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An exploration of the integration and design of virtual reality for mathematics education

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VIRTUAL REALITY IN MATHEMATICS EDUCATION (VRiME)

AN EXPLORATION OF THE INTEGRATION AND DESIGN OF VIRTUAL REALITY FOR MATHEMATICS EDUCATION

BY
LUI ALBÆK THOMSEN

DISSERTATION SUBMITTED 2023

AALBORG UNIVERSITY
DENMARK

Virtual Reality in Mathematics Education (VRiME)

An exploration of the integration and design of virtual reality for mathematics education

Ph.D. Thesis

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Abstract

The bounds of traditional learning environments are constantly challenged and reshaped by emerging technologies. The idea of an abstract mathematical concept, once confined to the pages of a textbook, can now be visualized and manipulated in a three-dimensional, immersive world thanks to the advent of Virtual Reality (VR). VR has the potential to not just supplement, but fundamentally transform our educational paradigms, transposing learners from the passivity of traditional lecture-based teaching to an interactive, experiential learning environment. Mathematics, often perceived as a difficult and abstract subject, stands to greatly benefit from such transformations. This thesis explores the integration of VR into mathematics education, demonstrating the potential of VR to act as a medium for deeper understanding, appreciation, and enjoyment of mathematics, thus painting a vibrant and immersive picture of abstract mathematical ideas. It is a journey towards envisioning a new reality for mathematics education.

This thesis explores the integration and design of VR for mathematics education. With its immersive and interactive capabilities, VR offers an innovative platform for teaching and learning mathematics. The main research questions addressed in this thesis are centred around the effective integration and design of VR for mathematics education. The work documented in this thesis involves analysis of existing research on VR in mathematics education, leading to the development of four novel VR prototypes designed for learning equations, geometry, collaboration, and vectors. Each of these designs was implemented and evaluated based on various parameters such as usability, user experience, learning outcomes, and applicability in the educational context. An important part of this thesis involves the presentation of six research articles, each produced during the PhD project, that further delves into the specifics of the research questions. These articles detail the development process of the prototypes, their experimental application, and subsequent results.

Paper A reviews the use of asymmetric VR technology in educational settings, particularly in mathematics classrooms. The paper discusses the considerations associated with integrating VR technology, underscoring its potential to promote collaborative learning. The review and discussions led to the publication of a taxonomy of asymmet-

ric interfaces for collaborative immersive learning.

Paper B examines how VR technology can be adapted to assist students with Autism Spectrum Disorder (ASD) in developing daily living skills that require mathematical knowledge. It suggests that VR, through avatar customization, communication channels, and collaboration mechanics, may be able to foster engagement in learning daily living skills and mathematics.

Paper C focuses on using VR technology to promote social inclusion among neurodiverse students. It demonstrates how VR can meet diverse student needs and potentially enhance the learning of daily living skills and basic mathematical concepts through interactive features.

Paper D describes a VR prototype designed to teach algebra, equations and equation-solving strategies. The paper finds that the immersive, exploratory learning approach facilitated by VR resulted in positive student responses while showing potential for the successful transfer of conceptual knowledge to traditional exercises.

Paper E highlights the potential of hand-tracking in VR for interactive learning in geometry. It discusses the design challenges related to hand-tracking technology, an approach of combining gesture-based interaction with dynamic geometry, and proposes a new taxonomy of learning environments for geometry education.

Finally, paper F explores the use of VR in teaching mathematical objects to software engineering students, demonstrating the mathematical foundations behind game engines and virtual environments. It suggests that visualisation techniques and game-based learning in VR can be leveraged to create a contextualised approach to learning mathematics.

Taken together, the results from this research indicate promising prospects for the design and potential for learning mathematics in VR. These findings could catalyze the transformation of traditional teaching methods and contribute to the evolution of mathematics education through VR technologies. This thesis aims to further our understanding of VR as an educational tool and serves as a stepping stone towards more immersive, engaging, and effective learning experiences in the field of mathematics. The findings may benefit educators, VR developers, and learners in mathematics.

Resumé

Grænserne for traditionelle læringsmiljøer bliver konstant udfordret og omformet af nye teknologier. Ideen om et abstrakt matematisk koncept, der engang var begrænset til siderne i en lærebog, kan nu visualiseres og manipuleres i en tredimensionel, fordybende verden takket være fremkomsten af Virtual Reality (VR). VR har potentiale til ikke blot at supplere, men fundamentalt transformere vores uddannelsesparadigmer, ved at omdanne elever fra passivitet i traditionel forelæsningsbaseret undervisning til et interaktivt, oplevelsesbaseret læringsmiljø. Matematik, der ofte opfattes som et svært og abstrakt fag, har meget at vinde fra sådanne transformationer. Denne afhandling udforsker integrationen af VR i matematikundervisningen, og demonstrerer VR's potentiale til at fungere som et medie for dybere forståelse, værdsættelse og begejstring af matematik, og dermed male et levende og fordybende billede af abstrakte matematiske ideer. Det er en rejse mod at forestille sig en ny virkelighed for matematikundervisningen.

Denne afhandling udforsker den lovende grænse af uddannelsesteknologier med fokus på integration og design af VR til matematikundervisning. Med sine fordybende og interaktive kapaciteter tilbyder VR en innovativ platform for undervisning og læring i matematik. De centrale forskningsspørgsmål behandlet i denne afhandling er centreret omkring den effektive integration og design af VR til matematikundervisning. Studiet involverer analyse af eksisterende forskning på VR i matematikundervisning, hvilket fører til udviklingen af fire nye VR-prototyper designet til at lære ligninger, geometri, samarbejde og vektorer. Hver af disse designs blev implementeret og evalueret baseret på forskellige parametre såsom brugervenlighed, brugeroplevelse, læringsresultater og anvendelighed i den pædagogiske kontekst. En vigtig del af denne afhandling involverer præsentationen af seks forskningsartikler, hver produceret i løbet af PhD-projektet, der yderligere uddyber specifikationerne af forskningsspørgsmålene. Disse artikler detaljerer udviklingsprocessen af prototyperne, deres eksperimentelle anvendelse og efterfølgende resultater.

Artikel A gennemgår brugen af asymmetrisk VR-teknologi i uddannelsesmæssige indstillinger, især i matematikklasser. Artiklen diskuterer de overvejelser, der er forbundet med integrationen af VR-teknologi, og understreger dens potentiale til at fremme samar-

bejdsbaseret læring. Gennemgangen og diskussionerne førte til offentliggørelsen af en taksonomi af asymmetriske interfaces til kollaborativ fordybende læring.

Artikel B undersøger, hvordan VR-teknologi kan tilpasses for at hjælpe studerende med Autismespektrumforstyrrelser (ASD) med at udvikle hverdagsfærdigheder, der kræver matematisk viden. Den antyder, at VR gennem avatar-tilpasning, kommunikationskanaler og samarbejdsmekanikker kan fremme elevers engagement i at lære hverdagsfærdigheder og matematik.

Artikel C fokuserer på brugen af VR-teknologi til at fremme social inklusion blandt neurodiverse studerende. Det demonstrerer, hvordan VR kan imødekomme forskellige elevbehov og forbedre indlæringen af hverdagsfærdigheder og grundlæggende matematiske koncepter gennem interaktive funktioner.

Artikel D beskriver en applikation designet til at undervise i algebra, ligninger og løsningsstrategier i VR. Artiklen finder, at den fordybende, udforskende læringsmetode, der faciliteres af VR, resulterede i positiv respons fra studerende og viste potentiale for overførbarheden af viden til traditionelle læringsøvelser.

Artikel E fremhæver potentialet i håndsporing i VR til interaktiv læring i geometri. Den diskuterer designudfordringer relateret til håndsporingsteknologi, en tilgang til at kombinere interaktion baseret på gestikulation med dynamisk geometri, og foreslår en ny taksonomi af læringsmiljøer til geometriundervisning.

Artikel F udforsker brugen af VR i undervisningen af matematiske objekter til software engineering-studerende, og demonstrerer de matematiske grundlag bag spilmotorer og virtuelle miljøer. Det antyder, at visualiseringsteknikker og spilbaseret læring i VR kan udnyttes til at skabe en kontekstualiseret tilgang til at lære matematik.

Resultaterne fra denne forskning indikerer lovende udsigter for designet og potentialet i at lære matematik i VR. Disse resultater kunne være katalysator for transformationen af traditionelle undervisningsmetoder og bidrage til udviklingen af matematikundervisningen gennem VR-teknologier. Denne afhandling har til formål at øge vores forståelse af VR som et uddannelsesværktøj og fungerer som et springbræt mod mere fordybende, engagerende og effektive læringsoplevelser inden for matematikområdet. Resultaterne kan potentielt gavne undervisere, VR-udviklere og elever i matematik.

Contents

Abstract iii

Resumé v

Thesis Details ix

Preface xi

I Introduction 1

Introduction 3

1 Introduction . . . 4
1.1 Educational Technologies in Mathematics Education . . . 5
1.2 Virtual Reality in the Classroom . . . 7
1.3 Summary of Introduction . . . 11
2 Background . . . 12
2.1 Instructional Design Models . . . 12
2.2 Learning Objectives . . . 14
2.3 Designing Representations of Mathematics . . . 15
2.4 Educational Approaches . . . 26
2.5 Measuring Outcome . . . 37
2.6 Summary and Model . . . 44
3 Related Research . . . 47
3.1 Summary of Related Research . . . 63
4 Classroom Integration . . . 65
4.1 Practical Implementation . . . 66
4.2 Technical Implementation . . . 68
5 Subject Matter . . . 73
5.1 Algebra and Equations . . . 73

5.2 Geometry 78
5.3 Vectors 80
6 Research Questions 83
7 Summary of Papers 85
7.1 Paper A: Asymmetric Collaboration in Virtual Reality: A Taxonomy of Asymmetric Interfaces for Collaborative Immersive Learning 85
7.2 Paper B: A Collaborative Virtual Reality Supermarket Training Application to Teach Shopping Skills to Young Individuals with Autism Spectrum Disorder 88
7.3 Paper C: Designing a Collaborative Virtual Reality System to Assess Social Inclusion among Neurodiverse Students 90
7.4 Paper D: Adapting the Balance Model for Equation Solving to Virtual Reality 92
7.5 Paper E: An Immersive Geometry Environment for Mathematics Education: Taxonomy and Preliminary Evaluation 97
7.6 Paper F: Real Mathematics in Virtual Worlds 100
8 Discussion and Conclusion 104
8.1 Research Questions 104
8.2 Contributions 106
8.3 Conclusion 107
8.4 Future Perspective 109
9 References 111
References 111

II Papers 125

Thesis Details

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Assoc. Prof. Niels Christian Nilsson, Aalborg University

The main body of this thesis consist of the following papers.

[A] Thomsen, Lui Albæk and Nilsson, Niels Christian and Nordahl, Rolf and Lohmann, Boris, "Asymmetric collaboration in virtual reality: A taxonomy of asymmetric interfaces for collaborative immersive learning," *Tidsskriftet Læring Og Medier (LOM)*, vol. 12, no. 20, 2019.

[B] Thomsen, Lui Albæk and Adjorlu, Ali, "A collaborative virtual reality supermarket training application to teach shopping skills to young individuals with autism spectrum disorder," *2021 IEEE Conference on Virtual Reality and 3D User Interfaces Abstracts and Workshops (VRW)*, pp. 50–55, 2021.

[C] Thomsen, Lui Albæk and Adjorlu, Ali, "Designing a collaborative virtual reality system to assess social inclusion among neurodiverse students," *2021 IEEE Conference on Virtual Reality and 3D User Interfaces Abstracts and Workshops (VRW)*, pp. 353–357, 2021.

[D] Elkjær, Morten and Thomsen, Lui Albæk, "Adapting the balance model for equation solving to virtual reality," *Digital Experiences in Mathematics Education*, vol.8, no.2, pp. 127–156, 2022.

[E] Thomsen, Lui Albæk and Nilsson, Niels Christian and Nordahl, Rolf and Støvelbæk, Kevin Baars and Mundbjerg-Sunne, Christoffer Bendig, "An Immersive Geometry Environment for Mathematics Education: Taxonomy and Preliminary Evaluation," *2023 IEEE Conference on Virtual Reality and 3D User Interfaces Abstracts and Workshops (VRW)*, pp. 513–518, 2023.

[F] Thomsen, Lui Albæk and Timcenko Olga, “Real mathematics in virtual worlds,” *9th International Research Symposium on Problem-Based Learning: Transforming Engineering Education*, 2023.

In addition to the main papers, the following publications have also been made.

[1] Thomsen, Lui Albæk and Elkjær, Morten “Equation Lab - Fixing the balance for teaching linear equations using Virtual Reality,” *Mathematics Education in the Digital Age (MEDA)*, pp. 311, 2020.

[2] Thomsen, Lui Albæk and Elkjær, Morten “An Immersive Learning Experience for Teaching Equations,” *International Congress on Mathematical Education (ICME)*, 2020.

This thesis has been submitted for assessment in partial fulfillment of the PhD degree. The thesis is based on the submitted or published scientific papers which are listed above. Parts of the papers are used directly or indirectly in the extended summary of the thesis. As part of the assessment, co-author statements have been made available to the assessment committee and are also available at the Faculty. The thesis is not in its present form acceptable for open publication but only in limited and closed circulation as copyright may not be ensured.

Preface

This thesis is a testament to my belief that technology, when used appropriately, has the potential to revolutionise education. It is the culmination of years of research, experimentation, and engagement with the aim to delve deeper into the potential of an innovative blend of Virtual Reality (VR) technology and mathematics education. Ever since my first encounter with VR, I was fascinated by its ability to blur the boundaries between reality and the digital world. The more I immersed myself in this technology, the more I became convinced of its untapped potential, particularly in the field of education. Meanwhile, the opportunity to apply and research VR in mathematics education, a subject often misunderstood and feared, gave me a new purpose: to find ways to make it more accessible, engaging, and effective for learners. The journey of writing this thesis was challenging, enlightening, and ultimately rewarding. It involved countless hours of research and deep diving into the literature on VR technology, interaction design, learning theories, pedagogical methods, and mathematics education. It required endless trials, user testing, feedback gathering, and revisions of the VR mathematics modules I developed. It demanded an immense amount of patience, persistence, and open-mindedness to not only embrace innovative ideas but also learn from the failures and setbacks that came along the way. This work attempts to push the boundaries of how we perceive and teach mathematics by utilising VR as a tool to inspire a deeper understanding and appreciation of the subject. Through theoretical examinations, experimental studies and qualitative analysis, it explores how a virtual environment can transform mathematics learning into an immersive and interactive experience, making complex concepts more digestible and mathematics less daunting for students. However, this thesis is only the tip of the iceberg in a vast ocean of possibilities. I believe that the future holds much more than what we can imagine today with regard to the intersection of VR and education. I am hopeful that my work will inspire others to further explore this realm and together we can transform the way we educate our future generations.

Thesis Roadmap

In this industrial PhD thesis, the focus will lie on the integration and design of VR in mathematics education. The Introduction will provide a broad perspective on educational technologies, particularly emphasising the role of VR in the classroom. The *Background* will explore pertinent theories and design frameworks related to VR, including the design of representations such as manipulatives, examples of VR applications, and the process of instrumental genesis. This section will also discuss various instructional designs like constructivism, situated learning, exploratory learning, embodied learning, gamification, serious games, and generative learning strategies. Further, methods for measuring learning outcomes and ensuring quality in VR education will be discussed. The thesis will then proceed to the *Related Research* section, where it will present an examination of the existing studies in this field. In the *Classroom Integration* section, both the practical and technical aspects of implementing VR technology in the classroom will be detailed. The *Learning Material* section will focus on designing learning material for VR, specifically addressing algebra and equations, geometry, and vectors. In the *Summary of Papers* section, a synopsis of six published papers associated with this PhD research will be presented, including the various VR applications developed for different educational contexts. The final part of the thesis, *Discussion and Conclusion*, will reflect on the insights gained from the research, considering the implications for VR in mathematics education and the potential directions for future research.

Acknowledgments

First and foremost, I want to express my profound gratitude to my academic supervisors, Prof. Rolf Nordahl and Prof. Niels Christian Nilsson. A special thanks to Rolf Nordahl for making this PhD project a possibility. Prof. Nilsson, your extensive knowledge and guidance have been key in directing this project toward its goal. Your willingness to share your expertise and provide constructive criticism have been invaluable in my personal, academic and professional development.

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I am deeply grateful for the camaraderie and thought-provoking discussions with my PhD colleague, Morten Elkjær. Your questioning spirit, optimism, and friendship have significantly eased and enriched this journey. Furthermore, your willingness to provide insightful didactic observations and constructive feedback when needed is something I value greatly.

My time at Aalborg University and the Multisensory Experience Lab would not have been the same without the incredible team I had the privilege to work with. I wish to extend my thanks to Stefania Serafin, Ali Adjorlu, Razvan Paisa, Emil Rosenlund Høeg, Silvin Willemsen, Jon Ram Bruun-Pedersen, Lars Koreska Andersen, and Nicklas Bundgaard Stavad Andersen. Your intellectual companionship, collegiality, and shared

enthusiasm have fostered a stimulating environment that has been indispensable to my growth as a researcher.

To my girlfriend, Maria Holm Nielsen, your unwavering love, encouragement, and understanding have been my anchor during this journey. Your patience, support, and belief in me have given me the strength to overcome obstacles and pursue my passion. I am grateful for your presence in my life.

Finally, to my dear family: mom, dad, sisters, and grandparents. Your enduring love, emotional support, and faith in my abilities have always lifted my spirits. Your belief in me has been the foundation upon which I have built this achievement. To you, I dedicate this work, as a testament to your constant love and support.

To everyone mentioned above, and to all those who have contributed in various ways but may not be named here, please accept my heartfelt appreciation for your significant contributions, guidance, and encouragement throughout this Industrial PhD project.

Thank you, everyone, for being part of this substantial milestone in my life. Each contribution, in its unique way, has helped shape my journey and has been instrumental in the successful completion of this project. I am truly fortunate to have been surrounded by such a supportive network.

Lui Albæk Thomsen
Aalborg University, June 1, 2023

Part I

Introduction

Introduction

1 Introduction

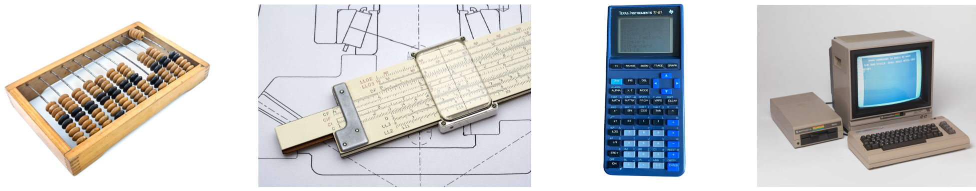
The rapid advancement of technology in modern times has paved the way for innovative approaches to education, with Virtual Reality (VR) emerging as a promising medium for teaching and learning. Tracing its roots back to the early attempts in the 1960s, such as Ivan Sutherland's development of the first head-mounted display (HMD) [166], VR technology has undergone significant evolutions over the decades. In the 1990s, VR gained public attention as a cutting-edge technology, but it wasn't until the 2010s that it finally gained more widespread acceptance and accessibility.

Examining the historical progression of VR technology, we can observe that its development has been shaped by many applications and innovations, ranging from entertainment to military training. As the technology matured, its potential for educational purposes began to gain traction. In recent years, VR has been applied to various fields of education, such as medical training, language learning, and even social skills development [46, 69, 80, 158].

In this section, I explore the role of educational technologies in mathematics education, followed by considerations on the integration of VR in the classroom. This investigation is structured into two main sections. The first section provides a historical overview of technological tools used in mathematics education, from the earliest calculating devices to modern adaptive learning systems. I explore the scientific and philosophical underpinnings of these technologies, understanding their development and impact on pedagogical strategies and learning outcomes. This exploration sets the foundation for my discussion on the potential of VR in mathematics education.

The second section examines the prospects of integrating VR into mathematics education. I explore VR's technological and psychological foundations, illuminating how these contribute to creating an immersive learning environment. I discuss the benefits and challenges of integrating VR into classroom settings, highlighting factors such as financial implications, logistical considerations, and the role of asymmetric VR. My investigation aims to provide educators, policymakers, and researchers with insights into the potential of VR in enriching mathematics education, fostering a more engaging and effective learning environment.

By examining the evolution of educational technologies and the potential of VR, this section contributes to the ongoing discourse on the future of mathematics education. My exploration is underpinned by the understanding that technology, in itself, is not a cure; it serves as an enabler that, when used appropriately, can augment teaching strategies and facilitate effective learning. Therefore, my focus is not merely on technological possibilities but also on pedagogical implications, ensuring that we move towards a future of mathematics education that is inclusive, engaging, and learner-centric.



(a) Abacus (b) Slide rule (c) Calculator (d) Personal computer

Fig. 1: Educational technologies (1)

1.1 Educational Technologies in Mathematics Education

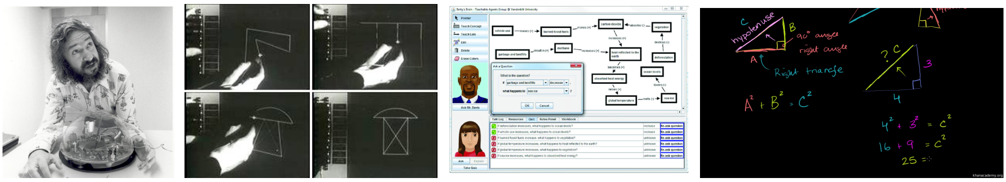
The landscape of mathematics education has been transformed by educational technologies, which have evolved to support the teaching and learning of mathematics throughout history. In this section, I outline the most significant historical events in the evolution of educational technologies in mathematics education, exploring both their scientific and philosophical underpinnings.

The earliest known calculating tool, the abacus, can be traced back to 2400 BCE in Sumeria, and its invention marked a milestone in the history of educational technologies [14]. The abacus enabled users to perform arithmetic operations more efficiently than mental calculations or using fingers, paving the way for the development of more advanced mathematical concepts. The slide rule, invented in the 17th century by William Oughtred [18], was another significant advancement. It allowed users to perform calculations such as multiplication, division, and exponentiation by utilising logarithmic scales. The slide rule remained a popular tool in mathematics and engineering education until the mid-20th century when electronic calculators began to emerge [14, 18] (see Figure 1). The invention of programmable computers in the mid-20th century revolutionised educational technologies in mathematics education. Pioneers such as Alan Turing and John von Neumann laid the groundwork for modern computing [59, 178]. As computers became more accessible, they were adopted for educational purposes, giving rise to computer-assisted instruction (CAI). The landmark study by Suppes and Morningstar [165] provided evidence of the effectiveness of CAI in teaching elementary school mathematics. CAI allowed for individualised instruction, immediate feedback, and data-driven adjustments to instructional strategies, which contributed to improved learning outcomes. The advent of personal computers in the late 20th century made computing technology more accessible, fostering the development of educational software for mathematics education.

Programs like Logo, developed by Papert and his colleagues at MIT [127], enabled students to explore mathematical concepts through programming and problem-solving activities. Logo's turtle graphics allowed students to engage with geometry in a creative and interactive manner, embodying Papert's constructionist learning philosophy [127].

The proliferation of educational software also facilitated the creation of dynamic geometry environments (DGEs), such as Geometer's Sketchpad [68] inspired by Ivan Sutherland's Sketchpad from 1963. DGEs allow students to manipulate geometric objects and explore mathematical relationships visually, potentially fostering a deeper understanding of geometric concepts. The rise of the internet in the late 20th and early 21st centuries opened new frontiers for educational technologies in mathematics education. Online learning environments, such as the Khan Academy [81] and the National Library of Virtual Manipulatives [35], provide students and educators with access to a wealth of digital resources and interactive learning tools (see Figure 2). These online environments facilitated individualised learning, collaborative problem-solving, and real-time feedback, contributing to improved learning outcomes [88]. MatematikFessor — a web-based e-learning platform developed and maintained by the industrial partner of the Industrial PhD project — has substantially influenced mathematics education in Denmark. Developed in Danish and tailored to the national curriculum, the platform incorporates gamified learning, videos, interactive exercises, and quizzes to engage learners from primary to upper-secondary levels. This adaptive learning tool adjusts content based on individual progress, facilitating a personalised learning journey. It also fosters a productive learning community, enabling students to collaborate and discuss problems, while providing tools for teachers and parents to monitor progress. Furthermore, Massive Open Online Courses (MOOCs) emerged as a platform for providing high-quality mathematics education to a global audience, democratising access to knowledge [16]. MOOCs allowed learners from diverse backgrounds to engage with mathematics content at their own pace, fostering a sense of autonomy and self-directed learning [16]. Incorporating artificial intelligence (AI) into educational technologies has led to the development of adaptive learning systems, which tailor instruction based on each student's individual needs, strengths, and weaknesses [184]. Systems such as ALEKS [39] and ASSISTments [56] employ AI techniques like knowledge tracing and constraint-based modelling to provide personalised feedback and support, enhancing students' understanding of mathematical concepts and promote mastery learning [172]. Integrating AI-driven technologies has also opened up new avenues for collaborative problem-solving and inquiry-based learning, as exemplified by Betty's Brain system [11]. This environment allows students to interact with intelligent agents, building and testing their understanding of mathematical concepts through guided exploration and reflection [11].

As educational technologies continue to evolve, we can anticipate further advancements in the ways mathematics is taught and learned. Virtual and augmented reality (AR) technologies have the potential to immerse students in rich mathematical environments, fostering deeper engagement with concepts and problem-solving tasks [111]. Additionally, advancements in brain-computer interfaces and neurofeedback technologies may enable more precise monitoring and modulation of students' cognitive and emotional states during mathematical learning, paving the way for truly personalised and adaptive instruction [179]. The historical evolution of educational technologies in



(a) Logo **(b)** Sketchpad **(c)** Betty's Brain System **(d)** Khan Academy

Fig. 2: Educational technologies (2)

mathematics education has been marked by significant milestones, from the abacus to AI-driven adaptive learning systems. As we reflect on these developments, we must be mindful of the philosophical underpinnings of our educational practices, ensuring that technology serves as an enabler rather than a barrier to meaningful learning experiences. With continued research, collaboration, and innovation, we can harness the power of educational technologies to create a brighter future for mathematics education.

The proliferation and evolution of educational technologies in mathematics education, from Logo's turtle graphics to AI-driven adaptive learning systems, have paved the way for more immersive and interactive learning experiences. As we look forward to the future of mathematics education, we are confronted with an exciting prospect: the integration of VR technologies. Leveraging the immersive nature of VR, we can revolutionise the way students interact with mathematical concepts, potentially enhancing their understanding and engagement. Not only does VR offer novel means for visualising and manipulating mathematical ideas, but it also brings a unique psychological element into the learning process. This dual technological and psychological nature of VR, encapsulated in Paul Milgram's Reality-Virtuality Continuum and Mel Slater's concept of presence, promises a transformative shift in mathematics education. By grounding mathematical concepts in a more concrete and experiential context, VR has the potential to augment traditional teaching methods, providing a richer, more engaging, and more effective learning experience for students.

1.2 Virtual Reality in the Classroom

VR is a technology-driven, immersive experience that simulates a three-dimensional (3D), interactive environment. This innovative medium leverages visual, auditory, and often haptic feedback to provide users with an engaging and realistic encounter. A useful framework for understanding VR's technological underpinnings is the Reality-Virtuality Continuum proposed by Paul Milgram [114]. This model represents a spectrum that extends from completely real environments to fully virtual ones, with augmented reality (combining real and virtual elements) and augmented virtuality (mainly virtual with some real elements) falling in between. On the psychological side, VR's power lies in its capacity to evoke a sense of presence. This concept, rooted in Mel Slater's work, refers to the subjective experience of "being there" within a virtual environment [156]. Slater delineates two types of presence: place illusion and plausibility illusion. Place illusion is the sensation of physically inhabiting the virtual space, achieved through perceptual fidelity and 360-degree, uninterrupted immersion. Plausibility illusion, on the other hand, is the believability of the scenario or events occurring within the VR environment, which requires consistency and coherence of the virtual world and its interaction with the user. The interconnectedness of technological and psychological factors in VR creates a unique opportunity for educational applications, such as in classroom settings and specifically within mathematics education. VR's immersive nature and high degree of interactivity can make abstract mathematical concepts more concrete and comprehensible. For instance, students can explore 3D geometric shapes, manipulate graphs in a virtual space, or visualise complex equations and their solutions in ways that are not possible with traditional teaching methods. The Reality-Virtuality Continuum allows for diverse implementations, from AR-enhanced textbooks to fully immersive VR lessons. Moreover, the experience of presence may enhance learning outcomes in mathematics. Place illusion may help students focus and fully engage with the material, reducing external distractions and creating a private, personalised learning environment. Plausibility illusion may foster deeper understanding and long-term retention by enabling learners to 'experience' mathematical concepts and processes, thereby linking abstract symbols and operations to more intuitive, tangible experiences. Thus, VR in mathematics education represents an exciting convergence of technology and psychology, offering significant potential to enrich teaching and learning in the digital age.

While the potential efficacy of VR technology in mathematics education may be considerable, the successful integration of this immersive tool into classroom settings is not without challenges. This transformation requires a thorough evaluation of numerous critical factors, encompassing financial, logistical, and resource considerations, to ensure the successful adoption and sustainability of such systems. The upfront costs associated with obtaining the necessary equipment represent merely the initial investment. Educational institutions must also consider the potential for additional expenditures, required training procedures, curriculum adaptation, and provisions for equitable access. The remaining section provides an exploration of the potential financial implications and related concerns linked to the incorporation of VR into a mathematics classroom setting.

Integrating VR into a school classroom for teaching and learning mathematics has the potential to improve student engagement, understanding, and overall learning outcomes. However, there are financial implications that need to be considered, such as the initial investment in equipment, like VR HMDs, controllers, and computers with sufficient processing power and graphics capabilities. These costs can vary significantly depending on the number of students and the quality of the equipment. Schools may also need to invest in specialised furniture or other modifications to create a suitable learning environment for VR-based lessons, as well as in acquiring the appropriate educational software, licensing, and content packages. Over time, schools may need to upgrade their equipment to keep up with advancements in VR technology, which can include updating software, purchasing new hardware components, or investing in newer HMD models. Additionally, schools must budget for ongoing maintenance, such as updating security features, troubleshooting technical issues, and repairing or replacing worn or damaged equipment. Providing equal access to VR technology for all students is crucial, but it can be challenging and costly to achieve. Schools may need to invest in additional equipment or resources to accommodate students who don't have access to VR systems at home or ensure that alternative learning experiences are available for those who cannot participate in VR-based lessons due to medical or other reasons. Teachers and staff must be trained to effectively use VR technology in the classroom, which can involve additional expenses such as professional development courses, workshops, or seminars. Providing ongoing support for educators is also essential, as they may encounter challenges or need assistance in integrating VR into their lesson plans effectively. Schools should consider allocating resources to maintain a support team or contracting with external support providers to ensure a seamless implementation of VR technology. Integrating VR into mathematics instruction may necessitate the development of new, VR-compatible curriculum materials, which can involve hiring subject matter experts, instructional designers, and programmers to create custom content or purchasing pre-made content packages from third-party providers. Schools must also consider the costs of updating and maintaining these materials over time to ensure their continued relevance and effectiveness. Lastly, high-speed internet connections are often required to support the smooth operation of VR systems. This may involve upgrading a school's existing network infrastructure to accommodate increased bandwidth demands, which can be a significant expense. Additionally, schools may need to invest in secure data storage solutions to protect student data and comply with data privacy regulations. Ultimately, schools must evaluate the cost-effectiveness of implementing VR technology in the classroom. While the immersive and engaging nature of VR has the potential to improve learning outcomes, it is essential to consider whether the financial investment will lead to significant improvements in student performance when compared to more traditional teaching methods. This analysis should take into account not only the direct costs of the technology but also the indirect costs related to training, support, and infrastructure.

