

## Robust Helicopter Stabilization in the Face of Wind Disturbance

A. Danapalasingam, Kumeresan; Leth, John-Josef; la Cour-Harbo, Anders; Bisgaard, Morten

*Published in:*  
I E E E Conference on Decision and Control. Proceedings

*DOI (link to publication from Publisher):*  
[10.1109/CDC.2010.5717418](https://doi.org/10.1109/CDC.2010.5717418)

*Publication date:*  
2010

*Document Version*  
Accepted author manuscript, peer reviewed version

[Link to publication from Aalborg University](#)

*Citation for published version (APA):*  
A. Danapalasingam, K., Leth, J.-J., la Cour-Harbo, A., & Bisgaard, M. (2010). Robust Helicopter Stabilization in the Face of Wind Disturbance. *I E E E Conference on Decision and Control. Proceedings*, 3832 - 3837.  
<https://doi.org/10.1109/CDC.2010.5717418>

### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal -

### Take down policy

If you believe that this document breaches copyright please contact us at [vbn@aub.aau.dk](mailto:vbn@aub.aau.dk) providing details, and we will remove access to the work immediately and investigate your claim.

# Robust Helicopter Stabilization in the Face of Wind Disturbance

Kumeresan A. Danapalasingam, John-Josef Leth, Anders la Cour-Harbo and Morten Bisgaard

**Abstract**—When a helicopter is required to hover with minimum deviations from a desired position without measurements of an affecting persistent wind disturbance, a robustly stabilizing control action is vital. In this paper, the stabilization of the position and translational velocity of a nonlinear helicopter model affected by a wind disturbance is addressed. The wind disturbance is assumed to be a sum of a fixed number of sinusoids with unknown amplitudes, frequencies and phases. An estimate of the disturbance is introduced to be adapted using state measurements for control purposes. A nonlinear controller is then designed based on nonlinear adaptive output regulations and robust stabilization of a chain of integrators by a saturated feedback. Simulation results show the effectiveness of the control design in the stabilization of helicopter motion and the built-in robustness of the controller in handling parameter and model uncertainties.

## I. INTRODUCTION

Autonomous helicopters are highly agile and has six degree of freedom maneuverability making them a favourite candidate for a wide range of practical applications including agriculture, cinematography, inspection, surveillance, search and rescue, reconnaissance, etc. For certain tasks the ability of a helicopter to follow a given state reference is crucial for a successful outcome; for instance when hovering over a ship for rescue operations or when flying close to power lines or wind turbines for inspections. In windy conditions this becomes a significant challenge for any pilot and hence an autopilot capable of accounting effectively for the wind disturbance is a realistic alternative. In this work, the authors present a control design for longitudinal, lateral, and vertical helicopter stabilization in the presence of a wind disturbance, with intrinsic robustness property in handling parameter and model uncertainties.

Firstly, some previous works are reviewed. In [10], a feedback-feedforward proportional differential (PD) controller is developed for heave motion control. With the assistance of a gust estimator, the controller is reported to be able to handle the influence from horizontal gusts. The effects of rotary gusts in forward and downward velocity of a helicopter is addressed in [6]. It is shown that via a state feedback law, the rotary gust rejection problem is always solvable. In a work by Wang et al., a multi-mode linear control strategy is designed for unmanned helicopters in the presence of model uncertainties, atmospheric disturbances and handling qualities requirements [9].

This work is supported by Universiti Teknologi Malaysia and Ministry of Higher Education Malaysia.

All the authors are currently with Section for Automation & Control, Department of Electronic Systems, Aalborg University, Denmark. {kad, jjl, alc, bisgaard}@es.aau.dk

In the present work, assuming that all the state variables (position, attitude and derivatives hereof) are available for measurements, a control strategy combining nonlinear adaptive regulation and robust stabilization of a chain of integrators by a saturated feedback is carried out (see for instance, [1], [2] and [3]). Guided by a control solution for vertical trajectory tracking presented in [3] and [4], the design technique presented therein is extended here where a robust longitudinal, lateral and vertical stabilizer capable of compensating for parameter and model uncertainties is developed for helicopter stabilization in the presence of wind disturbance in all three axes.

In the next section, mathematical model governing a model helicopter and a problem statement are given. This is followed by Section III which cover the vertical, longitudinal and lateral stabilization. After the presentation of simulation results in Section IV, conclusion and future works are discussed in Section V.

## II. PRELIMINARIES

In this section, mathematical model of a helicopter will be described after which the problem statement will be given.

### A. Helicopter Model

The motion of the center of mass of a helicopter is expressed in an inertial coordinate frame  $F_i$  as

$$M\ddot{p}^i = Rf^b,$$

where  $M$  is the mass and  $p^i = [x, y, z]^T \in \mathbb{R}^3$  is the position of the center of mass of the helicopter with respect to the origin of  $F_i$ . The rotation matrix  $R$  is parametrized in terms of unit quaternions  $\mathbf{q} = (q_0, \mathbf{q}) \in S_4$  where  $q_0$  and  $\mathbf{q} = [q_1, q_2, q_3]^T$  denote the scalar and the vector parts of the quaternion respectively (see Appendix A.2 in [3]). With the overall control inputs given by main and tail rotor thrusts,  $T_M$  and  $T_T$  respectively, and longitudinal and lateral main rotor tip path plane tilt angles,  $a$  and  $b$  respectively, a simplified resultant external force  $f^b$  in a body-fixed coordinate frame  $F_b$  is taken as

$$f^b = \begin{bmatrix} 0 \\ 0 \\ -T_M \end{bmatrix} + R^T \begin{bmatrix} 0 \\ 0 \\ Mg \end{bmatrix} + R^T \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}, \quad (1)$$

where  $d_x$ ,  $d_y$ ,  $d_z$  are wind disturbances that affect the helicopter motion in  $x$ ,  $y$  and  $z$  axis respectively. Assuming that the tilt angles  $a$  and  $b$  are small, the following equations of motion can be derived,

$$\ddot{x} = \frac{-(2q_1q_3 + 2q_0q_2)T_M}{M} + \frac{d_x}{M} \quad (2)$$

$$\begin{aligned}\ddot{y} &= \frac{-(2q_2q_3 - 2q_0q_1)T_M}{M} + \frac{d_y}{M} \\ \ddot{z} &= \frac{-(1 - 2q_1^2 - 2q_2^2)T_M}{M} + g + \frac{d_z}{M}.\end{aligned}\quad (3)$$

Also,

$$J\dot{\omega}^b = -S(\omega^b)J\omega^b + \tau^b, \quad \dot{\mathbf{q}} = \frac{1}{2} \begin{bmatrix} -q^T \\ q_0I + S(q) \end{bmatrix} \omega^b,$$

where  $S(\cdot)$  is a skew symmetric matrix,  $\omega^b \in \mathbb{R}^3$  represents the angular velocity in  $F_b$  and  $J$  is the inertia matrix. The external torques  $\tau^b$  expressed in  $F_b$  are given by the following equation

$$\tau^b = \begin{bmatrix} \tau_{f1} \\ \tau_{f2} \\ \tau_{f3} \end{bmatrix} + \begin{bmatrix} R_M \\ M_M + M_T \\ N_M \end{bmatrix}, \quad (5)$$

where  $\tau_{f1}$ ,  $\tau_{f2}$ ,  $\tau_{f3}$  are moments generated by the main and tail rotors and  $R_M$ ,  $M_M$ ,  $N_M$ ,  $M_T$  are moments of the aerodynamic forces [5]. With some approximations a compact torque equation is obtained,

$$\tau^b = A(T_M)\mathbf{v} + B(T_M), \quad \mathbf{v} := [a, b, T_T]^T, \quad (6)$$

where  $A(T_M)$  and  $B(T_M)$  are a matrix and a vector whose entries are functions of  $T_M$  and the above mentioned parameters. Since one of the objectives of the controller to be designed is to handle parameter uncertainties, all nominal values of the helicopter parameters are collected in  $\mu_0$  with its additive uncertainties  $\mu_\Delta$  ranging in a compact set. Also,  $M = M_0 + M_\Delta$ ,  $J = J_0 + J_\Delta$ ,  $A(T_M) = A_0(T_M) + A_\Delta(T_M)$  and  $B(T_M) = B_0(T_M) + B_\Delta(T_M)$ .

