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Simplicial models for trace spaces II: General higher dimensional automata

by

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SIMPLICIAL MODELS FOR TRACE SPACES II: GENERAL HIGHER DIMENSIONAL AUTOMATA

MARTIN RAUSSEN

ABSTRACT. Higher Dimensional Automata (HDA) are topological models for the study of concurrency phenomena. The state space for an HDA is given as a pre-cubical complex in which a set of directed paths (d-paths) is singled out. The aim of this paper is to describe a general method that determines the space of directed paths with given end points in a pre-cubical complex as the nerve of a particular category.

The paper generalizes the results from Raussen [19, 18] in which we had to assume that the HDA in question arises from a semaphore model. In particular, important for applications, it allows for models in which directed loops occur in the processes involved.

1. INTRODUCTION

1.1. **Background.** A particular model for concurrent computation in Computer Science, called *Higher Dimensional Automata* (HDA), was introduced by Pratt [15] back in 1991. Mathematically, HDA can be described as (labelled) pre-cubical sets (with n-dimensional cubes instead of simplices as building blocks; cf Brown and Higgins [2, 1]) with a preferred set of *directed paths* (respecting the natural partial orders) in any of the cubes of the model.

Compared to other well-studied concurrency models like labelled transition systems, event sturctures, Petri nets etc. (for a survey on those cf Winskel and Nielsen [23]), it has been shown by R.J. van Glabbeek [22] that Higher Dimensional Automata have the highest expressivity; on the other hand, they are certainly less studied and less often applied so far.

All concurrency models deal with sets of *states* and with associated sets of *execution paths* (with some further structure). The interest is mainly in the structure of the *spaces of execution paths*; typically, it is difficult to extract valuable information about the path space from the state space model. We use topological models for both state space and the execution (=path) space consisting of the *directed paths* (called d-paths) in state space. It is particularly important to know whether the path space is *path-connected*; and, if not, to get an overview over its path components: Executions in the same path component yield the *same result* (decision) in a concurrent computation; different components *may* lead to different results. From a topological perspective, the ultimate aim is to determine the *homotopy type* of these path spaces.

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Higher Dimensional Automata are prototypes of *directed* topological spaces, cf Grandis [12, 13]. General topological properties of spaces of d-paths and of *traces* (=d-paths up to monotone reparametrizations; cf Fahrenberg and Raussen [5, 16]) in pre-cubical complexes were investigated in Raussen [17]. The articles Raussen [19, 18] desccribe an algorithmic method to determine the homotopy types of trace spaces for Higher Dimensional Automata (and thus in particular to calculate and describe their components) through explicitly constructed *finite simplicial complexes* for a *restricted class* of model spaces:

(1) We had to stick to *semaphore* – or PV – models as described by Dijkstra [4] – an important but restricted class of HDA. Loosely speaking, a PV-model space is a hypercube I^n – with I the unit interval $[0, 1]$ – from which a number of n-dimensional hyperrectangles has been removed; cf Raussen [19].
(2) We only considered model spaces *without non-trivial directed loops*.

For these restricted class of models, the resulting algorithm has meanwhile been implemented with encouraging results, cf Fajstrup etal [8].

In the present paper, we propose an algorithm extending the framework to full generality yielding (generalized simplicial) models for spaces of traces in general pre-cubical complexes; hence we cover models for general (unlabeled) HDA. For these, the homotopy type of trace spaces between given end points is identified with an explicitly constructed complex (a generalization of a simplicial complex); all components of that complex are finite. Using this complex, topological invariants (eg homology) can be calculated.

A price has to be paid: the algorithm determining this complex is, at least in general, more intricate than in the semaphore case. Data structures can be much more complicated, and we have no experience with running times yet.

1.2. **Structure and overview of results.** Section 2 introduces pre-cubical complexes as HDA; we abstract away from labels. We introduce a signed L_1-arc length on *general* paths in a pre-cubical complex with positive or negative values extending the definition from Raussen [17] for d-paths. It is shown that this signed L_1-arc length is invariant under homotopy *with fixed end points* for *all* paths and that the range of the L_1-arc length map is *discrete* given a pair of end points.

We introduce the class of *non-branching* (and non-looping) pre-cubical complexes in Section 3. We show, that the space of traces between two points in such a complex is always *either empty or contractible*.

In the central Section 4, we consider traces in pre-cubical complexes *with branch points* but without non-trivial directed loops. We decompose such a complex into subcomplexes *without* branch points and such that the associated trace spaces cover the trace space corresponding to the entire complex. This decomposition can be quite complicated in the presence of higher order branch points. The nerve of the poset category associated to this cover is homotopy equivalent to trace space. Moreover, we construct a complex (with cones of products of simplices as building blocks) homotopy equivalent to trace space and "more economical" than this nerve.

In Section 5, we show that trace spaces for a pre-cubical complex *with non-trivial directed loops* can be analysed through trace spaces in an associated covering space in which lifts of paths depend on their L_1-arc length – and in which (non-trivial) d-loops lift to to non-loops.

In the final Section 6, we give a few hints about a possible implementation that, with a pre-cubical set as input and using an associated directed graph, allows to determine the poset category describing the associated trace space.

2. Pre-cubical complexes and length maps

2.1. **Directed paths and traces in a pre-cubical complex.** Properties of Higher Dimensional Automata (cf. Section 1.1) are intimately related to the study of directed paths in a pre-cubical set, also called a $\square$-set; this term (cf. [6]) is used in a similar way as a Δ-set – as introduced in [20] – for a simplicial set without degeneracies. We use $\square_n$ as an abbreviation for the n-cube $I^n = [0,1]^n$ with the product topology.

Definition 2.1. (1) A $\square$-set or pre-cubical complex X is a family of disjoint sets $\{X_n | n \geq 0\}$ with face maps $\partial_i^k : X_n \to X_{n-1}, \leq i \leq n, k = 0,1$, satisfying the pre-cubical relations $\partial_i^k \partial_j^l = \partial_{j-1}^l \partial_i^k$ for $i < j$.