While considering the initial expenses and ongoing costs associated with implementing VR technology in classrooms, a possible solution that combines the benefits of VR with cost-effectiveness is the integration of asymmetric VR. This innovative approach provides a middle ground that still harnesses the immersive and interactive capabilities of VR while mitigating the requirement for high-cost equipment for every student. Asymmetric VR allows for a different level of interaction and participation for each student in a shared virtual space, effectively reducing the total investment necessary for a full-scale implementation. The varying roles within the VR environment may enhance collaboration and participation, potentially fostering a deeper understanding of mathematical concepts. This model allows schools to gradually increase the adoption of VR, managing costs while still benefiting from this cutting-edge technology. Therefore, asymmetric VR could be the key to unlocking the educational potential of VR in mathematics education, while also addressing the significant financial implications.

Asymmetric VR denotes a VR experience in which multiple participants engage in a shared virtual environment, each possessing distinct roles or abilities. Within an asymmetric VR configuration, some users may fully immerse themselves in the virtual realm using VR HMDs and controllers, while others might employ conventional input devices like keyboards, mice, or game controllers to interact with or impact the experiences of the VR users. The designation "asymmetric" stems from the unequal or non-symmetrical nature of participants' experiences, roles, and interactions within the virtual space. This setup can yield novel and captivating gameplay or learning experiences by facilitating collaboration and cooperation among users with varying degrees of immersion and interaction capabilities. Asymmetric VR experiences hold potential for diverse applications, encompassing gaming, education, and training. For example, in an educational scenario, an instructor might manipulate the virtual environment using a tablet or computer, directing students wearing VR HMDs through a learning task. This approach may promote collaboration, problem-solving, and communication both among students and between students and their teachers. Asymmetric VR may provide numerous benefits for educational institutions, such as reduced equipment costs, flexibility in implementation, enhanced collaboration, and inclusivity. By allowing some participants to engage in the virtual environment without a full VR setup, schools can allocate resources more effectively, providing VR setups to those who need them most and using alternative devices for others. This flexibility enables schools to adopt VR technology in stages, gradually increasing adoption as budgets allow while still providing meaningful experiences for all students. In addition, asymmetric VR promotes teamwork, communication, and problem-solving by enabling participants with different roles and capabilities to collaborate within the virtual environment. This inclusivity ensures that students who may not be comfortable or able to use VR HMDs and controllers can still participate in the learning experience using alternative input devices. Asymmetric VR also supports differentiated instruction, shared learning experiences, and increased motivation and engagement. Teachers can use this technology to provide personalised guidance and support to individual students by monitoring progress, providing feedback, and manipulating the virtual environment through a computer or tablet while some students are fully immersed in the VR environment. This approach allows for the integration of different learning approaches, combining the immersive nature of VR with more traditional methods to create a rich learning environment that appeals to a wider range of students. Furthermore, the unique and engaging nature of asymmetric VR may capture students' interest and encourage participation, as they take on different roles within the virtual environment, leading to a sense of ownership and responsibility for their learning experience. In summary, using asymmetric VR to teach and learn mathematics may result in financial benefits such as reduced equipment costs and increased flexibility in implementation. Social benefits include enhanced collaboration, inclusivity, differentiated instruction, shared learning experiences, and increased motivation and engagement. These factors can contribute to a more effective and engaging learning environment for teaching mathematics.

1.3 Summary of Introduction

The exploration of VR in the classroom, particularly within the realm of mathematics education, presents an exciting convergence of technological innovation and psychological insight. VR's immersive and highly interactive nature can help transform abstract mathematical concepts into more tangible, comprehensible experiences. While the Reality-Virtuality Continuum allows for diverse applications, from augmented reality-enhanced textbooks to fully immersive VR lessons, the sense of presence evoked by VR—comprising of place illusion and plausibility illusion—may contribute to enhanced learning outcomes.

However, the integration of VR in classrooms is not without its challenges, with significant financial, logistical, and resource considerations to navigate. The initial investment for equipment, potential upgrades, maintenance costs, provision for equitable access, teacher training, and curriculum adaptation represent some of the hurdles schools may face. One cost-effective solution could be the application of asymmetric VR, which mitigates the need for high-cost equipment for every student, instead promoting differentiated interaction within a shared virtual environment.

Given the potential and challenges of VR in mathematics education, it becomes crucial to delve deeper into the theoretical underpinnings that govern its application in this sphere. The following section will explore the theoretical background of VR in mathematics education, setting a foundation to understand its potential, its constraints, and how it can be effectively leveraged in teaching and learning mathematical concepts.

2 Background

In this section, I present the theoretical foundations of utilising VR in mathematics education, exploring its potential to revolutionise how students engage with and learn mathematical concepts. As the field of VR continues to expand, researchers and educators alike are harnessing its immersive nature to create innovative learning environments that may enhance comprehension, motivation, and retention in mathematics. This section offers an overview of the various approaches to designing materials and instructions that could underpin the use of VR in mathematics education.

To better understand the instructional design of VR in mathematics education, I will first examine the educational approaches that should guide its integration and design. These approaches — constructivism, exploratory learning, situated learning, embodied learning, and game-based learning — emphasise the importance of active participation, physical interaction, and real-world context in the learning process. By aligning these approaches with the immersive and interactive capabilities of VR, educators may be able to create engaging experiences that foster a deeper understanding of mathematical concepts.

Next, I will explore the nuances of designing material for VR, addressing the unique challenges and opportunities that arise when creating educational experiences within this immersive medium. This includes considerations such as user interaction, spatial representation of mathematical concepts, and instructional scaffolding, all of which contribute to the development of effective and engaging VR-based learning environments. Moreover, this section will explore the methodologies relevant to measuring the outcome of VR in mathematics education, both from an affective and cognitive perspective. While discussing a range of generalised approaches, it will also aim to provide clarifications on how to approach and understand learning in mathematics education. Finally, this section will culminate in the synthesis of various approaches and considerations, resulting in an expanded version of the ADDIE model. This framework serves as a step-by-step guide for educators and researchers seeking to embark on the design and development of VR applications for mathematics education.

2.1 Instructional Design Models

As VR technology continues to advance, its applications in education become increasingly prominent. The development of effective VR learning materials requires a systematic approach through analysis of existing literature and with the aid of models such as ADDIE. These instructional design models guide the creation of tailored, student-centred experiences that effectively target specific learning objectives in mathematics.

The *Analysis, Design, Development, Implementation, and Evaluation* (ADDIE) model is a widely-used instructional design model consisting of five stages [44]. Each stage assists in developing successful learning materials, including VR content for mathematics education. The following will present the stages and reflect on VR in this context.

In the first phase, instructional designers identify learners' needs, establish learning objectives, and assess the feasibility of using VR for mathematics education. This process involves gathering data on learners' demographics, prior knowledge, learning preferences, and the specific mathematical concepts to be taught. Based on the analysis, designers create a blueprint outlining the learning experience's structure, content, and interactivity. For VR mathematics education, this may involve determining how learners will explore mathematical concepts in a 3D environment, interact with virtual manipulatives, and collaborate with peers. The design phase also includes determining the appropriate assessment methods for measuring learning outcomes. The development phase involves creating the VR learning materials, including 3D models, animations, simulations, and interactive elements. Programmers, graphic designers, and subject matter experts collaborate to ensure that the virtual environment is engaging, accurate, and pedagogically sound. The implementation phase consists of deploying the VR learning materials and integrating them into the mathematics curriculum. This process includes training teachers to use VR technology effectively and establishing a support system to address technical issues. In addition, it is crucial to monitor the learning experience to ensure it meets the intended objectives. Ongoing evaluation is essential for assessing the VR learning materials' effectiveness and identifying areas for improvement. This process includes collecting feedback from learners and teachers, analysing learning outcomes, and comparing results to the original learning objectives. Adjustments can then be made to improve the VR experience for future iterations. While the ADDIE model is highly popular, several alternative instructional design models can also be used to develop learning materials in VR for mathematics education.

Some of these include the Successive Approximation Model (SAM) and the Dick and Carey model [31, 75]. SAM is an iterative, agile approach to instructional design that emphasises rapid prototyping and continuous refinement [75]. This model may be well-suited for the dynamic nature of VR development and enables designers to adapt quickly to emerging technological advancements. The Dick and Carey model is a systematic approach that focuses on the relationship between learning objectives, instructional strategies, and assessment methods [31]. It may be particularly useful for designing VR experiences that align with specific mathematics curriculum standards. The use of instructional design models like ADDIE, SAM, and the Dick and Carey model should ensure the creation of effective and engaging VR learning materials for mathematics education. These systematic approaches facilitate the development of immersive experiences that may address learners' needs, enhance their understanding of mathematical concepts, and promote long-term retention. As VR technology continues to evolve, the application of these models will remain crucial in driving innovation and improving educational outcomes in mathematics.

2.2 Learning Objectives

In many cases, learning objectives are often established at a governmental level. In Danish mathematics education, learning goals are defined by "Fælles Mål" or "Common Goals" developed by The Ministry of Children and Education [1]. This curriculum embodies a comprehensive approach towards building foundational competencies in mathematics. The objective is not just to teach mathematical concepts, but also to equip students with the ability to apply these concepts in real-life situations. The approach differentiates and categorises various competency areas related to the subject matter, which are outlined in the following paragraphs.

Mathematical Competencies The first goal emphasises fostering mathematical competencies in students. Here, the student's ability to interact effectively with mathematical situations forms the core. This competency entails understanding and interpreting mathematical information, identifying and addressing mathematical problems, employing mathematical language and tools, and evaluating and validating results and arguments. VR can provide an immersive environment where students are placed in real-life situations requiring mathematical problem-solving. Through interactive scenarios, students may develop and practice their competencies, such as interpreting mathematical information, addressing mathematical problems, and validating results, in a more engaging and intuitive way.

Numbers and Algebra This goal focuses on developing the student's understanding of natural numbers and their properties. It encourages students to create methods for calculations with natural numbers, including addition, subtraction, multiplication, and division. The emphasis is on fostering a fundamental understanding of number systems, enabling students to solve problems in various contexts effectively. VR can offer a hands-on experience for understanding numbers and algebraic concepts. For example, students can manipulate virtual objects representing different numbers, perform operations on them, and observe the results. This would provide a more concrete understanding of abstract algebraic operations.

Geometry and Measurement The third goal is to instil a firm understanding of geometric concepts and measurements in students. Here, the focus is on the application of geometric concepts and understanding measurements to represent, model, and interpret phenomena. Students learn to analyse spatial structures and relationships, measure length, area, volume, and angle, and use geometric figures and their properties. VR truly shines when it comes to spatial understanding. Students can interact with 3D models of geometric shapes, explore their properties, and understand spatial

[1] Matematik Fælles Mål: `https://emu.dk/grundskole/matematik/faghaefte-faelles-maal-laeseplan-og-vejledning`

relationships more intuitively. They can also practice measurement skills within the virtual environment, such as using virtual rulers or protractors to measure the dimensions or angles of objects.

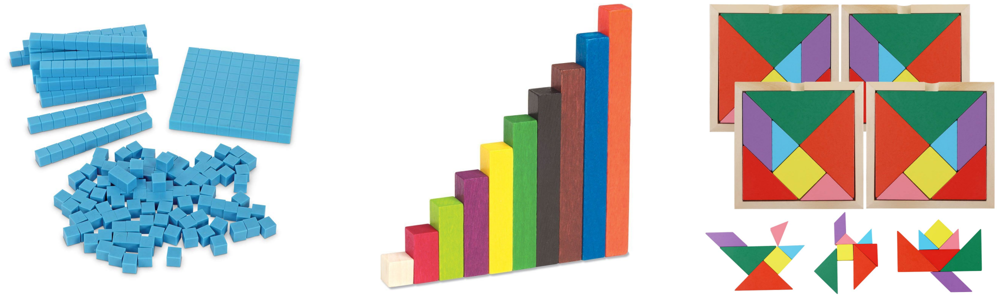
Statistics and Probability The final goal targets the realm of statistics and probability. The aim is to familiarise students with statistical investigations and intuitive probability measures. It allows students to collect, organise, and interpret data, make predictions based on data, understand the concept of chance and probability, and evaluate statistical arguments. VR can provide a visual and interactive approach to understanding statistics and probability. Students can participate in virtual simulations where they collect data, perform statistical analysis, and understand probability distributions. They can visualise complex data sets or probability concepts in three dimensions, which may enhance their understanding and intuition.

By focusing on these "Common Goals", Denmark's mathematics education aspires to create a mathematically literate society, equipped with the necessary skills to navigate an increasingly data-driven world. These goals not only foster a thorough understanding of mathematical concepts but also promote critical thinking, problem-solving abilities, and quantitative reasoning skills. Incorporating VR into mathematics education, as outlined in the "Fælles Mål", may provide a rich, immersive, and interactive experience for students. It can not only enhance students' understanding of mathematical concepts but also increase their engagement and motivation to learn.

Even though the individual components of the PhD project did not explicitly utilise the previously mentioned instructional design models, they were all fundamentally rooted in identifying and analysing academic subjects or skills where students typically face difficulties based on the objectives of "Fælles Mål" by the Danish Ministry of Children and Education. The objective was not only to craft tailored interventions for a broad spectrum of student needs through an iterative design process, but also to contribute to the growing body of research in immersive learning. This process involved continuous observations during both the development and evaluation stages as a key method for informing and refining future designs.

2.3 Designing Representations of Mathematics

The integration of VR technology in mathematics education presents a multitude of opportunities for enhancing the learning experience and deepening conceptual understanding. By leveraging the immersive and interactive nature of VR, complex mathematical concepts can be visualised and explored in 3D space, which may foster intuitive interactions and enable students to make connections between abstract ideas and tangible representations. The fusion of mathematics and VR holds great promise for revolutionising educational methods and empowering students to unlock their full potential in the captivating world of mathematical exploration.



(a) Base-ten blocks **(b)** Cuisenaire rods **(c)** Tangrams

Fig. 3: Concrete manipulatives

I will begin by examining manipulatives, including interactivity, user engagement, and pedagogical value. I will then explore the various types of virtual manipulatives that can be employed in mathematics education, ranging from basic number lines and geometric shapes to more advanced algebraic and calculus tools. I will also present several examples of digital and immersive learning tools that are available to the public. Furthermore, I will discuss the challenges and considerations involved in designing and implementing mathematical representations for VR. This will include addressing issues related to accessibility, usability, and technological constraints, as well as ensuring that the representations align with established educational objectives through instrumental genesis.

Manipulatives

Manipulatives in mathematics education refer to concrete or digital objects used to represent mathematical concepts and promote active learning. These objects are designed to help learners visualise abstract mathematical ideas, facilitate problem-solving, and enhance understanding of complex concepts [20]. This section will provide an overview of manipulatives in mathematics education, focusing on their various types, applications, and effects on learning outcomes.

Manipulatives can be categorised into three primary types: concrete, virtual, and digital [119]. Concrete manipulatives are physical objects that can be handled and manipulated by learners, such as base-ten blocks, Cuisenaire rods, and tangrams [150] (see Figure 3).

A body of research has investigated the effects of manipulatives on mathematics education. Studies have shown that the use of manipulatives can lead to improved conceptual understanding, problem-solving abilities, and overall achievement in mathematics [119]. Moreover, they have been found to be particularly effective in engaging diverse learners, including students with learning disabilities and English language learners [152, 153]. Several studies have focused on the comparative effectiveness of different types of manipulatives. For example, a meta-analysis by Carbonneau, Marley, and Selig [20] reported that both concrete and virtual manipulatives were effective in improving mathematics achievement, with no significant difference in effect sizes between the two. However, the authors noted that virtual manipulatives offer additional benefits, such as ease of access, reduced material costs, and instant feedback [20]. Manipulatives are used across various grade levels and mathematical domains. In early childhood education, they are often employed to teach foundational concepts such as counting, addition, and subtraction [150]. Manipulatives can also be integrated into problem-solving strategies, such as Polya's four-step approach [135]. This four-step approach to problem-solving is a classic tool in mathematics education that encourages students to approach problems in a structured, systematic way. Integrating virtual manipulatives may enhance this four-step approach. They can be integrated into each step of Polya's approach as follows:

1. **Understand the Problem:** This involves reading and re-reading the problem to ensure you understand what is being asked. You might need to identify the unknowns, the data, and the conditions. Manipulatives can be used to visually represent the problem. For instance, if a problem involves dividing a set of items into groups, manipulatives can be used to illustrate this.

2. **Devise a Plan:** Once you understand the problem, the next step is to plan how to solve it. This could involve drawing a diagram, creating a table, or devising a formula. Manipulatives can help students devise a plan by allowing them to experiment with different approaches. For example, they could use a number line or blocks to explore possible solutions.

3. **Carry Out the Plan:** After devising a plan, the next step is to carry it out. This might involve calculations, constructing a model, or performing a procedure. Students can use manipulatives to execute their plans. They can move, group, and rearrange digital objects to implement their solution strategy.

4. **Look Back:** Finally, after solving the problem, take a moment to reflect on the solution. Check the answer and consider alternative approaches to the problem. This reflection can lead to deeper understanding and better problem-solving skills in the future. Manipulatives can also be used in the reflection stage. Students can replay their actions, observe what they did, and think about alternative strategies.

By combining Polya's four-step approach with manipulatives, educators can create a dynamic, engaging problem-solving experience that allows students to visualise, experiment, and reflect on mathematical concepts.

Overall, manipulatives play an essential role in mathematics education by helping students visualise abstract concepts, enhance problem-solving skills, and engage diverse learners. Research has demonstrated the effectiveness of manipulatives in improving mathematics achievement, with concrete, virtual, and digital manipulatives all offering unique benefits. As educators continue to integrate manipulatives into their instructional practices, it is crucial to stay up-to-date with the latest research findings and explore innovative ways to utilise these tools for optimal learning outcomes. Educators should consider the unique needs of their students when selecting the most appropriate manipulatives for instruction. By aligning the type of manipulative with the intended learning goals, teachers can create engaging and effective learning experiences that promote deep understanding and long-term retention of mathematical concepts. Future research on manipulatives in mathematics education should focus on identifying the most effective strategies for integrating these tools into the curriculum, as well as exploring the potential benefits of emerging technologies, such as augmented reality (AR) and VR, in enhancing the learning experience. As a complement to physical concrete manipulatives, I would like to introduce the concept of virtual manipulatives.

Following a revision of their original definition from 2002, Moyer-Packenham and Bolyard presented in 2016 an updated definition of a virtual manipulative as "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" [118]. These virtual tools have the potential to not only promote a deeper understanding of mathematical principles but also to foster a greater appreciation for the beauty and elegance of mathematics itself. In the context of VR, these virtual manipulatives become more than just two-dimensional images on a screen; they become tangible, interactive elements within a fully immersive 3D environment. This allows for a more intuitive, engaging, and accessible approach to learning mathematics, particularly for those who may have struggled with traditional teaching methods. In geometry education, virtual manipulatives provide students with interactive tools for visualising and exploring geometric concepts and relationships, allowing them to construct, manipulate, and analyse geometric figures on a digital platform. These digital tools can be accessed through various devices, such as computers, tablets, and smartphones, and are designed to enhance students' understanding and engagement in learning geometry [117]. Several studies have investigated the impact of virtual manipulatives on students' learning outcomes and attitudes in geometry education. In one study, Moyer et al. [119] examined the impact of virtual manipulatives on students' geometry problem-solving skills [119]. The researchers found that students who used virtual manipulatives exhibited a higher level of conceptual understanding and were more successful in solving geometry problems compared to those who used physical manipulatives or no manipulatives. Overall, the research suggests that virtual manipulatives can have a positive impact on students' understanding, problem-solving skills, and engagement in geometry education.

Examples of Digital Learning Tools

Digital learning tools for mathematics education, including interactive simulations and virtual manipulatives, are digital resources designed to support students' learning, understanding, and engagement with various mathematical concepts and skills. These tools help students visualise abstract ideas, explore mathematical relationships, and practice problem-solving in a dynamic, interactive environment.

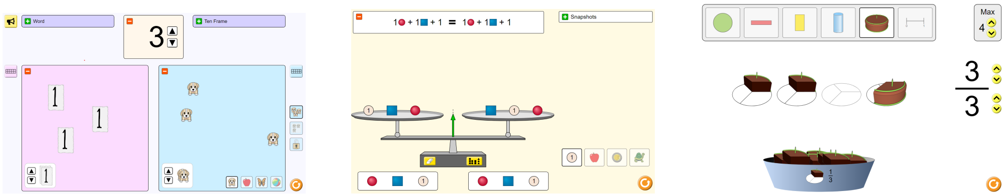
Interactive simulations for mathematics education are digital representations of mathematical concepts that allow learners to manipulate variables, test hypotheses, and observe the effects of their actions in real-time. These simulations provide a safe and controlled environment for students to experiment with mathematical principles, fostering a deeper understanding of the subject matter. Virtual manipulatives, on the other hand, are digital versions of physical manipulatives traditionally used in mathematics classrooms. They are designed to represent mathematical concepts concretely and enable students to interact with these concepts in a more tangible way. Virtual manipulatives can include digital objects such as blocks, tiles, number lines, and geometric shapes, which students can manipulate and arrange to explore mathematical ideas and relationships.

Examples of digital learning tools for mathematics education include the National Library of Virtual Manipulatives (NLVM) [2], PhET Interactive Simulations (PhET) [3], and Gizmos by ExploreLearning [4]. These platforms offer a wide range of resources that cover various mathematical topics and cater to different grade levels, making them valuable tools for both educators and students in enhancing mathematics education. PhET and NLVM are two popular online resources that offer interactive learning tools for science, technology, engineering, and mathematics (STEM) education. This section aims to provide an overview of PhET and NLVM, discuss their similarities and differences, and explore their advantages and disadvantages. Additionally, the section will present research articles that have studied the impact of these platforms on learning outcomes.

PhET, developed by the University of Colorado Boulder, is a collection of interactive simulations designed to enhance students' learning experiences in STEM subjects (see Figure 4). The simulations are designed to be intuitive, engaging, and grounded in research-based pedagogy [181]. They are available in multiple languages and are free to use for both educational and non-profit purposes.

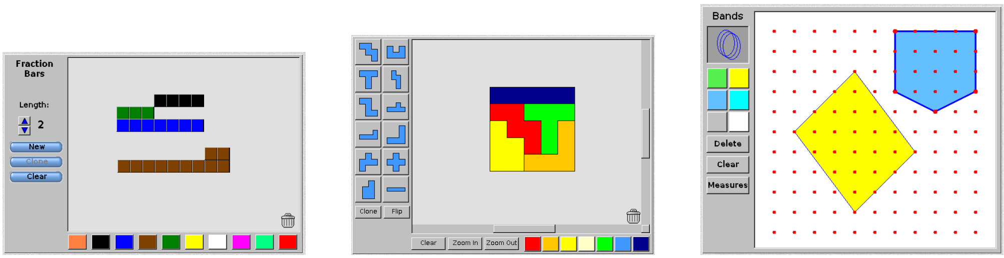
NLVM, developed by Utah State University, is an online library that offers interactive digital manipulatives for mathematics instruction (see Figure 5). It covers a wide range of topics, from numbers and operations to geometry and data analysis, for learners at various levels, from pre-kindergarten to grade 12 [119]. The resources are also available for free use by educators and students.

[2] National Library of Virtual Manipulatives: `http://nlvm.usu.edu/`
[3] PhET Interactive Simulations: `https://phet.colorado.edu/`
[4] Gizmos: `https://gizmos.explorelearning.com/`



(a) Number Play **(b)** Equality Explorer **(c)** Fractions

Fig. 4: PhET Simulations for Mathematics



(a) Fraction Bars **(b)** Pentominoes **(c)** Geoboard

Fig. 5: Virtual Manipulatives from NLVM

Both PhET and NLVM are designed to facilitate active learning and enhance student engagement in STEM subjects. They provide interactive digital resources that are research-based, free to use, and accessible on various devices, including computers, tablets, and smartphones. Furthermore, both platforms offer resources in multiple languages, catering to diverse learners worldwide. While PhET primarily focuses on simulations for physics, chemistry, biology, earth science, and mathematics, NLVM exclusively targets mathematics, offering a broad range of topics within the subject. Additionally, PhET simulations are typically more visually appealing and dynamic, making them more engaging for learners [115]. On the other hand, NLVM resources are more simplistic, focusing primarily on mathematical concepts and skills [119]. PhET simulations have been shown to enhance conceptual understanding, reduce misconceptions, and promote inquiry-based learning [41, 181]. However, they may require higher bandwidth for smooth functioning, and students might need some guidance to effectively use the simulations [115]. NLVM resources have been found to promote mathematical reasoning, problem-solving, and communication skills [119]. However, they might not be as engaging as PhET simulations due to their relatively simple interface, and the lack of support for subjects other than mathematics could be a limitation.

Both PhET and NLVM offer valuable interactive resources for STEM education. While PhET provides visually engaging simulations for various STEM subjects, NLVM focuses exclusively on mathematical concepts and skills. Educators and learners can benefit from these platforms by selecting the resources that best align with their learning objectives and teaching methods.

In Paper E, we introduced a taxonomy of learning environments within the realm of geometry education. This system incorporated a novel category referred to as the 'Immersive Geometry Environment', constructed upon a foundation of both research and advanced technology. For the purposes of this thesis, we describe this new class of environment as an IVE. It is characterised by the inclusion of interactive elements, devised as virtual manipulatives. We introduced a new term for these elements - *Immersive Virtual Manipulatives.* Immersive Virtual Manipulatives merge the tangible benefits associated with traditional, concrete manipulatives and the dynamic qualities intrinsic to their virtual counterparts. By integrating the tangible aspects of concrete manipulatives and the digital attributes of virtual manipulatives, IVMs may provide an interactive learning environment within a VR context. IVMs may bring together the best of both worlds, allowing users to actively manipulate virtual objects within an immersive VR environment. They may harness the engagement potential of VR, thus offering a highly interactive, immersive, and stimulating learning experience that can improve mathematics education.

One example of an immersive virtual manipulative for mathematics is a virtual geoboard. A geoboard is a mathematical manipulative tool used to explore basic concepts in plane geometry such as shapes, angles, and area. Within a VR environment, students can pull and stretch virtual rubber bands around pegs on a virtual geoboard, experiencing the spatial aspects of geometric shapes in a 3D setting. This may provide a more robust understanding of geometric concepts compared to traditional 2D representations. Another example is the use of immersive base-ten blocks for understanding place value, addition, and subtraction. In a VR setting, students can grasp, move, and assemble virtual blocks, simulating the experience of handling actual base-ten blocks, but with the added capability to easily alter their configurations or quantities. Virtual fractions circles provide another illustration of IVMs for mathematics education. Traditionally, fraction circles are physical manipulatives divided into equal parts to illustrate the concept of fractions. Using VR technology, students can manipulate these circles, combine fractional parts, or compare different fractions in an engaging and visually captivating way.

IVMs provide a wide range of opportunities for mathematics education. By embodying the advantages of both concrete and virtual manipulatives, they may offer enhanced interactivity, engagement, and flexibility. The dynamic and immersive nature of IVMs may help learners to better visualise and understand abstract mathematical concepts, fostering a deeper and more lasting learning experience. As VR technology continues to advance and become more accessible, the potential of IVMs in mathematics education will undoubtedly expand. In the upcoming section, I will present and compare a range of commercially available immersive learning tools specifically designed for mathematics education.

Examples of Immersive Learning Tools

Calcflow [5] is a unique immersive learning tool designed for visualising mathematical functions in three dimensions through a VR interface (see Figure 6). Its main strength lies in this immersive 3D environment, which bolsters understanding and interaction with mathematical constructs. In this interactive 3D space, complex concepts become more intuitive, enriching the user's comprehension. However, Calcflow's specialisation in mathematical functions also serves as a limitation as it does not extend to other subjects.

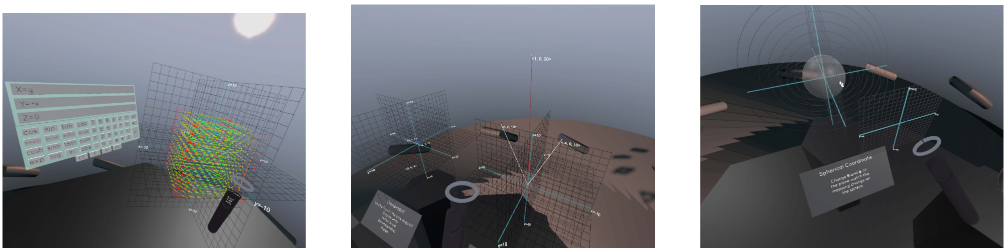


Fig. 6: CalcFlow

GeoGebra Mixed Reality [6], similar to Calcflow, focuses on the 3D visualisation of mathematical concepts, aiming to simplify the learning of complex mathematical topics (see Figure 7). Offering a range of tools for various mathematical disciplines like graphing, geometry, algebra, calculus, and probability, it extends the range of potential users. However, some users might find GeoGebra Mixed Reality's interface difficult to navigate, presenting a steep learning curve. While the application initially showed promise upon its release, it has since failed to deliver any updates and continues to fall short in terms of features when compared to the original GeoGebra.

Differing from Calcflow and GeoGebra Mixed Reality, Math World VR [7] explores the gamification of mathematical learning (see Figure 8). Rather than solely focusing on 3D visualisation, it integrates mathematical concepts into a game-like environment, aiming to make learning both enjoyable and engaging. The game-based design, with its multiple-choice questions and game mechanics such as shooting arrows or throwing balls, might appeal to a broad and younger audience.

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[5] Calcflow: https://store.steampowered.com/app/547280/Calcflow/

[6] GeoGebra Mixed Reality: https://store.steampowered.com/app/880270/GeoGebra_Mixed_Reality/

[7] Math World VR: https://www.oculus.com/experiences/quest/4923914040997217/

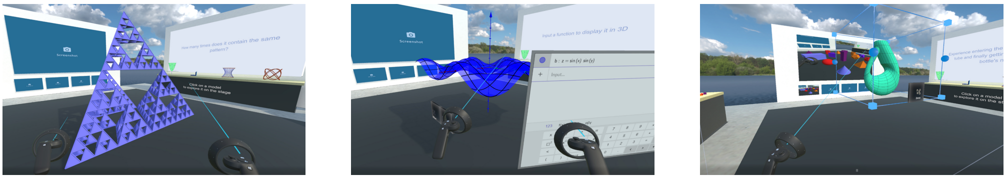


Fig. 7: GeoGebra Mixed Reality

Fig. 8: Math World VR

However, this approach has its drawbacks. The depth of learning in Math World VR could be compromised due to its primary focus on engagement and fun. Also, learners who might find the game mechanics challenging could struggle to focus on understanding the mathematical concepts, as their learning experience is closely tied to their performance in the game elements.

NeoTrie VR [8] is an extensive VR software designed to aid the teaching and learning of 2D and 3D geometry. This software offers a highly immersive and interactive environment where users can create, manipulate, and interact with various geometric objects and models, supplemented with text, photos, videos, and sounds (see Figure 9).