### B. Problem Statement

The objective of the controller is to stabilize a helicopter affected by a disturbance  $d = [d_x, d_y, d_z]^T$  in the presence of parameter and model uncertainties. The disturbance that affects the acceleration of the helicopter is modeled to be in the following form,

$$d_j = \sum_{i=1}^N A_{ji} \cos(\Omega_i t + \varphi_{ji}),$$

with unknown amplitude  $A_{ji}$ , phase  $\varphi_{ji}$  and frequency  $\Omega_i$ , for  $j = x, y, z$  and  $i = 1, \dots, N$ . It can be shown that the disturbance is generated by the following linear time-invariant exosystem,

$$\begin{aligned}\dot{w}_j &= S(\varrho)w_j \\ d_j &= RS^2(\varrho)w_j, \quad j = x, y, z,\end{aligned}\quad (7)$$

where  $w_j \in \mathbb{R}^{2N}$ ,  $\varrho = [\Omega_1, \dots, \Omega_N]^T$ ,  $S(\varrho) = \text{diag}(H(\Omega_1), \dots, H(\Omega_N))$  with

$$H(\Omega_i) = \begin{bmatrix} 0 & \Omega_i \\ -\Omega_i & 0 \end{bmatrix}, \quad i = 1, \dots, N,$$

and  $R = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \end{bmatrix}$  is a  $1 \times 2N$  matrix (see [3], pp. 89-90). It is assumed that the frequencies  $\Omega_i$  belong to a compact set. To note, the initial conditions  $w_j(0)$  of the exosystems

represent the amplitudes and phases of the disturbance. In the next section, it will be clear how the representation of the disturbance in such a form can be advantageous in the development of an internal model for stabilizing control input generation.

## III. CONTROLLER DESIGN

The control design is divided into two parts. In the first part, a stabilizing  $T_M$  is constructed for the vertical dynamics. This is then used together with virtual controls  $q_1$  and  $q_2$  for longitudinal and lateral stabilization.

### A. Stabilization of Vertical Dynamics

With reference to the vertical dynamics (4) given above, to counter the nominal effect of the gravity, the following preliminary control law is chosen,

$$T_M = \frac{gM_0 + u}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}, \quad (8)$$

where  $\text{sat}_c(s) := \text{sign}(s) \min\{|s|, c\}$  is a saturation function with  $0 < c < 1$ . This yields

$$\ddot{z} = -\frac{\psi_c^z(q)u}{M} + g\left(1 - \frac{M_0}{M}\psi_c^z(q)\right) + \frac{d_z}{M},$$

where

$$\psi_c^z(q) = \frac{1 - 2q_1^2 - 2q_2^2}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}$$

and  $u$  is an input to be designed.

If  $q(t)$  is small enough such that  $\psi_c^z(q) = 1$ , then a *feedforward*  $u$  needed for vertical stabilization is

$$c_u(w_z, \mu, \varrho) = gM_\Delta + d_z.$$

Note that if in steady-state the control  $c_u(w_z, \mu, \varrho)$  is achieved, zero  $z$  acceleration will be produced. Since  $c_u(w_z, \mu, \varrho)$  depends on  $M_\Delta \in \mu$  and  $d_z$  which is a function of  $\varrho$  and  $w_z(0)$ , it can be only implemented with an internal model. Hence, the desired control is rewritten as an output of a linear system

$$\begin{aligned}\frac{\partial \tau_z}{\partial w_z} S(\varrho)w &= \Phi(\varrho)\tau_z(w_z, \mu) \\ c_u(w_z, \mu, \varrho) &= \Gamma(\varrho)\tau_z(w_z, \mu),\end{aligned}\quad (9)$$

where

$$\tau_z(w_z, \mu) = \begin{bmatrix} gM_\Delta \\ w_z \end{bmatrix}, \quad \Phi(\varrho) = \begin{bmatrix} 0 & 0 \\ 0 & S(\varrho) \end{bmatrix},$$

$$\Gamma(\varrho) = [1 \quad \Gamma_2(\varrho)],$$

and

$$\Gamma_2(\varrho) = [-\Omega_1^2 \quad 0 \quad -\Omega_2^2 \quad 0 \quad \dots \quad -\Omega_N^2 \quad 0].$$

It has been shown in Lemma 3.3.1 in [3] that there exist controllable matrix pair  $(F, G)$  with dimensions  $(2N+1) \times (2N+1)$  and  $(2N+1) \times 1$  respectively, a unique  $1 \times (2N+1)$  vector  $\Psi_\varrho = [1 \quad \Psi_{2,\varrho}]$  and a  $(2N+1) \times (2N+1)$  nonsingular matrix  $T_\varrho$  such that  $\Phi(\varrho) = T_\varrho^{-1}(F+G\Psi_\varrho)T_\varrho$ ,  $\Gamma(\varrho) = \Psi_\varrho T_\varrho$

and

$$\begin{aligned}\frac{\partial \bar{\tau}_z}{\partial w_z} S(\varrho) w &= (F + G\Psi_\varrho) \bar{\tau}_z(w_z, \mu, \varrho) \\ c_u(w_z, \mu, \varrho) &= \Psi_\varrho \bar{\tau}_z(w_z, \mu, \varrho),\end{aligned}\quad (10)$$

where  $\bar{\tau}_z(w_z, \mu, \varrho) = T_\varrho \tau_z(w_z, \mu)$ .

Because  $c_u(w_z, \mu, \varrho)$  depends on unknown parameters and unmeasurable state  $w_z$ , the control is chosen as  $u = u_{\text{im}} + u_{\text{st}}$  which consists of an internal model control  $u_{\text{im}}$  and a stabilizing control  $u_{\text{st}}$ . Thus the controller can be taken as

$$\begin{aligned}\dot{\xi}_z &= (F + G\hat{\Psi})\xi_z + g_{\text{stz}} \\ u &= \hat{\Psi}\xi_z + u_{\text{st}},\end{aligned}\quad (11)$$

with the update law

$$\dot{\hat{\Psi}}_2 = \gamma \xi_{z2}^T (\dot{z} + k_1 z) - [\text{tas}_d(\hat{\Psi}_2)]^\top$$

where  $g_{\text{stz}} = G u_{\text{st}} + F G M_0 \dot{z}$ ,  $u_{\text{st}} = k_2(\dot{z} + k_1 z)$  for  $k_1, k_2 > 0$ ,  $\xi_z = [\xi_{z1} \ \xi_{z2}]^\top$  with  $\xi_{z1} \in \mathbb{R}$ ,  $\xi_{z2} \in \mathbb{R}^{2N}$ ,  $\hat{\Psi} = [1 \ \hat{\Psi}_2]$  with a  $1 \times 2N$  row vector  $\hat{\Psi}_2$ ,  $\gamma > 0$  and  $\text{tas}_d(\cdot)$  is a dead-zone function with  $d > \max_{i=1}^{2N} (|(\Psi_{2,\varrho})_i|)$ . Defining