(2) The geometric realization $|X|$ of a pre-cubical set X is given as the quotient space $|X| = (\coprod_n X_n \times \square_n)_{/\equiv}$ under the equivalence relation induced from

$$(\partial_i^k(x), t) \equiv (x, \delta_i^k(t)), \quad x \in X_{n+1}, \quad t = (t_1, \ldots t_n) \in \square_n$$

$$\text{with} \quad \delta_i^k(t) = (t_1, \ldots, t_{i-1}, k, t_{i+1}, \ldots, t_n).$$

(3) A pre-cubical complex M is called *non-self-linked* (cf. [9, 17]) if, for all n, $x \in M_n$ and $0 < i \leq n$, the $2^i\binom{n}{i}$ iterated faces
$\partial_{l_1}^{k_1} \cdots \partial_{l_i}^{k_i} x \in M_{n-i}, \quad k_i = 0,1, \quad 1 \leq l_1 < \cdots < l_i \leq n,$ are all different.

In the future, we will not distinguish between a pre-cubical complex X and its geometric realization and just write X for both. We will tacitly assume that all pre-cubical complexes are non-self-linked; if necessary, after a barycentric subdivision.

We are interested in *directed* paths in X. A continuous path within a cube $\square_n$ is a d-path, if all n component functions are (not necessarily strictly) *increasing*. A path in X is a d-path if it is the concatenation of d-paths within cubes; cf Definition 2.2 in Raussen [17] for details. The set of all d-paths in X will be denoted $\vec{P}(X) \subset X^I$ with subspaces $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ consisting of paths with $p(0) = \mathbf{c}$ and $p(1) = \mathbf{d}$. These spaces inherit a topology from the CO-topology on X^I (the uniform convergence topology).

Reparametrization equivalent d-paths [5] in X have the same directed image (= *trace*) in X. Dividing out the action of the monoid of (weakly-increasing) reparametrizations of the parameter interval $\vec{I}$, we arrive at trace space $\vec{T}(X)(\mathbf{c}, \mathbf{d})$, cf Fahrenberg and Raussen [5, 16]; it is shown in Raussen [17] to be homotopy equivalent to path space $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ for a far wider class of directed spaces X; in the latter paper, it is also shown

that trace spaces enjoy nice properties; e.g., they are metrizable, locally compact, locally compact, and they have the homotopy type of a CW-complex.

Notation: Within X and for $x \in X$, we let $\downarrow x := \{y \in X | \vec{P}(X)(y,x) \neq \emptyset\}$ denote the *past* of x.

2.2. **Length maps.** The L_1-arc length of a d-path in a pre-cubical complex was introduced and studied in Raussen [17]. The definition and important properties can be extended to general non-directed paths; for these the (signed) L_1-arc length may be negative. This goes roughly as follows:

The *signed* L_1-length $l_1^{\pm}(p)$ of a path $p : I \to \square_n$ within a cube $\square_n$is defined as $l_1^{\pm}(p) = \sum_{j=1}^{n} p_j(1) - p_j(0)$. For any path p, that is the concatenation of finitely many paths each of which is contained in a single cube, the signed L_1-length is defined as the sum of the lengths of the pieces; the result is independent of the choice of decomposition – and of the parametrization! Moreover, it is non-negative for every d-path and positive for every non-constant d-path.

This construction can be phrased more elegantly using differential one-forms on a cubical complex (a special case of the PL differential forms introduced by D. Sullivan [21] in his approach to rational homotopy theory, or of the closed one-forms on topological spaces by M. Farber [10, 11]): On an n-cube $e \simeq \square_n$, consider the particular 1-form $\omega_e = dx_1 + \cdots + dx_n \in \Omega^1(\square_n)$. It is obvious that $\omega_{\partial_i^k e} = (i_i^k)^* \omega_e$ with $i_i^k : |\partial_i^k e| \hookrightarrow |e|$ denoting inclusion. Pasting together, one arrives at a particular (closed!) 1-form ω_X on every pre-cubical set X – the one-form that reduces to ω_e on every cell e in X.

The signed length of a (piecewise differentiable) path γ on X can then be defined as $l_1^{\pm}(\gamma) = \int_0^1 \gamma^* \omega_X$ and extended to continuous paths using uniformly converging sequences of such piecewise differentiable paths. This length is invariant under orientation preserving reparametrization; it changes sign under orientation reversing reparametrization; it is additive under concatenation and non-negative for d-paths. It yields a *continuous* map $l_1^{\pm} \colon P(X)(x_0, x_1) \to \mathbf{R}$. An application of Stokes' theorem shows:

Proposition 2.2. *Two paths $p_0, p_1 \in P(X)(x_0, x_1)$ that are homotopic rel end points have the same signed length: $l_1^{\pm}(p_0) = l_1^{\pm}(p_1)$.* □

A more direct proof can be given along the lines of Raussen [17] using the continuous map $s : X \to S^1 = \mathbf{R}/\mathbf{Z}$ given by $s(e; x_1, \ldots, x_n) = \sum x_i \bmod 1$. It is clear from the construction, that $l_1^{\pm}(p) \equiv s(p(1)) - s(p(0)) \bmod 1$. As a consequence, $l_1^{\pm}(P(X)(x_0, x_1))$ is constant mod 1 and, in particular, a discrete subset of the reals. Hence, $l_1^{\pm}$ is constant on a connected component, i.e., a homotopy class of paths in $P(X)(x_0, x_1)$.

Remark 2.3. As remarked in Raussen [17], Remark 2.8, it is *not* possible to extend *non-negative* L_1-arc length continuously to non-directed paths.

3. Trace spaces for non-branching pre-cubical complexes

In the following two sections, we will only consider *non-looping* pre-cubical complexes. In such a complex X, the only *directed loops* are trivial, i.e., *constant*.

A (finite) such pre-cubical complex X will be called *non-branching* if it satisfies the following additional property

NB: Every vertex $v \in X_0$ is the lower corner vertex of a *unique maximal* cube c_v in X. This maximal cube c_v contains thus all cubes with lower corner vertex v as a (possibly iterated) lower face.

On a non-branching cubical complex, there is a *privileged directed* flow $F^X\colon X \times \mathbf{R}_{\geq 0} \to X$: Every element $x \in X$ is contained in the interior or the lower boundary of a uniquely determined maximal cube, i.e., the maximal cube c_v of its lowest vertex v. On the interior and the lower faces of such a cube c_v, this flow is locally given by the *diagonal* flow:

$$F^X_c(c;(x_1,\dots,x_n);t) = (c;x_1+t,\dots,x_n+t) \text{ for } 0 \leq t \leq 1 - \max_{1\leq i\leq n} x_i. \tag{3.1}$$

On a maximal vertex v_1 with $c = c_{v_1} = v_1$ – a deadlock – , F^X_c is defined to be constant in t for $0 \leq t$.