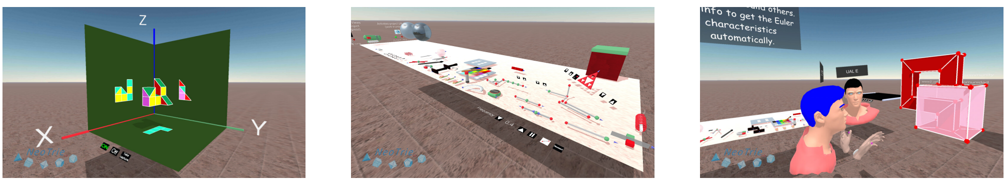


Fig. 9: NeoTrie VR

NeoTrie VR's objectives span from visualising plane geometry in three dimensions to stimulating deductive reasoning and fostering cooperative work among students. The

[8]NeoTrie VR: https://store.steampowered.com/app/878620/Neotrie_VR_Multiplayer/

platform covers a wide range of geometric topics such as angles, volumes, polyhedra, and Euler and Hamiltonian graphs in space. Unique features like a 3D graphic calculator and the manipulation of round bodies enrich the learning experience. Key features of NeoTrie VR include galleries of pre-designed geometric figures, a speech recognition system for hands-free figure insertion, and a multiplayer tool for collaborative learning. Additionally, users can import-export objects from other software and document their learning process through photos and video recordings within the VR scene. In summary, NeoTrie VR is transforming geometry and 3D modelling education by transforming abstract concepts into tangible, immersive experiences, heralding a new era in virtual reality education.

Calcflow, GeoGebra Mixed Reality, Math World VR, and NeoTrie VR each offer a distinct approach to mathematical education. While Calcflow and GeoGebra Mixed Reality strive to provide a detailed, immersive experience through 3D visualisation, Math World VR leans into the gamification of learning, providing a fun and engaging way to grasp mathematical concepts. Adding to these tools, NeoTrie VR offers the most extensive feature set of VR software for geometry and 3D modelling. Users can interact with various geometric objects and models, fostering deductive reasoning, 3D visual skills, and collaborative learning among students. The best fit among these products would depend on student needs, learning objectives and resource availability. In the forthcoming section, I will introduce the concept of instrumental genesis, which holds the potential to assist educators in crafting impactful learning tools.

Instrumental Genesis

Instrumental genesis is a vital concept in mathematics education, particularly concerning the use of digital technologies and mathematical tools. Developed by scholars such as Luc Trouche [169] and Paul Drijvers [36], it investigates how learners develop or generate mathematical knowledge and techniques using these tools, and how the tools themselves are adapted and modified by the users over time [7]. The term instrumental genesis arises from the amalgamation of two primary concepts: *instrumentalization* and *instrumentation*. Instrumentalization refers to the technical dimension: the transformation of an artefact into an instrument of a specific mathematical activity [169]. It entails gaining proficiency in the technical operations needed to employ the tool effectively. On the other hand, instrumentation addresses the psychological and cognitive dimension [36]. It involves developing particular mathematical strategies, insights, and understanding through interaction with the instrument [138]. Together, these two processes lead to what is known as *instrumental orchestration* [36]. This is the manner in which educators design and implement the use of technological tools in the classroom to support learning and teaching activities. Successful instrumental orchestration allows learners to explore, conjecture, experiment, and prove in mathematical problem-solving situations. The instrumental genesis process is dynamic and relies on various factors, including the mathematical task at hand, the tool's characteristics, the student's prior knowledge and experience, and the learning context [176]. This concept is particularly relevant in today's classrooms, where technology has an instrumental role in education [7]. Understanding instrumental genesis can guide educators in crafting effective teaching strategies and learning environments that leverage digital tools while fostering profound mathematical understanding. However, it is vital to acknowledge that the process of instrumental genesis is not always linear. Learners can encounter instrumental obstacles or difficulties in morphing the artefact into an instrument for mathematical problem-solving [169]. These obstacles could be related to the tool itself, the learner's perceptions and attitudes, or the nature of the task. Recognising these hurdles and devising strategies to overcome them is an integral aspect of effective mathematics instruction. Ultimately, mastering instrumental genesis could empower learners to adapt to new tools and technologies, broadening their mathematical horizons and equipping them with the skills needed for the digital age [36].

VR technology presents a new and promising field for the application of instrumental genesis in mathematics education. VR, by offering an immersive, interactive, and 3D learning environment, may amplify the potential for instrumental genesis, enhancing learners' comprehension and engagement with mathematical concepts. When considering instrumental genesis in VR environments, the interplay between instrumentalization and instrumentation remains central. Instrumentalization involves learning to manipulate VR tools effectively, while instrumentation involves developing strategies for leveraging VR tools to enhance mathematical understanding. In VR settings, instrumentalization might involve learners becoming familiar with VR hardware, such as HMDs, hand controllers, and haptic devices. It also involves understanding how to navigate and interact within VR environments, including how to use virtual tools to manipulate mathematical objects or datasets. Instrumentation, on the other hand, may involve learning to apply these tools and interactions to solve mathematical problems. For instance, a student might use a virtual tool to visually manipulate a 3D geometric object, fostering a deeper understanding of its properties. The concept of instrumental orchestration also has implications for VR-based learning. Educators will need to design virtual learning environments that facilitate effective interactions and exploration [36]. This might involve considering the layout of the virtual space, the types and accessibility of virtual tools, and how virtual interactions can be guided to promote mathematical learning. However, instrumental genesis in VR also presents unique challenges. Users might encounter novel instrumental obstacles related to the immersive nature of VR. For example, navigating in three dimensions or manipulating virtual objects can be initially disorientating or challenging for some learners. Overcoming such obstacles will require innovative instructional strategies and ongoing user support. Despite these challenges, the potential for VR to enhance instrumental genesis is fascinating. By creating immersive and interactive learning experiences, VR can potentially facilitate a deep, intuitive understanding of mathematical concepts. It can also encourage engagement and motivation, making learning more enjoyable and rewarding. Further research is needed to understand the specific processes and impacts of instrumental genesis in VR settings. However, the existing body of knowledge provides a strong theoretical framework for guiding this exploration and for informing the design of effective, VR-based learning tools and environments. However, designing effective learning experiences in VR also relies on the principles of instructional design. In the upcoming section, I will present instructional design strategies relevant to VR.

2.4 Educational Approaches

In this section of the thesis, I explore the rich landscape of educational approaches to learning in VR, with a crucial focus on the roles of pedagogy and didactics in mathematics education. The immersive nature of VR presents a myriad of opportunities for educators to revolutionise the way mathematics is taught and understood. Pedagogy, embodying diverse teaching strategies to promote mathematical comprehension and problem-solving skills, and didactics, centred on the precise methods of teaching mathematics, form the backbone of these innovative practices.

As I investigate the potential of these virtual spaces, I will concentrate on five prominent educational approaches: constructivism, situated learning, exploratory learning, embodied learning, and game-based learning. Each of these approaches, influenced by the constructivist theory which suggests learners construct their own understanding of mathematical concepts based on their existing knowledge, offers unique insights into how learners can actively engage with mathematical concepts in a VR context. Additionally, I will present strategies related to generative learning, aimed at optimising the learning outcomes of mathematics education in VR.

Didactics also involves the use of manipulatives—tangible objects that aid the understanding of abstract mathematical notions—which are potentially brought to life in new ways within virtual environments. By examining the theoretical foundations, practical applications, and existing research on these approaches, I aim to shed light on the potential benefits and challenges of incorporating them into mathematics education within virtual environments. Ultimately, this section will provide an overview of the educational approaches to designing VR experiences for mathematics education.

Constructivism

Constructivism is a widely adopted learning theory that posits that learners actively construct their own knowledge by interacting with the environment and reflecting upon their experiences [128, 133, 180]. Rooted in Papert's [128] work, constructionist learning theory supports knowledge development through the hands-on creation of physical or digital artefacts for shared experiences. This approach motivates learners to design within immersive environments, leading to heightened engagement and retention. For instance, in a classroom, students might design a "monster truck" for each planet in our solar system, learning about planetary characteristics and engineering principles in the process [102]. This interactive learning style, which leverages tutorials and educational materials from immersive technology manufacturers, encourages self-authorship and can be initiated at younger ages than traditionally expected. However, teachers' proficiency in relevant technology, effective instructional strategies, and a curriculum that ties constructionist experiences to academic objectives are vital, echoing Laurillard's argument on the importance of pedagogy in technology-enhanced learning [95]. Therefore, constructionist learning, with its synergy between creativity, immersive experiences, and academic rigour, offers a potent, engaging approach to education.

In the context of mathematics education, constructivism emphasises problem-solving, student-centred learning, and the development of conceptual understanding [151]. The emergence of VR technology has provided a novel platform for implementing constructivist approaches to mathematics education. This section presents recent research on the application of constructivist principles in the context of learning mathematics in VR environments. VR, as an immersive and interactive technology, provides a unique opportunity to foster constructivist learning in mathematics education. According to Dalgarno and Lee [27], VR environments allow learners to experience complex phenomena and abstract concepts in a more concrete and accessible way. Research indicates that embodiment plays a critical role in mathematical learning, particularly in areas such as geometry and spatial reasoning [94]. VR technology, which allows for natural and intuitive interactions with digital objects, has the potential to support the development of embodied mathematical understanding.

In conclusion, VR offers a promising platform for implementing constructivist approaches to mathematics education. The immersive, interactive, and embodied nature of VR environments can facilitate student-centred learning, collaboration, and the development of conceptual understanding. Future research should continue to explore the potential constructivist approaches to VR in mathematics education, as well as address the challenges and limitations associated with its implementation.

Situated Learning

Situated learning is an educational approach that emphasises the importance of context, social interaction, and authentic activities in the learning process [96, 102]. This approach, where knowledge acquisition occurs within a relevant context, has seen applications across various fields, from medical internships to mathematics education. In recent years, the application of VR technology has gained traction in education, offering new ways to engage students and facilitate situated learning experiences [111]. The principle of situated learning facilitates a more authentic, comprehensive, and effective learning experience by integrating theoretical concepts and their real-world applications. In mathematics education, the potential of situated learning lies in bridging the gap between abstract mathematical concepts and their practical implementations. This can be achieved by situating mathematical instruction within the context of real-world scenarios, enabling students to visualise and apply these concepts. Additionally, collaborative problem-solving activities provide a social context that promotes communication, logical reasoning, and potentially a deeper understanding of mathematical principles. The advent of VR technology has provided a new platform for the implementation of situated learning. With its capacity to replicate realistic, immersive environments, VR may amplify the learning experience in both medical training and mathematics education. In medical training, VR can simulate the high-pressure environment of a surgical operating room, allowing interns to practice procedures and develop crucial tacit skills in a safe and controlled setting. Conversely, in mathematics education, VR can create interactive scenarios that situate mathematical concepts in engaging, real-world contexts. For instance, a virtual supermarket can help students understand the application of percentages and fractions, while a virtual construction site can provide a practical setting for exploring geometric principles.

Despite its promising potential, integrating VR into situated learning presents challenges such as resource requirements, the need for educator training, and the development of suitable pedagogical strategies. Therefore, careful planning, innovative teaching methods, and empirical evaluation of its effectiveness will be necessary to optimise its implementation. In essence, situated learning, whether in a physical setting or within a VR environment, emphasises context and practical application. By bridging the gap between theory and practice, it offers a robust framework for fostering a more comprehensive learning experience, thus revolutionising traditional teaching and learning methods across various domains.

Exploratory Learning

Exploratory learning in VR offers a transformative educational experience, combining immersive technology with hands-on discovery. By simulating diverse environments and scenarios, VR enables learners to actively explore and interact with their surroundings, potentially fostering curiosity, engagement, and deep understanding. This innovative approach encourages experimentation, problem-solving, and critical thinking, empowering individuals to learn through direct experience and trial-and-error. With its ability to transport learners to new realms and provide dynamic feedback, exploratory learning in VR revolutionises traditional education, opening up endless possibilities for experiential learning and knowledge acquisition.

Kolb's experiential learning cycle, a pedagogical model highlighting the importance of experience in learning, comprises four stages: concrete experience, reflective observation, abstract conceptualisation, and active experimentation [90]. Learners first engage in a direct experience, reflect on it, develop theories based on these reflections, and then apply these theories in new situations, illustrating learning as an iterative process. VR may enhance each stage of this cycle: it offers immersive simulations for direct experiences, tools for reviewing interactions to aid reflection, visualisation of abstract concepts for better understanding, and an environment for testing theories in real-time. Hence, VR offers a realistic, safe, and controllable way to enrich experiential learning.

An extended version of Kolb's original cycle has been developed to establish a stronger connection between the experiential learning cycle and VR and immersive learning.

The Exploratory Learning Model (ELM), proposed by Freitas and Neumann [28], presents a fresh perspective on the use of immersive learning in virtual environments. It is grounded in the belief that the process of learning can be enhanced by providing learners with opportunities for exploration and discovery, thereby facilitating a more engaging and interactive learning experience. The ELM emphasises the importance of open-ended exploration, where learners are allowed and encouraged to freely navigate and interact with their learning environment. The model asserts that this exploratory approach can lead to a deeper understanding of the subject matter, as it encourages learners to construct their own knowledge and understanding through self-directed investigation and inquiry. In the cycle of learning proposed by ELM (see Figure 10), the first step is the immersion of the learner in a richly designed experience. This experience is situated within a virtual environment where learners can freely interact, manipulate virtual objects, navigate, and communicate, thus actively engaging with the learning material. Following the immersive experience, the second step is exploration, where learners undertake a self-guided journey of discovery within the virtual environment. This open-ended exploration encourages the learners to construct their own knowledge and understanding. Next, the third step is reflection, which involves learners contemplating their actions, experiences, and the feedback received during the exploration phase. This act of reflection cultivates the development of critical thinking skills, as learners have the opportunity to analyse their experiences and consider their implications. Building on the reflection step, the fourth stage involves forming abstract ideas. During this phase, learners are encouraged to consolidate their experiences and reflections into abstract concepts, thereby deepening their understanding of the subject matter. This step promotes the cognitive process of linking new information to pre-existing knowledge, thus enabling learners to create a broader and more integrated understanding of the topic. The final step in this cycle is testing the abstract ideas in different environments. Here, learners apply their newly formed knowledge and skills in varied contexts, reinforcing their understanding and enhancing the transferability of what they've learned. This step not only reinforces the learning but also provides an opportunity for learners to validate and refine their understanding, thus completing the learning cycle and preparing for a new cycle, to begin with further experience.

ELM may serve as a powerful pedagogical tool for mathematics education. Its constructivist approach, which emphasises self-directed exploration and discovery, aligns well with contemporary goals for math education. Rather than having students memorise formulas and procedures, the model's open-ended exploration encourages learners to construct their own understanding of mathematical concepts, thereby fostering a deeper comprehension of mathematics. The virtual environment can be customised to present complex mathematical models and theories in an accessible and interactive format, enabling learners to manipulate virtual objects and scenarios that embody these concepts.

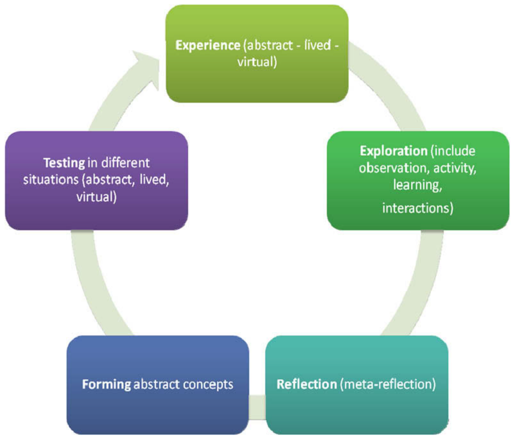


Fig. 10: The Exploratory Learning Model

This promotes an active learning experience, where learners are directly engaging with the math material. The reflection step in ELM can be used to nurture critical thinking skills in mathematics, by encouraging learners to contemplate and analyse their actions, observations, and the feedback they receive during the exploration phase. As they form abstract ideas, learners consolidate their experiences and reflections into mathematical concepts, deepening their understanding of the subject matter and establishing connections with their prior knowledge. The final step, testing abstract ideas in different environments, allows learners to apply mathematical concepts and skills in diverse situations. This may not only reinforce their understanding but also enhance the transferability of mathematical knowledge, skills, and dispositions beyond the classroom. Therefore, the ELM provides a useful framework for immersive learning in mathematics education, potentially creating a more engaging, effective, and meaningful learning experience.

Embodied Learning

Embodied learning in mathematics education is a pedagogical approach that involves the mind and body in the learning process. This theory is based on cognitive science, which posits that cognition is not solely situated within the brain, but also involves the body and its interactions with the environment [182]. This approach to mathematics education aims to translate abstract mathematical concepts into tangible and accessible experiences through physical manipulation and activity. Students engage in activities that link mathematical ideas to bodily movements and sensations, like gestures, manipulatives, and physical interactive technology such as touchscreens or VR [4]. For instance, students might demonstrate the concept of a slope using their hands, or use their fingers to model multiplication [125]. Real-world contexts that can be physically experienced are also incorporated, providing a tangible representation of mathematical concepts [110]. For instance, teaching fractions might involve dividing a pizza or a pie, providing students with a visual, tactile experience that reinforces the concept. Technological platforms, such as VR, are becoming instrumental in facilitating embodied learning [29]. These platforms provide immersive and interactive experiences, allowing students to manipulate and visualise mathematical concepts in 3D space. Research indicates that this approach may improve mathematical learning, especially for concepts traditionally challenging to grasp through conventional teaching methods [1]. The embodied learning approach aids in connecting abstract mathematical concepts to tangible bodily experiences, fostering a deeper understanding and promoting the retention of mathematical knowledge. However, it is important to acknowledge that effective embodied learning requires careful planning and implementation. Educators need to design activities that accurately represent mathematical concepts, are aligned with students' cognitive development, and be cautious about potential misconceptions that could stem from oversimplified bodily representations [122].

VR applications can be designed to represent mathematical concepts in ways that encourage students to use their bodies to interact with the virtual environment. For instance, students can move along a virtual number line, physically shift geometric shapes in space, or manipulate graphs of functions with their hands. Such experiences may foster a deeper understanding of mathematical concepts by allowing students to directly perceive and explore their spatial and dynamic properties [73]. Moreover, VR can facilitate the use of multimodal representations—visual, auditory, and kinesthetic—which can enhance students' comprehension and recall of mathematical concepts [116]. For example, in a VR learning environment, students could hear a tone that changes pitch according to the value of a function while they see and manipulate its graph. VR also enables educators to design learning experiences that would be difficult or impossible in the real world, helping students explore mathematical concepts from novel perspectives. For example, students can "walk inside" a 3D graph or manipulate geometric objects in ways that defy the laws of physics, providing a fresh perspective on familiar mathematical ideas [160]. Despite these potential benefits, the implementation of VR in mathematics education should be carefully planned to avoid cognitive overload, which can occur when the learner is presented with too much information at once [167]. Instructional design principles should be adhered to, ensuring that the VR environment effectively facilitates learning and does not distract students with irrelevant information.

Gesture-based interactions offer a natural, intuitive way of exploring complex mathematical concepts. Combining hand-tracking and VR technology may be a powerful approach for supporting embodied learning of mathematics, as it allows students to actively engage with mathematical concepts in a way that is more concrete and meaningful. Hand-tracking and VR technology could be used to create virtual manipulatives, which are digital versions of physical manipulatives that students can use to explore mathematical concepts and solve problems. These virtual manipulatives can be manipulated using hand gestures, allowing students to feel as though they are interacting with real objects in a virtual environment. Hand-tracking and VR technology can also be used to create immersive environments in which students can explore and manipulate 3D objects and spaces. This can be especially useful for teaching spatial visualisation skills, as it allows students to experience and manipulate geometric objects and concepts in a way that is more concrete and meaningful. Hand-tracking and VR technology may also be used to support collaborative learning in mathematics through non-verbal communication. For example, students can work together in a VR environment to solve problems, share ideas, and communicate their reasoning to each other through gesturing combined with speech. This could be especially useful for developing problem-solving skills and collaborative learning skills. Overall, combining hand-tracking and VR technology may be an effective approach for supporting embodied learning of mathematics, as it allows students to actively engage with mathematical concepts and explore them in a way that is more concrete and meaningful.

In the following paragraphs, I will propose several design ideas for gesture-based interactions with mathematical objects in VR environments.

Manipulating 3D Graphs: VR offers an immersive and interactive way for users to engage with 3D graphs of functions, such as surfaces or parametric curves. By using hand gestures to grab, rotate, scale, and translate these objects, users gain a deeper understanding of the underlying mathematical structures and relationships. The ability to view graphs from various angles and distances allows users to observe the behaviour of functions across different domains, providing a more comprehensive and tangible understanding of the mathematical concepts at play.

Constructing Geometric Objects: VR tools empower users to build and manipulate geometric objects, such as triangles, polyhedra or tessellations, through gesture-based interactions. These interactions might involve selecting vertices or edges with a pinch gesture, stretching or rotating objects by grabbing and moving or snapping objects together with a flick of the wrist. These hands-on experiences facilitate the development of a deeper understanding of geometric transformations, symmetries, and spatial reasoning by allowing users to directly engage with the geometric concepts.

Proof Visualisation: VR can be an invaluable tool for visualising and interacting with visual proofs for mathematical theorems. By manipulating geometric constructions, such as moving points or lines, users can observe how the properties of the proof change. Gestures like dragging, rotating, and scaling allow for the rearrangement of proof elements, enabling users to better comprehend the logical structure and the underlying mathematical principles that form the foundation of the proof.

Collaborative Problem Solving: Multi-user VR environments provide a unique opportunity for groups of users to work together in solving mathematical problems. By jointly manipulating objects and sharing visualisations, users can draw attention to specific elements using gestures like pointing, circling, or tapping. Natural hand movements may facilitate collaborative exploration and communication, potentially fostering a sense of teamwork, social interaction, and shared understanding as users tackle complex mathematical challenges.

Dynamic Calculus Demonstrations: Gesture-based interactions offer an engaging method for controlling dynamic visualisations of calculus concepts, such as the behaviour of functions, derivatives, and integrals. Users can adjust parameters, like the position of a tangent line or the bounds of an integral, using simple hand movements. This hands-on approach helps users build an intuitive understanding of the relationships between functions and their derivatives, as well as the concept of the area under a curve, by providing a tangible way to engage with these abstract mathematical ideas.

Exploring Fractals: Fractals are complex geometric shapes that display self-similarity at different scales. VR environments enable users to navigate through these fascinating structures using hand gestures, such as pinching to zoom in and out or swiping to pan and rotate the view. This interactive exploration of fractals cultivates an intuitive grasp of concepts like recursion, infinite detail, and the relationship between scale and dimension, as users can physically engage with the structures and observe their properties firsthand.

In summary, gesture-based interactions with mathematical objects in VR environments offer a powerful and intuitive way to explore complex mathematical concepts. By providing users with the ability to manipulate objects and visualise relationships in a hands-on manner, VR has the potential to revolutionise the way we teach and learn mathematics.

An alternative instructional design approach, with the potential to complement and enhance other instructional strategies, involves game-based learning. This innovative approach incorporates elements of gaming into the educational process, making it engaging and immersive for learners. In the subsequent section, I will further explore this approach.

Game-based Learning

The utilisation of game-based learning in education has gained attention in recent years, offering innovative approaches for enhancing student engagement and learning outcomes. Incorporating this approach into mathematics education could transform the way students perceive mathematical concepts, making them more tangible and engaging through the immersive experience offered by VR. It would provide an environment where abstract mathematical concepts could be visualised, thus enhancing understanding and retention.

Serious games, as defined by Michael and Chen [112], are digital games used for purposes other than mere entertainment, particularly for education and training. These games have been adopted in various educational contexts, from K-12 education to higher education and professional training, demonstrating positive outcomes in fostering students' problem-solving abilities, critical thinking skills, and collaborative skills [141]. In mathematics education, serious games could serve as a practical tool to facilitate the learning of complex mathematical problems. Through the manipulation of virtual manipulatives in mathematical situations, students may be able to actively explore mathematical principles and engage in problem-solving activities, hence promoting experiential learning.

On the other hand, gamification, the application of game-design elements in non-game contexts, has been widely implemented in education to motivate and engage students [30]. The application of gamification elements such as points, badges, leaderboards, and quests has been found to promote student engagement and motivation, enhance peer interaction, and improve learning outcomes [51, 52]. When applied to mathematics education, gamification could help in overcoming the common perception of mathematics as a daunting subject. Gamifying math learning experiences can motivate students to challenge themselves, cooperate with peers, and maintain a positive attitude towards the subject, which is essential for their mathematical proficiency.

Recent studies have emphasised the potential of serious games and gamification in facilitating personalised learning experiences. For instance, Barata et al. [10] highlighted the use of gamified systems to adapt to individual learners' needs, thereby increasing their intrinsic motivation and enhancing learning experiences. Likewise, serious games, when designed to adjust according to the learner's progress, can cater to individual learning needs, fostering self-directed learning [170]. This personalisation is particularly valuable in mathematics education, where the learning curve can differ greatly among students. A tailored VR math game can adapt to the learner's pace and proficiency, offering a flexible and inclusive learning environment that caters to diverse student needs.

However, the integration of serious games and gamification in education is not without challenges. Issues related to design, implementation, and evaluation persist, with the need for further research on their long-term effects on learners, potential risks, and ethical considerations [149]. In the context of VR in mathematics education, it is vital to address these challenges effectively to ensure the optimal use of these technologies. Researchers and educators need to consider how best to design these applications to align with curriculum standards, address potential equity issues around access to technology and assess their long-term effectiveness in improving students' math proficiency.

The proliferation of digital technologies and the ongoing evolution of pedagogical methods suggest a promising future for serious games and gamification in education.

As these approaches become increasingly intertwined with educational practice, further empirical research and comprehensive analyses are warranted to provide insights into their full potential and limitations. This applies to mathematics education as well; as we venture into this new era of digital learning, the potential of VR to transform mathematics teaching and learning is immense. However, thorough research and careful implementation are crucial to fully realise these opportunities and navigate potential pitfalls.

A method for effectively navigating pitfalls is to prioritise the cultivation of knowledge and its application across different contexts. This approach aligns with generative learning strategies, which, when employed correctly, have the potential to optimise the learning outcomes associated with using VR in mathematics education.

Generative Learning Strategies

Generative learning strategies are an innovative approach to education that seeks to foster learning through active engagement. Rather than the traditional model of passive absorption, these strategies aim to involve learners in the production of their own knowledge, leading to a greater depth of understanding [43]. Several generative learning strategies are known to promote the active processing of information. These include *summarizing*, *drawing*, *self-questioning*, *teaching*, and *enacting*. For instance, the strategy of summarizing information encourages learners to condense the main ideas from a text into their own words. This process helps learners to engage actively with the material, thereby improving retention and understanding [171]. Drawing is another effective generative learning strategy. This involves creating visual representations of the material, which can help to clarify complex concepts and processes. A study by Fiorella and Mayer [42] demonstrated that learners who observe an instructor drawing diagrams promote learning through principles of multimedia learning and social cues associated with the presence of the instructor's hands. Furthermore, generative strategies also include teaching and enacting. When students teach the material to their peers, they are forced to deepen their understanding and organise their knowledge into a coherent and understandable format [148]. Similarly, enacting - physically acting out the learned material - can increase comprehension, particularly for kinesthetic learners [104].

Fiorella and Mayer's eight strategies of generative learning include *summarizing*, *mapping*, *drawing*, *imaging*, *self-testing*, *self-explaining*, *teaching*, and *enacting* [43]. These strategies may be possible to innovatively apply in VR for the purpose of fostering active engagement and improving learning outcomes. In the subsequent list, I will attempt to exemplify how these strategies may be incorporated into VR learning experiences.

1. **Summarizing**: After an immersive VR experience, learners can be prompted to summarise the events or information. The spatial, multisensory nature of VR can aid memory recall and help to facilitate effective summarisation.

2. **Mapping**: VR allows learners to map out concepts in a 3D space, which can enhance the understanding of complex systems or structures. For example, in biology, students can map the structure of a cell or a DNA molecule within a VR environment.

3. **Drawing**: VR, especially with hand-tracking interfaces, can offer opportunities for learners to draw and illustrate concepts in a 3D space. This can aid in understanding geometrical concepts, architectural designs, or molecular structures.

4. **Imaging**: In VR, learners can use the technology to imagine and create visual scenarios. For instance, VR can allow students to walk through a historical event, providing a rich sensory experience that can be recalled and explained later.

5. **Self-Testing**: VR can provide interactive quizzes and simulations where learners can test their knowledge in an immersive environment. This can help learners monitor their understanding and get immediate feedback.

6. **Self-Explaining**: Learners can verbalise their thought processes or explain what they are learning while engaged in VR experiences. This can be facilitated by VR environments that support audio recording or text input.

7. **Teaching**: Multi-user VR environments can provide platforms for students to teach their peers in immersive settings. By using VR tools and objects to illustrate concepts, learners can reinforce their own understanding.

8. **Enacting**: VR is an ideal platform for embodied learning. Learners can physically enact or simulate scenarios in the virtual world, enhancing their grasp of certain concepts, especially those involving movement or space.

In a series of studies, the implications of integrating VR with generative learning strategies were examined for their effectiveness in education. Klingenberg et al. [85] focused on the use of VR and generative learning strategies in teaching biochemistry to undergraduates. Their experiment's findings highlighted the improved learning transfer, retention, and self-efficacy in VR, especially when paired with teaching, compared to non-immersive virtual environments. Students preferred VR when they switched media, revealing improved motivation, enjoyment, and presence. Similarly, Andreasen et al. [6] probed the efficacy of immersive VR versus video as teaching tools, especially when used in conjunction with the enactment method. Although there were no significant differences in declarative knowledge or knowledge transfer across the groups, they found that the group using VR and the enactment method demonstrated superior procedural knowledge and enjoyed the VR learning experience more than the video. Adding a psychological dimension, Yang and Wang [185] investigated how generative learning strategies in a VR-based environment interact with cognition and emotion. Participants studied a chemical lesson under three conditions: VR alone, VR with summarizing, and VR with self-testing. The results demonstrated that self-testing as a generative learning strategy in VR environments elicited more positive emotions during the learning process, led to more positive post-learning ratings, and resulted in higher memory test scores. The summarising strategy also induced positive emotions, although it led to lower immediate memory scores.

The integration of VR with generative learning strategies, as suggested by various studies, offers a promising enhancement to different aspects of learning. This spans from knowledge retention and transfer to the enjoyment and emotional experience of the learning process. Learners can augment their understanding post-VR experience by engaging with these generative strategies, for instance, summarising the learned content, self-testing their retention, or even teaching others. Additionally, they can revisit the VR environment to indulge in self-explaining, enacting, or drawing, hence capitalising on the immersive and interactive nature of VR. This could potentially deepen comprehension and improve retention. However, further research is necessary to optimise these strategies across diverse subject matters and student populations. Ultimately, generative learning strategies represent a powerful toolset for educators looking to inspire learners and foster their conceptual understanding, and their intersection with VR technology may offer novel ways to achieve these educational goals.

To gain an understanding of the impacts of integrating generative learning strategies, it is crucial to delve into the assessment and measurement of learning experiences and outcomes in VR. In the forthcoming section, I will shed light on this topic by presenting pertinent research on various metrics. These metrics encompass usability, user experience, immersive learning, and mathematical competencies, thereby elucidating the significance of assessing and quantifying the effectiveness of VR-based learning.

2.5 Measuring Outcome

VR has increasingly been employed as an educational and training tool due to its immersive and interactive nature, which has the potential to improve knowledge transfer compared to traditional learning methods [111]. In order to evaluate the effectiveness of VR-based instruction, it is crucial to examine the methodologies employed for metrics such as user experience, knowledge transfer and competency development.