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \hat{\Psi}_2^T \end{bmatrix}, \text{ with } \eta_1 = \begin{bmatrix} \chi \\ z \\ \zeta \end{bmatrix},$$

where the change of coordinates are set as

$$\begin{aligned}\chi &:= \xi_z - \bar{\tau}_z(w_z, \mu, \varrho) + G M \dot{z} \\ \hat{\Psi}_2 &:= \hat{\Psi}_2 - \Psi_{2,\varrho} \\ \zeta &:= \dot{z} + k_1 z,\end{aligned}$$

the following equations are obtained,

$$\begin{aligned}\dot{\eta}_1 &= \left( A + A_1(\psi_c^z(q) - 1) \right) \eta_1 \\ &\quad - \left( \frac{1}{M} b + B(\psi_c^z(q) - 1) \right) \xi_{z2}^\top \eta_2 - B \rho \\ \dot{\eta}_2 &= \gamma \xi_{x2} b^T \eta_1 - \text{tas}_d(\eta_2 + \Psi_{2,\varrho}),\end{aligned}\quad (12)$$

where  $\rho = (\psi_c^z(q) - 1)(g M_0 + \Psi_\varrho \bar{\tau}_z(w, \mu, \varrho))$ ,

$$\begin{aligned}A &= \begin{bmatrix} F & k_1 F G M_\Delta & -F G M_\Delta \\ 0 & -k_1 & 1 \\ -\frac{1}{M} \Psi_\varrho & -k_1(\Psi_\varrho G + k_1) & (\Psi_\varrho G + k_1 - \frac{1}{M} k_2) \end{bmatrix}, \\ A_1 &= \begin{bmatrix} -G \Psi_\varrho & -G \Psi_\varrho G M k_1 & G \Psi_\varrho G M - G k_2 \\ 0 & 0 & 0 \\ -\frac{1}{M} \Psi_\varrho & -k_1 \Psi_\varrho G & (\Psi_\varrho G - \frac{1}{M} k_2) \end{bmatrix}, \\ b &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} G \\ 0 \\ \frac{1}{M} \end{bmatrix}.\end{aligned}$$

Using the same arguments as in Proposition 3.3.2 in [3], it can be shown that there exists  $k_2^* > 0$  such that if  $k_2 > k_2^*$ ,  $A$  is a Hurwitz matrix. Subsequently when  $q$  is small enough such that  $\psi_c^z(q) = 1$ , Proposition 3.4.2 guarantees that if the initial states of the exosystems belongs to a compact set  $\mathcal{W}$  defined therein, then system (12) is globally asymptotically and locally exponentially stable. Note that even though the

system equations are different than that of in [3] due to the addition of a disturbance and a different vertical control, the two propositions still apply.

It is only natural now to ensure that the condition  $\psi_c^z(q) = 1$  can be achieved in finite time. Setting

$$\mathbf{v} = A_0(T_M)^{-1}(\tilde{\mathbf{v}} - B_0(T_M))$$

yields the torque equation

$$\tau^b(\tilde{\mathbf{v}}) = L(T_M)\tilde{\mathbf{v}} + \Delta(T_M),$$

where  $\tilde{\mathbf{v}}$  is an additional control input to be determined,  $L(T_M) = I + A_\Delta(T_M)A_0^{-1}(T_M)$  and  $\Delta(T_M) = B_\Delta(T_M) - A_\Delta(T_M)A_0^{-1}(T_M)B_0(T_M)$ . Dropping the superscript  $b$  in  $\omega^b$ , choose

$$\tilde{\mathbf{v}} = K_P(\eta_1 - K_D(\omega - \omega_d)),$$

where  $K_P, K_D > 0$  are design parameters,

$$\eta_1 := q_r - q \quad \text{and} \quad (13)$$

$$q_r = q^* + q_d. \quad (14)$$

The expressions for the desired angular velocity  $\omega_d$  and desired quaternions  $q_r$  will be given in the next subsection. In Appendix A, the existence of a Lyapunov function for the attitude dynamical system is shown. This condition is a key element in the proof of Proposition 5.7.1 in [3]. Consequently for the problem in hand, it could be stated that for any  $T^* > 0$ , there exist  $K_D^* > 0$ ,  $\lambda_3^*(K_D) > 0$ ,  $K_d^*(K_D) > 0$  and  $K_P^*(K_D) > 0$ , such that for all positive  $K_D \leq K_D^*$ ,  $\lambda_3 \leq \lambda_3^*(K_D)$ ,  $K_d \leq K_d^*(K_D)$  and  $K_P \geq K_P^*(K_D)$ , the trajectory of the attitude dynamical system is bounded,  $q_0(t) > \varepsilon$  for arbitrary  $0 < \varepsilon < 1$ ,  $t > 0$  and  $\psi_c^z(q(t)) = 1$  for all  $t \geq T^*$ , with initial conditions  $(q(0), \omega(0)) \in \mathcal{Q} \times \Omega$ , where  $\mathcal{Q}$  and  $\Omega$  are compact sets (cf. Prop. 5.7.1 and see definition of  $\mathcal{Q}$  in [3]).

## B. Stabilization of Longitudinal and Lateral Dynamics

Now, we will show that with an appropriate selection of design parameters, virtual controls  $q_1$  and  $q_2$  can be manipulated to produce horizontal stability. The control law (8) that has been designed for vertical stabilization also appears in the longitudinal (2) and lateral (3) dynamics. The numerator of (8) can be expanded as

$$g M_0 + u = g M + d_z + y_{\mathbf{z}}(\mathbf{z}, w),$$

where  $y_{\mathbf{z}}(\mathbf{z}, w) = \tilde{\Psi} \bar{\tau}_z(w, \mu, \varrho) + (\tilde{\Psi} + \Psi_\varrho)(\chi - G M \dot{z}) + k_2(\dot{z} + k_1 z)$  and  $\mathbf{z} = \eta$ . Subsequently, (2) and (3) can be written as

$$\begin{aligned}\ddot{x} &= \frac{-\tilde{d}(\mathbf{q}, t)q_2 + m(\mathbf{q}, t)q_1q_3 + n_x(\mathbf{q})y_{\mathbf{z}}(\mathbf{z}, w)}{M} + \frac{d_x}{M} \\ \ddot{y} &= \frac{\tilde{d}(\mathbf{q}, t)q_1 + m(\mathbf{q}, t)q_2q_3 + n_y(\mathbf{q})y_{\mathbf{z}}(\mathbf{z}, w)}{M} + \frac{d_y}{M},\end{aligned}$$

where

$$\tilde{d}(\mathbf{q}, t) = \frac{2(g M + d_z)q_0}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)},$$

$$\begin{aligned}
m(\mathbf{q}, t) &= -\frac{2(gM + d_z)}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}, \\
n_x(\mathbf{q}) &= \frac{-(2q_1q_3 + 2q_0q_2)}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}, \\
n_y(\mathbf{q}) &= \frac{-(2q_2q_3 - 2q_0q_1)}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}.
\end{aligned}$$

Introducing the bank of integrators  $\dot{\eta}_x = x$ ,  $\dot{\eta}_y = y$  and  $\dot{\eta}_q = q_3$ , the following new state variables are defined

$$\begin{aligned}
\zeta_1 &:= \begin{bmatrix} \eta_y \\ \eta_x \end{bmatrix} \\
\zeta_2 &:= \begin{bmatrix} y \\ x \end{bmatrix} + \lambda_1 \sigma\left(\frac{K_1}{\lambda_1} \zeta_1\right) \\
\zeta_3 &:= \begin{bmatrix} \dot{y} \\ \dot{x} \\ \eta_q \end{bmatrix} + P_1 \lambda_2 \sigma\left(\frac{K_2}{\lambda_2} \zeta_2\right),
\end{aligned}$$

where

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By adopting the following 'nested saturated' control law

$$q^* = -P_2 \lambda_3 \sigma\left(\frac{K_3}{\lambda_3} \zeta_3\right), \quad (15)$$

where the function  $\sigma(\cdot)$  is a vector-valued saturation function of suitable dimension and,  $K_i$  and  $\lambda_i$ ,  $i = 1, 2, 3$  are design parameters, the time derivatives can be written as

$$\begin{aligned}
\dot{\zeta}_1 &= -\lambda_1 \sigma\left(\frac{K_1}{\lambda_1} \zeta_1\right) + \zeta_2 \\
\dot{\zeta}_2 &= -\lambda_2 \sigma\left(\frac{K_2}{\lambda_2} \zeta_2\right) + P_0 \zeta_3 + K_1 \sigma'\left(\frac{K_1}{\lambda_1} \zeta_1\right) \dot{\zeta}_1 \\
M \dot{\zeta}_3 &= -\tilde{D}(t) P_2 \lambda_3 \sigma\left(\frac{K_3}{\lambda_3} \zeta_3\right) + \tilde{D}(t) q_d \\
&\quad + M K_2 P_1 \sigma'\left(\frac{K_2}{\lambda_2} \zeta_2\right) \dot{\zeta}_2 - \tilde{D}(t) \eta_1 + p + d,
\end{aligned} \quad (16)$$

where

$$\begin{aligned}
\tilde{D}(t) &= \begin{bmatrix} \tilde{d}(\mathbf{q}, t) & m(\mathbf{q}, t) q_3 & 0 \\ m(\mathbf{q}, t) q_3 & -\tilde{d}(\mathbf{q}, t) & 0 \\ 0 & 0 & M \end{bmatrix}, \quad d = \begin{bmatrix} d_y \\ d_x \\ 0 \end{bmatrix} \\
\text{and } p &= \begin{bmatrix} n_y(\mathbf{q}) y_z(\mathbf{z}, w) \\ n_x(\mathbf{q}) y_z(\mathbf{z}, w) \\ 0 \end{bmatrix}.
\end{aligned}$$

