerically.

In the numerical analysis we then show that only in a part the frequency range where the Euler-Bernoulli and Timoshenko theory agrees, their optimal profile also agrees. For higher frequencies the optimal profile found using Euler-Bernoulli then stops being "smooth" and instead only continuous, while the optimal profile using Timoshenko theory stays "smooth" for much higher frequencies, before it shows the same phenomenon as in the Euler-Bernoulli setting and becomes only continuous.

The numerical investigations of the second wave predicted by Timoshenko theory turned up to be much more involved and is not considered in this paper. An interesting future project is to study this second wave in more details and make a comparison between the optimal profile coming from the first and the second wave in Timoshenko theory.

5 Stochastic variables in base-q expansion

We move on to the last of the topics which has been studied in this dissertation, concerning stochastic variables. We are here interested in a stochastic variable $X \in [0,1]$, which is represented as

$$X = \sum_{n=1}^{\infty} X_n q^{-1},$$

where $q \in \mathbb{N} = \{1, 2, \ldots\}$ and $\{X_n\}_{n \geq 1}$ is a stochastic process taking values in $\{0, \ldots, q-1\}$. An interesting question is then, what one can say about the cumulative distribution function (CDF)

$$F(x) := P(X \leq x),$$

when you imposes some specific conditions on $\{X_n\}_{n \geq 1}$.

One of the simplest cases is when X_n is independent identically distributed (IID), this case is also known as the Bernoulli scheme. If we consider the dyadic case $q = 2$, with $P(X_n = 0) = p_0$ and $P(x_n = 1) = 1 - p_0$ it is well known, see [12], that the CDF F is then continuous and strictly increasing on $[0,1]$. Furthermore, if $0 \leq k < 2^n$, is an integer, then there exists some $(\mu_1, \ldots, \mu_n)$, with $u_i \in \{0,1\}$ such that $\sum_{i=1}^{n} = \mu_i 2^{-i}$ and hence

$$F\left(\frac{k+1}{2^n}\right) - F\left(\frac{k}{2^n}\right) = P\left(\frac{k}{2^n} < X < \frac{k+1}{2^n}\right) = P(X_i = \mu_i, i \leq n). \tag{5.1}$$

If $p_0 = p_1 = \frac{1}{2}$, then it follows from (5.1) that the CDF $F(x) = x$ on $[0,1]$ and if $p_0 \neq p_1$ then one can show that $F'(x) = 0$ for almost all $x \in [0,1]$ [12]. Thus for $p_0 \neq p_1$ we end up with a function that is continuous, strictly increasing and whose derivative is equal to zero almost everywhere. Such a function is called a singular function, since the associated measure (which exists since F is nondecreasing and right-continuous on $\mathbb{R}$ [12])

$$\mathrm{d}F(a,b] = F(b) - F(a)$$

is singular with respect to the Lebesgue measure. This gives a complete characterization of the CDF in the dyadic case when the X_n's are IID. This characterization can be generalized to general Bernoulli schemes, in the sense that for $q \geq 2$, the measure dF is singular, unless the probabilities $P(X_n = j) = p_j$ are given by $p_0 = p_1 = \cdots = p_n = \frac{1}{n}$, for which the measure is absolutely continuous with respect to the Lebesgue measure.

In [34] a characterization of the CDF is made, under the assumptions that the X_n's are assumed to be stationary and of a mixing type. This characterization states that either

1. $F(x) = x$ on $[0,1]$,

2. F is a discrete CDF,

3. F is singular.

In [22] a similar characterisation is made if the X_n's are stationary and ergodic.

With these characterizations, it would seem that singular functions occurs quite frequently. This is also the case, eg. in [79] it is shown that "most monotone functions are singular" and in [70] it is shown that such functions also often occur in mathematical physics. Some specific examples of a singular function, is the Cantor function [13], [21], the Minkowski question-mark function [54] and the Riesz-Nagy functions [67], [68], [75]. In [68] and [67] one can even find a method to construct Riesz-Nagy functions (see Paper E for some plots of singular functions).