Pre- and post-test assessments are a common method for measuring knowledge acquisition in VR-based training [108]. These assessments evaluate the participants' knowledge before and after the VR training, allowing researchers to determine the extent of learning that has occurred. The advantage of this method is that it provides a direct measure of learning outcomes. However, pre- and post-test assessments may not fully capture the transfer of skills to real-world situations. Skill transfer tests assess the participant's ability to apply the skills learned in the virtual environment to real-world tasks [47]. These tests measure performance in terms of accuracy, efficiency, and speed. Skill transfer tests provide a more direct measure of knowledge transfer, as they assess the application of learned skills in real-world contexts. However, the design and implementation of these tests can be complex and resource-intensive. Self-reported questionnaires ask participants to rate their perceived learning, confidence, and the ease of transferring the knowledge gained in VR to real-world tasks [15]. This method provides insights into the subjective experiences of learners and may reveal factors that contribute to successful knowledge transfer. However, self-reported measures can be influenced by social desirability and may not always accurately represent actual learning outcomes. Physiological measurements, such as heart rate variability, skin conductance, and eye tracking, can be employed to assess the cognitive load, engagement, and learning during VR-based training. These measures provide objective data on the learner's physiological responses to the virtual environment, which can be correlated with knowledge transfer. However, interpreting physiological data can be challenging, and the relationship between physiological measures and knowledge transfer may not be straightforward.

Despite the variety of methodologies available for assessing knowledge transfer in VR environments, there is no one-size-fits-all solution. Researchers should carefully consider the objectives of their studies, the populations they are investigating, and the specific characteristics of the VR environments used. Combining multiple measures in a mixed methods research design may provide a more comprehensive understanding of knowledge transfer and help identify the most effective strategies for promoting learning in VR.

Usability and User Experience

VR applications are becoming increasingly popular in various domains such as gaming, education, and training [46]. Ensuring positive user experience and high usability is crucial to the success and adoption of VR applications [98]. This section will describe and discuss methodologies for measuring usability and user experience in VR applications.

Usability refers to the extent to which a system can be used by specified users to achieve specified goals with effectiveness, efficiency, and satisfaction in a specified context of use [67]. Several methodologies can be employed to measure the usability of VR applications.

- **Questionnaires:** Self-report questionnaires, such as the System Usability Scale (SUS) [17], provide quantitative data on users' perception of usability. These questionnaires typically assess usability dimensions such as learnability, efficiency, and satisfaction.
- **Task Performance Metrics:** Task completion time, task success rate, and error rates are common metrics used to assess the effectiveness and efficiency of VR applications [98]. For instance, VR systems can be tested with participants performing specific tasks, and their performance can be compared to a predefined benchmark.

- **Expert Evaluations:** Heuristic evaluations, cognitive walkthroughs, and pluralistic walkthroughs are expert-based usability inspection methods that can be applied to VR applications [123]. These methods involve experts assessing the system based on established usability principles or heuristics.

User experience (UX) refers to a person's perceptions and responses resulting from the use or anticipated use of a product, system, or service. UX is a broader concept than usability, encompassing aspects such as emotions, expectations, and aesthetics [55]. Measuring UX in VR applications can be challenging due to its multifaceted nature; however, several methodologies can be applied:

- **Questionnaires:** Self-report questionnaires such as the Slater-Usoh-Steed (SUS) Questionnaire, Witmer and Singer's Presence Questionnaire (PQ), Lombard and colleagues' questionnaire, and Schubert, Friedmann, and Regenbrecht's Igroup Presence Questionnaire (IPQ). These questionnaires approach presence from different angles, ranging from sensory aspects to the role of the social actor and medium, and are designed for use in diverse virtual environments. In addition, some of these tools, specifically the Witmer-Singer Presence Questionnaire, the ITC-Sense of Presence Inventory (ITCSOPI), and the IPQ, also incorporate elements of involvement or engagement.
- **Interviews and Focus Groups:** Qualitative methods such as semi-structured interviews and focus groups can provide rich, in-depth insights into users' experiences with VR applications [23]. These methods allow researchers to explore emotions, motivations, and preferences that may not be captured by quantitative methods.
- **Physiological Measures:** Physiological data such as heart rate, skin conductance, and facial expressions can provide objective measures of users' emotional responses to VR applications [109]. These measures can be used to complement self-report questionnaires and qualitative methods.

Measuring the usability and user experience of VR applications is critical to ensure their success and adoption. A combination of quantitative and qualitative methods, including questionnaires, task performance metrics, expert evaluations, interviews, focus groups, and physiological measures, can provide a comprehensive understanding of the VR application's usability and user experience. Further research is required to develop and validate new methodologies tailored specifically to the unique challenges posed by VR environments.

Learning Outcome

The emerging role of VR in education, particularly in mathematics, necessitates effective methods to measure learning outcomes. This section provides an overview of such approaches. In the first part, I introduce the Cognitive Affective Model of Immersive Learning (CAMIL) to understand learning in VR in general. The second part shifts focus to the objectives of learning in general and in mathematics education, examining processes such as competency development or conceptual fields. These metrics allow us to gauge VR's effectiveness within this field. By exploring these measures, I aim to enhance our understanding of VR's impact on mathematics education, potentially guiding its optimal use in teaching this discipline.

Learning Outcome of VR CAMIL proposed by Makransky and Petersen [107] is an innovative model that aims to explain the underlying mechanisms by which immersive learning occurs in VR environments [107]. This comprehensive model integrates various technological factors, such as immersion, control, and representational fidelity, with VR affordances like presence and agency, and affective and cognitive factors, including interest, motivation, self-efficacy, cognitive load, and self-regulation (see Figure 11). Ultimately, these components influence learning outcomes, such as factual, conceptual, and procedural knowledge, and transfer of learning.

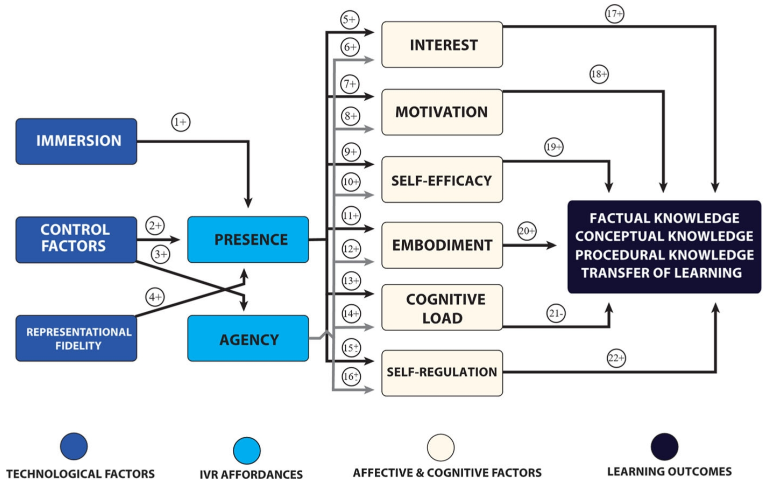


Fig. 11: The Cognitive Affective Model of Immersive Learning (CAMIL)

The technological factors of the CAMIL model are crucial in determining the effectiveness of immersive learning experiences. Immersion is achieved through the combination of VR technology and the sensory stimuli it offers, creating a sense of being physically present in the virtual environment [107]. Control factors refer to the extent of user control within the VR environment, including interaction and navigation capabilities, which can impact learning outcomes. Representational fidelity is the accuracy of the virtual environment's representation of real-world phenomena, which can influence the learner's perception of the virtual experience as authentic [107]. VR affordances, such as presence and agency, are also essential components of the CAMIL model. Presence refers to the subjective experience of "being there" in the virtual environment, which can enhance learning by promoting increased attention and engagement [107]. Agency, on the other hand, relates to the user's perceived control over their actions within the virtual environment, fostering a sense of autonomy that can positively impact motivation and learning. Affective and cognitive factors play a critical role in the CAMIL model, as they influence how learners process and engage with the virtual environment. Interest and motivation are central to the learning process, as they drive learners to actively engage with the material and persist in challenging situations. Self-efficacy, or the learner's belief in their ability to successfully complete tasks, also contributes to the overall learning experience [107]. The model acknowledges that cognitive load, or the mental effort required to process information, can influence learning outcomes. Effective VR environments should balance the cognitive load to prevent overloading the learner while maintaining engagement. Self-regulation, which encompasses goal setting, monitoring, and adjusting behaviour, is another essential cognitive factor in the CAMIL model, as it supports learners in managing their learning experience effectively. The ultimate goal of the CAMIL model is to facilitate various learning outcomes. Factual knowledge, or the recall of discrete pieces of information, is a fundamental outcome in many learning scenarios. Conceptual knowledge, which involves understanding relationships between ideas and organising information into mental frameworks, is another critical outcome. Procedural knowledge refers to the ability to apply specific procedures or strategies to solve problems or complete tasks. Lastly, the transfer of learning, or the ability to apply knowledge and skills to new and diverse situations, is a highly desirable outcome that demonstrates the effectiveness of the learning experience [107].

In conclusion, CAMIL provides a comprehensive framework for understanding the complex interplay between technological factors, immersive VR affordances, and affective and cognitive factors in the context of immersive learning environments. By considering these components, the model can inform the design of effective VR experiences that foster meaningful learning outcomes, including factual, conceptual, and procedural knowledge, as well as the transfer of learning.

Bloom and Krathwohl's Taxonomy Bloom's taxonomy is a classification framework for learning objectives developed by Benjamin Bloom and his colleagues in the mid-20th century. The taxonomy is traditionally organised into a hierarchical pyramid consisting of six levels of understanding: remember (knowledge), understand (comprehension), apply, analyse, evaluate, and create (synthesis) [12]. This framework aims to promote higher forms of thinking in education, going beyond mere rote memorisation. Parallel to Bloom's cognitive process dimension, David Krathwohl proposed a two-dimensional taxonomy that further segments the knowledge aspect. Krathwohl's knowledge types distinguish four categories of knowledge: factual, conceptual, procedural, and metacognitive [91]. Factual knowledge includes essential facts and terminology, conceptual knowledge involves understanding of principles and theories, procedural knowledge refers to methods and techniques, and metacognitive knowledge encompasses awareness of one's own cognition [91]. Together, Bloom's and Krathwohl's taxonomies provide educators with a comprehensive system for designing, assessing, and refining educational outcomes. In mathematics education, Bloom's taxonomy and Krathwohl's knowledge types offer a structured approach to designing lessons and assessing student understanding. For instance, the *remember* and *understand* stages of Bloom's taxonomy might involve students memorising multiplication tables (factual knowledge) and grasping the principles of multiplication (conceptual knowledge). *Application* could involve solving basic arithmetic problems (procedural knowledge), while *analysis* and *evaluation* might require students to tackle complex word problems and verify the correctness of their solutions. Ultimately, the *create* stage could see students developing their own mathematical proofs or problem-solving strategies, demonstrating metacognitive knowledge as they reflect on and refine their thinking process. This integrated approach to the taxonomy and knowledge types provides a roadmap for educators to gradually escalate cognitive demands, ensuring that students are not just memorising formulas, but truly understanding mathematical concepts, applying them effectively, and engaging in higher-order mathematical thinking.

Mathematical Competencies Mathematical competency development, as defined by Niss and Højgaard [124], consists of eight distinct competencies: problem-solving, mathematical modelling, mathematical reasoning, representation, communication, mathematising, using aids and tools, and dealing with symbolic language. VR platforms can be used to create immersive problem-solving scenarios, allowing educators to assess students' problem-solving and mathematical modelling competencies in real-time. For example, students can be tasked with solving mathematical problems that require the construction of 3D models, giving educators the opportunity to observe their thought processes and strategies as they work through the problem. Mathematical reasoning and representation competencies may be assessed through interaction analysis within the virtual environment [76]. By tracking students' interactions with virtual objects, educators can infer the development of their reasoning and representation skills. Interaction analysis can be combined with think-aloud protocols, where students verbally express their thought process while solving problems, to provide a more comprehensive understanding of their reasoning abilities [38]. Collaborative virtual environments can be used to evaluate students' communication and mathematising competencies. By participating in group tasks, students are required to communicate their mathematical ideas and negotiate shared understandings, allowing educators to assess their ability to express and comprehend mathematical concepts effectively. Virtual environments can be designed to incorporate various aids and tools, such as virtual calculators, protractors, and graphing utilities [78]. By observing how students utilise these tools in the virtual environment, educators can assess their competency in using aids and dealing with symbolic language.

While VR may offer several benefits for assessing mathematical competencies, it also presents challenges. For instance, the design of virtual environments may influence students' performance and learning outcomes [113]. Moreover, assessing mathematical competencies in VR may be limited by the availability of appropriate tools and methodologies, requiring research and development to refine assessment techniques. Further research is necessary to address the challenges and limitations associated with VR-based assessment methodologies for mathematical competency development.

Conceptual Fields Conceptual fields theory, developed by Gérard Vergnaud, is an influential cognitive theory that emphasises the importance of interconnected mathematical learning [174]. This theory posits that understanding in mathematics is facilitated through the development and connection of concepts, procedures, and situations within conceptual fields [175]. Vergnaud suggested that mathematical knowledge is built through the recognition and formulation of connections within these conceptual fields [174]. Central to these fields are three key components. Firstly, concepts, which are abstract entities encapsulating fundamental mathematical ideas, encompass objects like numbers and shapes along with properties and relationships. Secondly, situations, that serve as contexts where these mathematical ideas are employed, provide a meaningful environment for the development and application of mathematical understanding. Lastly, there are procedures, sequences of actions or operations carried out to solve mathematical problems. These enable learners to operationalise their understanding of mathematical concepts.

The theory of conceptual fields has profound implications for mathematics education. It underscores the necessity of introducing students to a broad range of related concepts, procedures, and problem situations within a particular mathematical field, fostering interconnected learning and understanding [48]. Vergnaud's theory implies that instruction should involve exploring various situations where a mathematical concept or procedure can be applied, urging students to identify the breadth of contexts in which their mathematical knowledge is applicable [173]. Additionally, it encourages active engagement in mathematical problem solving, aiding students in applying their understanding in practice [7]. The theory of conceptual fields also offers a useful framework for diagnosing students' misconceptions and difficulties, enabling educators to identify specific areas of struggle within a broader conceptual field context [154]. To conclude, Vergnaud's theory underscores the complexity and interconnectedness of mathematical knowledge. It proposes a comprehensive approach to understanding how students learn mathematics, highlighting the need for an interconnected, context-rich learning environment that facilitates deep understanding and application of mathematical concepts [53].

VR provides a unique opportunity to apply and further investigate Vergnaud's theory of conceptual fields in mathematics education. The immersive nature of VR can be used to facilitate the exploration and understanding of mathematical concepts by providing a realistic, interactive, and engaging learning environment.

1. **Exploring Concepts:** VR can help students visualise and manipulate abstract mathematical concepts. For instance, a student studying geometry could explore a 3D geometric shape within VR, manipulate it, and observe it from different perspectives to understand its properties. This provides an experiential and intuitive way to understand mathematical concepts, resonating with Vergnaud's emphasis on the interconnectedness of mathematical knowledge within conceptual fields.

2. **Application in Situations:** VR can simulate different real-world or abstract scenarios where mathematical concepts and procedures can be applied. These simulated environments can help students recognise the breadth of contexts where their mathematical knowledge is relevant, directly corresponding to Vergnaud's emphasis on situations within conceptual fields.

3. **Performing Procedures:** Within VR environments, students can engage in active problem-solving, using virtual tools to operationalise their understanding of mathematical procedures. For example, a student studying statistics could gather, graph, and analyse data within a VR environment, enabling a deeper understanding of statistical procedures.

4. **Assessment and Feedback:** VR technology can also provide real-time feedback and assessment, allowing educators to identify students' misconceptions or difficulties within a conceptual field. These interactive assessments provide an opportunity for students to learn from their mistakes and correct their understanding.

In conclusion, VR has the potential to enhance the application of the theory of conceptual fields in mathematics education by creating an interactive, immersive, and engaging learning environment. However, further research is required to fully understand the best practices for integrating VR technology into mathematics education, ensuring it is used effectively to promote deep, meaningful learning.

2.6 Summary and Model

Based on the analysis of this section, I propose an extended version of the ADDIE model aimed towards advancing the field of mathematics education by designing VR experiences. This approach offers a shift from traditional instructional design, leveraging VR's unique capabilities to facilitate constructivist, situated, exploratory, and/or embodied learning. My objective is to integrate virtual manipulatives and models as central pedagogical tools, fostering potentially enhanced comprehension and interaction with mathematical concepts. Additionally, to ensure a robust and holistic learning experience, the VR engagement is complemented by generative learning strategies. These post-VR exposure activities aim to bolster learners' understanding and retention by encouraging them to analyse, reflect on, and apply the knowledge they've acquired. The framework's novelty lies not only in its design but also in its evaluation strategy. Utilising the theory of instrumental genesis, I aim to investigate learners' utilisation and evolution of virtual tools towards achieving learning objectives. By synergising the best of VR technology and pedagogical strategies, this extended model makes a contribution to mathematics education. It presents a potential pathway to enhance learner engagement and comprehension, as well as providing a means for continually improving the instructional design based on actual usage and learning outcomes.

1. **Analysis**
    - **Needs Analysis:** Identify the learner's needs, their prior knowledge, and potential gaps. Understand the learners' familiarity with VR technology.
    - **Learning Objectives:** Define the learning outcomes that the VR experience should achieve. These could include understanding specific mathematical concepts, building mathematical competencies, developing problem-solving skills, improving spatial understanding, etc. in accordance with national goals, such as the aforementioned "Fælles Mål" in Denmark.
    - **Task Analysis:** Define the mathematical concepts and problems to be learned and how these could be best represented in a VR environment.
    - **Technical Analysis:** Evaluate the technology requirements for the VR experience, including hardware and software capabilities, and the limitations or constraints these might impose on the design.
2. **Design**
    - **Learning Theories:** Incorporate the theories of constructivism, situated learning, exploratory learning, or embodied learning in the design of the VR experiences.
    - **Immersive Virtual Manipulatives and Models:** Design and decide on the virtual manipulatives and models that will be used to represent the mathematical concepts.
    - **Gameplay Mechanics:** Determine the mechanics of the game which are conducive to learning. The mechanics should encourage interaction, exploration, and engagement with the mathematical models.
    - **Generative Learning Activities:** Design post-VR activities that prompt learners to process and reflect on the information learned, reinforcing and extending the learning process.

3. **Development**
    - **Prototype:** Develop a working prototype of the VR experience. This should include the virtual environment, manipulatives, and models, as well as the gameplay mechanics.
    - **User Interface:** The UI should be intuitive, and incorporate elements of instructional design to guide the learner through the VR experience.
    - **Generative Learning Materials:** Develop the supporting materials or resources for the generative learning activities.
4. **Implementation**
    - **Pilot Testing:** Before fully implementing the VR experience, pilot test it with a small group of learners to identify potential issues and make necessary adjustments.
    - **Training:** Provide training for learners and educators to use the VR technology, understand the gameplay mechanics, and know how to engage with the post-VR generative learning activities.
    - **Roll-out:** Deploy the VR experience in the learning environment.
5. **Evaluation**
    - **Instrumental Genesis Theory:** Apply the theory of instrumental genesis to evaluate how the virtual tools (manipulatives and models) are used by learners and how their use evolves to achieve the learning objectives.
    - **Feedback and Revision:** Use feedback from learners and educators to make ongoing revisions and improvements to the VR experience.
    - **Assessment of Learning Objectives:** Evaluate to what extent the learning objectives are being met. This could be through tests, quizzes, or other formative and summative assessments.
    - **Learning Analytics:** Use the data from the VR environment (e.g., interaction data, progress data) to understand how learners are interacting with the game and where potential difficulties lie.

While the sections examined provide an understanding of teaching and learning mathematics in VR, they represent a starting point for investigation. As we move forward in this exciting new era of education, it is important to ground our research and practice in existing studies. This brings us to our next section, where I provide an overview of the existing body of research related to using VR in mathematics education. By drawing upon this research, we can better understand the successes and challenges encountered thus far, refine our own practices, and continue to push the boundaries of what is possible in mathematics education.

3 Related Research

In recent years, VR technology has emerged as a promising and innovative tool in the field of education, offering new avenues for teaching and learning experiences. One area, in which VR has gained significant attention, is mathematics education, where it holds the potential to transform the way students perceive and interact with mathematical concepts. This section provides a summary of the existing research on the effects of using VR in mathematics education, drawing from a selected pool of 21 articles identified through a search on Google Scholar.

The keywords "virtual reality," "mathematics education," and "user study" were utilised to initially identify a total of 117 articles. To ensure the relevance and rigour of the selected studies, a set of criteria was applied, reducing the number of articles to 21. The articles were selected by the exclusion criteria of being concerned with mathematics education and being focused on VR systems, such as Meta Quest, HTC Vive, and Valve Index, which offer a rich and immersive learning environment for the users. The studies encompass a wide range of educational levels, from primary school to university, ensuring a comprehensive understanding of the impact of VR in mathematics education across various age groups and academic stages. Below is a histogram showcasing the number of publications per year, encompassing the years 2006 to 2023, based on the selected articles (see Figure 12). The graph shows 2019 and 2020 being the years with the highest number of publications. While it remains speculation, the subsequent decrease in the number of publications could be ascribed to the onset of the COVID-19 pandemic. The restrictions necessitated by this health crisis may have impeded researchers' capabilities to execute user studies, potentially leading to a decline in scholarly output in recent years. It can also be noted that without the exclusion of the current PhD thesis studies from the research summaries, the collective output from recent years nearly matches that of 2019 and 2020.

In the following subsections, the findings of these articles are synthesised and discussed, highlighting key insights into the effectiveness of VR in mathematics education, the challenges and limitations associated with its implementation, and the potential future directions for research and development in this domain. Through this summary, I aim to provide a valuable resource for educators, researchers, and policymakers interested in understanding the current state of VR applications in mathematics education and their implications for enhancing the learning experience. In conclusion, I will provide a summary of the presented research by organising the information into a table categorised by research design, methodology, technology, subject area, and educational level.

The seminal contributions to the understanding of VR in relation to mathematics education can be traced back to the research conducted by Winn and Bricken [183]. During this period, VR had not yet achieved its current commercial availability, and the definition they provided did not encompass the immersive experience offered by

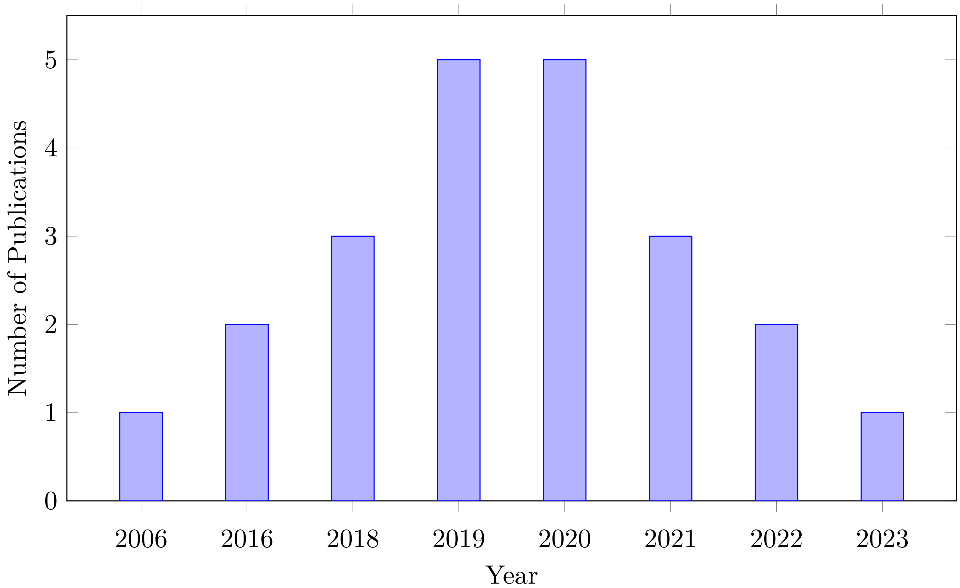
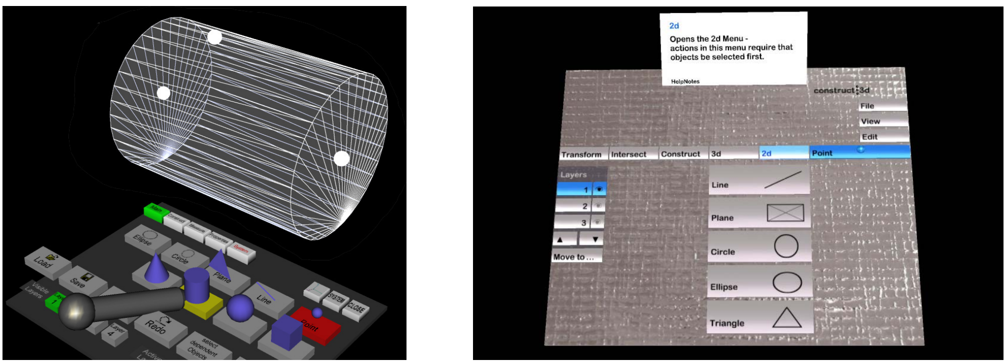


Fig. 12: Histogram of scientific publications per year.

HMDs. Rather, they described VR as "a computer-generated, multidimensional, and inclusive environment that is perceived as legitimate by the user." Despite the primary focus of the subsequent sections being on VR, it is valuable to mention Winn and Bricken's work due to its significance in the context of algebra and equation-solving within virtual settings.

The article explores the use of VR to improve students' learning of elementary algebra [183]. Their project is focused on applying current research on learning to create a more effective classroom experience. They discovered the importance of how abstract objects and processes are visually represented in a virtual world and developed the *Spatial Algebra* system to represent algebraic concepts in a way that is easier to understand than traditional text-based symbolism. This new system can be embedded into the behaviour of virtual objects in a VR environment, enabling students to construct their own understanding of the subject matter. In conclusion, Winn and Bricken suggest that VR can be a powerful tool in the classroom and that the benefits of learning theory and spatial algebra should be taken into account to ensure its successful implementation.

Kaufmann and Schmalstieg [79] introduces *Construct3D*, a system for geometry education that uses collaborative AR and 3D dynamic geometry to facilitate mathematics learning (see Figure 13). Despite the primary focus of the system being on AR technology, I have included this study and the subsequent one due to their significance in the history of immersive technology for mathematics education. Additionally, the emphasis on virtual environments in these studies may have potential applications in similar VR contexts. The authors describe improvements in the user interface and visual design of the system and report on practical experiences with using the system for the actual teaching of high school students. They also present initial quantitative data on the educational value of the approach. The system described in the study is innovative in its use of collaborative AR and 3D dynamic geometry, both of which are relatively new to geometry education. The authors highlight the potential benefits of these technologies, including the ability to facilitate collaborative learning and to provide a more engaging and interactive learning experience for students. The study also presents practical experiences with using the system in a high school classroom setting. The authors report positive feedback from both students and teachers, suggesting that the system is effective in promoting engagement and improving learning outcomes. The initial quantitative data presented in the study also supports this conclusion, although more research is needed to fully evaluate the educational value of the system. The study represents an important contribution to the field of geometry education, highlighting the potential of new technologies to improve learning outcomes and engage students in the learning process. The practical experiences and initial quantitative data presented in the study provide valuable insights into the effectiveness of the system and paved the way for further research in this area.



(a) Cylinder

(b) Menu system

Fig. 13: Images from Construct3D by Kaufmann and Schmalstieg [79]

In another article, Kaufmann [77] discusses the historical significance of visualisations in mathematics and geometry education, which have been employed since ancient times to describe, study, and teach mathematical concepts. In contemporary education, visualisations continue to be utilised to facilitate teaching, engage students, and enable them to visualise abstract mathematical ideas. The advent of VR has introduced an innovative and highly motivating tool for educators, allowing students to experience mathematics in three dimensions. The article provides an overview of various immersive virtual environments developed over the past decade to support mathematics and geometry education. It focuses on a specific advanced application for geometry education that has been used and evaluated with over 500 students throughout the years. The findings and teaching experiences related to this application are described and discussed, highlighting the potential of VR as a powerful educational tool in the realm of mathematics and geometry.

Lai et al. [93] presents *Geometry Explorer*, a VR system designed to enhance spatial understanding in geometry education (see Figure 14). The system employs the Samsung Gear VR to allow users to view and manipulate 3D shapes, and a game-based approach where users aim to manipulate shapes to reach target volumes. The authors conducted informal evaluations of the initial prototype using the Rapid Iterative Testing and Evaluation (RITE) approach and reported modifications based on the usability issues identified during testing. The authors discuss the future of the Geometry Explorer, highlighting the potential for the system to improve geometry education by leveraging the benefits of VR for spatial understanding. The study suggests that immersive VR technology can be beneficial for enhancing spatial understanding and supporting the learning of high-level geometry concepts. The Geometry Explorer system presents a novel approach to geometry education that leverages the affordances of VR for the visualisation and manipulation of 3D shapes. The use of game-based learning may also increase engagement and motivation for learners. The informal evaluations conducted using RITE testing provided valuable insights into the usability of the initial prototype, which were used to inform modifications to improve the system's effectiveness. Overall, the study highlights the potential for VR technology to improve learning outcomes in geometry education and points towards future research directions in this field.

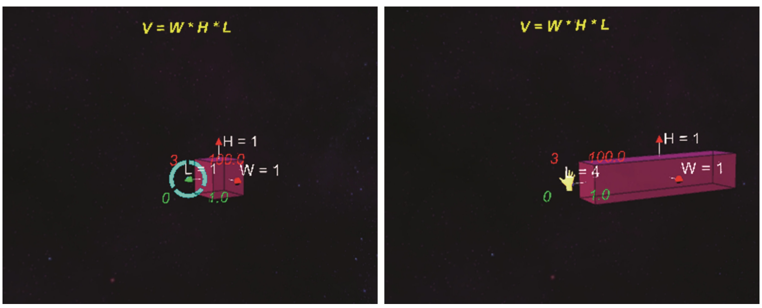


Fig. 14: Images from Geometry Explorer by Lai et al. [93]

Liu et al. [103] discusses the potential of VR in enhancing basic education, particularly in primary mathematics. VR technology allows students to immerse themselves in interactive learning environments, overcoming barriers of distance, time, and safety. The article proposes a practical VR course focused on the Mobius strip for primary students and outlines a detailed teaching procedure to effectively utilise VR in the learning process. This study aims to offer educators and researchers insights into integrating VR technology in primary mathematics classrooms to improve the overall learning experience.

Stranger-Johannessen [162] investigates the potential benefits of using VR to improve learning outcomes in mathematics, specifically multiplication, for fifth-grade students (see Figure 15). The intervention involved using a VR environment where students practised multiplication by buying items from shops, and they collected credits for correct answers and attempts to develop dragons. The control group received regular math instruction. The study used a quasi-experimental design and analysed the data of 79 students who used HMDs and 37 students in the control group. The preliminary results showed an increase in post-test scores for boys who used HMDs, but the sample size of the control group was too small for statistical significance at the .05 level. The study highlights the potential benefits of using VR and gamification to improve learning outcomes in mathematics. The use of HMDs in a virtual environment offers an immersive experience that can increase student motivation and engagement. The findings also suggest that boys may benefit more from this intervention than girls, which is an interesting avenue for further research. However, the sample size of the control group is small, which limits the generalisability of the findings. Moreover, the study did not account for other factors that may have contributed to the difference in learning outcomes between the two groups, such as the quality of instruction or prior knowledge. Despite these limitations, the study provides valuable insights into the potential of VR and gamification to enhance mathematics learning, and it highlights the need for further research in this area.