Note that if one can set  $q_d = -\tilde{D}^{-1}(t)d$ , the disturbance  $d$  can be completely eliminated from subsystem (16) and shown to be *input-to-state stable* (ISS) with restrictions on the inputs  $(\eta_1, p)$  and linear asymptotic gains (see Lem. 5.7.4 in [3]). However, since  $\tilde{D}$  is a function of uncertain  $M$  and unknown  $d$ , it is not entirely possible to have such a  $q_d$ . As a result, the following is proposed,

$$q_d = -K_d \tilde{D}_0^{-1}(t) \hat{d}, \quad (17)$$

where  $K_d > 0$  is another design parameter,  $\hat{d} = [\hat{d}_y, \hat{d}_x, 0]^\top$  is a disturbance estimate to be adapted and

$$\tilde{D}_0^{-1}(t) = \begin{bmatrix} \tilde{d}_0(\mathbf{q}, t) & m_0(\mathbf{q}, t) q_3 & 0 \\ m_0(\mathbf{q}, t) q_3 & -\tilde{d}_0(\mathbf{q}, t) & 0 \\ 0 & 0 & \frac{1}{M_0} \end{bmatrix}, \quad (18)$$

with

$$\begin{aligned}
\tilde{d}_0(\mathbf{q}, t) &= \frac{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}{2(q_0^2 + q_3^2)(gM_0 + \hat{\Psi}_2 \xi_{z2})}, \\
m_0(\mathbf{q}, t) &= -\frac{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}{2(q_0^2 + q_3^2)(gM_0 + \hat{\Psi}_2 \xi_{z2})}.
\end{aligned}$$

It is important to notice at this point that a constraint on  $d_z$  should be imposed. To have  $T_M > 0$  (see (8) and (9)) and to avoid singularities in (18), it is required that  $|d_z(t)| < gM$  for all  $t > 0$ . Note that from (7), (9), (10) and (11),  $\hat{d}$  can be taken as

$$\hat{d} = K_d P e_\xi + P \bar{\tau} + \tilde{P}_2 \xi_2, \quad (19)$$

where

$$P = \begin{bmatrix} \Psi_\varrho & 0 \\ 0 & \tilde{\Psi}_\varrho \\ 0 & 0 \end{bmatrix}, \quad \tilde{P}_2 = \begin{bmatrix} \tilde{\Psi}_2 & 0 \\ 0 & \tilde{\Psi}_2 \\ 0 & 0 \end{bmatrix}, \quad e_\xi = \begin{bmatrix} e_{\xi y} \\ e_{\xi x} \end{bmatrix},$$

$$\bar{\tau} = \begin{bmatrix} \bar{\tau}_y \\ \bar{\tau}_x \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} \xi_{y2} \\ \xi_{x2} \end{bmatrix},$$

and

$$e_{\xi i} = \frac{\xi_i - \bar{\tau}_i}{K_d}, \quad (20)$$

with  $\xi_i = [\xi_{i1}, \xi_{i2}]^\top$ , for  $i = x, y$ . To note,  $e_\xi$  is governed by

$$\dot{e}_\xi = K_d \tilde{F} e_\xi + \tilde{G} \tilde{P}_2 \xi_2 + g_{\text{st}}, \quad (21)$$

where

$$\tilde{F} = \begin{bmatrix} F + G \Psi_\varrho & 0 \\ 0 & F + G \Psi_\varrho \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}$$

and

$$g_{\text{st}} = k_4 \tilde{G} \begin{bmatrix} \dot{y} \\ \dot{x} \end{bmatrix} + k_4 k_3 \tilde{G} \begin{bmatrix} y \\ x \end{bmatrix}, \quad k_3, k_4 > 0. \quad (22)$$

Now in addition to (13), set

$$\begin{aligned}
\eta_2 &:= \omega - \omega_d - \frac{1}{K_D} \eta_1 \\
\eta_3 &:= P_3 \eta_1 - e_\xi,
\end{aligned}$$

where

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

is a  $(4N + 2) \times 3$  matrix and

$$\omega_d = Q_d \dot{q}_d \quad (23)$$

with

$$Q_d = \frac{2}{\varepsilon} \begin{bmatrix} \varepsilon^2 + q_{d1}^2 & q_{d1}q_{d2} & -\varepsilon q_{d2} \\ q_{d1}q_{d2} & \varepsilon^2 + q_{d2}^2 & \varepsilon q_{d1} \\ \varepsilon q_{d2} & -\varepsilon q_{d1} & \varepsilon^2 \end{bmatrix},$$

keeping in mind that  $q_d = [q_{d1}, q_{d2}]^\top$ . Taking the time derivatives,

$$\begin{aligned} \dot{\eta}_1 &= -\frac{1}{2}(q_0 I + S(q))(\eta_2 + \frac{1}{K_D}\eta_1 + \omega_d) + \dot{q}_r \\ J\dot{\eta}_2 &= -S(\omega)J(\eta_2 + \frac{1}{K_D}\eta_1 + \omega_d) - K_P K_D L(T_M)\eta_2 \\ &\quad - \frac{1}{K_D}J\dot{\eta}_1 + \Delta(T_M) - J\dot{\omega}_d \\ \dot{\eta}_3 &= P_3\dot{\eta}_1 - K_d \tilde{F}e_\xi - \tilde{G}\tilde{F}_2\xi_2 - g_{st}. \end{aligned} \quad (24)$$

We will now study the stability of feedback interconnection of subsystems (16) and (24). Subsystem (16) is a system with state  $(\zeta_1, \zeta_2, \zeta_3)$  and inputs  $(\eta_1, \eta_3, p, d, \bar{\tau})$ , while subsystem (24) is a system with state  $(\eta_1, \eta_2, \eta_3)$  and inputs  $(y_{\zeta 0}, y_{\zeta 1}, I_{\eta 1}, I_{\eta 2}, I_{\eta 3})$ . Let  $M^L$ ,  $M^U$  and  $d^L$ ,  $d^U$  be such that for all  $t > 0$ ,  $M^L \leq M \leq M^U$  and  $0 < d^L \leq \tilde{d}(\mathbf{q}, t) \leq d^U$  respectively. Additionally, assume that  $q_0(t) > \varepsilon > 0$ ,  $\|\omega\| \leq m_\omega$ ,  $\|\omega_d\| \leq m_{\omega_d}$ ,  $\|D_0^{-1}\hat{d}\| \leq m_{q_d}$ ,  $\|Q_d\| \leq m_{Q_d}$  and  $\|e_\xi\| \leq m_{e_\xi}$  for all  $t > 0$  and some positive  $m_\omega$ ,  $m_{\omega_d}$ ,  $m_{q_d}$ ,  $m_{Q_d}$  and  $m_{e_\xi}$ . With that in mind, the following proposition is stated.