On cubes, that are not lower boundaries of others, these flow lines are the gradient lines of the 1-form ω_X from Section 2.2; this is *not* true on such lower boudaries. Piecing together these local flows so that they satisfy the flow semi-group property yields a piecewise-linear (hence Lipschitz continuous) global flow all of whose flow lines are d-paths; note from the construction that this flow can only have equilibria at deadlocks.

Remark 3.1. At a branching vertex v_0 in a general (branching) pre-ubical complex X, it is not possible to construct such a flow. Diagonal flows on several maximal cubes do not fit together on their intersections.

Lemma 3.2. *A finite non-branching* connected *pre-cubical complex X has a* uniqe *maximal vertex v_1.*

Proof. First of all, there is at least one maximal vertex. Otherwise, one would have d-paths of arbitrary length in X; hence X – without non-trivial loops – could not be finite.

Suppose $v_1, v_2, \dots, v_k \in X, k > 1$, is a list of *all* maximal vertices. Consider the maximal vertices in the common past subcomplexes $\downarrow v_i \cap \downarrow v_j, i \neq j$, and choose among those the *maximal* ones (that cannot reach any of the others). Pick such a maximal vertex v and consider the the associated maximal cube c_v.

There is at least one edge in c_v with v as lower boundary from which one can reach v_i and not v_j; likewise another edge from which one can reach v_j and not v_i. From the top edge of c_v, at least one of the v_r in the list is reachable. As a consequence, from at least one of the two edges mentioned before, *two* maximal vertices can be reached. Contradiction to maximality! □

The key Proposition 2.8 from Raussen [19] generalizes as follows:

Proposition 3.3. *For every pair of elements $x_0, x_1 \in X$ in the geometric realization X of a pre-cubical* non-branching *complex X, trace space $\vec{T}(X)(x_0, x_1)$ is either* empty *or* contractible.

Proof. We assume that $\vec{T}(X)(x_0, x_1) \neq \emptyset$ and, without restriction, that x_1 is *the* maximal vertex in X; in general, just replace X by $\downarrow x_1 \subset X$, still a non-branching complex; without deadlocks and unsafe regions.

The directed flow line corresponding to the flow F^X, cf (3.1), starting at $x \in X$ and ending at x_1 (after linear renormalization so that its domain becomes the unit interval I) will be called $p_x \in \vec{P}(X)(x, x_1)$.

A contraction $H : \vec{P}(X)(x_0, x_1) \times I \to \vec{P}(X)(x_0, x_1)$ to the flow path $H_0 = p_{x_0}$ is constructed as follows: For $p \in \vec{P}(X)(x_0, x_1)$, let $H(p,t)(s) = \begin{cases} p(s) & t \leq s \\ p_{p(t)}(\frac{s-t}{1-t}) & s \leq t \end{cases}$.
Remark that $H_1 = p$ and that an intermediate d-path H_t follows p until $p(t)$ and then it follows the flow line starting at $p(t)$ automatically ending at x_1.

Finally, the quotient map $\vec{T}(X)(x_0, x_1) \to \vec{P}(X)(x_0, x_1)$ is a homotopy equivalence, cf [17]. □

This proof, using the diagonal flow F^X, is different from the one given in Raussen, [19, Proposition 2.8] for the special case of cubical complexes arising from semaphore models; but it is certainly similar in spirit.

4. Trace spaces for non-looping pre-cubical complexes

In this section, we study traces in a more general finite pre-cubical complex X; still *without non-trivial loops*, but allowing for branch points: How to find subcomplexes $Y \subseteq X$ satisfying (NB)? Investigating the space of d-paths between $\mathbf{x}_0$ and $\mathbf{x}_1$ in X, we assume that $X = [\mathbf{x}_0, \mathbf{x}_1] = \uparrow \mathbf{x}_0 \cap \downarrow \mathbf{x}_1$. In particular, X contains neither unsafe nor unreachable regions. We start with an abstract description:

4.1. **An abstract simplicial model.** The subcomplex given by the *carrier sequence* corresponding to any directed path, cf Fajstrup [6], the sequence of cubes containing segments of that path, is obviously a subcomplex satisfying (NB).

One may order (NB) subcomplexes of X by inclusion – chains are of bounded length since there are only finitely many cubes – and focus on the *maximal* non-branching subcomplexes. Every d-path with a given start point is contained in a maximal (NB) subcomplex, that is in general not uniquely determined. Traces contained in maximal (NB) subcomplexes cover thus the space of all d-paths (with given start and end point).

Lemma 4.1. *An intersection $S = \bigcap X_i$ of subcomplexes X_i each satisfying (NB) satisfies (NB) as well. Hence the space of traces $\vec{T}(S)(\mathbf{x}_0, \mathbf{x}_1)$ in S is either contractible or empty.*

Proof. The intersection of maximal cubes at every vertex will be the maximal cube in the intersection and hence unique. For contractibility, use Proposition 3.3. Empty path spaces may arise when S is not connected. □

The subcomplexes $X_i \subset X, i \in I$, satisfying (NB) that are *maximal* with respect to inclusion give thus rise to a covering $\vec{T}(X_i)(x_0, v_1)$ of trace space $\vec{T}(X)(x_0, v_1)$ by contractible sets; in fact:

Theorem 4.2. *For a finite pre-cubical complex X, trace space $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homotopy equivalent to the nerve of the covering given by the subspaces $\vec{T}(X_i)(\mathbf{x}_0, \mathbf{x}_1)$.*

Proof. The theorem is an almost immediate consequence of the nerve lemma, cf Kozlov [14]. Since the subspaces X_i in general are not open, one has to enlarge them a little to get open path spaces homotopy equivalent to the original ones (and hence contractible, as well); this has been described in detail in a particular case in Raussen [19]. □

4.2. **An index category.**

4.2.1. *(Higher order) branch points.* The maximal subcomplexes X_i from Theorem 4.2 may be very difficult to identify for a complex X with many cells. In the following, we describe an algorithmic method that determines an index category $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$ that can be represented by a complex $\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ which is homotopy equivalent to trace space $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$. The building blocks of the complex $\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ are products of simplices and of cones of such spaces. The construction is similar in spirit to that in Raussen [19], albeit, in the details slightly more complicated.