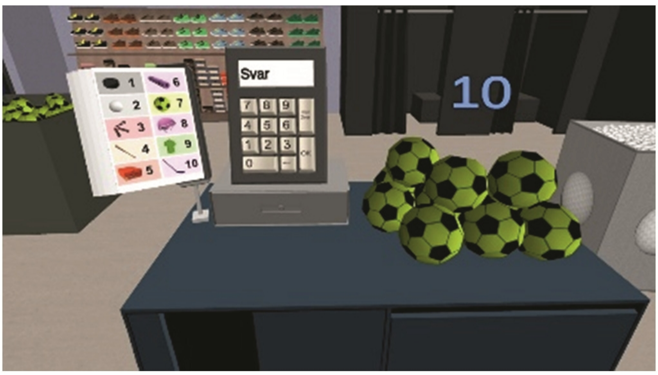
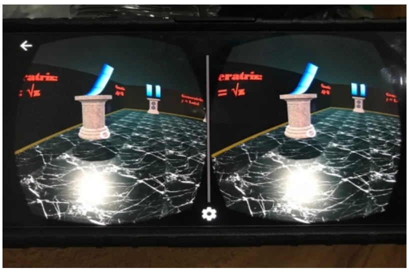


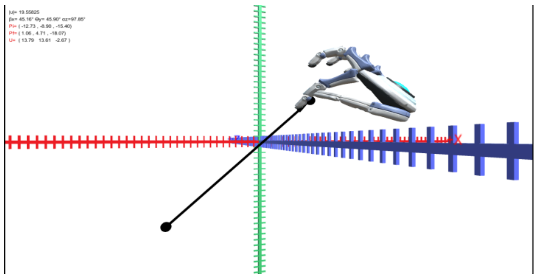
Fig. 15: Image of intervention by Stranger-Johannessen [162]

Nathal et al. [120] discusses the use of technology to support the teaching and learning of mathematics, specifically focusing on the learning of vector themes and 3D space.

The authors highlight the challenges faced by students in learning mathematical concepts and argue that the use of technology can provide a supportive learning environment. They describe an instructional scenario that uses physical and virtual objects to promote the understanding of mathematical concepts and the interpretation of figures such as solids and cylinders (see Figure 16). The paper presents the results of a study that investigated the effectiveness of this instructional design, including the conceptual elements involved in the learning process and the effects of the design on the learning of mathematical concepts. The study found that virtual scenarios for the manipulation of mathematical objects can be introduced in the classroom and can promote the development of learning materials mediated by new technologies for the learning of mathematics, which can contribute to strengthening the construction of mathematical knowledge of students. Overall, the paper highlights the potential of technology to support and enhance the teaching and learning of mathematics.



(a) Quadratic surfaces and cylinders



(b) 3D Euclidean space

Fig. 16: Images from Nathal et al. [120]

Rodríguez et al. [144] presents the use of the VR software NeoTrie VR for teaching geometry to children aged 11-14 years old (see Figure 17). The software was tested as a part of math lessons at a school in Żernica, Poland, and the results showed that the use of Neotrie helped to eliminate some of the problems faced by the students in the early stages of geometry learning. The participating students were able to solve geometric tasks more quickly, especially those that require spatial imagination and those with a higher degree of difficulty. The study also found that the use of Neotrie made students more active and prone to cooperation, and to formulate conclusions. Overall, Neotrie was found to be a useful tool for teaching geometry, helping students to better organise their geometric knowledge in a ludic way, to increase their spatial reasoning and creativity. The use of VR software for teaching geometry is a promising approach to improve students' learning outcomes. By providing an immersive and interactive learning experience, VR can enhance students' understanding of complex geometric concepts and promote active engagement in the learning process. The results of this study highlight the potential of the NeoTrie VR software as a useful tool for teaching geometry to children. However, further research is needed to investigate the long-term effects of using VR in math education and to determine the most effective ways to integrate this technology into the classroom.

(a) Measuring an angle of a triangle

(b) Painting edges

Fig. 17: Images from NeoTrie VR by Rodríguez et al. [144]

Putman and Id-Deen [137] describes how VR can improve the learning experience of students by immersing them in a structured learning environment that blocks out distractions. The authors focus on the use of VR HMDs to teach math concepts to 5th graders at a school in Kentucky. The study highlights the benefits of using VR in education, particularly in math learning, and emphasises the availability of VR applications that can enhance learning experiences. The study's findings suggest that VR can be an effective tool for teaching math and improving students' understanding of key concepts. The authors argue that VR applications have the potential to revolutionise education by creating engaging and interactive learning experiences that can help students learn more effectively. The study emphasises the potential of VR to transform education and improve students' learning outcomes in math and other subjects.

Pearl et al. [131] discusses the potential benefits of using VR software for educational purposes, specifically in teaching mathematical concepts in three dimensions. The authors aim to extend their VR software to support multiple users in collaborative graphing and problem-solving and to evaluate the effectiveness of high school mathematics units focused on drawing, transforming, and manipulating vectors and objects in 3D space. The authors argue that VR software can build upon the benefits presented by non-immersive educational software and take advantage of principles of embodied cognition, providing users with a tangible experience of constructing concrete vectors and objects in the 3D world around them. This is made possible by the use of 6-degree-of-freedom HMDs and controllers that allow the tracking of rotational orientation as well as position. The study also highlights the potential of VR for creating more intuitive conceptual representations that can make complex mathematical concepts more accessible. Overall, the study suggests that VR software has the potential to improve learning outcomes and enhance students' understanding of mathematical concepts. The authors' findings could be useful for educators and developers looking to create effective educational VR software that can leverage the benefits of embodied cognition and provide more intuitive representations of abstract concepts.

Crutchley et al. [26] aims to investigate the potential benefits of using VR technology to aid in the teaching of college-level calculus (see Figure 18). The authors built a VR system that included visualisations of solutions to equations from calculus textbooks used at Shepherd University. They surveyed three mathematics professors who reported positive reactions to the system and expressed support for its use in the classroom. The authors plan to continue developing the system by adding features such as projections of curves onto any plane and testing it in a classroom setting. The study highlights the potential of VR as an augmentative technology for classroom learning, particularly for mathematics education. The use of VR technology allows for 3D visualisation of mathematical concepts, which may aid in understanding and improve student engagement. The positive response from the mathematics professors suggests that there is potential for VR technology to be integrated into the classroom and used as a teaching aid. However, the study is limited in its sample size and scope, and further research is needed to determine the effectiveness of VR technology in improving student learning outcomes in mathematics education.

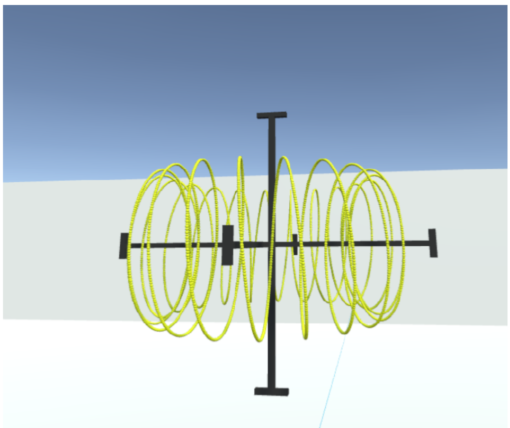
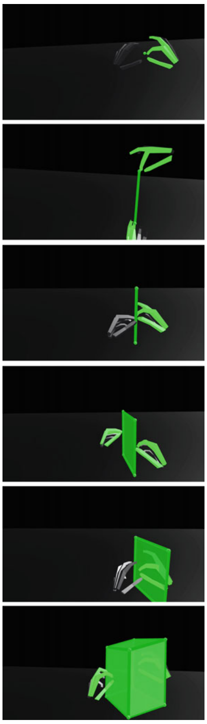


Fig. 18: Toroidal spiral

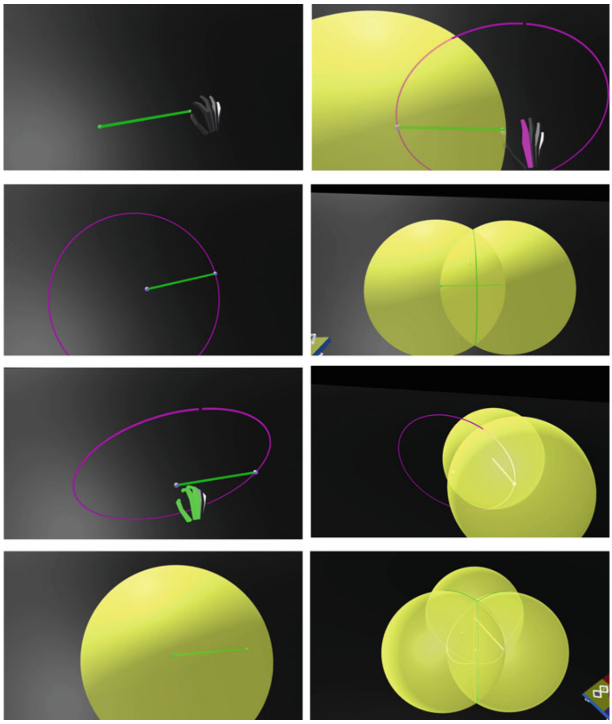
Dimmel and Bock [34] discusses the design and development of *HandWaver*, a gesture-based mathematical making environment for use with immersive, room-scale VR (see Figure 19). The goal of the project was to harness the modes of representation and interaction available in virtual environments to create experiences where learners use their hands to make and modify mathematical objects. The sandbox construction environment was created where learners can construct geometric figures using a series of gesture-based operators, such as stretching figures to bring them up into higher dimensions, or revolving figures around axes that learners can position by dragging and locking. The study also discusses plans for future development and research. The use of VR and gesture-based interaction has the potential to revolutionise the way we teach mathematics by providing learners with a more immersive and interactive experience. The sandbox construction environment in *HandWaver* allows learners to experiment with mathematical concepts in a way that is difficult to achieve in traditional classroom settings. The use of gesture-based operators also provides learners with a more intuitive way to manipulate mathematical objects, which can help to increase understanding and retention of mathematical concepts. The study highlights the potential benefits of using VR and gesture-based interaction in mathematics education, but further research is needed to fully evaluate the effectiveness of these tools. In addition, future development of HandWaver and similar tools could expand on the capabilities of the sandbox construction environment to provide learners with even more opportunities for experimentation and exploration. Overall, the study provides an exciting glimpse into the possibilities of using VR for mathematical education.

Hsu [64] highlights the global trend of integrating information technology into education, with a focus on the application of VR in teaching high school mathematics (see Figure 20). The study aims to examine the impact of VR-based instruction on students' learning motivation and effectiveness when learning the system of linear equations in three unknowns. Utilising a quasi-experimental research design and a questionnaire survey method, the researchers developed VR tutorial materials for high school mathematics. The experimental group of students was exposed to both VR-based learning and traditional teaching methods. Pre- and post-experiment assessments were conducted using the Attention, Relevance, Confidence, and Satisfaction (ARCS) Model and Bloom's taxonomy to quantitatively analyse the data. Additionally, simple interviews were carried out with the participants and teachers to obtain qualitative insights. The results of the experiment indicate that incorporating VR in digital teaching for mathematics has a positive effect on students' learning motivation and overall effectiveness.

Price et al. [136] examines the potential of digital designs in fostering sensorimotor experience and meaningful movement in geometry learning and spatial thinking among elementary children. The study involves a learning environment that utilises a 3D immersive virtual environment where one child collects flowers from target coordinates selected by another child using a 2D visual representation of the virtual garden and person's location in space (see Figure 21). The evaluation involves twenty-one children aged 8-9 years, and a qualitative, multimodal analysis was conducted to examine collaborative interaction among the children. The study finds that the use of different representations (tangible and visual 2D screen-based) situates the meaning-making process in a space where children, using their bodies, crafted connections between the different representations and used transcending objects to facilitate the integration of the different perspectives. The findings suggest that digital designs can support mean-

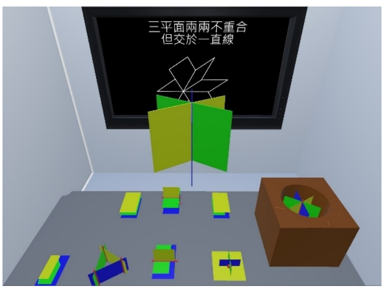


(a) Stretch operator

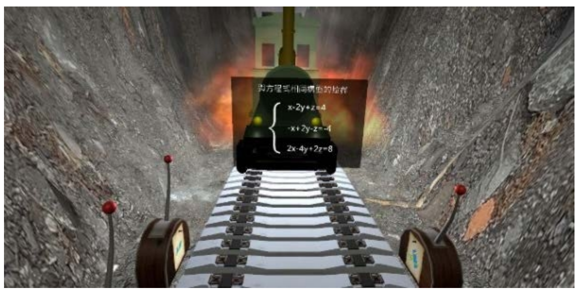


(b) Arctus tool

Fig. 19: Images from HandWaver VR by Dimmel and Bock [34]



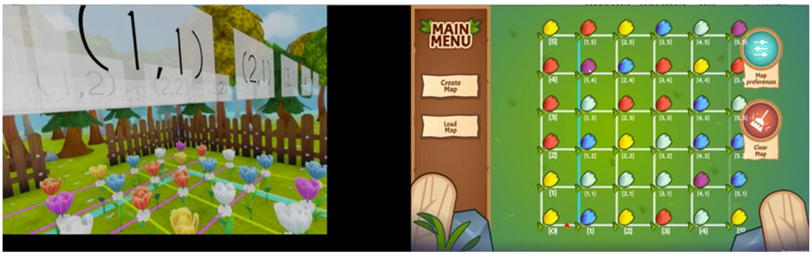
(a) Three planes



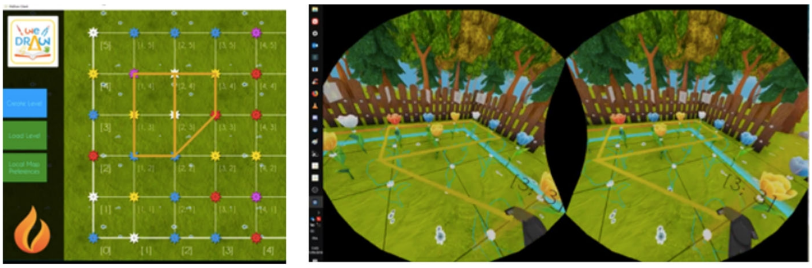
(b) Equation-solving quiz

Fig. 20: Images from Hsu [64]

ingful mathematical movement through the use of shared alternative representations that allow for bodily experience and visible body movement, position, and orientation, fostering geometry learning and spatial thinking among elementary children.



(a) VR and 2D representation



(b) Shape creation

Fig. 21: Images from Cartesian Garden by Price et al. [136]

Rosa and Pinheiro [146] investigates the integration of VR technology in mathematical activities, focusing on the impact on the education and professional development of mathematics teachers. The concept of Cybereducation is introduced, which is the core of the study, and is particularly concerned with the mathematical dimension in the context of mathematics teachers. The research presents the perspective of work-with-VR and its implications for the formation and action of mathematics teachers. The article initially outlines the concept of Cybereducation for mathematics teachers and discusses the implementation of VR technologies in the classroom. It then concentrates on the use of mathematical activities that incorporate VR. The study examines the experiences of teachers who participated in an extension course centred on Cybereducation, which featured mathematical activities using VR. A key finding from the study is that the use of VR in mathematical activities enhances spatial recognition and increases the development of mathematical conjectures. The immersive nature of VR alters the perception of time and space, offering alternative possibilities for mathematical exploration and understanding. This ultimately has an impact on the education and professional growth of teachers who teach mathematics, expanding their ability to think, know, and engage with mathematical concepts in novel ways through the use of VR technology.

Takac [168] presents the development of a web-based VR (WebVR) application called

MathworldVR, aimed at teaching higher mathematics concepts that require spatial abilities (see Figure 22). The paper discusses the use of instrumental interaction within VR through direct visual manipulation of input variables of parametrised functions to help students understand the underlying principles of a given mathematical theory. The authors also discuss the advantages, disadvantages, and limitations of developing VR applications for the web in contrast to native desktop applications. The MathworldVR application was built using the A-Frame framework together with the JavaScript library React for fast VR prototyping. The study presents the individual components that define the MathworldVR 3D user interface within the virtual environment (VE) and how various two-handed interaction techniques help in the process of teaching mathematics. The authors claim that MathworldVR shortens the time needed for students to understand higher mathematics concepts by providing an immersive learning experience that allows for a better understanding of spatial relationships and visualisation of mathematical concepts. The study also identifies some limitations of WebVR applications, such as the lack of support for haptic feedback and lower-quality graphics compared to native desktop applications. Nevertheless, the authors argue that the advantages of WebVR applications, such as their accessibility and ease of deployment, outweigh these limitations. The authors conclude that MathworldVR provides an effective tool for teaching higher mathematics concepts and that the use of WebVR applications has the potential to revolutionise the way mathematics is taught in the future.

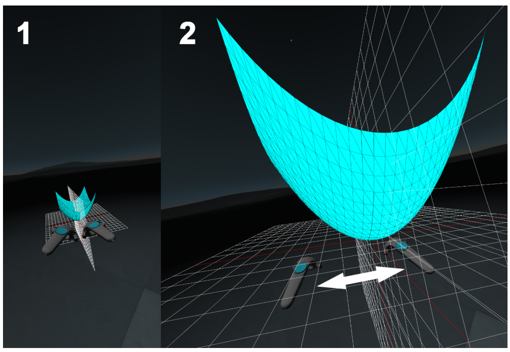
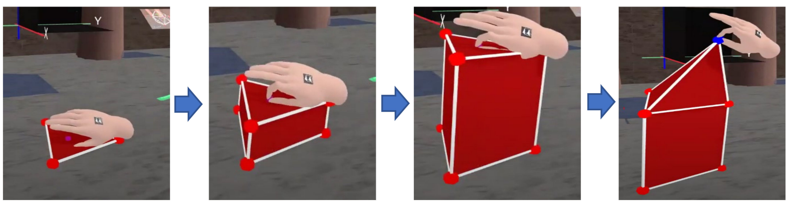


Fig. 22: MathworldVR by Takac [168]

Akman and Çakır [3] discusses the current state of VR in education, particularly its potential to improve student learning experiences. The study in question focused on the effects of an educational VR game, *Keşfet Kurtul*, on fourth-grade students' academic achievement in fractions and engagement in mathematics. The research employed a quasi-experimental design, with a treatment group using the VR game and a comparison group using the school's standard method, primarily involving mobile applications developed for teaching fractions. The results demonstrated that the educational VR game increased academic achievement and maintained student engagement in mathematics at a level comparable to the traditional method used in the comparison group. However, in terms of the social sub-dimension of student engagement, the VR game was found to be more effective than the standard method. In summary, the study suggests that VR has the potential to enhance students' learning experiences and academic achievement, particularly in the early stages of adoption for education. The educational VR game was shown to be at least as effective as the traditional method in improving academic performance and engagement in mathematics, with added benefits in the social sub-dimension of student engagement.

Rodríguez et al. [145] explores the potential of VR technology in teaching geometry and presents the results of introducing NeoTrie VR software in real classrooms (see Figure 23). The authors use a Design Research framework and present qualitative observational data to report the improvements made to the software's design and the mathematical activity that students have access to. The study highlights the collaborative efforts between the software development company, university researchers, and schools to improve the software's geometrical content, representations, and how teachers conceive and manage the teaching of geometry. The study provides insights into the potential of immersive VR software in enhancing the teaching and learning of geometry and emphasises the importance of collaboration among different stakeholders to optimise the design and implementation of such software in real classrooms.



(a) Triangle extrusion

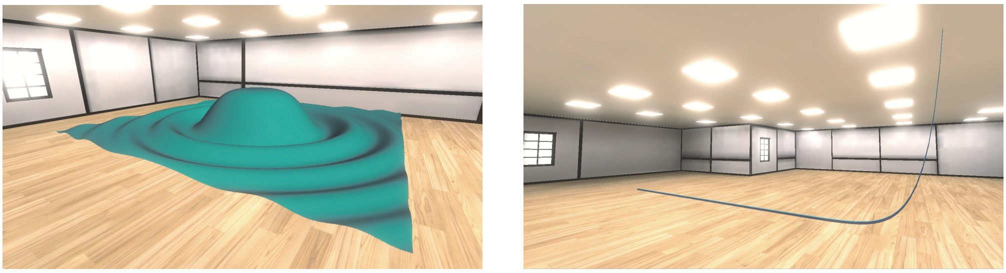
(b) Menu and tools

Fig. 23: Images from NeoTrie VR by Rodríguez et al. [145]

Cangas et al. [19] reports on the use of the NeoTrie VR software package for teaching geometry to students at various educational levels, ranging from primary to university levels. The researchers found that the use of NeoTrie VR led to an increase in students' motivation, organisation of geometric knowledge, creativity, spatial reasoning, and understanding of the properties of 3D structures. The software was also found to enhance students' understanding of geometric concepts in the math curriculum and improve their math vocabulary and geometric proficiency. VR technology made it easy to teach 3D geometry in distance learning, with pre-recorded mono and stereoscopic videos directly in NeoTrie. Overall, the study highlights the potential benefits of using VR technology for teaching geometry and mathematics in general. The use of NeoTrie VR allowed for a more interactive and engaging learning experience, which appears to have resulted in improved learning outcomes. The findings suggest that VR technology can be an effective tool for addressing some of the challenges associated with teaching and learning mathematics. However, more research is needed to explore the effectiveness of VR technology for teaching other subjects and to better understand the specific factors that contribute to its effectiveness.

Perri et al. [132] presents a learning system that incorporates VR and AR to help teach analytical-geometric structures, a topic found in high school math and physics curricula (see Figure 24). The authors suggest that an immersive educational setting can offer several benefits over conventional 2D methods, such as textbooks or computer screens. These benefits include a better spatial comprehension of the concepts presented, increased peripheral awareness, and a noticeable reduction in information scattering. However, this does not imply that the proposal aims to replace traditional methods entirely; instead, it is positioned as a strong supplementary learning tool. The initial phase of the research aimed to identify which mathematical elements and tools could be enhanced through VR and AR techniques, thereby demonstrating that these technologies can significantly boost students' comprehension of the mathematics being taught. The system, which involves integrating both hardware and software elements, was later trialled by a representative sample of students who subsequently gave their feedback via a questionnaire.

Rahmawati et al. [140] aims to investigate the effectiveness of VR-based mathematics learning media with an ethnomathematical approach in improving student learning outcomes. The research was conducted on high school students who were selected using purposive sampling. The study followed the ADDIE model, which involves analysis, design, development, implementation, and evaluation stages. In this article, the implementation and evaluation stages are described. The experimental class received the VR-based mathematics learning media with an ethnomathematical approach, while the control class received traditional mathematics teaching. The study found that the average learning outcomes of the experimental class were better than those of the control class. Additionally, the experimental class showed higher results for learning mastery than the control class. The study concludes that VR-based mathematics learning media



(a) Damped sine plot

(b) Exponential function plot

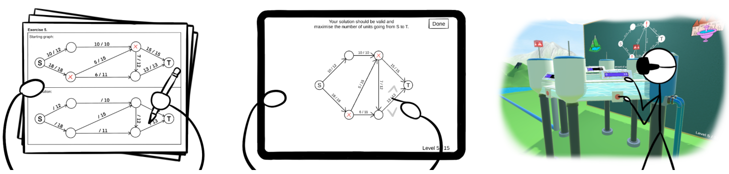
Fig. 24: Images from Perri et al. [132]

with an ethnomathematical approach can be effectively used for high school students. Overall, the study provides evidence for the potential of VR-based mathematics learning media with an ethnomathematical approach to improve student learning outcomes. This approach may be particularly useful in helping students apply mathematical concepts to solve everyday problems, which is often a challenge in mathematics education. The study also highlights the importance of innovative learning media in improving students' understanding of mathematical concepts. However, further research is needed to explore the long-term impact of this approach on student learning outcomes and its applicability in different contexts.

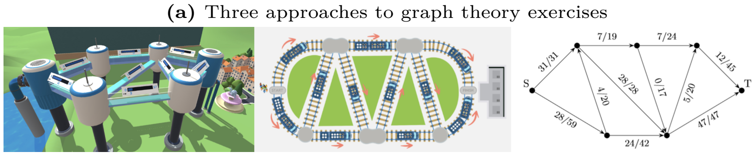
Dilling and Sommer [33] explores the potential of VR technology in the context of mathematics education. Despite the novelty and innovation of VR, there has been limited research and development in this area so far. The article explores the opportunities and challenges associated with implementing VR in mathematics education, utilising a multiview projection VR application as a case study. The authors highlight the potential of VR technology to revolutionise mathematics education by offering immersive, engaging, and interactive learning experiences. The multiview projection VR application serves as an example of how VR can facilitate the understanding of complex mathematical concepts through visualisation and manipulation in a 3D environment. However, the authors also identify several challenges in integrating VR into mathematics education. These include the need for extensive research on the efficacy of VR in promoting mathematical understanding, addressing the limitations of current VR hardware, and overcoming potential barriers related to accessibility and cost. Furthermore, the development of pedagogical strategies and curricula tailored to VR learning environments is crucial to ensure successful implementation. In summary, this article emphasises the promise of VR as a tool for mathematics education, while acknowledging the need for further research and development to address the associated challenges. The multiview projection VR application serves as an example of how VR can enhance the learning process, but more work is required to fully realise its potential in this field.

Villena-Taranilla et al. [177] highlights the growing popularity of VR in educational settings and the need for meta-analyses to assess its impact on learning outcomes. Focusing on K-6 students, the study examines variables such as the level of immersion, intervention length, and knowledge domain. After analysing 21 experimental studies from a pre-selected pool of 4658 references published between 2010 and 2021, the results indicate that VR significantly improves student learning compared to control conditions, with an effect size of 0.64. The observed effects in mathematics education were classified to be medium-sized based on five studies. The study also finds that immersive VR is more effective than semi-immersive and non-immersive systems. Importantly, this positive effect is consistent across different educational levels and across most knowledge domains. Moreover, shorter interventions (less than 2 hours) yield more effective outcomes compared to longer interventions.

Chatain et al. [21] investigates how concreteness in mathematics education can help learners grasp abstract concepts (see Figure 25). The authors explore the use of embodiment, or the grounding of abstract mathematics in concrete experience, as a form of concreteness. They designed and evaluated an embodied learning activity on graph theory and compared it to two other approaches: abstraction and manipulated concreteness. The study involved 89 participants, and the results showed that both forms of concreteness increased learners' perceived attention, confidence, and satisfaction compared to abstraction. However, only embodied concreteness increased perceived relevance and grounding. Moreover, unlike manipulated concreteness, embodied concreteness did not impair learning outcomes or transfer abilities. The findings suggest that embodied concreteness can effectively ground abstract mathematical concepts and enhance learners' understanding and engagement without compromising learning outcomes or transfer abilities. This study has implications for mathematics education and the design of learning activities that integrate embodiment as a form of concreteness.



(a) Three approaches to graph theory exercises



(b) Post-test representations

Fig. 25: Images from Chatain et al. [21]

3.1 Summary of Related Research

The growing body of research presented here provides evidence for the potential benefits of using VR technology in mathematics education. These studies showcase various applications of VR in areas such as geometry, calculus, primary mathematics, and spatial reasoning. The immersive and interactive nature of VR environments has been shown to increase student engagement, motivation, and learning outcomes across different domains of mathematics and at various educational levels.

Some studies emphasise the benefits of collaborative AR, 3D dynamic geometry, and game-based learning in promoting student engagement and improving learning outcomes [79]. Others highlight the potential advantages of immersive VR systems like Geometry Explorer in enhancing spatial understanding and supporting the learning of high-level geometry concepts [93]. Furthermore, the use of technologies like NeoTrie VR has been demonstrated to improve student understanding of geometric concepts alongside increased motivation, organisation of geometric knowledge, and spatial reasoning [144].

Despite these promising results, the generalisability of the findings can be limited due to small sample sizes or specific contexts in individual studies. Additionally, many studies call for further research to more extensively evaluate the long-term effects of using VR in mathematics education, as well as determine the most effective ways to integrate VR technology in classroom settings.

In terms of pedagogical implications, the use of VR technology in mathematics education presents opportunities for educators to design and implement innovative learning experiences that capitalise on the unique affordances of immersive and interactive environments. By leveraging the potential of VR technology through collaboration with software developers, researchers, and other stakeholders, educators can pave the way for more effective and engaging mathematics education that caters to the diverse needs of learners. However, care must be taken to address potential challenges, such as the development of appropriate pedagogical strategies, ensuring accessibility and affordability of VR tools, and overcoming technological limitations.

Overall, the research reviewed here provides a solid case for the value of VR technology in mathematics education, both as a means of enhancing student engagement and understanding and as a valuable tool for the professional development of mathematics educators. As VR technology continues to advance and become more accessible, we can anticipate further innovation and research into its use in mathematics education, ultimately paving the way for more effective and engaging teaching and learning experiences in the field.

The table below categorises the studies according to their research design, methodology, the technology used, targeted subject area, and the age and educational level of the study population (see Table 1).

Reference	Research design	Methodology	Technology	Subject Area	Educational Level
Kaufmann and Schmalstieg (2006)	Exploratory/Implementation	Mixed method: Practical experience, initial quantitative data	Augmented Reality, 3D Dynamic Geometry	Geometry	High School
Kaufmann (2011)	Overview/Discussion	Descriptive, qualitative	VR	Geometry	Not specified
Lai et al. (2016)	Exploratory/Implementation	Usability Testing: Rapid Iterative Testing and Evaluation (RITE) approach	VR	Geometry	Not specified
Liu et al. (2018)	Proposal/Discussion	Descriptive, proposal for a VR course	VR	Primary Mathematics	Primary School
Stranger-Johannessen (2018)	Quasi-Experimental	Quasi-experimental design, data analysis	VR	Multiplication	Fifth-grade
Nathal et al. (2018)	Empirical/Investigative	Detailed analysis of instructional design, learning outcomes	Physical and Virtual Objects	Vector Themes and 3D Space	Not specified
Rodríguez et al. (2019)	Implementation/Experimental	Practical implementation, user experience feedback	VR (Software: NeoTrie)	Geometry	Children aged 11-14 years old
Putman and Id-Deen (2019)	Discussion/Proposal	Argumentative, proposes VR use in math education	VR	Mathematics	Fifth-grade
Pearl et al. (2019)	Proposal/Discussion	Descriptive, proposal for VR application development	VR	Vectors and Objects in 3D Space	High school students
Crutchley et al. (2019)	Developmental	Surveys	VR	Calculus	College students
Dimmel and Bock (2019)	Developmental	Design and development discussion	VR with gesture-based interaction	Mathematics	Not specified
Hsu (2020)	Quasi-experimental	Questionnaire survey, pre- and post-experiment assessments, interviews	VR	Linear equations	High school students
Takac (2020)	Developmental	Design and development discussion	VR (WebVR)	Spatial abilities	High school to university students
Price et al. (2020)	Qualitative, multimodal	Observation and analysis of interaction	VR (Game: Cartesian Garden)	Geometry and Spatial Thinking	Primary students, ages 8-9
Rosa and Pinheiro (2020)	Qualitative	Observation and analysis of experiences	VR	Mathematics teacher education	Adult, Professional Development
Akman and Çakır (2020)	Quasi-experimental	Comparison of VR game to traditional methods	VR (Game: Keşfet Kurtul)	Fractions	Fourth-grade students
Rodríguez et al. (2021)	Design research	Qualitative observational data	VR (Software: NeoTrie)	Geometry	Not specified
Cangas et al. (2021)	Implementation research	Analysis of software application	VR (Software: NeoTrie)	Geometry	Primary to university students
Perri et al. (2021)	Empirical study	Survey (questionnaire) feedback	VR	Geometry	High school students
Rahmawati et al. (2022)	Quasi-experimental (ADDIE model)	Comparison of VR-based learning to traditional teaching	VR (ethnomathematical approach)	Mathematics education	High school students
Dilling and Sommer (2022)	Implementation research	Case study	VR (Application: Multiview projection)	Mathematics education	Not specified
Villena-Taranilla et al. (2022)	Meta-analysis	Analysis of multiple studies	VR (Level of immersion)	Education (various subjects)	K-6 students
Chatain et al. (2023)	Implementation research	Comparison of different learning methods	VR (Embodied learning)	Graph theory	Not specified

Table 1: Summary of related research

4 Classroom Integration

This section investigates the integration of VR technology in mathematics education, specifically focusing on the practical implementation in classroom settings and the technical implementation of VR applications designed for learning mathematics.