We end this section by an interesting observation regarding the measure dF, when F is the CDF of some stochatic process $\{X_n\}_{n\geq 1}$. A finite measure whose characteristic function

$$\mathbb{E}(e^{itX}) \to 0 \quad \text{as} \quad t \to \pm\infty$$

is called a Rajchman measure [52]. If the $X_n's$ are IID, then it is well-known that the CDF F is the uniform CDF if and only if the corresponding measure dF is a Rajchman measure [39].

5.1 Overview of Paper D and E

In Paper D we make a characterization of stationarity of $\{X_n\}_{n\geq 1}$ by considering the corresponding CDF F. We first show that stationarity of $\{X_n\}_{n\geq 1}$ is equivalent to F solving

$$F(x) = F(0) + \sum_{j=0}^{q-1} \left(F\left(\frac{x+j}{q}\right) - F\left(\frac{j}{q}\right) \right), \tag{5.2}$$

for every base-q fractions in $(0,1)$. We then show that if we assume stationarity of $\{X_n\}_{n\geq 1}$, then the X_n's being IID is equivalent to F satisfying

$$F(x) = \sum_{j=0}^{q-1} P(X_1 = j) F(qx - j),$$

for all $x \in [0,1]$.

Since a CDF is almost everywhere differentiable, then (5.2) also gives a functional equation for the derivative of F under the assumption of stationarity. By using this functional equation, we then show that dF is purely absolutely continuous if and only if dF is the Lebesgue measure. Furthermore, we show that, under stationarity, dF is a Rajchman measure (see Section 5 for a definition) if and only F is the uniform CDF on $[0,1]$.

We then shows that F satisfying (5.2) is equivalent to F being a mixture, i.e.

$$F = \theta_1 F_1 + \theta_2 F_2 + \theta_3 F_3,$$

where $\theta_i \geq 0$ for $i \in \{1,2,3\}$, $\theta_1 + \theta_2 + \theta_3 = 1$ and F_1, F_2, F_3 are CDF's satisfying that

1. $F_1(x) = x$, for all $x \in [0,1]$,
2. F_2 is a mixture of specific step functions,
3. F_3 is singular,

and all three satisfies (5.2).

In Paper E we then give a characterization of some well-known stationary stochastic processes $\{X_n\}_{n\geq 1}$. We first show that if $\{X_n\}_{n\geq 1}$ is a stationary Markov chain, then either $F(x) = x$ for all $x \in [0,1]$ or F has derivative equal to zero almost everywhere and if F satisfies (5.2), then it can be approximated in the uniform norm by stationary Markov chains. We then proceed to show similar characterizations when the CDF comes from a stationary renewal process, a mixture of Markov chains of fixed order and mixtures of renewal processes.

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Part II

Papers

Paper A

Magnetic pseudodifferential operators represented as generalized Hofstadter-like matrices

Horia D. Cornean, Henrik Garde, Benjamin B. Støttrup, Kasper S. Sørensen

The paper has been published in the
Journal of Pseudo-Differential Operators and Applications Vol. 10, pp. 307–336, 2019
Paper available at DOI:10.1007/s11868-018-0271-y

Paper B

Bulk-edge correspondence for unbounded Dirac-Landau operators

Horia D. Cornean, Massimo Moscolari, Kasper S. Sørensen

The paper has been submitted to
Journal of Mathematical Physics
Preprint available at arxiv:2208.02218

Paper C

Optimal profile design for acoustic black holes using Timoshenko beam theory

Kasper S. Sørensen, Horia D. Cornean, Sergey Sorokin

The paper has been submitted to
The Journal of the Acoustical Society of America

Paper D

Characterization of random variables with stationary digits

Horia D. Cornean, Ira W. Herbst, Jesper Møller, Benjamin B. Støttrup, Kasper S. Sørensen

The paper has been accepted for publication in
Journal of Applied Probability
Preprint available at arXiv:2001.08492

Paper E

Singular distribution functions for random variables with stationary digits

Horia D. Cornean, Ira W. Herbst, Jesper Møller, Benjamin B. Støttrup, Kasper S. Sørensen

The paper has been submitted to
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