A vertex $v \in X_0$ is called a *branch point* if there are *more than one* maximal cube c having v as lower vertex (i.e., an iterate of ∂^0_* yields v). The set of all such maximal cells with lower vertex v is called the *branch set* B_v with $|B_v| > 1$.

Remark 4.3. Let $v \in X_0$ be a branch point with several maximal cubes $c_1, \ldots, c_r$ with lower vertex v. Obviously, at most one of the cubes c_j can be contained in every of the (NB) subcomplexes X_i from Section 4.1.

For a cell c in X, we denote by c^- the geometric realization of c and of all (iterated) *lower* boundaries – not including mixed or upper boundaries. Hence, $|c^-| \cong [0,1[^n$ for an n-cell c. For a fixed branch point v and a branch cell $c_j \in B_v$, let

$$X^v_j := \downarrow c^-_j \cup \mathcal{C}(\downarrow \bigcup_{c_i \in B_v} c^-_i)$$

consist of all points that, as far as they can reach any branch in B_v, they have to stay in the past of the particular branch c_j; $\mathbf{C}$ denotes the complement within X. Clearly, $X = \bigcup_{c_j \in B_v} X^v_j$.

Lemma 4.4. $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1) = \bigcup_{c_j \in B_v} \vec{T}(X^v_j)(\mathbf{x}_0, \mathbf{x}_1)$ *for every branch point v.*

Proof. We need to show that $\vec{T}(\bigcup_{c_j \in B_v} X^v_j)(\mathbf{x}_0, \mathbf{x}_1) = \bigcup_{c_j \in B_v} \vec{T}(X^v_j)(\mathbf{x}_0, \mathbf{x}_1)$: Every d-path from $\mathbf{x}_0$ to $\mathbf{x}_1$ starts in the (past closed) set $\downarrow \bigcup_{c_j \in B_v} c^-_j$ and then leaves it for its (future closed) complement. The sets $\downarrow c^-_j$ are all past closed; a d-path p that has left one of these sets will never get back to it. In particular, there is at least one (last) set $X^v_j, c_j \in B_v$, containing p. □

Contrary to the special situation of pre-cubical complexes arising from PV-protocols discussed in [19], it is not enough to consider only (1. order) branch points as the following example (cf Figure 1) shows:

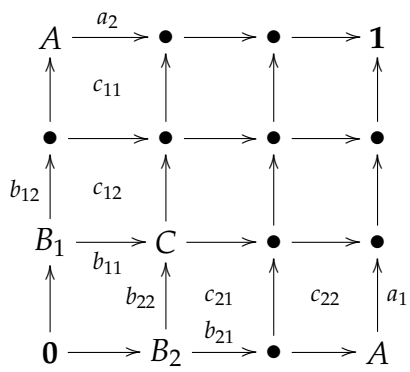

FIGURE 1. Branch points in a 2D complex

Example 4.5. The complex X to be discussed arises from the 9 planar 2-cubes in Figure 1 by identifying the two vertices denoted A. Remark the two special "horizontal" and "vertical" d-paths from $\mathbf{0}$ to $\mathbf{1}$ through A. The vertex A is the only branch point in X; it has branch set $B^A = \{a_1, a_2\}$. The subcomplex X_1^A arises from X by crossing out the two cells c_{11} and c_{12} – apart from the left hand boundary 1-cells. Likewise, for X_2^A, the cells c_{21} and c_{32} – apart from the lower boundary 1-cells – have to be deleted. The first subcomplex has a secondary branch point B_1 with branches b_{11} and b_{12}. Likewise, the second one has a secondary branch point B_2 with branches b_{21} and b_{22}.

The homotopy type of the trace space $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$ will be identified in Example 4.11 below.

4.2.2. *The index category $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$.* Hence, it is necessary to consider secondary, and in general higher order branch points, as well:

- The original space X comes with a set $BP = \{b^1, \ldots, b^l\}$ of branch points and associated maximal branch cubes $BC = \{c_j^i\}$ and a surjective map $p : BC \downarrow BP; p(c_j^i) = b^i$.
- For every section $s(1) : BP \uparrow BC$ of the map p, consider the subcomplex $X_{s(1)} = \cap_{b_i \in BP}(\downarrow c^{i-}_{s(1)(i)} \cup \mathcal{C}(\bigcup_{p(j)=b_i} \downarrow c_j^{i-}) \subset X$. $X_{s(1)}$ is in fact a subcomplex of X since the branch cubes are all maximal. It is a proper subcomplex containing $\mathbf{x}_0$ and $\mathbf{x}_1$.
- Such a complex $X_{s(1)}$ may have (second order) branch points $b^i(s(1)) \in BP_{s(1)}$ and branch cubes $c_j^i(s(1)) \in BC_{s(1)}$ coming with a projection $p(s(1)) : BC_{s(1)} \downarrow BP_{s(1)}$ and sections $s(2) : BP_{s(1)} \uparrow BC_{s(1)}$.

- Iterate: Given subsequent sections $s(1), \dots, s(r)$ define a proper subcomplex $X_{s(r)} \subset X_{s(r-1)}$ as

$$X_{s(r)} = \bigcap_{0 \le k \le r} \bigcap_{b_i \in BP(s(k))} (\downarrow c^{i-}_{s(k)(b_i)} \cap \mathcal{C}(\bigcup_{p(k)(j)=b_i} \downarrow c^{i-}_j ; X_{s(k-1)}))$$

  – with the convention that $p(s(0)) : BC(s(0)) \downarrow BP(s(0))$ is the original projection map $p : BC \downarrow BP$. This complex $X_{s(r)}$ may give rise to sets of new branch cells $BC_{s(r)}$ and branch points $BP_{s(r)}$ with a projection map $p(s(r)) \colon BC_{s(r)} \downarrow BP_{s(r)}$.
  Since these subcomplexes become smaller and smaller under iteration in the finite complex X, every such iteration will ultimately end in a subcomplex $X_{s(r)}$ *without branch points*.
- A subsequent sequence of sections $s(k) : BP_{s(k)} \uparrow BC_{s(k)},\ k \le r$, is called *coherent and complete* if $X_{s(r)}$ satisifes property (NB), cf. Section 3. The set of all such coherent and complete sequences will be called $CCS(X)$.
- To a coherent and complete sequence $s \in CCS(X)$, we may associate the set of branch points $BP(s) = \bigcup_{k=0}^{r} BP_{s(k)}$ and branch cubes $BC(s) = \bigcup_{k=0}^{r} BC_{s(k)}$, the projection $p(s) : BC(s) \downarrow BP(s)$, and the "tautological" section $\bar{s} \colon BP(s) \uparrow BC(s)$; there are no branch cells to choose at depth s!