From a commercial perspective, the global VR market in education is projected to grow over the next decade. This growth is fueled by the increasing demand for experiential learning and the expanding technological infrastructure supporting VR. An industrial PhD focused on the technical aspects of VR development will not only provide insight into the practical applications of this technology but also equip the scholar with the skills to design and develop VR solutions. These skills are highly sought after in the growing VR market, thus enhancing the commercial viability of the research. In addition, understanding the implementation of VR in a classroom setting is a substantial commercial asset. Knowing how to seamlessly integrate VR into existing educational structures can help to bridge the gap between technology and pedagogy. This integration can lead to the development of products that are user-friendly and effective, thereby increasing their market potential.

From an academic standpoint, a focus on the practical and technical implementation of VR in education is an important area of research. It provides valuable insights into how these emerging technologies can be harnessed to enhance learning outcomes. Given the dynamic nature of VR technology, this research contributes to the evolving discourse on educational technology, influencing future pedagogical strategies and policies. Moreover, this research can stimulate interdisciplinary dialogue, bridging the gap between technology, pedagogy, and cognitive science. By understanding how VR can be effectively utilised in education, academics can explore its effects on cognitive processes such as memory, attention, and learning. This research can consequently inform the design of more effective learning environments, contributing to the academic field's understanding of how technology can augment human cognition.

Asymmetric VR offers a unique approach to incorporating VR into the learning process, as it enables multiple users with different roles to engage simultaneously in a shared virtual environment. In the context of mathematics education, this approach allows teachers and students to explore mathematical concepts together, fostering collaboration and providing a more immersive and interactive learning experience. The potential benefits of this technology are immense, ranging from increased student motivation and engagement to deeper conceptual understanding and improved problem-solving skills.

In addition to exploring the classroom integration of asymmetric VR, this section also sheds light on the technical development of VR applications designed for mathematics education. These applications incorporate advanced visualisation techniques, haptic feedback, and user-friendly interfaces, allowing students to manipulate and explore complex mathematical concepts in a manner that is both intuitive and engaging. By breaking down barriers to understanding, these applications have the potential to reshape the landscape of mathematics education and pave the way for a new era of learning.

4.1 Practical Implementation

The integration of VR into the classroom involves several considerations, including practical, social, ethical, and financial aspects. This section will provide a comprehensive analysis of these factors, drawing upon relevant research articles to substantiate claims and statements.

Implementing VR technology in classrooms requires careful planning and consideration of the physical environment, technological infrastructure, and support from educational stakeholders [46]. Teachers need adequate training to effectively utilise VR as an instructional tool and integrate it into their pedagogical approaches. Moreover, the selection of appropriate VR content, aligned with curriculum goals and learning objectives, is critical for successful integration [66]. Additionally, maintaining and troubleshooting VR equipment can be time-consuming and require technical expertise [37]. Ensuring compatibility and seamless integration of VR devices and applications with existing classroom technologies is also essential to minimise disruptions to the learning process [72]. The social aspects of VR integration into the classroom encompass collaboration, communication, and social presence [70]. Research suggests that collaborative VR learning experiences can foster social interaction and cooperation among students, leading to improved learning outcomes [32, 106]. However, some studies have raised concerns about the potential isolation of students who are immersed in individual VR experiences, which may impact group dynamics and classroom cohesion [129]. To address these concerns, educators should consider implementing collaborative VR applications that promote teamwork and communication, while also balancing the use of VR with traditional, face-to-face instructional methods [99]. Ethical considerations in the use of VR technology in the classroom include privacy, data security, and potential negative psychological effects [159]. Protecting students' personal information and privacy in the digital age is of utmost importance, especially considering the vast amount of data collected by VR applications and devices [157]. Ensuring data security and adhering to relevant data protection regulations is a crucial aspect of ethical VR implementation [105]. Furthermore, potential negative psychological effects, such as simulator sickness or increased anxiety, must be taken into account [97, 143]. Educators and researchers should be cautious when exposing vulnerable populations to potentially distressing VR content and monitor students' well-being during VR experiences [9]. The cost of VR hardware and software can be a significant barrier to widespread adoption in educational settings [8]. The financial burden of equipping classrooms with VR HMDs, controllers, and high-performance computers may be prohibitive for many schools, particularly those with limited budgets [111]. Moreover, the rapid advancement of VR technology may necessitate frequent updates and replacements, further straining financial resources [5]. However, the decreasing cost of consumer VR devices and the development of more affordable VR solutions (e.g., Google Cardboard) have made the technology more accessible for educational use [2]. Furthermore, research has shown that the potential benefits of VR-based learning, such as increased student engagement and improved learning outcomes, may outweigh the initial financial investment [139]. As the cost of VR technology continues to decline, it is expected that more schools will be able to incorporate VR into their classrooms, potentially leading to long-term educational benefits [101].

In summary, the integration of VR into the classroom as a teaching tool requires careful consideration of practical, social, ethical, and financial aspects. By addressing these concerns and adopting a balanced approach, educators can harness the transformative potential of VR technology to enhance teaching and learning experiences. As research in this area continues to grow, future studies should focus on evaluating the effectiveness of VR-based learning across various educational contexts and exploring strategies for overcoming the challenges associated with VR implementation in the classroom.

Revision of our Taxonomy

Ouverson and Gilbert [126] presents a refinement of our taxonomy on asymmetry in VR contexts. Our taxonomy focused on interaction design, specifically the timing of actions and the direction of dependence within an asymmetric system. The revised taxonomy extends this perspective by incorporating a wider range of potential asymmetric experiences in VR, including the Collaborative Asymmetric Virtual Reality (CAVR) dimension of team interdependence [126] (see Figure 26).

The revision maintains the definitions of dependence direction, including mirrored, unidirectional, and bidirectional dependence, and the classification of action timing as coincident, sequential/disjoint, concurrent, asynchronous, and expectant. However, the revised taxonomy cross-references this with the work of other researchers in the field, such as Harris et al. [54] and Johansen [71], bringing in the concepts of concurrent synchronised, serial, mixed, and unsynchronised timing. This revision is valuable because it enables a more detailed analysis of the interaction between dependence and timing in asymmetrical VR environments. By incorporating the extended timing categorisations into the existing taxonomy, the researchers provide a more nuanced understanding of these factors, highlighting how different action timings might be better supported by a particular asymmetry. The revised taxonomy also suggests the importance of considering the degree of asymmetry in different VR settings. High asymmetry refers to situations where one actor is uniquely dependent on another, and low asymmetry describes situations where actors depend on each other in similar ways. Medium asymmetry supports positive interdependence, typical of collaborative teams. In the context of VR design, these variations can have implications for the user experience and should inform interface and perceptual access decisions. Lastly, the paper underscores the practical implications of the revised taxonomy for VR designers. Aligning the timing

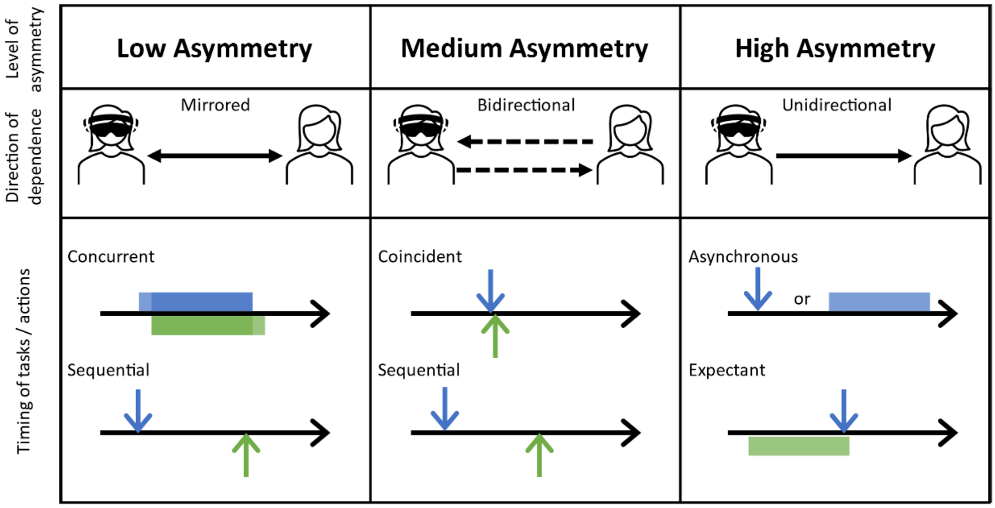


Fig. 26: Composite framework for Asymmetric VR (CAVR)

and dependence of interactions can aid in the development of more efficient and effective asymmetric VR systems. In turn, this can improve user experiences and facilitate smoother, more natural interactions within these environments.

The revised taxonomy offers a more comprehensive and nuanced understanding of asymmetry in VR, enriching our work by incorporating a wider range of asymmetric experiences and their interrelation with dependence and timing. It provides a conceptual framework that not only benefits researchers in the field but also offers practical insights for VR designers, aiding in the creation of improved VR experiences.

4.2 Technical Implementation

In recent years, the rapid advancement of VR technology has expanded the horizons of immersive experiences, transforming not only the gaming industry but also various other fields, such as education, healthcare, and entertainment. Developing robust, responsive, and high-quality VR applications demands the seamless integration of multiple technologies, including game engines, interaction frameworks, and networking frameworks. Game engines, the backbone of VR applications, provide a comprehensive development platform, offering a suite of tools, libraries, and assets that enable developers to build rich and interactive 3D worlds. By leveraging the power of popular game engines, such as Unity and Unreal Engine, developers can capitalise on built-in physics systems, rendering pipelines, and audio management to create visually stunning and realistic VR experiences. Interaction frameworks play a critical role in bridging the gap between users and the virtual world, allowing for natural, intuitive interactions within the VR environment. By integrating advanced input systems, such as hand tracking and haptic feedback, these frameworks facilitate the design of compelling user interfaces and the implementation of complex interactions, elevating the sense of presence and immersion. Networking frameworks, on the other hand, are essential to enable communication and synchronisation between multiple users in a shared virtual space. By incorporating real-time networking solutions, such as Photon and Mirror, developers can create collaborative and social experiences that transcend geographical boundaries, fostering a sense of togetherness and community within the virtual realm.

In this technical section, I describe the intricacies of employing these tools to create immersive, interactive, and connected virtual environments.

Game Engines

VR technology has seen a surge in demand across industries, including gaming, healthcare, and education. As VR continues to gain traction, it is essential for developers to select the appropriate game engine to create immersive, high-quality VR experiences. This section of the research article provides a detailed comparison of the three widely used game engines — Unity [9], Unreal [10], and Godot [11] — focusing on their capabilities in developing VR applications.

Unity, Unreal Engine, and Godot are all prominent game engines used in VR development. Unity is widely recognised for its versatility, user-friendly nature, and robust support for various VR platforms, including Oculus Rift, Meta Quest, HTC Vive, and PlayStation VR. Its large community, extensive documentation, and a vast collection of pre-built assets in the Asset Store, paired with the ease of using the C# programming language, significantly aid developers, though it falls short in graphical fidelity and performance optimisation compared to Unreal Engine. Unreal Engine, developed by Epic Games, stands out for its superior visuals and high-performance capabilities, making it ideal for visually intensive VR experiences. Its advanced graphics, built-in visual scripting system (Blueprints), and VR-specific tools balance its steeper learning curve and smaller community size. Lastly, Godot, an open-source, lightweight engine, supports VR development with its dedicated VR module and features an active community that continuously improves the platform. Despite being less feature-rich and possessing lower graphical and performance capabilities, its efficiency and GDScript language, similar to Python, make it suitable for smaller projects or those with limited resources.

Selecting the right game engine for developing VR applications depends on various factors, such as the desired level of graphical fidelity, performance requirements, and ease of use. Unity offers a versatile and user-friendly solution, with extensive support and resources. Unreal Engine excels in visual quality and performance optimisation, making

[9] Unity: `https://unity.com/`
[10] Unreal: `https://www.unrealengine.com/`
[11] Godot: `https://godotengine.org/`

it suitable for visually intensive VR experiences. Godot is an efficient, lightweight, and open-source alternative, ideal for smaller projects or those with limited resources. Each engine has its strengths and weaknesses, and the choice ultimately depends on the specific needs of the developer and their VR project.

Interaction Frameworks

As the VR industry continues to grow, the need for efficient and versatile development tools is more crucial than ever. This section aims to provide a comparison of four widely used interaction frameworks for developing VR applications in the Unity game engine: Virtual Reality Toolkit (VRTK) [12], XR Interaction Toolkit [13], Oculus Integration [14], and SteamVR Plugin [15]. I will compare these frameworks based on their features, ease of use, compatibility, and community support.

VRTK is an open-source VR development framework for Unity, offering extensive features like object interactions and UI interaction, though its community-driven updates might result in slower bug fixes. The XR Interaction Toolkit is an official Unity package for developing XR applications, with clear documentation and regular updates, yet it has a more limited feature set and less mature community support. Oculus Integration is a Unity package from Oculus for creating optimised VR experiences for the Oculus platform with features like hand tracking and spatial audio, but its platform-specific nature restricts its cross-platform applicability. Lastly, the SteamVR Plugin developed by Valve supports the creation of VR applications for various HMDs like HTC Vive and Valve Index, but its exclusive optimisation for the SteamVR platform hampers cross-platform development and may require more setup compared to other frameworks.

In conclusion, each of these four interaction frameworks offers a unique set of features and benefits for developing VR applications in Unity. VRTK is an excellent choice for developers seeking an open-source, cross-platform solution, while the XR Interaction Toolkit is a good option for those who prioritise official Unity support. Oculus Integration and SteamVR Plugin are tailored for their respective platforms and provide optimised development experiences. Ultimately, the choice of the framework will depend on the specific requirements and target platforms of a given project. Developers should consider each framework's features, ease of use, compatibility, and community support when making their decision.

[12] Virtual Reality Toolkit (VRTK): `https://www.vrtk.io/tilia.html`
[13] XR Interaction Toolkit: `https://docs.unity3d.com/Packages/com.unity.xr.interaction.toolkit@2.3/manual/index.html`
[14] Oculus Integration: `https://assetstore.unity.com/packages/tools/integration/oculus-integration-82022`
[15] SteamVR Unity Plugin: `https://valvesoftware.github.io/steamvr_unity_plugin/`

Networking Frameworks

The development of multiplayer VR applications has grown significantly in recent years, fueled by advancements in VR technology and increasing user demand. To facilitate the creation of these experiences, various networking frameworks have emerged, each with its own set of features and capabilities. In this section, I will provide a detailed comparison of three popular networking frameworks — Photon [16], Normcore [17], and Mirror [18] — focusing on their suitability for developing multiplayer VR applications within the Unity game engine.

Photon, Normcore, and Mirror are prominent networking frameworks for Unity, each offering unique advantages. Photon, by Exit Games, is widely used for real-time multiplayer applications, boasting cross-platform compatibility, seamless integration with Unity, and key components such as the globally distributed Photon Cloud and the customisable Photon Server. On the other hand, Normal's Normcore framework emphasises user-friendly design with minimal coding, featuring a "zero-config" philosophy, automatic server discovery, real-time physics, and voice chat support, making it ideal for multiplayer VR experiences. Meanwhile, Mirror, a community-driven fork of Unity Networking (UNET), provides an improved, stable alternative to its predecessor with a simple and familiar API. It offers a high-level API for common networking tasks, a low-level API for advanced users, and supports multiple transport layers like Telepathy, KCP, and WebSockets, ensuring adaptability for diverse networking needs.

In summary, Photon, Normcore, and Mirror are three viable networking frameworks for developing multiplayer VR applications in Unity. Photon stands out for its cross-platform compatibility, global server infrastructure, and extensive feature set, making it an excellent choice for large-scale, commercial projects. Normcore's simplicity and focus on real-time physics and voice chat make it an attractive option for smaller-scale projects or developers who prefer a more straightforward approach. Lastly, Mirror's familiarity and compatibility with the deprecated UNET system, along with its high-level and low-level APIs, offer a balanced and flexible solution for a wide range of applications. Ultimately, the choice of networking framework will depend on the specific needs and preferences of the developer and the project at hand.

Implementations of the Project

In the exploration of educational VR, I developed four prototypes for mathematics learning in VR. These prototypes were created using the Unity game engine, leveraging its robust capabilities and the C# programming language to script gameplay. I utilised the Tilia package to handle cross-platform VR setup and configurable interaction design, thereby rendering the user experience more intuitive and engaging. To facilitate

[16] Photon: `https://www.photonengine.com/pun`
[17] Normcore: `https://normcore.io/`
[18] Mirror: `https://github.com/MirrorNetworking/Mirror`

collaborative learning and interaction, I incorporated multiplayer functionality with the aid of the Normcore package. The culmination of my efforts led to the deployment of these games on the Oculus Quest 1 and 2, thereby making our educational VR games accessible to a broad spectrum of users. Through these prototypes, I aimed to investigate how VR can revolutionise the teaching approaches in mathematics education, potentially fostering deeper understanding and engagement among learners.

5 Subject Matter

The landscape of mathematics education is undergoing a profound transformation with the integration of VR technology. As educators and content creators, we have the exciting opportunity to revolutionise the way students engage with complex mathematical concepts such as algebra and equations, geometry, and vectors. By leveraging the immersive and interactive capabilities of VR, we can create learning experiences that are not only captivating but also highly effective in fostering a deep understanding of these foundational topics.

This section explores the process of designing learning material for specific subject matter that harnesses the full potential of VR technology to enhance the teaching and learning of algebra, geometry, and vectors. I will explore the unique opportunities offered by VR, such as spatial visualisation, experiential learning, and embodiment, which can help students develop intuition, hone their problem-solving skills, and foster a growth mindset.

5.1 Algebra and Equations

The study of algebra is a crucial component of mathematics education, with a significant impact on students' problem-solving abilities and career opportunities in STEM fields. However, a growing body of research has documented the difficulties students face when learning algebra. In this section, I will discuss the main challenges students encounter as identified in the research.

One of the most significant hurdles students face when learning algebra is understanding and manipulating symbolic representations. According to Kieran [82], students often struggle to make sense of algebraic symbols and the relationships they represent. The difficulties in comprehending abstract symbols can lead to misconceptions and hinder the development of problem-solving skills [13]. This issue is further complicated by the fact that students may be proficient in arithmetic operations but have difficulty transitioning to the symbolic nature of algebra [57]. Algebra learning involves the development of both procedural and conceptual knowledge. Procedural knowledge refers to the ability to execute mathematical operations and manipulate expressions, while conceptual knowledge refers to the understanding of underlying mathematical principles and relationships [142]. Students frequently struggle with distinguishing and integrating these two types of knowledge, leading to challenges in solving complex algebraic problems [58]. Moreover, students often prioritise procedural knowledge over conceptual understanding, which can limit their ability to generalise and apply algebraic concepts to novel situations [155]. Another challenge in algebra learning is the ability to represent and interpret mathematical problems. Students may have difficulty interpreting word problems and translating them into algebraic expressions or equations [121]. This can stem from inadequate linguistic skills, lack of contextual understanding, or difficulties in connecting real-world situations to abstract algebraic concepts [24]. As a result, students may struggle to grasp the practical relevance of algebra and find it challenging to engage with the subject [89]. Cognitive factors, such as working memory capacity, also play a crucial role in students' ability to learn algebra. Sweller [167] suggests that learning algebra involves the processing and manipulation of information within working memory, which has a limited capacity. When students attempt to solve algebraic problems, they may experience cognitive overload due to the complex relationships between variables, expressions, and equations. This can result in difficulties in retaining and applying the necessary information to solve problems successfully [130]. Given the multitude of challenges students face in learning algebra, targeted interventions are necessary to support their understanding and mastery of the subject. Strategies such as using multiple representations, encouraging the development of both procedural and conceptual knowledge, and providing real-world problem contexts can help students overcome these difficulties [83, 163]. Additionally, incorporating technology and adaptive learning platforms can offer personalised learning experiences, allowing students to develop a deeper understanding of algebraic concepts at their own pace [147].

The rapid advancement of technology has revolutionised education, and algebra learning is no exception. Recent research demonstrates the potential of technology-enhanced learning environments to improve students' understanding and performance in algebra. In this section, I present a brief overview of existing research on the use of technology in algebra education.

Computer Algebra Systems (CAS) have been identified as powerful tools for learning and teaching algebra. CAS, such as Mathematica, Maple, and GeoGebra, can manipulate algebraic expressions, plot graphs, and solve equations symbolically, enabling students to explore algebraic concepts interactively [60]. A study by Pierce et al. [134] revealed that students who used CAS in their algebra classes demonstrated better conceptual understanding and increased algebraic proficiency compared to those who did not. Intelligent Tutoring Systems (ITS) are adaptive, computer-based learning environments designed to provide individualised instruction and feedback based on students' needs. Research indicates that ITS, such as Cognitive Tutor Algebra (CTA) and ALEKS, can improve students' algebraic problem-solving skills and overall performance [40, 87]. These systems can identify students' misconceptions, provide targeted feedback, and scaffold the learning process, making them effective instructional tools in algebra education. Online learning platforms, such as Khan Academy [19], offer students access to a wealth of algebra resources, including instructional videos, practice exercises, and quizzes. Research has shown that students who engage with these platforms can improve their algebra skills and performance [100]. Furthermore, incorporating gamification elements into online learning platforms can increase students' motivation, engagement, and learning outcomes [50]. Collaborative learning technologies, such as virtual manipulatives and shared whiteboards, can facilitate algebra learning by pro-

[19]Khan Academy: https://www.khanacademy.org/

moting group problem-solving and peer interactions. Studies have demonstrated that students who engage in collaborative learning environments exhibit improved conceptual understanding and algebraic reasoning [25, 84]. These technologies encourage students to work together, exchange ideas, and explore algebraic concepts in a supportive social context.

Models

Mathematics often involves complex abstract concepts, and one of the effective methods of making these concepts accessible and understandable, especially to students, is by using models. A model in this context serves as a simplified representation that highlights essential aspects of the concept in a tangible or visual way. One such model is the balance model, commonly used to visualise the properties of an equation. However, the balance model is just one among many such tools utilised in mathematics education. The following are seven additional models frequently employed to illustrate various mathematical ideas and operations.

Number Line Model: Number lines are a common tool used in elementary school classrooms to help students understand the concept of order and magnitude, the difference between numbers, and operations like addition and subtraction. They can be extended to integers, fractions, and even irrational numbers in higher levels of math. VR can turn the number line into a walkable path, where students can physically move to different positions, experiencing concepts like addition, subtraction, and the idea of negative numbers in a new and tangible way.

Area Model: Area models are often used for multiplication and division problems, especially when dealing with multi-digit numbers or fractions. They work by breaking each number down into its component parts and representing these as lengths and widths of rectangles, whose area represents the product or quotient. With VR, the area model can become a 3D volume model. Students can manipulate shapes to understand multiplication, division, and distributive property in a more tactile and intuitive manner.

Fraction Bars or Circles: These are physical or visual representations of fractions. By showing fractions as part of a whole (e.g., a bar or a circle), students can better understand the concept of fractions, compare fractions, and perform operations like addition and subtraction. VR can allow students to manipulate virtual fraction bars or circles, adjusting their sizes, comparing them, and observing the impact of operations like addition or subtraction of fractions.

Pie Chart or Circle Graph: These models are used to represent proportions or percentages. For example, if a student is learning about percentages, they might use a pie chart to represent a certain percentage of a population or a set. Students can step inside a large-scale, 3D pie chart or circle graph in a VR environment. This can help them better understand fractions, percentages, and their relationship to the whole.

Coordinate Plane: The coordinate plane, or Cartesian plane, is a model used to graphically illustrate relations or functions and to understand the concept of coordinates and dimensions. It is particularly useful for visualising algebraic equations, geometric transformations, or for introducing calculus concepts. A VR coordinate plane could be explored in three dimensions, where students can plot points, draw lines, and visualise equations in 3D space. This adds depth to the understanding of coordinates and their relationships.

Tree Diagrams: Tree diagrams are a visual representation used in probability to demonstrate all possible outcomes or combinations of an event. VR can turn tree diagrams into actual "decision trees" that students can traverse, helping them understand concepts of probability and combinatorics in a dynamic and interactive manner.

Manipulatives: Manipulatives are physical objects that help students visualise and understand abstract mathematical concepts. For example, base ten blocks can be used to understand place value and carry out addition, subtraction, multiplication, or division. In a VR environment, students can handle and manipulate virtual objects that mimic physical manipulatives, providing an intuitive and engaging way to explore a variety of mathematical concepts.

Algebra Tiles: These are manipulatives used to represent and manipulate algebraic expressions and equations, particularly in the learning of integer operations and distributive property. In VR, students can manipulate algebra tiles in a 3D space, providing a hands-on approach to understanding and solving algebraic expressions and equations.

These models offer different perspectives on mathematical ideas, allowing students to connect with the material in various ways that match their learning preferences. They help convert abstract mathematical concepts into visual, tactile experiences, making math more accessible and engaging, and fostering a deeper understanding. Each model is tailored to elucidate certain aspects of mathematics, making them particularly effective in their respective areas.

Balance Model

The balance model is an invaluable tool in mathematics education, specifically in elucidating the principles of equations and equalities. By employing a physical metaphor, the balance model aids students in visualising and understanding these abstract concepts. It equates an equation to a balance scale, where the two sides need to be equal for the balance to be maintained. This provides a tactile and intuitive means to grasp and apply the principles of equality and operations.

This model is not limited to the most fundamental aspects of equations; it can be expanded and adapted to illustrate a variety of mathematical concepts and approaches. Here are seven ways that the balance model can be utilised to enhance understanding of different mathematical principles:

Equality: The balance model helps to visualise the concept of equality. An equation is like a balance scale, where the left side and the right side need to be equal for the scale to balance. This is a fundamental concept in mathematics and a foundational element in understanding equations [86]. VR could create a dynamic balance scale that users can interact with. This would help students to physically see and understand the concept of equality, by adding or removing weights to balance the scale.

Addition/Subtraction: The balance model allows for an intuitive understanding of addition and subtraction. If you add (or remove) the same amount from both sides of a balance, it stays balanced. This mirrors the property of equations where you can add or subtract the same quantity from both sides without changing the solution of the equation [86]. Using the VR balance scale, students could see how adding or subtracting the same amount from both sides maintains the balance. They could perform these operations themselves and see the impact in real-time.

Multiplication/Division: Similar to addition and subtraction, if you multiply or divide both sides of a balance by the same non-zero amount, it remains balanced. This concept is useful when solving equations [86]. The VR balance model could show the effect of multiplying or dividing both sides of an equation. For example, the weights on the balance scale could be visually altered to represent multiplication or division.

Variables: The balance model is a good way to introduce the concept of variables. You can say that a variable is similar to a box on the balance that you do not know the weight of, and the goal is to figure out what weight (value) would make the balance scale balanced [164]. The VR balance model could use various shapes or objects to represent unknown variables. This could provide a more tangible way for students to understand and manipulate variables in equations.

Transforming Equations: The balance model also provides a tangible way to demonstrate how equations can be transformed. For instance, if you have the equation $2x + 3 = 9$, you can subtract 3 from both sides (removing the same weight from both sides of the balance), which gives $2x = 6$. Then you can divide by 2 (splitting the remaining weights in half), to get $x = 3$ [45]. VR could allow students to physically manipulate the elements of an equation on the balance scale. They could, for instance, move variables or constants from one side to the other and see the effect on the balance.

Inequalities: The balance model can also be extended to illustrate the concept of inequalities. For instance, if one side of the scale is heavier, it represents a 'greater than' inequality, while if it is lighter, it represents a 'less than' inequality [86]. The VR balance model could be adapted to visualise inequalities, perhaps tilting the scale to one side or the other to represent 'greater than' and 'less than' relationships.

Algebraic Expressions: More complex algebraic expressions can also be represented with the balance model. For instance, if you have the equation $3(x + 2) = 15$, you can imagine each side of the balance having three bags, each containing 'x+2' units of weight. This allows students to concretely understand the distributive property and the process of solving more complex equations [45]. More complex algebraic expressions could be represented in VR, allowing students to visualise and manipulate these expressions on the balance scale. This could provide a better understanding of how more complex equations work.

Each of these seven approaches leverages the balance model to convert abstract ideas into tangible, understandable concepts. This not only makes mathematics more accessible and engaging but also fosters deeper understanding and retention.

5.2 Geometry

DGEs represent a revolutionary tool in the field of mathematics education, providing an interactive, visual, and explorative framework for both teaching and learning [63]. By leveraging technology to create manipulable mathematical objects and models, DGEs allow students to explore complex mathematical concepts intuitively and deepen their understanding beyond traditional pedagogical methods [60].

At their core, DGEs allow users to construct, manipulate, and observe geometric shapes and figures in a responsive setting. They offer an interactive, malleable space where users can adjust parameters and witness changes in real-time. DGEs represent mathematical relationships visually, making abstract concepts more tangible and comprehensible. For instance, in a DGE, students can adjust the sides of a triangle and watch how the angles respond accordingly. They can witness the Pythagorean theorem in action by dragging the corners of a right-angled triangle and observing how the square of the hypotenuse always equals the sum of the squares of the other two sides.

The introduction of DGEs into classrooms has had a profound impact on teaching methods and student outcomes [92]. Traditionally, mathematics has been taught in a static, rote manner, with students passively receiving information. DGEs shift this paradigm towards a more active and explorative learning process, where students can experiment, observe, and draw conclusions by themselves. The dynamic nature of

DGEs fosters a deeper understanding of mathematical principles. By allowing students to 'see' and 'touch' the mathematics, DGEs create a more engaging and immersive learning experience. Højsted [62] reviewed the literature on the use of DGEs in mathematics education and identified four key potentials: *feedback*, *dragging*, *measuring*, and *tracing*. It provides guidelines for enhancing students' reasoning skills, considering student cognition, task design, and the teacher's role, drawing from various theoretical models. The article also discusses the potential of these guidelines to boost mathematical reasoning competency. DGEs can also help students develop their spatial reasoning skills, understand mathematical proofs, and appreciate the beauty and interconnectedness of mathematical concepts [74]. Moreover, DGEs are instrumental in differentiating instruction, as they cater to a broad range of learning needs. Visual learners, in particular, can benefit from the graphical representations and real-time manipulations that these tools offer.