*Proposition 1:* Let  $K_D$  be fixed and let  $K_i^*$  and  $\lambda_i^*$ ,  $i = 1, 2, 3$ , be such that the following inequalities are satisfied

$$\begin{aligned} \frac{\lambda_2^*}{K_2^*} &< \frac{\lambda_1^*}{4}, \quad \frac{\lambda_3^*}{K_3^*} < \frac{\lambda_2^*}{4}, \quad 4K_1^*\lambda_1^* < \frac{\lambda_2^*}{4}, \\ 4K_2^*\lambda_2^* &< \frac{d^L}{M^U} \frac{\lambda_3^*}{8}, \quad 24\frac{K_1^*}{K_2^*} < \frac{1}{6} \\ \text{and } 24\frac{K_2^*}{K_3^*} &< \frac{1}{6} \frac{d^L}{d^U} \frac{M^L}{M^U}. \end{aligned}$$

Moreover let  $R_{\eta 1}$ ,  $R_{\eta 2}$  and  $R_{\eta 3}$  be arbitrary positive numbers. Then, there exist positive numbers  $K_P^*$ ,  $K_d^*$ ,  $\epsilon^*$ ,  $R_{\zeta 1}$ ,  $R_{\zeta 2}$ ,  $R_{\zeta 3}$ ,  $\gamma_{\zeta 1}$ ,  $\gamma_{\zeta 2}$ ,  $\gamma_{\zeta 3}$ ,  $\gamma_{\eta 1}$ ,  $\gamma_{\eta 2}$  and  $\gamma_{\eta 3}$  such that, taking  $\lambda_i = \epsilon^{i-1}\lambda_i^*$  and  $K_i = \epsilon K_i^*$ ,  $i = 1, 2, 3$ , for all  $K_P \geq K_P^*$ ,  $K_d \leq K_d^*$  and  $0 < \epsilon \leq \epsilon^*$ , the feedback interconnection of subsystems (16) and (24) is ISS without restrictions on the initial state, restrictions  $(\epsilon^2 R_{\zeta 1}, \epsilon^2 R_{\zeta 2}, \epsilon^2 R_{\zeta 3}, R_{\eta 2}, R_{\eta 1}, R_{\eta 3})$  on the inputs  $(p, d, \bar{\tau}, I_{\eta 1}, I_{\eta 2}, I_{\eta 3})$  and linear asymptotic gains; in particular, if  $\|p\|_\infty < \epsilon^2 R_{\zeta 1}$ ,  $\|d\|_\infty < \epsilon^2 R_{\zeta 2}$ ,  $\|\bar{\tau}\|_\infty < \epsilon^2 R_{\zeta 3}$ ,  $\|I_{\eta 1}\|_\infty < R_{\eta 1}$ ,  $\|I_{\eta 2}\|_\infty < R_{\eta 2}$  and  $\|I_{\eta 3}\|_\infty < R_{\eta 3}$ , then  $(\zeta_1(t), \zeta_2(t), \zeta_3(t), \eta_1(t), \eta_2(t), \eta_3(t))$  satisfies the following asymptotic bound

$$\begin{aligned} &\|(\zeta_1, \zeta_2, \zeta_3, \eta_1, \eta_2, \eta_3)\|_a \leq \\ &\max \{ \gamma_{\zeta 1} \|p\|_a, \gamma_{\zeta 2} \|d\|_a, \gamma_{\zeta 3} \|\bar{\tau}\|_a, \\ &\gamma_{\eta 1} \|I_{\eta 1}\|_a, \frac{\gamma_{\eta 2}}{K_P} \|I_{\eta 2}\|_a, \gamma_{\eta 3} \|I_{\eta 3}\|_a \}, \end{aligned}$$

where  $\|\cdot\|_\infty$  and  $\|\cdot\|_a$  denote the  $\mathcal{L}_\infty$  norm and asymptotic  $\mathcal{L}_\infty$  norm respectively [8].

*Proof:* The proof of Proposition 1 involves showing that the two subsystems are ISS separately, and that the composed system satisfies the *small gain theorem* (see for instance, [2]). In this paper however, only subsystem (24) is shown to be ISS and hence the following lemma is presented.

*Lemma 1:* Let  $K_D$  be fixed and assume that  $q_0(t) > \varepsilon > 0$  and  $L(T_M(t)) \geq l_1$ , for all  $t \geq 0$ . There exist positive numbers  $\bar{\lambda}_3^*$ ,  $\bar{K}_3^*(K_D)$ ,  $\bar{K}_P^*(K_D)$ ,  $\bar{K}_d^*(K_D, K_3)$  and  $r_{\zeta 0}, r_{\zeta 1}, r_{I1}, r_{I2}, r_{I3}$  such that, for all  $\lambda_3 \leq \bar{\lambda}_3^*$ ,  $K_3 \leq \bar{K}_3^*(K_D)$ ,  $K_P \geq \bar{K}_P^*(K_D)$  and  $K_d \leq \bar{K}_d^*(K_D, K_3)$ , system (24) is ISS, without restrictions on the initial state and on the inputs  $(y_{\zeta 0}, y_{\zeta 1}, I_{\eta 1}, I_{\eta 2}, I_{\eta 3})$  and with linear asymptotic gains; in particular, for all bounded inputs  $y_{\zeta 0}, y_{\zeta 1}, I_{\eta 1}, I_{\eta 2}$  and  $I_{\eta 3}$ , the state  $(\eta_1, \eta_2, \eta_3)$  satisfies the asymptotic bound

$$\begin{aligned} \|(\eta_1, \eta_2, \eta_3)\|_a &\leq \max \{ r_{\zeta 0} \|y_{\zeta 0}\|_a, r_{\zeta 1} \|y_{\zeta 1}\|_a, \\ &\quad r_{I1} \|I_{\eta 1}\|_a, \frac{r_{I2}}{K_P} \|I_{\eta 2}\|_a, r_{I3} \|I_{\eta 3}\|_a \}. \end{aligned}$$

*Proof:* See Appendix B.

Note that using the same arguments as in [3] and [4], subsystem (16) can be shown to be ISS without restrictions on the initial state, restrictions on the inputs and with linear asymptotic gains. Subsequently, also based on discussion therein, since the feedback interconnection of subsystems (16) and (24) can be proven to satisfy all the conditions of the small gain theorem, the proof of Proposition 1 is completed.

Recall that earlier in the vertical dynamics stabilization,  $K_P$  is required to be arbitrarily large while  $\lambda_3$  and  $K_d$  are needed to be arbitrarily small, which is also part of the requirements for longitudinal and lateral dynamics stabilization as suggested by Proposition 1. To be noted, only restrictions on inputs  $(p, I_{\eta 1}, I_{\eta 2}, I_{\eta 3})$  can be always fulfilled in finite time as  $p(t)$  is asymptotically vanishing (since  $y_z(\mathbf{z}, w)$  is) and,  $R_{\eta 1}$ ,  $R_{\eta 2}$  and  $R_{\eta 3}$  can be chosen arbitrarily large. Consequently, only for sufficiently small magnitude of disturbance, for any initial conditions  $w(0) \in \mathcal{W}$ ,  $\mathbf{z}(0) \in \mathcal{Z}$ ,  $(x(0), \dot{x}(0), y(0), \dot{y}(0)) \in \mathbb{R}^4$ ,  $(q(0), \omega(0)) \in \mathcal{Q} \times \Omega$  with  $q_0(0) > 0$ , by choosing  $K_P$  and  $K_d$  sufficiently large and small respectively,

$$\begin{aligned} \lim_{t \rightarrow \infty} \|(x(t), \dot{x}(t), y(t), \dot{y}(t))\| &\leq R_S(\|d\|_a), \quad \text{and} \\ \lim_{t \rightarrow \infty} \|(z(t), \dot{z}(t))\| &= 0, \end{aligned}$$

where  $\mathcal{Z}$  is a compact set and  $R_S$  is a class  $\mathcal{K}$  function (see [2] for the definition of a class  $\mathcal{K}$  function).