Definition 4.6. (1) The poset category $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ has as
objects: all pairs of the form (S, C) with $\emptyset \neq S \subset CCS(X)$ and C a set of the form $C = \prod_{b_i \in \bigcup_{s \in S} BP(s)} C_i,\ \bar{s}(b_i) \in C_i \subset BC(b_i)$; with $\bar{s}$ denoting the tautological section from Section 4.2.2.
morphisms: $(S, C) \le (S', C') \Leftrightarrow S \subseteq S', \forall s \in S, b_i \in BP(s) \colon C_i \subseteq C'_i$.
Note that the minimal objects (S, D) of this category are composed of a set S with precisely one element s and such that $b_i \in BP(s) \Rightarrow |C_i| = 1$.
(2) To a section $s \in CCS(X)$, a branch point $b_i \in BP(s(k))$ and a branch cube $c_i \in BC(s(k))$, we associate the subspace $(X_{s(k)})^{c_i}_{b_i} \subset X$. To an object (S, C) in $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ we associate the subspace $X_{(S,C)} := \bigcap_{s \in S, b_i \in BP(s(k))} (X_{s(k)})^{c_i}_{b_i} \subset X$.
(3) The category $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is the full subcategory of $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ whose objects (S, C) are characterized by the fact that $\vec{T}(X_{(S,C)})(\mathbf{x}_0, \mathbf{x}_1)$ is *non-empty*.

Proposition 4.7. (1) $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1) = \bigcup_{(S,C)} \vec{T}(X_{(S,C)})(\mathbf{x}_0, \mathbf{x}_1)$*; the union extends over all objects of the category* $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$*.*
(2) *For every object* (S, C) *of the subcategory* $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$*, the subspace* $\vec{T}(X_{(S,C)})(\mathbf{x}_0, \mathbf{x}_1)$ *is contractible.*

Proof. (1) follows from Lemma 4.4 by induction.
(2) According to Proposition 3.3, the trace space $\vec{T}(X_{s(r)})(\mathbf{x}_0, \mathbf{x}_1)$ is empty or contractible for every complete coherent sequence of branches and branch points.

For every object (S, C) of the category $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$, the space $X_{(S,C)}$ is a finite intersection of spaces of type $\vec{T}(X_{s(r)})(\mathbf{x}_0, \mathbf{x}_1)$, $s \in S$. Apply Lemma 4.1.

□

Remark 4.8. An algebraic representation of the category $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is not as straightforward as for semaphore models that resulted in a poset category of binary matrices (cf Raussen [19]). In the general case, one has to consider combinations of all branch point sequences (of different lengths) and the occuring branches – which may have different cardinalities.

4.3. **Homotopy equivalences.** We will now present a description of the homotopy type of $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ as a colimit of simple spaces (generalizing the prodsimplicial complex for traces arising from semaphore models, cf Raussen [19]). For simplicity, we will describe only the case with branch points of order at most two.

In this case, to every first order branch point $b^i \in BP$, we associate the branch cells $BC^i = p^{-1}(b^i)$; a section $s : BP \uparrow BC$ gives rise to 2nd order branch points in $BP(s)$ and, to every of the 2nd order branch points $b^i(s) \in BP(s)$, a set of branch cells $BC^i(s)$. Then, the total complex corresponding to *all* objects (S, C) corresponds to

$$\Delta^{|BC^1|-1} \times \cdots \times \Delta^{|BC^l|-1} \times \prod_{s:BP\uparrow BC} C(\Delta^{|BC^1(s)|-1} \times \cdots \times \Delta^{|BC^{l(s)}(s)|-1}).$$

Here CX denotes the *cone* over X.

A subset $S = S_1 \times \cdots \times S_l \subseteq BC^1 \times \cdots \times BC^l$ corresponds to a product of simplices $\Delta_S = \Delta^{|S_l|-1} \times \cdots \times \Delta^{|S_1|-1}$. A subset $C(s) = \prod_{b^i(s)\in BP(s)} C^i(s), s : BP \uparrow BC$, of products of first and second order branch cells corresponds to

$$\Delta^C(s) := \Delta^{|C^1(s)|-1} \times \cdots \times \Delta^{|C^{l(s)}(s)|-1} \subseteq \Delta^{|BC^1(s)|-1} \times \cdots \times \Delta^{|BC^{l(s)}(s)|-1}.$$

An object (S, C) in $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ corresponds to

$$\Delta(S, C) := \Delta_S \times \prod_{s\in S} C(\Delta^C(s)) \times \prod_{s\notin S} *_s.$$

with $*_s$ the cone point in $C(\Delta^C(s))$. Morphisms $(S, C) \leq (S', C')$ correspond to inclusions $\Delta(S, C) \hookrightarrow \Delta(S', C')$.

Definition 4.9. The complex $\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is defined as the colimit

$$\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1) := \mathbf{colim}_{\mathcal{C}(X)(\mathbf{x}_0,\mathbf{x}_1)} \Delta(S, C).$$

Theorem 4.10. *Trace space $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homotopy equivalent to*

(1) *the nerve $\Delta(\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1))$ of the poset category $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$, and*
(2) *the complex $\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$.*

Proof. The proof of Theorem 4.10 is analogous to that of Raussen [19, Theorem 3.5]: The homotopy colimit of the functor associating the contractible spaces $\vec{T}(X)_{(S,C)}(\mathbf{x}_0, \mathbf{x}_1)$, resp. $\Delta(S, C)$ to an object (S, C) in $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homotopy equivalent to the functor

associating the same point to every (S, C), ie to the nerve of that category. Homotopy colimit and colimit of the first two functors are also homotopy equivalent; by Lemma 4.4, this colimit is the entire trace space; resp. by definition the complex $\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$. $\square$

4.4. **Example: Trace spaces for some 2-dimensional pre-cubical complexes.**

Example 4.11. First, we look at the case of the space X described in Example 4.5 and Figure 1. There are four coherent and complete sequences of sections:

$$\begin{aligned} s_1(A) &= a_1, \quad s_1(1)(B_1) = b_{11} \\ s_2(A) &= a_1, \quad s_2(1)(B_1) = b_{12} \\ s_3(A) &= a_2, \quad s_3(1)(B_2) = b_{21} \\ s_4(A) &= a_2, \quad s_4(1)(B_2) = b_{22} \end{aligned} \tag{4.1}$$

corresponding to minimal objects $(S_i = \{s_i\}, C_i)$ with C_i the one-element set given by the branches chosen by the section.