There are several DGE software applications available, each with unique features. Geometer's Sketchpad, GeoGebra, and Cabri Geometry are among the most widely used throughout history. These platforms offer a suite of tools for creating and manipulating geometric figures, graphing functions, and exploring algebraic structures. GeoGebra, for instance, is a free and open-source platform that combines geometry, algebra, spreadsheets, graphing, calculus, and statistics in one easy-to-use package [61]. It provides an expansive environment where students can explore a multitude of mathematical concepts and phenomena.

Despite the potential of DGEs, their implementation in classrooms is not without challenges. There is a need for adequate teacher training to ensure the tools are used effectively [22]. The integration of DGEs into the curriculum should be done thoughtfully, aligning with learning objectives and complementing rather than replacing traditional teaching methods. Furthermore, as technology advances, the capabilities of DGEs continue to expand. Future iterations may incorporate features like VR for even more immersive experiences or machine learning algorithms that adapt the environment to individual learners' needs. As such, ongoing research is essential to understand how to best utilise these tools and adapt to their evolving potential. In conclusion, DGEs are a powerful asset in mathematics education. They provide an interactive, explorative, and visually rich approach to learning that can enhance students' understanding and appreciation of mathematics. The continued exploration and integration of these tools in educational settings is a promising avenue for improving mathematics education.

The intersection of DGEs and VR showcases the potential for revolutionising the way mathematical and spatial concepts are taught and understood. DGEs, traditionally used to manipulate geometric constructions and provide a deeper understanding of abstract mathematical ideas, are evolving into an immersive 3D learning experience through the incorporation of VR technology. These novel VR-enabled DGEs offer a hands-on, intuitive exploration of geometrical concepts, promoting a highly engaging platform for learners. By embodying geometric figures and their transformations within a virtual space, students can experience and manipulate mathematical relationships in ways previously confined to the two-dimensional plane. This paradigm shift in instruction fosters a more comprehensive, contextual, and visceral understanding of geometry, potentially reducing the cognitive load associated with abstract mathematical reasoning.

5.3 Vectors

Vectors, fundamental elements in fields such as physics, engineering, and computer science, often pose a challenge for students to visualise and understand. They represent quantities that have both direction and magnitude, which can be difficult to convey in traditional two-dimensional diagrams. This is where VR can play a pivotal role, enabling 3D representation and interaction with vectors, providing a more intuitive understanding of vector operations and properties.

One of the most effective uses of VR in vector visualisation may be the representation of vector fields. In a vector field, each point in a 3D space is associated with a vector. VR can render these fields in a 3D space, allowing students to explore the field from any angle. They can zoom in and out, move around, and even follow individual vectors to see how they interact with the rest of the field. This immersive experience may provide a deeper understanding of complex concepts like divergence, curl, and gradient. VR technology can also uniquely illustrate the concepts of cross product and dot product, operations that may otherwise seem abstract on a two-dimensional plane. For the cross product, VR allows the visualisation of the right-hand rule and the resultant vector orthogonal to the plane formed by the two original vectors. By manipulating the input vectors in real-time, learners can see the resultant vector's direction and magnitude change dynamically, providing an intuitive understanding of the geometric interpretation of the cross product. In the case of the dot product, VR can highlight its scalar nature and its relationship with the angle between two vectors. By changing the angle between two vectors, students can see how the dot product changes, emphasising its connection with the cosine of the angle between vectors. Additionally, the concept of orthogonality (when the dot product equals zero) becomes immediately clear as learners can manipulate the vectors to a 90-degree angle and observe the resulting null dot product. VR, therefore, provides an innovative way to visually and intuitively understand these vector operations, fostering a deeper comprehension of the principles underlying them. VR can also be used to illustrate vector operations interactively. For example, students can manipulate vectors in a 3D space to perform operations such as addition, subtraction, and cross products. They can visually see how the resultant vector changes as they alter the magnitude and direction of the original vectors. This hands-on approach may enhance comprehension and retention of these operations. Dynamic systems, such as fluid dynamics or electromagnetic fields, can be challenging to visualise on a static, two-dimensional plane. With VR, these systems can be represented as they are: constantly changing and moving. Students can observe how vectors in these systems interact in real-time, giving them a more accurate picture of these complex systems. VR's ability to provide haptic feedback can also enhance the learning experience. Students could feel the force exerted by a vector, or the resistance resulting from a vector operation, giving them another sensory channel through which to understand vectors.

3Blue1Brown [20], a mathematics education channel by Grant Sanderson, revolutionises learning by employing innovative visualisations to simplify complex mathematical topics. These animations, characterised by their intuitive and colourful designs, supplement traditional learning methods and have inspired educators and curriculum designers to adopt similar approaches. 3Blue1Brown has been instrumental in transforming the landscape of mathematics education, particularly in the field of vector visualisation. A vector, in mathematical terms, is a quantity defined by both a magnitude and a direction. Traditionally, understanding vectors has been a challenging prospect for many students, given their abstract nature. In 3Blue1Brown's video series on linear algebra, vectors are visualised in a tangible and intuitive way. By using colours and moving graphics to represent vectors, 3Blue1Brown provides a visual intuition which has been shown to improve comprehension and retention of these complex concepts. Dynamic visualisations allow learners to observe and understand the properties of vectors, their operations, and how these operations can influence the magnitude and direction of resultant vectors.

Considering this novel approach to vector visualisation, the next logical step could be the incorporation of VR technologies. VR, with its immersive, 3D environment, could enhance the experience of visual learning. For instance, imagine a VR application that allows learners to interact with vectors in a 3D space. They could physically manipulate vectors, seeing the effects of addition, subtraction, and scaling in real-time. Students could walk around the vectors, gaining different perspectives and a deeper understanding of the spatial relations between them. Concepts like dot product and cross product could be made more intuitive by physically showing the projection of one vector onto another, or the perpendicular vector resulting from a cross product. Such a VR platform would take the visualisation approach employed by 3Blue1Brown to another level, providing a fully immersive and interactive learning experience. It could lead to a major breakthrough in mathematics education, creating an even more engaging and effective tool for understanding mathematical concepts, particularly in the field of vector analysis. Furthermore, it could democratise access to these resources, enabling more learners worldwide to grasp the beauty and complexity of mathematics in an intuitive way.

In conclusion, merging 3Blue1Brown's approach to vector visualisation with the continually advancing VR technology may enhance mathematics education. This blend leverages the interactive capabilities of VR, providing a range of techniques that may improve the understanding and retention of mathematical concepts, including vectors.

[20] 3Blue1Brown: https://www.3blue1brown.com/

As VR technology becomes increasingly accessible, the potential for its use in educational settings is expected to expand. Thus, this fusion could lead to the creation of a deeply immersive learning tool that not only inspires curiosity but also fosters a profound understanding of mathematical concepts.

6 Research Questions

The immersive, engaging, and interactive features of VR have the potential to redefine pedagogical approaches and learning experiences, enhancing the overall quality of education. The current research aims to explore the integration and design of VR in mathematics education, with particular emphasis on algebra, equation-solving strategies, fostering collaboration, embodied learning experiences, and the visualisation and contextualisation of mathematical concepts. Based on these objectives, I have formulated and categorised five research questions aimed at addressing the various areas pertaining to the integration and design of VR in mathematics education:

- **Technology Integration in Mathematics Education**
  - What are effective ways of integrating VR technology into mathematics education within the classroom setting?
- **Pedagogical Strategies for VR in Mathematics Education**
  - What approaches to teaching algebra, equations, and equation-solving strategies to students can be used in VR?
- **Collaborative and Inclusive VR for Mathematics Education**
  - How can VR be tailored to accommodate students with diverse needs, fostering collaboration in learning daily living skills and mathematical concepts?
- **Interaction Design and Embodiment in VR for Mathematics Education**
  - In what ways can hand-tracking capabilities in VR support gesture-based interaction for embodied learning experiences in geometry education?
- **Visualisation and Contextualisation of Mathematics in VR**
  - How can visualisation strategies and gamification principles be leveraged in VR to contextualise mathematical concepts?

The first research question examines the integration of VR into mathematics education within a classroom setting. Despite the rising trend of VR applications in various educational fields, its implementation in mathematics classrooms remains largely underexplored. This study intends to investigate potential methodologies and strategies to incorporate VR in this specific learning environment. This integration could facilitate a comprehensive understanding of complex mathematical concepts, foster active learning, and engender a heightened sense of engagement among students.

The second research question pertains to the design of VR to aid in teaching algebra, equations, and equation-solving strategies. Algebra can be a challenging domain for many students due to its abstract nature and the high level of cognitive demand it entails. By leveraging VR's capabilities, it could be possible to render these abstract concepts into more tangible and interactive forms, making the learning process more intuitive and engaging. The design of such a VR system will be a primary focus of this research.

The third research question investigates the design of VR tools that foster collaboration between students with diverse needs. Collaboration is an essential aspect of the contemporary educational landscape, fostering critical thinking and problem-solving skills. However, the varying needs of students can sometimes hinder this collaborative process. Therefore, this research aims to leverage the potential of VR in creating an inclusive, adaptive, and collaborative virtual learning environment that can accommodate diverse student needs.

The fourth research question explores the design of gesture-based interaction in VR to support embodied learning experiences in mathematics education. Embodied learning, where learners use their bodies in the learning process, is shown to enhance understanding and retention of information. VR, with its immersive nature, offers a unique platform to incorporate gesture-based interaction, enabling students to physically engage with mathematical concepts. This question seeks to understand how such interactions can be effectively designed to maximise learning outcomes.

The final research question looks at how VR can be designed to contextualise mathematical concepts through visualisation and game-based learning. Contextualising abstract mathematical concepts can make learning more engaging and meaningful for students. The use of visualisation can help in illustrating these concepts in a more relatable way, while gamification can make the learning process more enjoyable and motivating. The potential of VR in this regard is enormous, but how to effectively harness it for the benefit of mathematics education is a question that warrants further exploration.

In conclusion, the proposed research questions aim to investigate the promising intersection of VR and mathematics education. The findings could serve to guide the design of future VR tools for educational purposes and contribute to the broader discussion on innovative pedagogical approaches in the digital age.

7 Summary of Papers

This section presents summaries of six research articles that provide an exploration of various facets of VR technology in the context of education and collaboration, particularly focusing on their applications in mathematics education and supporting a wide range of educational levels and neurodiverse needs.

The first article investigates the unique aspects of asymmetric collaboration in VR environments for learning purposes. The second article expands the discussion to the application of VR in fostering independent living skills among individuals with ASD. The third study furthers the exploration of VR in educational settings, specifically in promoting social inclusion for neurodiverse students. The fourth article narrows the focus to mathematics education, outlining an innovative VR-based approach for equation solving. In the fifth article, we offer insights into a VR model specifically for geometry learning and provide an initial evaluation of its effectiveness. Finally, the sixth article, ties together the threads of mathematics education and VR, exploring how real-world mathematical concepts can be effectively taught in a virtual environment. Collectively, these summaries highlight the progress being made in the intersection of VR and education, painting a picture of potential future pathways in this exciting field.

7.1 Paper A: Asymmetric Collaboration in Virtual Reality: A Taxonomy of Asymmetric Interfaces for Collaborative Immersive Learning

The Industrial PhD project initiated its exploratory phase by conducting a comprehensive survey of ten Danish schools to assess the availability and accessibility of VR technology. Despite the promising potential of VR in education, the collected data revealed a stark reality – VR access was close to non-existent in the educational context. Libraries, however, often showcased VR systems, serving as the few instances where this emerging technology could be found. During this period, VR technology was still in its developmental stages, with most of its latest generation of equipment yet to hit the market. This situation thus highlighted an untapped demand for applications that could facilitate shared experiences between immersed and non-immersed users. A glimmer of hope for harnessing VR in education was spotted in the commercial sector. Asymmetric VR games, exemplified by popular titles such as "Keep Talking and Nobody Explodes", rapidly gained traction and demonstrated the potential to be repurposed for educational use.

Summary and Future Perspective

This study's primary objective was to investigate the integration of VR technology in educational settings. The research solution sought to address the practicalities, societal ethics, and financial aspects of incorporating VR into classrooms, using the research question, "What are the most effective methods for integrating VR technology into mathematics education within the classroom setting?" The study examined various frameworks, such as Gutwin and Greenberg's [49] seven collaborative mechanics and The Mechanics, Dynamics, and Aesthetics (MDA) Framework [65], to answer this question. In addition, it scrutinised research on collaboration, game design, and immersive learning and presented a taxonomy of asymmetric interfaces for immersive collaborative learning. This taxonomy encompassed three degrees of asymmetry, serving as blueprints for future user studies, and differentiated between interaction techniques and interfaces. The primary focus of this research was asymmetrical immersive VR in educational settings and its impact on learners' communication and collaboration. VR has proven an effective educational tool with its unique qualities of situating learners in the desired knowledge context and enabling interactions with virtual simulations. In addition, asymmetrical immersive VR facilitates collaborative work among multiple students and is particularly beneficial in mathematics education. A narrative literature review was conducted to understand how asymmetric game mechanics influence collaboration. This review served as a basis for developing a taxonomy to map the dynamics of learning activities involving asymmetrical immersive VR. The taxonomy helped to classify and analyse different aspects of these activities, including the level of immersion, the nature of collaboration, and the effectiveness of student communication. Through the narrative literature review and taxonomy development, this research paper aimed to enhance understanding of how VR, specifically asymmetrical immersive VR, can be effectively utilised to foster better collaboration and communication among learners. By highlighting the benefits and challenges of incorporating such technology in educational settings, this study contributes to the broader discourse on integrating VR into education.

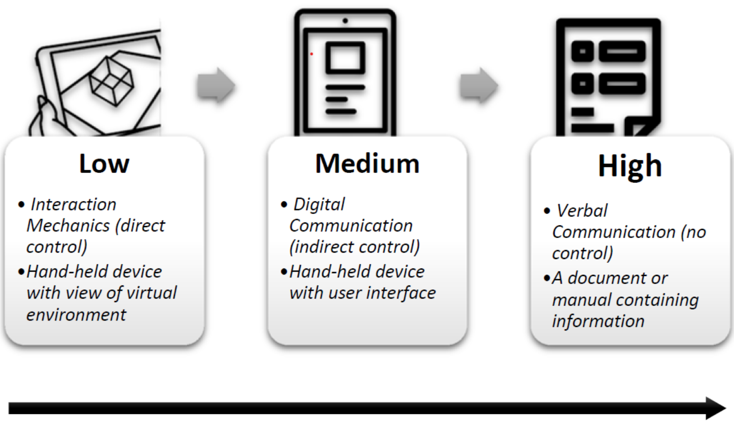


Fig. 27: Taxonomy of Asymmetric Interfaces for Collaborative Learning

The taxonomy developed in this study sets the stage for a detailed exploration of asymmetrical immersive VR, focusing on its potential in fostering collaboration and communication among learners. An interesting development in the field is the composite framework by Ouversson and Gilbert [126], which could be a valuable addition to this area of study. Built upon the earlier foundations, including our taxonomy, this composite framework presents a comprehensive and refined approach towards understanding and employing asymmetrical immersive VR. A future direction for our research could be to reconcile and integrate the concepts and principles from our taxonomy and the Ouversson and Gilbert framework. This amalgamation can provide a richer, more sophisticated lens through which we can examine the use of VR in education.

While our research focused on asymmetrical interfaces, it's essential to draw a clear distinction between asymmetric interfaces and asymmetric abilities. Asymmetric interfaces refer to the different interaction possibilities provided by the VR system to the users. For example, one student might manipulate objects in the virtual environment using hand-tracking technology, while another might use traditional game controllers. On the other hand, asymmetric abilities refer to the different roles, powers, or capabilities that each user has within the virtual environment itself. For instance, in a mathematics learning game, one student might have the ability to draw shapes while another has the power to manipulate their properties. By creating this distinction, future research can delve deeper into understanding how these elements individually and collectively contribute to the educational experience. Additionally, it can provide insights into the design principles that should guide the development of future VR educational tools. Applying roles to abilities and evaluating the user experience is another compelling perspective for future research. By attributing specific abilities to individual roles, we can stimulate different forms of collaboration, perhaps even promoting a more profound understanding of the subject matter. This process can be optimised by evaluating the user experience, such as investigating learners' levels of engagement, motivation, and satisfaction. This research pathway can help illuminate the ways in which asymmetrical abilities and roles might boost or hinder learning and collaboration. By coupling this with our earlier research on asymmetrical interfaces, we can gain a holistic view of the learning process in a VR environment.

In conclusion, the future perspective of this research project opens exciting avenues to investigate the complexities of VR in education further. It not only promises to enhance our understanding of technology-integrated learning but also has the potential to revolutionise educational strategies and practices.

7.2 Paper B: A Collaborative Virtual Reality Supermarket Training Application to Teach Shopping Skills to Young Individuals with Autism Spectrum Disorder

This subproject was started in partnership with colleague and fellow Multisensory Experience Lab member Ali Adjorlu. Ali is an assistant professor in Media Technology and holds a PhD in VR for the training of daily living skills in people with ASD. The later detailed balance model was discovered to be appropriate in the field of collaborative daily living skills training in the setting of grocery shopping. A networked VR prototype with asymmetric gameplay was produced and evaluated in partnership with the company Specialisterne [21], which works with training and recruiting individuals with neurodevelopmental problems in the workforce.

Summary and Future Perspective

The research paper aimed to introduce a collaborative VR prototype utilising HMDs as a novel method to teach shopping skills to children and adolescents diagnosed with ASD. Our study was driven by the understanding that ASD, a neurodevelopmental disorder characterised by difficulties in social interaction, communication, and repetitive behaviours, can impact an individual's ability to learn and perform daily living skills, such as shopping. In pursuing this goal, the research team utilised a method incorporating technology, specifically VR, an area that has seen increased exploration as a potential aid to support skill acquisition for those with ASD. The chosen materials for the study included a collaborative VR prototype specifically designed to teach shopping skills to children and adolescents with ASD and HMDs to facilitate an immersive experience. This approach allowed students and teachers to inhabit the same virtual supermarket and work together to complete a shopping trip, providing a safe and controlled environment for skill practice (see Figure 28 and Figure 29). The study was carried out with eight adolescents diagnosed with ASD, where the effectiveness of the VR prototype was evaluated. The method involved participants being guided by their teachers in the virtual supermarket, and their performance and experiences were duly assessed. The study aimed to gather preliminary data on the potential benefits of this intervention, even though the sample size was small. The findings of the exploratory study suggested that the collaborative VR prototype could indeed be beneficial in teaching shopping skills to adolescents with ASD. The evidence was drawn from improved performance in shopping tasks displayed by the participants and teachers' reports stating that the virtual environment was a valuable opportunity for practising these skills in a controlled setting. Additionally, the study underscored the importance of collaboration between students and teachers in the learning process, as it facilitated engagement and communication. The paper concludes by discussing the future potential of VR in-

[21] Specialisterne: `https://www.dk.specialisterne.com/`

terventions in teaching daily living skills to individuals with ASD. This may include personalising the virtual environment to the unique needs of each participant, integrating additional life skills, and examining the long-term effects of such interventions. We acknowledge the need for more extensive research to validate and generalise the findings of the exploratory study. However, our initial results affirm the motivation behind the study, positing that collaborative VR applications hold promise as a valuable tool in the education and skill development of children and adolescents with ASD.

(a) Overview **(b)** Shopping list **(c)** Scale **(d)** Self-checkout

Fig. 28: Collaborative Virtual Supermarket (1)

(a) Payment **(b)** Finding products **(c)** Classroom **(d)** Training area

Fig. 29: Collaborative Virtual Supermarket (2)

This exploratory study sets the foundation for the future direction of this line of research, especially considering how VR technology can be leveraged to enhance the learning process for children and adolescents with ASD. As the study concludes, the collaborative VR prototype appears promising as a tool for skill development, including practical life skills such as shopping. However, the research scope can be expanded to cover a broader range of skills and evaluate its effectiveness over a more extended period.

Longitudinal studies on collaboration using this VR prototype could delve into areas like learning algebra and other daily living skills. Mathematics, specifically algebra, is a field that might benefit from the immersive and interactive environment provided by VR. While it may require the development of new VR materials or adaptations to the existing application, creating a virtual algebra classroom where students and teachers can interact could provide a uniquely engaging and tailored learning experience. The use of VR may help demystify abstract concepts, making them more accessible and comprehensible for students with ASD. Long-term data gathered from such a study could provide insights into whether these virtual lessons improve the students' understanding and retention of algebraic concepts over time. Additionally, longitudinal studies can be used to assess the development of daily living skills beyond shopping, such as cooking, cleaning, personal hygiene, or navigating public transport. Monitoring progress over a longer duration would allow researchers to evaluate the sustained impact of VR-based training on the independence and quality of life of individuals with ASD. Future research could also explore variations in the experimental design of the collaborative VR prototype. Implementing different asymmetric interfaces might alter the dynamics of interaction between the student and teacher, potentially offering new avenues for engagement and learning. For example, interfaces that allow teachers more direct control over the virtual environment might enhance their ability to guide and instruct, while those that grant students more autonomy could encourage independent decision-making. Moreover, increasing the number and variety of roles in the VR scenario may enhance the realism and applicability of the training. In the context of a shopping trip, this could involve adding roles such as a cashier, other shoppers, or a store assistant. These additional roles can help simulate more real-world experiences and teach students how to interact with different individuals, which is a crucial social skill for those with ASD.

In conclusion, the potential of collaborative VR applications in education and skill development for ASD is vast. While this initial study is promising, additional research is necessary to explore the many dimensions and implications of this innovative approach fully. Through longitudinal studies and experimental variation, researchers can gain a more comprehensive understanding of the efficacy and adaptability of this technology for teaching and learning among individuals with ASD.

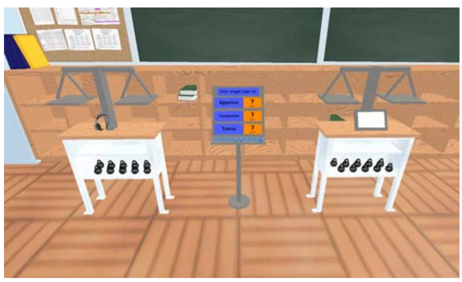
7.3 Paper C: Designing a Collaborative Virtual Reality System to Assess Social Inclusion among Neurodiverse Students

The research article describes the development of a collaborative VR system to assess social inclusion among neurodiverse students. The system allows students to engage in virtual environments, such as a supermarket shopping trip, to promote communication and collaboration. The study also presents a virtual classroom to introduce students to the task and relevant concepts. The article provides an overview of related literature on inclusion research and discusses potential applications of the VR system in future studies.

Summary and Future Perspective

Social inclusion is a critical aspect of the overall well-being and development of neurodiverse students. VR technology has emerged as a promising tool for assessing and promoting social inclusion among these students. This research article focuses on designing and implementing a collaborative VR system that aims to facilitate communication and collaboration among neurodiverse students in virtual environments, such as a supermarket shopping trip. The article provides an overview of relevant literature in the field of inclusion research, highlighting the importance of social inclusion for neurodiverse students. Previous studies have shown that VR technology can be effectively used to create immersive and engaging environments for individuals with diverse needs. In addition, VR may be a valuable tool in assessing and promoting social inclusion among neurodiverse populations. The article also discusses the potential benefits of using a collaborative VR system, such as enhanced engagement and more accurate assessment of social inclusion levels. A virtual classroom environment is developed to facilitate the introduction of the shopping task and relevant concepts (see Figure 30). The virtual classroom serves as a platform to familiarise students with the basic principles of shopping, such as identifying items, reading price tags, and making purchases. This preparatory stage ensures that students clearly understand the task before entering the virtual supermarket environment. The core of the research is the development of a collaborative VR system that allows neurodiverse students to participate in a virtual supermarket shopping trip. The system is designed to promote communication and collaboration among students, offering a safe and controlled environment to practice social skills. In this virtual environment, students can interact with one another, work together to complete the shopping task and learn from each other's experiences. The article concludes by discussing the potential applications of the VR system in future studies aimed at measuring social inclusion among neurodiverse students. The immersive and interactive nature of virtual environments can provide valuable insights into the social dynamics and interactions of neurodiverse individuals. Furthermore, the collaborative aspect of the VR system can help researchers identify areas of improvement for fostering social inclusion and develop targeted interventions. Overall, this collaborative VR system offers a promising avenue for further research and practical applications in the field of social inclusion for neurodiverse populations.

(a) Avatar customisation



(b) Extended training area

Fig. 30: Collaborative Virtual Supermarket (3)

The research outlined above has set an exciting precedent in the utilisation of VR technology as a tool for promoting social inclusion among neurodiverse students. As we move forward, the potential applications and improvements of this collaborative VR system pose an intriguing proposition for future research.

A longitudinal study investigating the impact of collaboration within the VR environment on social inclusion represents a crucial future perspective of this research project. As the current research focuses on the implementation and immediate effects of the system, a long-term study could illuminate the sustained impact of the VR experience on students' social behaviour and inclusion. Understanding the long-term effects would provide insights into the durability of the social skills acquired during VR interactions and their transferability to real-life situations. This could reveal the true potential of the VR tool in facilitating lasting change in social inclusion among neurodiverse students. Additionally, the evaluation of variations in the experimental design will provide valuable insights into optimising the VR system for a diverse range of neurodiverse populations. The implementation of different asymmetric interfaces, for example, could cater to various communication styles and cognitive abilities, enhancing the accessibility and inclusivity of the VR experience. This could involve designing interfaces that are more tailored to the needs and capabilities of different individuals, promoting their engagement and active participation in the virtual tasks. Increasing the number and variety of roles within the VR environment may also enhance the collaborative aspect of the system. By offering diverse roles, students can explore different responsibilities, social interactions, and perspectives, fostering a richer understanding of social dynamics. This aspect can also help in fostering a sense of agency and self-efficacy among students, contributing to enhanced social engagement and inclusion.

In conclusion, the future perspectives of this research project entail exploring the longitudinal effects of the collaborative VR system and evaluating modifications in the experimental design. These endeavours will deepen our understanding of the potential of VR as a tool for promoting social inclusion among neurodiverse students, guiding the development of more effective, tailored interventions in the future. As we advance in this exploration, it is imperative that we maintain a learner-centric approach, ensuring that the needs and experiences of neurodiverse students remain at the heart of this research.

7.4 Paper D: Adapting the Balance Model for Equation Solving to Virtual Reality

This research article was a joint effort involving Morten Elkjær, a PhD student in the didactics of mathematics from Aarhus University, and the author of the thesis. Elkjær's work primarily investigates the detection, diagnosis, and treatment of students' difficulties in mathematics education, specifically with concepts in algebra and equations. During our shared tenure as industrial PhD students at the now-terminated company

EduLab, we worked together on the VR-based Equation Lab research initiative. Several additional research papers documented the design and development process (see “Equation Lab: fixing the balance for teaching linear equations using Virtual Reality” and "An Immersive Learning Experience for Teaching Equations" in Thesis Details). The original conception of the project was as a serious VR game centred around the history of mathematics, aligning with the business objectives of EduLab, which had lately increased its focus on story-telling within its e-learning platform. The design was set to include an initial location styled as a museum, allowing students to virtually transport themselves to various eras in the timeline of mathematical discovery (see Figure 31).

Fig. 31: Museum area with video-based story-telling

The debut version was primarily based on Thales’ theorem, incorporating numerous narrative and instructional videos within the virtual setting. Post video-guided learning, students were invited to engage with and construct dynamic geometric models to solve problems related to Thales’ theorem (see Figure 32).

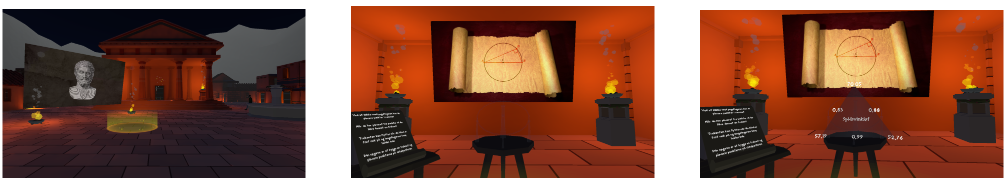


Fig. 32: Task area with instructional video

This prototype was demonstrated and evaluated at a series of industry exhibitions in Denmark’s educational sector. Observations gathered from these trials led me to reconsider the original design, shifting away from a narrative and interest-stimulating approach in mathematical education towards a more neutral virtual setting that accentuates the conveyance of mathematical knowledge via compelling task design.

Summary and Future Perspective

The research article discusses the development and implementation of a VR prototype for teaching linear equations using a modified balance model. The study aimed to analyse the user experience and potential learning outcome of the VR prototype in a

Danish grade 7 classroom and reported promising results, with students showing positive affective responses and some students being able to apply concepts taught in VR to traditional pen-and-paper exercises. The article also presents an in-depth analysis of a specific student case to explore the potential of VR in teaching equation solving. Linear equations are a fundamental concept in mathematics education, and finding effective teaching methods for these equations is crucial for students' understanding and success. Traditional teaching methods can sometimes struggle to convey the abstract concepts behind linear equations, especially when dealing with negative numbers. This research investigates the potential of VR to address these challenges and enhance the learning experience. We developed a VR prototype utilising a modified balance model to teach linear equations (see Figures 33 - 37). This model, which alters the physics behaviour in the virtual environment, was designed to strengthen students' understanding of solving linear equations and help them adapt to situations involving negative numbers and mathematical abstraction (see Figure 38 and Figure 39). By making the experience more immersive and interactive, the VR prototype aimed to promote a deeper understanding of the concepts. The study involved ten students and their mathematics teacher from a Danish grade 7 class (13-14 years of age). We conducted an exploratory study to analyse and evaluate the effect of teaching with the modified balance in the VR prototype. In addition, we aimed to assess the potential of VR in teaching linear equation solving and identify any new strategies that the virtual environment might enable. The findings revealed positive prospects for using VR in teaching linear equation solving. Most students responded positively to the experience, and they could apply the ideas from the VR experience to post-experience pen-and-paper exercises. This demonstrates that the VR experience engaged the students and had a tangible impact on their problem-solving abilities. The research also included an in-depth analysis of a student who exhibited interesting behaviour and reasoning during the study. Albert, a student participating in the study, demonstrated changes in his equation-solving strategies. Before the VR experience, he struggled with abstract equations, particularly those involving negative numbers, and largely relied on guess-and-check methods. However, through the VR teaching sequence, he learned to use the "invert mechanic," which helped him handle more abstract equations. His math teacher noticed Albert applying this new strategy in class after the VR teaching sequence, indicating the translation of VR learning into traditional math problem-solving. Despite these improvements, Albert still struggled with understanding the role of negative numbers in certain contexts. This experience suggests VR can provide innovative teaching methods that enhance problem-solving strategies, but these tools must also carefully address key mathematical concepts and be supplemented by traditional teaching. This study aimed to provide further insight into the benefits and future possibilities of using VR to teach equation solving. By examining this student's experience closely, we hoped to identify areas where VR could be particularly effective in supporting learning and understanding. This research article demonstrates the potential benefits of using VR in teaching linear equation-solving.

Fig. 33: Equation terms as immersive virtual manipulatives

Fig. 34: Moving manipulative to transformation area

Fig. 35: Different options depending on the number of terms

Fig. 36: Transforming a term

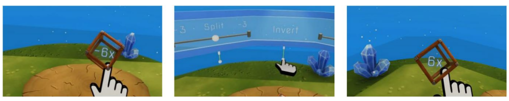


Fig. 37: Inverting a term

The modified balance model and immersive virtual environment provided students with an engaging and effective learning experience, as evidenced by their positive affective responses and ability to apply VR concepts to traditional exercises. The in-depth case study further supports the potential of VR as a valuable teaching tool for linear equations and other abstract mathematical concepts. Future research could explore the broader implications of VR in mathematics education and identify additional strategies and techniques to optimise the learning experience.