#### IV. SIMULATION RESULTS

Hover flight of an autonomous helicopter equipped with the proposed autopilot and influenced by a wind disturbance is simulated. The designed controller that makes a stable hover possible can be summarized as follows,

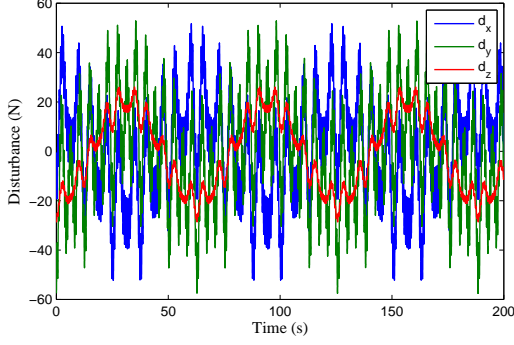


Fig. 1. Wind disturbance

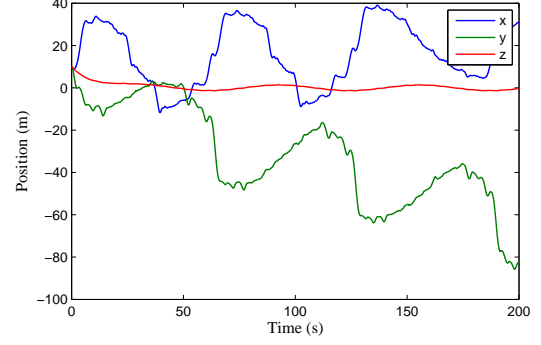


Fig. 2. Position when disturbance adaptation is turned off.

### 1) Vertical dynamics stabilizer

$$\dot{\xi}_z = (F + G\hat{\Psi})\xi_z + k_2G(\dot{z} + k_1z) + FGM_0\dot{z} \quad (25)$$

$$\begin{aligned} \dot{\hat{\Psi}}_2 &= \gamma\xi_{z2}^T(\dot{z} + k_1z) \\ T_M &= \frac{gM_0 + \hat{\Psi}\xi_z + k_2(\dot{z} + k_1z)}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)} \end{aligned} \quad (26)$$

### 2) Longitudinal and lateral dynamics stabilizer

$$\begin{aligned} \dot{\eta}_x &= x \\ \dot{\eta}_y &= y \\ \dot{\eta}_q &= q_3 \\ \dot{\xi}_x &= (F + G\hat{\Psi})\xi_x + k_4G(\dot{x} + k_3x) \end{aligned} \quad (27)$$

$$\dot{\xi}_y = (F + G\hat{\Psi})\xi_y + k_4G(\dot{y} + k_3y) \quad (28)$$

$$\dot{d}_x = \hat{\Psi}\xi_x$$

$$\dot{d}_y = \hat{\Psi}\xi_y$$

$$\mathbf{v} = A_0(T_M)^{-1}(K_P(q_r - q) - K_P K_D(\omega - \omega_d) - B_0(T_M)),$$

where  $\omega_d$  is given by (23) and  $q_r$  is as defined by (14), with  $q^*$  and  $q_d$  given by (15) and (17) respectively.

The simulation results presented here are based on a model of a small autonomous helicopter from [7]. To test the robustness property of the controller, parameter uncertainties are taken up to 30% of the nominal values. Even though the controller is designed based on simplified force and torque equations as described by (1) and (6) respectively, the helicopter model assumes full torque, (5) and full force equations. The wind disturbance shown in Fig. 1 is presented to the helicopter as a persistently acting external force generated by a 8-dimensional neutrally stable exosystem with  $\varrho = (1, 1.5, 0.1, 10)$ ,  $w_x(0) = (20, 1, 4, 0, -1800, 0, -0.1, -0.02)$ ,  $w_y(0) = (10, 2, 10, 2, 1500, 0, 0.1, 0)$  and  $w_z(0) = (5, 0, 1, 0, 2000, 0, 0.01, 0.01)$ . To further challenge the controller, only 4-dimensional internal models (27), (28) and (25) are used. Positions of the helicopter in the face of the wind disturbance without ( $\gamma = 0$ ) and with disturbance adaptation are given in Fig. 2 and 3 respectively.

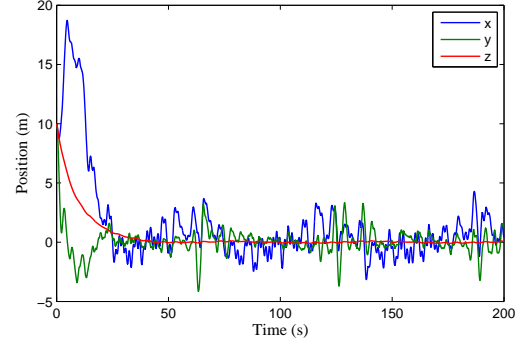


Fig. 3. Position when disturbance adaptation is turned on.

Without disturbance adaptation, while the controller fails to stabilize the  $x$  and  $y$  positions,  $z$  does converge fairly close to zero as could be seen in Fig. 2. Apparently,  $T_M$  is still capable of acting as a vertical stabilizer to a certain degree although the disturbance adaptation is turned off due to the presence of other terms in (26). The importance of information on the disturbance to the longitudinal/lateral stabilizer is demonstrated in Fig. 3. Now that the disturbance adaptation is turned on,  $z$  converges to zero and,  $x$  and  $y$  converge to a small neighbourhood of the origin as guaranteed by Proposition 1.

## V. CONCLUSIONS AND FUTURE WORKS

A robust controller for helicopter stabilization to reject wind disturbance is presented. The wind disturbance affecting the helicopter is assumed to be a function of time of a fixed structure with unknown parameters. By designing an internal model that estimates the disturbance, a control design is carried out for longitudinal, lateral and vertical dynamics stabilization. Despite the presence of helicopter parameter and model uncertainties, simulation results clearly demonstrates the effectiveness of the control technique. As future works, indoor and outdoor flights are to be carried out to test the feasibility of the proposed controller. That gives an immediate challenge caused by the presence of servo dynamics and limitations on wind disturbance that could be handled.

## APPENDIX

### A. Appendix A

Defining  $\tilde{\omega} := \omega + \bar{K}_D q$  where  $\bar{K}_D := \frac{1}{K_D}$ , the chosen control law becomes

$$\tilde{\mathbf{v}} = K_P q_r - \frac{K_P}{\bar{K}_D} \tilde{\omega} + \frac{K_P}{\bar{K}_D} \omega_d.$$

Consider a Lyapunov function candidate

$$V(q_0, \tilde{\omega}) = \frac{1 - q_0}{q_0 - \varepsilon} + \frac{1}{2} \tilde{\omega}^\top J \tilde{\omega}$$

defined on the open set  $(\varepsilon, 1] \times \mathbb{R}^3$ . Thus,

$$\begin{aligned} \dot{V} = & -\frac{(1-\varepsilon)}{2(q_0-\varepsilon)^2} \bar{K}_D \|q\|^2 \\ & + \tilde{\omega}^\top \left( \frac{(1-\varepsilon)}{2(q_0-\varepsilon)^2} I - \bar{K}_D^2 S(q) J - \frac{\bar{K}_D^2}{2} q_0 J \right) q \\ & + \tilde{\omega}^\top \left( \bar{K}_D S(q) J - \frac{K_P}{\bar{K}_D} L(T_M) + \frac{\bar{K}_D}{2} q_0 J \right. \\ & \left. + \frac{\bar{K}_D}{2} J S(q) \right) \tilde{\omega} + K_P \tilde{\omega}^\top L(T_M) q_r \\ & + \frac{K_P}{\bar{K}_D} \tilde{\omega}^\top L(T_M) \omega_d + \tilde{\omega}^\top \Delta(T_M). \end{aligned}$$

It is shown in [3], that a  $T^* > 0$  can be chosen such that if  $\psi_c^z(q(t)) = 1$  for all  $t \geq T^*$ , then  $2l_1 I \leq L(T_M(t)) + L^T(T_M(t))$ ,  $\|L(T_M(t))\| \leq l_2$  and  $\|\Delta(T_M(t))\| \leq \delta$  for all  $t > 0$ . Since  $\|q^*\| \leq \sqrt{3}\lambda_3$ ,  $\|q_d\| \leq K_d m_{q_d}$  and  $\|\omega_d\| \leq K_d m_{\omega_d}$ ,

$$\begin{aligned} \dot{V} \leq & -\bar{K}_D \frac{(1-\varepsilon)}{2(q_0-\varepsilon)^2} \|q\|^2 + (2\bar{K}_D c_2 - \frac{K_P}{\bar{K}_D} l_1) \|\tilde{\omega}\|^2 \\ & + \left( a_3 + \frac{3}{2} \bar{K}_D^2 c_2 + K_P l_2 (\sqrt{3}\lambda_3 + K_d m_{q_d}) + \delta \right. \\ & \left. + \frac{K_P}{\bar{K}_D} K_d l_2 m_{\omega_d} \right) \|\tilde{\omega}\|, \end{aligned}$$

where  $0 < c_1 \leq \|J\| \leq c_2$  and,  $a_1$ ,  $a_2$  and  $a_3$  are as defined in [3]. To show that  $\dot{V}$  can be made negative definite, it is desired to have

$$\begin{aligned} & (2\bar{K}_D c_2 - \frac{K_P}{\bar{K}_D} l_1) \|\tilde{\omega}\|^2 + \left( a_3 + \frac{3}{2} \bar{K}_D^2 c_2 \right. \\ & \left. + K_P l_2 (\sqrt{3}\lambda_3 + K_d m_{q_d}) + \delta + \frac{K_P}{\bar{K}_D} K_d l_2 m_{\omega_d} \right) \|\tilde{\omega}\| \\ & \leq -\bar{K}_D \frac{1-\varepsilon}{4} c_2 \|\tilde{\omega}\|^2. \end{aligned}$$