The only non-trivial intersection occurs for $\vec{T}(X_A^{a_1} \cap X_{B_1}^{b_{11}})(\mathbf{0}, \mathbf{1})$ and $\vec{T}(X_A^{a_2} \cap X_{B_2}^{b_{22}})(\mathbf{0}, \mathbf{1})$ giving rise to the object $(S_{14} = \{s_1, s_4\}, C_{14} = \{a_1, a_2\} \times \{b_{11}\} \times \{b_{22}\})$. In this case, the complex $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$ is thus a disjoint union of the cones on two vertices (each of this edges corresponds to one of the special horizontal and vertical traces) and product of an edge with the cone on a vertex.

Using Theorem 4.10, we can conclude: $\vec{T}(X)(\mathbf{0}, \mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0}, \mathbf{1}) \cong I \sqcup I \sqcup I^2$ is homotopy equivalent to a set of three disjoint points.

Example 4.12. The 2-dimensional complex X in Figure 2 below arises from glueing the boundaries $\partial\square^3$ of two 3-cubes $\square^3$ along a common face $\square^2$. Its trace space has previously been studied by Bubenik [3]. The complex has two branch points $\mathbf{x}_0$ and A and *no higher order* branch points.

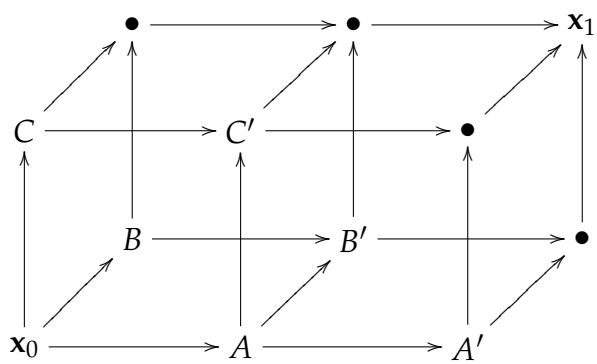

FIGURE 2. The complex X: Boundaries of two cubes glued together at common square $AB'C'\bullet$

Bubenik's "necklace" model yields a simplicial complex consisting of 16 triangles, 46 edges and 29 vertices which is seen to be homotopy equivalent to $S^1 \vee S^1$. The complex $\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ – a prodsimplicial in the terminology of Kozlov [14] since there are no higher order branch points – homotopy equivalent to $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$, cf Theorem 4.10 above and the construction in Raussen [19], is shown in Figure 3 below:

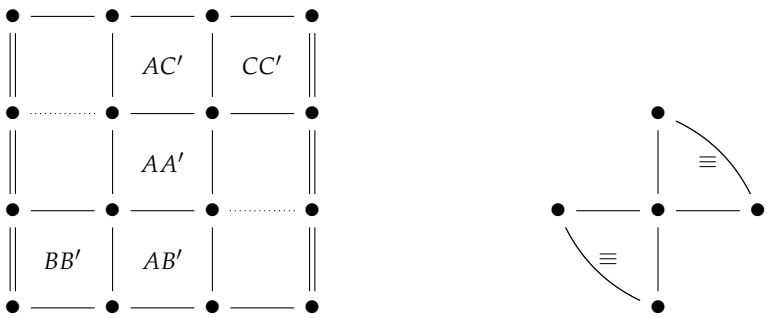

FIGURE 3. Prodsimplicial complex homotopy equivalent to the trace space for X – and a homotopy equivalent complex

It consists of the five named squares in the nine square decomposition of a 2-torus $\Delta^1 \times \Delta^1$ – identify boundary edges as usually – to which a full triangle has been attached along the circle on the vertical (left=right) triangle (marked by double lines =). The nine vertices correspond to 3×3 combinations of paths staying "under" the three branches (2-cubes) corresponding to the two branch points x_0 and A in Figure 2.

The labels in the five marked squares refer to paths staying under *two* branches for each of the branch points. They are marked by labels referring to traces through the points mentioned. For example, AC' denotes the space of all traces from $\mathbf{x}_0$ to A (front *and* bottom of the first cube) and then C' (front *and* left of the second cube) and from C' as an arbitrary d-path on the top square of the second cube to $\mathbf{x}_1$. It is easily seen, that there are no d-paths corresponding to the remaining four squares BC', BA', CA' and CB'.

Furthermore, there is a full triangle (marked with =): Every trace entering the interior of the "left" square $x_0 B \bullet C$ in Figure 2 leaves the past of the union of the branches corresponding to branch point A; such a path is therefore contained in *all three* sets X_j^A – giving rise to a full triangle $\Delta^0 \times \Delta^2$.

The complex $\mathbf{T}(X)(x_0, x_1)$ in Figure 3 consists thus of six 2-cells (five of type $\Delta^1 \times \Delta^1$, one of type $\Delta^0 \times \Delta^2$), 16 edges (all but the two stipled ones) and of all 9 vertices; it has Euler characteristic -1. A contraction of the full triangle can be extended to a contraction of the entire space to a union of two full triangles (shown on the right hand side of Figure 3) with three vertices (opposite vertices are identified). That simplicial 2-complex contracts to a 1-complex $S^1 \vee S^1$.

From Theorem 4.10, we may conclude: $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1) \simeq S^1 \vee S^1$.