The research project's future perspective looks promising, considering the numerous areas that could be further developed and improved upon based on this preliminary study.

One important area to explore is the *history of transformations*. The VR prototype could be updated to include a feature that tracks and displays the sequence of steps or

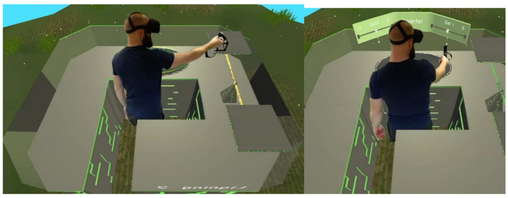


Fig. 38: Mixed Reality view of the environment

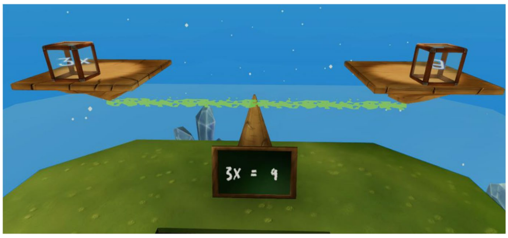


Fig. 39: Overview of environment

transformations made by a student when solving a linear equation. This would allow students to retrace their steps and understand the progression of their problem-solving process, promoting metacognitive skills. Moreover, educators could use this data to identify where students are making mistakes or getting confused.

The concept of *persistence vs. impersistence* of terms after transformation is another fascinating element to investigate. Should terms remain visible after a transformation has been made, or should they disappear to avoid potential confusion? There could be an adaptive mechanism based on the student's comfort level. For novice learners, persisting terms may prevent them from getting lost during the process. As students gain proficiency, they could be given the option to have the terms disappear after transformation, thus preparing them for traditional pen-and-paper equations where no such persistence exists.

The addition of an undo function could also enhance the VR prototype's usability. This feature would allow students to revert their actions, encouraging them to experiment and learn from their mistakes without fear of irreversible errors. It would also add an element of flexibility to the learning process, making it a more student-driven experience.

Introducing a new display with mathematical notation could bridge the gap between the VR environment and traditional mathematical learning. By seeing the mathematical notation associated with their VR actions, students can more easily connect their immersive experiences with abstract mathematical concepts, facilitating their transition to pen-and-paper tasks.

Direct transformation compared to the transformation area warrants further exploration. Should students perform transformations directly on the equation, or should there be a designated transformation area where the manipulation takes place? The choice between these two options could have an impact on students' understanding and sense of control over the transformations.

Finally, the introduction of a collaborative feature in the VR prototype could enhance the learning process. For example, this feature could be designed to allow multiple students to work in the same or mirrored VR space, with access to one side of the equation. In this context, collaboration could potentially promote a deeper understanding of the subject matter by enabling students to learn from each other, discuss strategies, and solve problems collectively.

In conclusion, the future of this VR prototype for teaching linear equations is vast, with numerous areas available for further development and improvement. By continuing to refine and expand upon this VR prototype, we can continue to explore and harness the immense potential of VR in mathematics education.

7.5 Paper E: An Immersive Geometry Environment for Mathematics Education: Taxonomy and Preliminary Evaluation

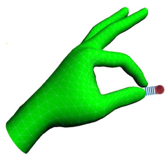
The sub-project was started through the supervision of the two Medialogy master's degree students, Kevin Baars Støvelbæk and Christoffer Bendig Mundbjerg-Sunne. As the final project of their master's degree, an immersive virtual environment was developed. At the time, embedded hand-tracking was introduced in the Oculus Quest 1, and the project sought to investigate this use in VR for geometry education. As a result, prior research was analysed, and a design for gesture-based interaction with dynamic geometry in VR was created. However, current interfaces, primarily based on hand-held controllers, frequently do not match real-world interactions' finesse, detail, and intuitiveness, separating the user and the virtual environment. To bridge this gap further, we set out to create a VR experience by establishing an immersive geometry environment that exploits gesture-based interaction via embedded hand-tracking in VR devices.

Summary and Future Perspective

The paper presents a taxonomy of learning environments in geometry education, focusing on the potential of VR as a teaching tool. A study involving usability tests and heuristics evaluation with expert users was conducted to evaluate VR-based geometry instruction (see Figure 40 and Figure 41). The study's findings led to design considerations and suggestions for future research. VR technology has emerged as a promising avenue for enhancing education, particularly in mathematics. Geometry, as a fundamental aspect of mathematics, may benefit from VR's immersive and interactive nature. This paper investigates the potential of VR-based geometry instruction by presenting a taxonomy of learning environments, conducting a study on a novel prototype for an immersive geometry environment with gesture-based interaction, and offering design considerations on gesture-based interaction with geometry based on the study's findings. To better understand the landscape of geometry education, the paper presents a taxonomy that categorises various learning environments. This taxonomy provides a framework for evaluating and comparing different approaches to geometry instruction, including traditional classroom settings, computer-based learning environments, and VR-based learning environments. In addition, it considers factors such as the degree of interactivity, immersion, and collaboration in each domain. The paper describes a study involving a usability test and a heuristics evaluation with expert users to assess the effectiveness of VR-based geometry instruction. The usability test examined factors like ease of use, user satisfaction, and learning outcomes. At the same time, the heuristics evaluation focused on the design principles and guidelines used in creating the VR learning environment. The study's findings revealed insights into the use of VR in geometry education. It demonstrated the potential for VR to provide an engaging and immersive learning experience that may promote a deeper understanding of geometric concepts. However, it highlighted areas where improvements could be made, such as user interface design, feedback mechanisms, and collaborative learning opportunities. Based on these findings, the paper offers a series of design considerations for future VR-based geometry instruction. The paper concludes by suggesting several directions for future research in the realm of VR-based geometry education. These include exploring the long-term effects of VR-based instruction, investigating the optimal balance between immersion and instructional design, and examining the potential for adaptive learning algorithms to personalise the learning experience for each student. In summary, this paper highlights the potential of VR as a powerful tool for geometry education by presenting a taxonomy of learning environments, conducting a user study and heuristics evaluation, and offering design considerations based on the study's findings. The insights and suggestions provided in this paper pave the way for further research and development in VR-based geometry instruction.

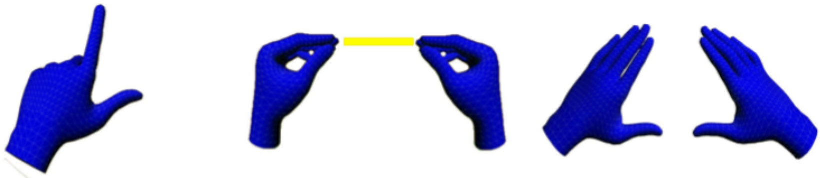
The research project offers a promising look into the potential of VR as a tool for geometry education. Given its immersive and interactive nature, VR brings a new perspective to educational strategies, potentially enhancing understanding and engagement. However, there is much room for growth and further exploration within this domain. Several future directions for this research project could be considered, each one adding a new layer of complexity and opportunity.

One important aspect to consider is extending the construction mode to other shapes and polygons. While the current work has primarily dealt with basic geometric concepts, expanding these principles to more complex geometric figures can provide a more comprehensive understanding of geometry. This might involve the development of tools or features within the VR environment that support the creation and manipulation of these complex shapes, giving users a hands-on understanding of how these figures are constructed and the properties they possess.



(a) Construct point

(b) Manipulate point or area



(c) Measure angles, lengths, and areas

Fig. 40: Gestures for construct, manipulate, and measure

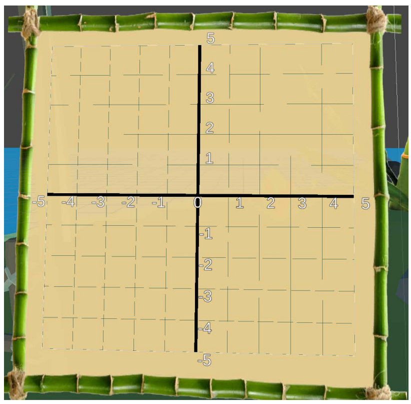


Fig. 41: Grid

Adding another dimension to the VR environment by involving 3D object interactions and 3D grids would also be a step forward. This could enhance spatial awareness and comprehension of 3D geometry. Interacting with 3D objects might give students a deeper understanding of concepts such as volume, surface area, and the relationship between different dimensions.

There is also potential in allowing tracing utilising the affordances of dynamic geometry environments. This feature could allow students to actively trace and interact with geometric shapes and figures, thus helping them better understand these concepts' properties and behaviour. The dynamic nature of this environment could also help demonstrate the effects of various transformations on these shapes, which might be difficult to visualise in a traditional 2D setting.

Moreover, adding gamification elements to VR-based geometry instruction could further motivate and contextualise learning. The competitive and engaging nature of games can create a learning environment where students are encouraged to explore and understand geometric principles independently. Points, leaderboards, badges, or story-driven missions could be integrated to make learning more engaging and meaningful for students.

Collaboration should be considered by creating roles for the different modes within the VR environment. This could involve designing tasks or activities where students need to work together to solve geometry problems or construct geometric figures. For example, each student could be given a specific role in the creation of a complex geometric figure, encouraging communication and teamwork.

In conclusion, the discussed research project has laid a solid foundation for VR's use in geometry education. There is a lot of potential in this field, and the proposed future directions only scratch the surface of what could be accomplished. By continually iterating and expanding on these ideas, VR can potentially revolutionise how we approach geometry education.

7.6 Paper F: Real Mathematics in Virtual Worlds

This subproject was done in collaboration with Associate Professor Olga Timcenko of Aalborg University, an experienced university-level mathematics teacher and researcher in mathematics education. Over the years, she has specialised in teaching the connection between mathematics, graphics, and games - areas where students frequently struggle. Recognising this, we saw it as an excellent chance to capitalise on the unique characteristics of VR. We wanted to improve the depiction of mathematical concepts by placing them in problem-solving settings within an immersive VR environment.

Summary and Future Perspective

The research article addresses software engineering students' challenges in learning mathematics and transferring their knowledge to problem-based projects (See Figure 42 and Figure 43). The paper suggests exposing students to mathematical objects in VR, using their fascination with modern technology to motivate them to learn mathematics and apply it in their work. Although only a pilot study has been conducted so far, initial findings indicate a potential for this approach, which requires further optimisation for classroom or project-room use. Software engineering students often face challenges learning mathematics and applying their knowledge to problem-based projects. However, traditional teaching methods may only partially engage students or help them make connections between mathematical concepts and their practical applications in software engineering. This research explores an approach involving exposing students to mathematical objects in VR to address these issues and improve motivation to learn and apply mathematics. The paper proposes using VR technology to expose students to mathematical objects and demonstrate the underlying mathematical foundation behind popular game engines and virtual environments. By utilising students' fascination with modern technology, we aimed to enhance their motivation to learn and apply mathematical concepts. The approach is still in its early stages, with only a pilot study completed. Initial indications from the pilot study suggest that the proposed VR-based approach could improve students' motivation to learn mathematics and apply their knowledge to problem-based projects. However, more work is needed to optimise this approach for classroom or project-room use. Using VR technology to expose software engineering students to mathematical objects offers a unique and engaging way to bridge the gap between mathematics learning and its practical application in their field. By leveraging students' interest in modern technology and showing them the mathematical foundations of their favourite game engines and virtual environments, we hope to inspire students to learn and apply mathematics more effectively. While the pilot study has shown promising results, further research is needed to refine the approach, optimise it for classroom or project-room use, and determine its long-term effectiveness in improving student's motivation and learning outcomes. Additionally, future studies could explore other emerging technologies, such as augmented reality, to further enhance the learning experience and facilitate the transfer of mathematical knowledge to problem-based projects. This research proposes an innovative approach to enhancing mathematics learning for software engineering students by explicitly exposing them to mathematical objects in VR. Although still in its early stages, the initial findings indicate that this approach could improve students' motivation to learn and apply mathematics in their projects. Future research is needed to develop and optimise this method for broader implementation.

The research project provides an intersection between education, technology, and software engineering. By using VR to expose students to mathematical objects, it explores a novel approach to enhancing mathematical understanding and application in problem-based projects. The initial findings suggest that the use of VR can improve students' motivation to learn mathematics, thereby facilitating a more effective transfer of their knowledge to real-world projects.

Looking towards the future, extending gameplay to a sandbox-like open environment could be an interesting area of development. In this kind of setting, students would have the freedom to manipulate mathematical objects, create new scenarios, and explore the consequences of their actions in real-time. This could provide students with a more engaging and immersive learning experience. The open-ended nature of sandbox environments could also promote creativity and problem-solving skills, as students would need to navigate complex systems and deal with unexpected challenges.

In addition to extending the gameplay environment, incorporating more advanced tasks could further enhance the educational value of this approach. These tasks could be

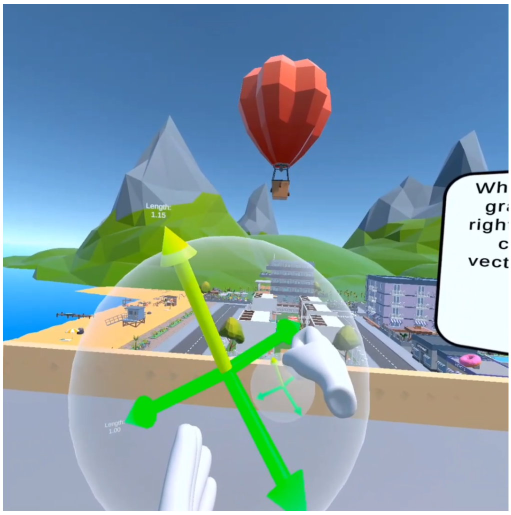
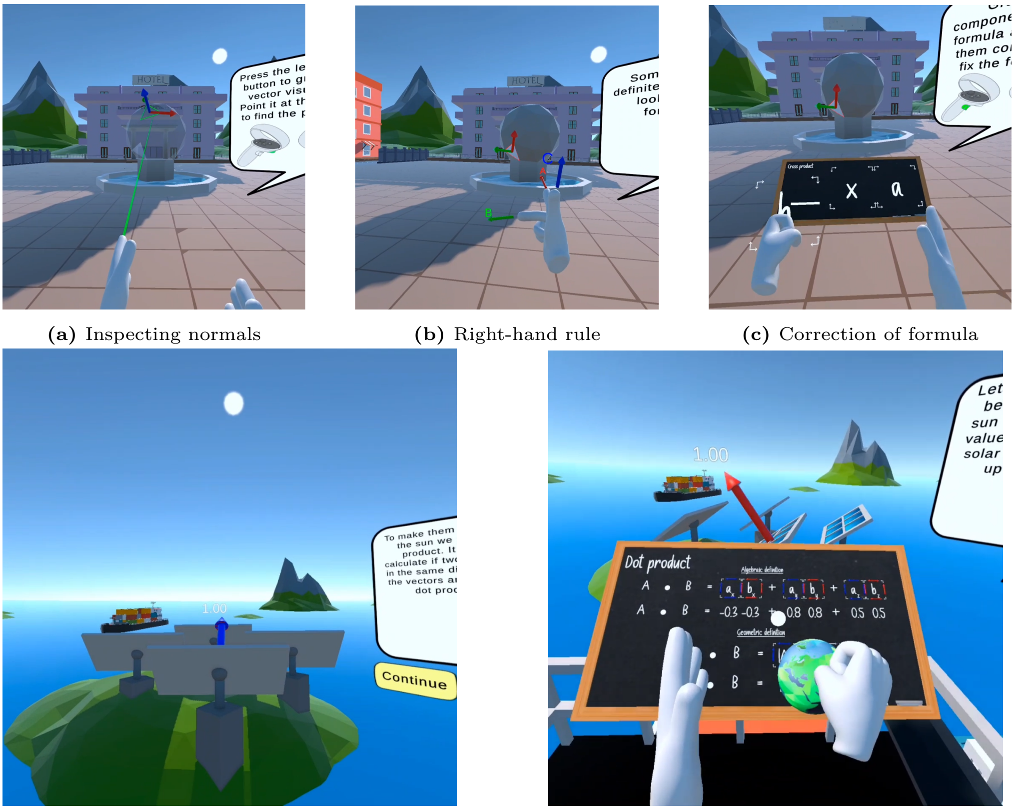


Fig. 42: Unit vector

designed to challenge students' understanding and application of various mathematical concepts, thereby reinforcing their learning and improving their ability to apply their knowledge in different contexts. This kind of active learning experience could potentially increase student engagement and motivation even further, as well as help to build critical thinking and problem-solving skills.

Finally, the role of the visualisation system in this approach could be considered in two key ways: as a tool for contextualisation and as a standalone visualisation system. As a contextualisation tool, the VR environment provides a rich context in which mathematical objects can be explored and understood. This can help students to see the relevance and applicability of mathematical concepts in the real world. As a standalone visualisation system, VR can enable students to interact with and manipulate mathematical objects in a way that isn't possible with traditional teaching methods. This could lead to a deeper and more intuitive understanding of these concepts.

As the research project continues to develop, it would be interesting to explore these areas in more depth. For example, how do students' learning outcomes and motivation levels change when they're given more freedom and complex tasks in a sandbox-like environment? How do the benefits of contextualisation compare to those of standalone visualisation? By examining these questions, this research could further our understanding of how technology can be used to enhance mathematics education, specifically within the realm of software engineering.



(a) Inspecting normals (b) Right-hand rule (c) Correction of formula

(d) Solar panels (e) Dot product

Fig. 43: Tasks on cross product and dot product

8 Discussion and Conclusion

The PhD project primarily investigated two aspects of using VR in mathematics education: integration and design. Integration focused on incorporating VR into educational settings such as classrooms, with a particular emphasis on using asymmetrical gameplay. Meanwhile, the sections on subject matter examined strategies for designing learning materials in VR specifically for mathematics education, focusing on creating designs for algebra, geometry, and vectors.

In this final section of my thesis, I revisit the proposed research questions, highlighting how each has been addressed in the associated research articles. I also underscore the broader contributions of this thesis to mathematics education, particularly within the VR context. I will also reflect on the strengths and weaknesses of my work, ensuring an evaluation and identification of areas for further refinement. Lastly, I will outline the future prospects of VR in mathematics education, setting the groundwork for future research. This conclusion aims to offer a summary of my project and a launchpad for further explorations.

8.1 Research Questions

All six research papers focus on the integration and design of VR for educational settings, with a particular emphasis on mathematics education and special education needs. They offer insights into the implementation of VR technology in educational settings, strategies for teaching mathematics in a VR environment, accommodating diverse needs within VR design, incorporating gesture-based interaction to create embodied learning experiences, and leveraging visualisation strategies and game-based learning in VR environments. In the subsequent paragraphs, I will outline each research question, while highlighting the studies conducted to address them.

What are effective ways of integrating VR technology into mathematics education within the classroom setting? In answering the first research question, Paper A investigates and reviews the integration of VR technology through asymmetric VR. This study explores the integration of VR in educational settings, looking at the practicalities, societal ethics, and financial aspects of this technological inclusion. It particularly highlights the potential of asymmetric VR to support collaboration among students in a mathematics class. This research could act as a starting point for educators and policymakers in devising effective strategies to integrate VR into the mathematics classroom. Similarly, Papers D, E, and F explore VR's potential in teaching specific mathematical concepts, namely linear equations, geometry, and vectors, respectively. These articles showcase how specific mathematical topics may be effectively taught using VR, thereby adding more depth to the broader discussion in Paper A.

What approaches to teaching algebra, equations, and equation-solving strategies to students can be used in VR? The second research question is most directly addressed by Paper D. In this research, a VR prototype was developed to teach equations and equation-solving strategies. Rather than relying on guess-and-check strategies, the design focused on potential competency development using an exploratory learning approach using immersive virtual manipulatives, a transformation area, and a modified balance model. We found that this VR-based approach resulted in positive affective responses from students and enabled some students to apply the VR-taught concepts to traditional pen-and-paper exercises. This suggests that VR can teach algebra and equations in an engaging, immersive way that may have an impact on students' learning.

How can VR be tailored to accommodate students with diverse needs, fostering collaboration in learning daily living skills and mathematical concepts? The third research question is primarily addressed by Papers B and C. Both studies focus on the use of VR technology to assist students with ASD and neurodiverse students in developing daily living skills and promoting social inclusion, respectively. These papers demonstrate how VR may be adapted and utilised to meet the diverse needs of students through collaboration mechanics, communication channels, and avatar customisation, all while fostering collaboration and engagement in learning daily living skills and basic mathematical concepts.

In what ways can hand-tracking capabilities in VR support gesture-based interaction for embodied learning experiences in geometry education? The fourth research question is addressed by Paper E, which presents a promising potential for hand-tracking in VR to enhance interactive learning in geometry. By directly manipulating geometric shapes in VR, students may be able to better understand geometric concepts. Nevertheless, challenges related to the design and accuracy of hand-tracking technology were noted, indicating room for refinement. Furthermore, the paper introduces a new taxonomy of learning environments in geometry education, expanding the possibilities for educators to design and develop immersive geometry environments.

How can visualisation techniques and gamification principles be leveraged in VR to contextualise mathematical concepts? Lastly, regarding the fifth research question, Paper F offers a promising starting point. It explores the use of VR to expose software engineering students to mathematical objects, demonstrating the mathematical foundations behind popular game engines and virtual environments. This implies a gamification approach to learning mathematics that can leverage VR's visualisation capabilities. Paper E also touches on this topic, highlighting the potential of VR for teaching geometry by visualising abstract concepts in an engaging, immersive environment.

In conclusion, the papers provide insights into how VR technology can be integrated into mathematics education to potentially enhance learning and teaching experiences. They collectively highlight the diverse applications of VR, from teaching algebra and geometry to catering to students with diverse learning needs. However, more research is needed, particularly concerning the long-term effects of VR in mathematics education with an emphasis on knowledge transfer, competency development, and skills development.

8.2 Contributions

The contributions of this PhD thesis on the integration and design of VR in mathematics education are multifaceted and have the potential to impact academic research, educational practice and commercial strategies.

From an academic perspective, the PhD project contributes to the evolving body of research on the application of VR in education. The thesis pushes the boundaries by not only investigating the use of VR as a teaching tool but also conceptualising an instructional design model grounded in the ADDIE model (see Section 2.1). This offers a perspective that goes beyond single-case usage of VR, providing a more systemic view on how VR can be integrated into the teaching and learning process.

The thesis also explores how VR can facilitate constructivist, situated, exploratory, and embodied learning in mathematics. This may add depth to the current understanding of learning theories by elucidating how they can be operationalised in VR in the context of mathematics education. Furthermore, the application of the theory of instrumental genesis in the evaluation of learning provides novel insights into how learners utilise and evolve virtual tools to achieve learning objectives.

From an educational practice perspective, the project established an extended version of the ADDIE model specifically for mathematics education, potentially assisting in the implementation of a new approach in the field. This model is intended not only to enhance comprehension and interaction with mathematical concepts through virtual manipulatives and models but also to leverage game-based learning and post-VR generative learning strategies to potentially bolster learners' understanding and retention of knowledge defined in learning objectives. The model could serve as a starting point for educators and instructional designers who wish to integrate VR into their curriculum, providing practical guidance from the analysis phase through to design, development, implementation, and evaluation. This systemic approach has the potential to improve the instructional design process of VR in mathematics education. In addition, the evaluation strategy, particularly the application of learning analytics and feedback revision, promotes a culture of continuous improvement. By analysing interaction data and learner feedback, educators can keep refining and optimising the VR experiences, ensuring the learning objectives are effectively met and continually improving upon them. Moreover, the thesis provides an overview of existing studies on the use of VR in mathematics education. The assessment of the current landscape, including its successes and challenges, could aid future research in this area by identifying areas that require further investigation, encouraging continued exploration and development of effective VR practices in mathematics education. This, in turn, has the potential to enhance learner engagement, improve comprehension, and bolster overall learning outcomes in mathematics education.

Lastly, the use of asymmetric VR in education, where not all users need VR devices, may lower financial barriers and increase accessibility. This could boost demand from a wider customer base, including institutions with different budget capacities. From a commercial perspective, this could lead to higher sales of software for asymmetric VR platforms and attract investment for affordable, scalable VR solutions. Businesses might adopt similar strategies for cost-effective internal training or collaborations. Hence, commercial strategies should focus on flexible solutions to broaden the VR market reach, creating growth opportunities in the industrial sector.

In sum, the thesis contributes to both the theoretical understanding, practical application, and commercial aspects of VR in mathematics education, offering new perspectives and solutions to advance the field.

8.3 Conclusion

The PhD project on VR in mathematics education offers contributions to both the academic and industrial aspects of this emergent field. Across the six articles, the project effectively explores the utility and application of VR technology in diverse educational settings, offering insights into the potential benefits, limitations, and future directions of this innovative pedagogical tool.

The strengths of the project may lie in its multidimensional approach to the application of VR in education. The studies range from examining the practicalities of integrating VR in the classroom setting (Paper A), exploring the use of VR in mathematics for special education needs (Papers B and C), and investigating the application of VR in teaching specific mathematical concepts (Papers D, E, and F). This broad exploration allows for an understanding of VR's potential in various contexts, helping to pave the way for future research and application in this area. Another strength of the project is the applied nature of the research. The majority of the papers present preliminary studies, offering data to support their discussions. This evidentiary basis not only lends credibility to the project but also provides actionable insights for practitioners, policymakers, and educators interested in integrating VR into their instruction.

In any scientific endeavour, challenges often surface as an integral part of the journey. These hurdles can also become potential weaknesses when they impede the project's progression, affecting the attainment of its initial goals. The course of an industrial PhD project is often influenced by a variety of factors, external and internal. In this particular project, a series of unprecedented events and organisational circumstances imposed challenges that affected the project direction and outcomes. The industrial partner associated with this PhD project underwent severe organisational changes during the course of the research. Driven by financial challenges, the company underwent multiple rounds of layoffs which influenced the project management. As a result, the project experienced a turnover in company supervisors, with five changes over the course of the project. Such frequent changes could have potential implications for the project's continuity, knowledge transfer, and overall progress. In the midst of these organisational transformations, the company was ultimately acquired by a larger entity. The acquisition posed new questions regarding the project's future, considering the potential restructuring of roles, changes in strategic priorities, and adaptation to the new organisation's culture and policies. Simultaneously, the project was further disrupted by the global COVID-19 pandemic. As part of the research was planned to involve longitudinal studies with Danish schools, the restrictions imposed due to the pandemic critically affected this portion of the project. Contamination risks associated with sharing VR HMDs led to an indefinite hold on these large-scale studies, adding to the project's overall uncertainties and hindrances. Smaller-scale user studies in isolated environments were made possible by following strict health and ethics protocols involving the use of VR research guidelines [161] and Ultraviolet-C (UVC) disinfection machines, such as CleanBox [22]. These challenges imposed by external factors required a dynamic adaptation strategy. The focus of the project had to be redirected from primarily conducting user studies to generating more theoretical contributions. This shift, although deviating from the initial project plan, opened new avenues for theoretical exploration and innovation. While these circumstances were challenging, they provide an insightful case on how external factors can influence the trajectory of an industrial PhD project. They underscore the need for flexibility in research, the ability to adapt to changes, and the importance of resilience in overcoming unexpected obstacles. Moreover, they offer an opportunity to evaluate the strengths and weaknesses of the industrial PhD model under such trying circumstances, prompting further reflections on its potential improvements for future resilience.

On a different note, the project could benefit from a more focused participant base. The studies predominantly focused on the Danish school system (Paper D), students with ASD (Paper B), neurodiverse students (Paper C), and software engineering students (Paper F). While these specific populations are important to study, a more focused participant group could help to generalise the findings and broaden the applicability of the results. Lastly, while the project examines VR's impact on learners and learning outcomes, there is less emphasis on teacher perspectives, a crucial component of any educational intervention. Including more teacher perspectives could provide more holistic insights into the benefits and challenges of VR implementation in classrooms.

Despite these areas of improvement, the project as a whole represents a contribution to the field of VR in mathematics education. It provides valuable insights into the

[22] CleanBox: https://cleanboxtech.com/

application of VR in various educational contexts, and while the project could explore certain technical aspects of VR more deeply and diversify its participant base, the theoretical considerations and empirical evidence it presents may help in bridging the gap between theory and practice in this field. This project effectively positions VR as a tool that can potentially revolutionise how mathematics is taught and learned, setting the stage for future research and innovation.

8.4 Future Perspective

Based on the results of this PhD project, several avenues for future research and development can be explored.

Further research could examine the long-term effects of using asymmetric VR in mathematics education by conducting longitudinal studies. This would allow for a deeper understanding of how students' conceptual knowledge and competencies develop over an extended period. Moreover, it would be beneficial to explore the potential of asymmetric VR in other STEM subjects or across a more comprehensive array of educational contexts, which could help determine this technology's generalisability and potential impact on education. These longitudinal studies, while still requiring a mixed-methods research design, would benefit from a heightened focus on gathering quantitative data pertaining to students' prior knowledge, knowledge transfer to new situations, daily living skills, inclusion and their interactions within the virtual environment.

Future research endeavours should also aim to comprehensively compare the outcomes of traditional and experimental strategies incorporating VR. For instance, this comparison should encompass both the degree of asymmetry indicated by our taxonomy and the diverse learning environments available for geometry education. In such instances, an in-depth analysis is warranted to evaluate the differential effects experienced by students utilising non-immersive 2D and 3D e-learning platforms as opposed to VR. This analysis should explore factors such as usability, cognitive load, performance, and learning outcomes, facilitating a comprehensive understanding of the comparative benefits.

Future work might also focus on developing advanced and immersive virtual learning environments. These environments could incorporate adaptive learning algorithms, AI-driven tutoring systems, and analytics to enhance the learning experience.

In addition, developing effective teacher training programs and support systems is crucial to ensure the successful implementation of asymmetric VR in the classroom.

Lastly, creating new assessment and evaluation methodologies tailored to VR learning experiences is essential. By doing so, we can better measure the impact of VR on student performance and understanding, ultimately providing valuable insights into the effectiveness of these technologies in education.

By exploring these future research avenues, the field can continue to advance our understanding of the potential benefits and challenges associated with integrating VR in mathematics education and beyond, ultimately contributing to developing more effective, engaging, and inclusive learning experiences.

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Part II

Papers

SUMMARY

This thesis explores the use of Virtual Reality (VR) in mathematics education. Four VR prototypes were designed and developed during the PhD project to teach equations, geometry, and vectors and facilitate collaboration.

Paper A investigates asymmetric VR for classroom integration and collaborative learning and presents a new taxonomy of asymmetric interfaces. Paper B proposes how VR could assist students with Autism Spectrum Disorder (ASD) in learning daily living skills involving basic mathematical concepts. Paper C investigates how VR could enhance social inclusion and mathematics learning for neurodiverse students. Paper D presents a VR prototype for teaching algebra and equation-solving strategies, noting positive student responses and the potential for knowledge transfer. Paper E investigates gesture-based interaction with dynamic geometry in VR for geometry education and presents a new taxonomy of learning environments. Finally, paper F explores the use of VR to visualise and contextualise mathematical concepts to teach software engineering students.

The thesis concludes that VR offers promising avenues for transforming mathematics education. It aims to broaden our understanding of VR's educational potential, paving the way for more immersive learning experiences in mathematics education.

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