Rearranging,

$$\begin{aligned} & \frac{K_P}{\bar{K}_D} (l_1 a_1 - \sqrt{3}\lambda_3 \bar{K}_D l_2 - K_d (\bar{K}_D m_{q_d} + m_{\omega_d}) l_2) \\ & \geq \left( a_3 + \frac{3}{2} \bar{K}_D^2 c_2 + \delta \right) + \bar{K}_D c_2 \left( \frac{9-\varepsilon}{4} \right) a_2. \end{aligned}$$

If  $\lambda_3 \leq \lambda_3^*(K_D)$ ,  $K_d \leq K_d^*(K_D)$  and  $K_P \geq K_P^*(K_D)$ , where

$$\lambda_3^*(K_D) = \frac{l_1 a_1}{4\sqrt{3}\bar{K}_D l_2}, \quad K_d^*(K_D) = \frac{l_1 a_1}{4(\bar{K}_D m_{q_d} + m_{\omega_d}) l_2},$$

$$K_P^*(K_D) = \frac{3c_2}{a_1 l_1} \bar{K}_D^3 + \frac{(9-\varepsilon)a_2 c_2}{2a_1 l_1} \bar{K}_D^2 + \frac{2(a_3 + \delta)}{a_1 l_1} \bar{K}_D,$$

then

$$\dot{V}(q_0, \tilde{\omega}) < -\bar{K}_D \frac{1-\varepsilon}{2} \left( \frac{\|q\|^2}{(q_0-\varepsilon)^2} + \frac{c_2}{2} \|\tilde{\omega}\|^2 \right)$$

for a fixed  $\bar{K}_D \geq \bar{K}_D^*$  and all  $(q_0, \tilde{\omega}) \in \mathcal{S}$ , where  $\bar{K}_D^*$  and  $\mathcal{S}$  is a positive number and a compact set respectively as defined in [3].

### B. Appendix B

Define an ISS-Lyapunov function candidate  $V(\eta_1, \eta_2, \eta_3) = \eta_1^\top \eta_1 + \frac{1}{2} \eta_2^\top J \eta_2 + \eta_3^\top \eta_3$ . From (19)-(24) and taking  $g_{st} = k_4 \tilde{G}(\zeta_2 - K_1 \sigma'(\frac{K_1}{\lambda_1} \zeta_1) \dot{\zeta}_1 + k_3 \dot{\zeta}_1)$ , the time derivative of  $V(\eta_1, \eta_2, \eta_3)$  can be written as

$$\begin{aligned} \dot{V} \leq & (\ell_1 + \delta_0 (\frac{\ell_4 + \ell_7}{2} + 2) + \ell_5 \frac{\delta_1}{2}) \|\eta_1\|^2 \\ & + (\ell_2 + \ell_4 \frac{1}{2\delta_0} + \ell_6 \frac{1}{2\delta_2} + \frac{\delta_3}{2} (\frac{1}{\bar{K}_D} \|J\| + \ell_8 + 1)) \|\eta_2\|^2 + (\ell_3 + \ell_5 \frac{1}{2\delta_1} + \delta_2 (\frac{\ell_6 + \ell_9}{2} + 2)) \\ & \|\eta_3\|^2 + (\frac{1}{\delta_0} + \frac{1}{2K_D \delta_3} \|J\| + \frac{1}{\delta_2}) \|y_{\zeta_0}\|^2 \\ & + \frac{1}{2} (\frac{\ell_7}{\delta_0} + \frac{\ell_8}{\delta_3} + \frac{\ell_9}{\delta_2}) \|y_{\zeta_1}\|^2 + \frac{1}{\delta_0} \|I_{\eta_1}\|^2 \\ & + \frac{1}{2\delta_3} \|I_{\eta_2}\|^2 + \frac{1}{\delta_2} \|I_{\eta_3}\|^2, \text{ where} \\ y_{\zeta_0} = & -K_3 P_2 \sigma'(\frac{K_3}{\lambda_3} \zeta_3) (-\frac{1}{M} \tilde{D}(t) P_2 \lambda_3 \sigma(\frac{K_3}{\lambda_3} \zeta_3) + \\ & K_2 P_1 \sigma'(\frac{K_2}{\lambda_2} \zeta_2) \dot{\zeta}_2 + \frac{p}{M} + \frac{d}{M} - \frac{K_d}{M} L_d G_4 \bar{\tau}), \\ y_{\zeta_1} = & \tilde{G}(\dot{\zeta}_2 - K_1 \sigma'(\frac{K_1}{\lambda_1} \zeta_1) \dot{\zeta}_1 + k_3 \dot{\zeta}_1), \\ I_{\eta_1} = & -K_d \left( I - \frac{1}{2} (q_0 I + S(q)) Q_d \right) (G_6 \bar{\tau} + \tilde{D}_0^{-1} P \dot{\tau}), \\ I_{\eta_2} = & K_d \left( S(\omega) J Q_d + \frac{1}{K_D} J \left( I - \frac{1}{2} (q_0 I + S(q)) \right. \right. \\ & \left. \left. Q_d \right) \right) (G_6 \bar{\tau} + \tilde{D}_0^{-1} P \dot{\tau}) + \Delta - J \dot{\omega}_d, \\ I_{\eta_3} = & K_d \left( -\frac{1}{2K_D} P_3 (q_0 I + S(q)) - K_d P_3 \left( I - \frac{1}{2} \right. \right. \\ & \left. \left. (q_0 I + S(q)) Q_d \right) G_5 P_3 + K_3 P_3 Q_0 - K_d G_7 P_3 \right) \\ & P_4 e_\xi - K_d P_3 \left( I - \frac{1}{2} (q_0 I + S(q)) Q_d \right) (G_6 \bar{\tau} + \\ & \tilde{D}_0^{-1} P \dot{\tau}) - K_d \tilde{G} \tilde{P}_2 P_6 \bar{\tau} \text{ and} \\ Q_0(t) = & \frac{1}{M} P_2 \sigma'(\frac{K_3}{\lambda_3} \zeta_3) (\tilde{D} + K_d L_d G_4 P_3), \\ Q_1(t) = & \frac{1}{M} P_2 \sigma'(\frac{K_3}{\lambda_3} \zeta_3) L_d G_4, \\ L_d(t) = & I + \tilde{D}_\Delta(t) \tilde{D}_0^{-1}(t), \\ G_0(t) = & P G_7, \quad G_1(t) = \tilde{P}_2 P_6 G_7 + \dot{\tilde{P}}_2 P_6, \\ G_2(t) = & (K_d P \tilde{G} \tilde{P}_2 + \dot{\tilde{P}}_2) P_6 + \tilde{P}_2 P_6 G_7, \\ G_3(t) = & K_d P + \tilde{P}_2 P_6, \quad G_4(t) = P + \dot{\tilde{P}}_2 P_6, \end{aligned}$$



$$\begin{aligned} G_5(t) &= \dot{D}_0^{-1}G_4 + \tilde{D}_0^{-1}(K_d G_0 + G_1), \\ G_6(t) &= \dot{D}_0^{-1}G_4 + \tilde{D}_0^{-1}G_2, \quad G_7(t) = \tilde{F} + \tilde{G}\tilde{P}_2 P_6. \end{aligned}$$