5. Trace spaces for general pre-cubical complexes

In this section we outline, how the methods previously explained can be adapted to trace spaces in a general pre-cubical complex X that may allow directed loops using suitable *coverings* of the complex X:

5.1. **Non-looping length coverings.** We exploit the d-map (directed map) $s : X \to \vec{S}^1 \cong \mathbf{R}_{/\mathbf{Z}}$ introduced in Raussen [17]: just glue the maps $s(x_1, \ldots, x_n) = \sum x_i \bmod 1$ on individual cubes. Consider the pullback $\tilde{X}$ in the pullback diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "S"] \arrow[d, "\pi"'] & X \times \mathbf{R} \arrow[d, "id \times \exp"] \\ X \arrow[r, "id \times s"'] & X \times S^1 \end{tikzcd} .$$

The map π is a covering map with unique path lifting. Since exp can be interpreted as a semi-cubical map, $\tilde{X}$ can be conceived as a semi-cubical complex: every cube e in X is replaced by infinitely many cubes $(e, n), n \in \mathbf{Z}$ with boundary maps given as $\partial_-(e, n) = (\partial_- e, n), \partial_+(e, n) = (\partial_+ e, n + 1)$.

The directed paths on $\tilde{X}$ are those that project to directed paths in X under the projection map π. Remark that the maps exp and s – and hence π and $\pi_2 \circ S$ – preserve the signed L_1-arc lengths from Section 2.2. Moreover, the L_1-length $l_1^{\pm}(p)$ of a path p in X – cf Section 2.2 – with lift $\tilde{p}$ can be expressed via the d-map $S : \tilde{X} \to X \times \mathbf{R}$ in the pullback diagram as follows:

Lemma 5.1. (1) $l_1^{\pm}(p) = \pi_2(S(\tilde{p}(1))) - \pi_2(S(\tilde{p}(0)))$.
(2) $\tilde{X}$ *has* only trivial *directed loops.*

Proof. (1) This is clearly true locally in any cell as long as start and end point have an L_1-distance less than one. Sum up and cancel!
(2) A directed path in $\tilde{X}$ projects to a directed path in X with positive L_1-length, unless it is constant. Apply (1).

□

Another method to construct this covering is to consider the homotopical length map $\pi_1(X) \xrightarrow{l_1^{\pm}} \mathbf{Z} \to 0$ (cf Proposition 2.2) from the non-directed classical fundamental group of the cubical complex X. Consider the cover $\tilde{X} \downarrow X$ with fundamental group $\pi_1(\tilde{X}) = K \trianglelefteq \pi_1(X)$ the kernel of the homotopical length map. It can be given the structure of a pre-cubical complex, and every element $\mathbf{x}$ in X has lifts $\mathbf{x}^n \in \tilde{X}$, $n \in \mathbf{Z}$. The projection map $\pi : \tilde{X} \downarrow X$ preserves signed L_1-arc length. A path in $\tilde{X}$ is directed if its projection to X is. There are no non-trivial directed loops in $\tilde{X}$ – these need to have L_1-length 0!

Example 5.2. (1) Consider a torus $T = \partial\Delta^1 \times \partial\Delta^1$ as a pre-cubical set consisting of nine 2-cubes. The length cover $\tilde{T}$ can then be modeled as an infinte strip of width 3 with identifications $(x, 3) \sim (x + 3, 0)$:

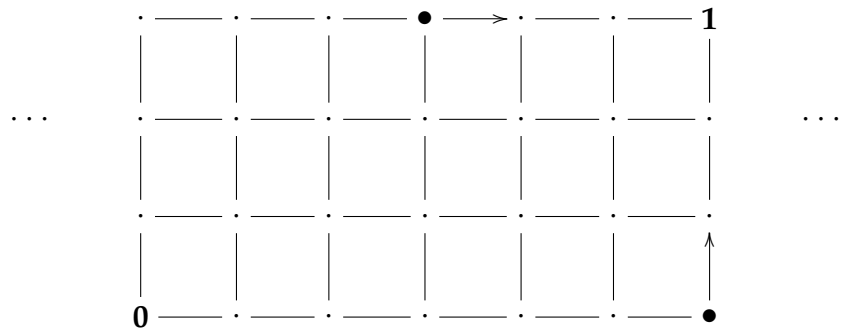

FIGURE 4. Length cover of a torus

The subcomplex in Figure 4 between an initial vertex **0** and a second final vertex **1** at length distance $3n, n > 0$, has exactly one branch vertex $\bullet$ (3 to the left $\sim$ below the final vertex). The algorithm deriving the homotopy type of all d-paths of length $3n$ between these two vertices (consisting of $n+1$ contractible components corresponding to pairs (k, l) of non-negative integers with $k + l = n$) runs through many higher order branch points and removes only few cells at a time.

(2) Now consider the space $\bar{T}$ arising from removing a (middle) cell in T. The length cover of $\bar{T}$ arises from $\tilde{T}$ by removing every third cell (marked X) in the middle strip; cf Figure 5. The lower corner vertices of the removed cells are all branch vertices.

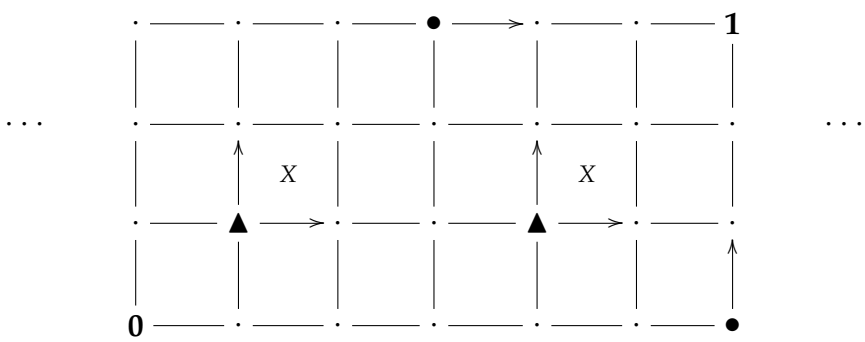

FIGURE 5. Length cover of torus with removed cell

For this space, no higher order branch points arise; the 2^n contractible components of the trace space correspond to the sequences of length n on the letters r, u – right and up.

5.2. **A decomposition of the trace space.** For a general pre-cubical complex X with length cover $\tilde{X}$ we obtain:

Proposition 5.3. *For every pair of points $\mathbf{x}_0, \mathbf{x}_1 \in X$, trace space $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homeomorphic to the disjoint union $\bigsqcup_{n\in\mathbf{Z}} \vec{T}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n)$.*

Proof. An inverse to the projection map $\Pi : \bigsqcup_{n\in\mathbf{Z}} \vec{T}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n) \to \vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ induced by the covering projection $\pi : \tilde{X} \downarrow X$ is given by unique lifts of the directed paths representing traces. Remark that many of the spaces $\vec{T}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n)$ may be empty for specific $n \in \mathbf{Z}$. □

Since the covering $\tilde{X}$ has *only trivial loops*, Proposition 5.3 allows us to apply the methods from Section 4 in order to describe the homotopy type of trace spaces $\vec{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ in an arbitrary pre-cubical complex X. It is of course desirable to exploit periodicity properties in the comparson of spaces $\vec{T}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n)$ for different values of n.

Remark 5.4. Simple semaphore models with loops can be constructed from spaces of the form $X = T^n \setminus F$ with $T^n = (S^1)^n$ an n-torus and F a collection of forbidden hyperrectangles. For such a space, one may consider the (sub)covering

$$\begin{tikzcd} \tilde{X} \arrow[r, hook] \arrow[d] & \mathbf{R}^n \arrow[d, "\exp"] \\ X \arrow[r, hook] & T^n \end{tikzcd}$$

of the universal covering of X – a far bigger gadget. It has the property that (d-)paths, that are not homotopic in the torus T^n, lift to (d-)paths with different end points. The methods from [19] can be applied to $\tilde{X}$ immediately. It is probably easier to get hold on periodicity properties in this setting. This line is currently investigated by several colleagues, cf Fajstrup [7] and Fajstrup etal [8].

6. Implementation issues

6.1. **A directed graph asscioated with a cubical complex.** To a cubical complex X, one may assoicate – forgetting dimensions and the pre-cubical relations – a *directed* graph $\vec{\Gamma}(X)$: the vertices are the cubes in X: $V(\vec{\Gamma}(X)) = \bigcup_n X_n$; to every vertex = cube $c \in X_n$, we associate arcs from c to $\partial_i^1 c$ and from ∂_i^0 to c. The past $\downarrow c \subset X$ is then the union of all *predecessors* of c regarded as a vertex in $\vec{\Gamma}(X)$; likewise, its future $\uparrow c \subset X$ is the union of all *successors* of c. Both can be determined recursively. Moreover, for a set C of cells, $\downarrow C = \bigcup_{c\in C} \downarrow c$.

6.2. **Steps in the determination of a trace complex.** In this section, we collect a few ideas on how to start the design of an algorithm determining the complex $\mathbf{T}(X)(\mathbf{x}_0, \mathbf{x}_1)$ associated to a non-looping pre-cubical complex X and two vertices $\mathbf{x}_0, \mathbf{x}_1 \in X_0$:

- The lower corner $L(c) \in X_0$ of an n-cell $c \in X_n$ can be determined as $L(c) = (\partial_0^0)^n(c)$. Altogether, this recipe defines a map $L : X \to X_0$.

- A *maximal* cell $c \in X$ has *no coface* under ∂^0_*; maximal is to be understood with respect to a lower vertex. The set M of maximal cells is thus of the form $M = \bigcup_n X_n \setminus \bigcup_{0\leq i\leq n} \partial^0_i(X_{n+1})$. The restriction of the map L to M denoted by $L_M : M \to X_0$ associates to a maximal cell c its lower corner vertex in X_0.
- A branch point $v \in X_0$ is characterized by $|L_M^{-1}(v)| > 1$. Given L_M, it is easy to determine the set of branch points $BP \subset X_0$ and the set of branch cells $BC = L^{-1}(BP) \subset M \subset X$.

Using the directed graph $\vec{\Gamma}(X)$ from Section 6.1, one can determine consecutively

- the pasts $\downarrow c_i$ for all branch cells $c_i \in B$;
- the unions $\bigcup_{L(c_i)=v_j} \downarrow c_i$ for every branch point $v_j \in BP$ and their complements $\mathcal{C}(\bigcup_{L(c_i)=v_j} \downarrow c_i) = \bigcap_{L(c_i)=v_j} \mathcal{C}(\downarrow c_i)$; such a complement is a pre-cubical complex since the cells c_i are maximal.
- the pre-cubical complex $X_i = \downarrow c_i \cup \mathcal{C}(\bigcup_{L(c_k)=L(c_i)} \downarrow c_k)$ for every branch cell c_i.

The next step is the investigation of higher order branch points and branch cells:

- A section $s : BP \uparrow BC$ fixes one maximal branch cell c^i_j for every branch point $b_i \in BP$. Form the intersection subcomplexes $X_s \subset X$ via intersections of subgraphs of $\vec{\Gamma}(X)$).
- For each of these subcomplexes as point of departure, iterate to determine second order branch points and branches and associated subcomplexes. Iterate to determine higher order ones.

By recursion, we arrive at the set of *all* – i.e., including higher order – branch points and branch cubes and thus to the objects of the category $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$; moreover, for every such object (S, C) the associated non-branching (!) subcomplex $X_{(S,C)} \subset X$.

To find out whether (S, C) is an object of $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$,we have to investigate whether there is a d-path from $\mathbf{x}_0$ to $\mathbf{x}_1$ in $X_{(S,C)}$. Since this space is non-branching, the future $\uparrow \mathbf{x}_0$ of $\mathbf{x}_0$ within it has a unique maximal element by Lemma 3.2. It is therefore enough to find out whether $\mathbf{x}_1$ is the *only* maximal vertex in $X_{(S,C)}$ or whether there is at least one other maximal vertex $\mathbf{v}$. Such a "deadlock" vertex $\mathbf{v}$ has no arrow with tail $\mathbf{v}$ in $\vec{\Gamma}(X_{(S,C)})$ in that sub-complex.

6.3. **Final comments.** Although each single of the steps to be taken is quite easy to implement, the number of steps can be enormeous. In particular, if higher order branch points arise, the categories $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1) \subseteq \mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ may be huge, even for a cubical complex HDA X of moderate size. As in the semaphore case in Raussen [19, 18], it is enough to determine the *minimal* "dead" objects (S, C) with deadlocks in $X_{(S,C)}$. Still, the determination of the category $\mathcal{C}(X)(\mathbf{x}_0, \mathbf{x}_1)$ describing the homotopy type of the trace space of a pre-cubical complex may need a lot of time and memory.

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