Note that

$$P_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ & \vdots & & \ddots & \\ & & & & \ddots \end{bmatrix} \text{ and } P_6 = \begin{bmatrix} P_5 & 0 \\ 0 & P_5 \end{bmatrix}$$

is a  $4N \times (4N+2)$  matrix and a  $\mathbb{R}^{2N+1} \rightarrow \mathbb{R}^{2N}$  projection matrix respectively. In addition, for all  $t \geq 0$ ,  $\|Q_i(t)\| \leq m_i$  and  $\|G_j(t)\| \leq g_j$  for some  $m_i > 0$ ,  $i = 0, 1$  and  $g_j > 0$ ,  $j = 0 \dots 7$  respectively. The arbitrary positive numbers  $\delta_0$ ,  $\delta_1$ ,  $\delta_2$  and  $\delta_3$  are obtained from Young's inequalities and

$$\begin{aligned} \ell_1 &:= -\frac{\varepsilon}{K_D} + \sqrt{3}\frac{\lambda_3}{K_D} + 2K_3m_0 + K_d\left(\frac{m_{qd}}{K_D} + 2(1+m_{Q_d})g_5\right) \\ \ell_2 &:= -K_P K_D \ell_1 + (m_\omega + \frac{1}{K_D})\|J\| \\ \ell_3 &:= -\frac{\varepsilon}{K_D} + \sqrt{3}\frac{\lambda_3}{K_D} + 2K_3m_0 + K_d\left(\frac{m_{qd} + m_{e\xi}}{K_D} + 2K_3m_1 + 4g_7 + 4(1+m_{Q_d})g_5\right) \\ \ell_4 &:= \left(\frac{m_\omega}{K_D} + \frac{1}{K_D^2} + \frac{K_3}{K_D}m_0 + K_d(m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D}g_5)\right)\|J\| + 2 \\ \ell_5 &:= 2K_d((1+m_{Q_d})g_5 + K_3m_1) \\ \ell_6 &:= K_d((m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D}g_5 + \frac{K_3}{K_D}m_1)\|J\| + 2 \\ \ell_7 &:= 2k_4 K_d(1+m_{Q_d})m_{\tilde{D}_0}g_3 \\ \ell_8 &:= k_4 K_d(m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D}g_5)m_{\tilde{D}_0}g_3\|J\| \\ \ell_9 &:= 2k_4 K_d(1+(1+m_{Q_d})m_{\tilde{D}_0}g_3). \end{aligned}$$

Set now

$$\begin{aligned} \bar{\lambda}_3^* &= \frac{\varepsilon}{5\sqrt{3}}, \quad \bar{K}_3^*(K_D) = \frac{\varepsilon}{10K_D m_0}, \\ \bar{K}_d^*(K_D, K_3) &= \min\{\bar{K}_{d1}^*(K_D, K_3), \bar{K}_{d2}^*(K_D, K_3)\}, \end{aligned}$$

where

$$\begin{aligned} \bar{K}_{d1}^*(K_D, K_3) &= \frac{\varepsilon}{5K_D\left(\frac{m_{qd}}{K_D} + 3(1+m_{Q_d})g_5 + K_3m_1\right)}, \\ \bar{K}_{d2}^*(K_D, K_3) &= \frac{\varepsilon}{5K_D\left(\frac{m_{qd} + m_{e\xi}}{K_D} + 5(1+m_{Q_d})g_5 + 3K_3m_1 + 4g_7\right)}. \end{aligned}$$

Next, fix

$$\begin{aligned} \delta_0 &= \frac{2\varepsilon}{5K_D(\ell_4 + \ell_7 + 4)}, \quad \delta_1 = 1, \\ \delta_2 &= \frac{2\varepsilon}{5K_D(\ell_6 + \ell_9 + 4)}, \\ \delta_3 &= \frac{\ell_1 K_P K_D}{\frac{1}{K_D}\|J\| + \ell_8 + 1}. \end{aligned}$$

Moreover let,

$$\begin{aligned} \ell_4^*(K_D) &:= \left(\frac{m_\omega}{K_D} + \frac{1}{K_D^2} + \frac{\bar{K}_3^*}{K_D}m_0 + \bar{K}_d^*(m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D}g_5)\right)\|J\| + 2 \\ \ell_6^*(K_D) &:= \bar{K}_d^*\left((m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D}g_5 + \frac{\bar{K}_3^*}{K_D}m_1)\|J\| + 2\right) \\ \ell_7^*(k_4) &:= 2k_4 \bar{K}_d^*(1+m_{Q_d})m_{\tilde{D}_0}g_3 \\ \ell_8^*(k_4, K_D) &:= k_4 \bar{K}_d^*(m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D}g_5)m_{\tilde{D}_0}g_3\|J\| \\ \ell_9^*(k_4) &:= 2k_4 \bar{K}_d^*(1+(1+m_{Q_d})m_{\tilde{D}_0}g_3) \\ \delta_0^*(k_4, K_D) &:= \frac{2\varepsilon}{5K_D(\ell_4^* + \ell_7^* + 4)} \\ \delta_2^*(k_4, K_D) &:= \frac{2\varepsilon}{5K_D(\ell_6^* + \ell_9^* + 4)} \\ \delta_3^*(k_4, K_D) &:= \frac{\ell_1 K_P K_D}{\frac{1}{K_D}\|J\| + \ell_8^* + 1}. \end{aligned}$$

It can shown that there exist  $\bar{K}_P^* > 0$  such that for all  $K_P \geq \bar{K}_P^* \frac{1}{K_D}$  and for all  $\lambda_3 \leq \bar{\lambda}_3^*$ ,  $K_3 \leq \bar{K}_3^*$  and  $K_d \leq \bar{K}_d^*$ ,

$$\begin{aligned} \dot{V} &\leq -\frac{\varepsilon}{5K_D}(\|\eta_1\|^2 + \|\eta_2\|^2 + \|\eta_3\|^2) + \left(\frac{1}{\delta_0^*} + \frac{1}{2K_D \delta_3^*}\right. \\ &\quad \left.\|J\| + \frac{1}{\delta_2^*}\right)\|y_{\zeta 0}\|^2 + \frac{1}{2}\left(\frac{\ell_7^*}{\delta_0^*} + \frac{\ell_8^*}{\delta_3^*} + \frac{\ell_9^*}{\delta_2^*}\right)\|y_{\zeta 1}\|^2 + \\ &\quad \frac{1}{\delta_0^*}\|I_{\eta 1}\|^2 + \frac{1}{2\delta_3^*}\|I_{\eta 2}\|^2 + \frac{1}{\delta_2^*}\|I_{\eta 3}\|^2. \end{aligned}$$

Hence, subsystem (24) is ISS (see for instance, [2]).

## REFERENCES

- [1] Alberto Isidori, *Nonlinear control systems*, third ed., Communications and Control Engineering Series, Springer-Verlag, Berlin, 1995. MR MR1410988 (97g:93003)
- [2] ———, *Nonlinear control systems. II*, Communications and Control Engineering Series, Springer-Verlag London Ltd., London, 1999. MR MR1717603 (2000m:93004)
- [3] Alberto Isidori, Lorenzo Marconi, and Andrea Serrani, *Robust autonomous guidance*, Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2003.
- [4] ———, *Robust nonlinear motion control of a helicopter*, IEEE Trans. Automat. Control **48** (2003), no. 3, 413–426. MR MR1962249 (2004f:93090)
- [5] T. John Koo and Shankar Sastry, *Output tracking control design of a helicopter model*, In Proceedings of the 37th Conference on Decision and Control, 1998, pp. 3635–3640.
- [6] F. N. Koumboulis, M. G. Skarpetis, and M. K. Gripaparis, *Rotary gust rejection for helicopters*, In Proceedings of the IEEE International Conference on Control Applications, 1998, pp. 103–107.
- [7] Omid Shakhnia, Yi Ma, T. John Koo, T. John, and Shankar Sastry, *Landing an unmanned air vehicle: Vision based motion estimation and nonlinear control*, Asian Journal of Control **1** (1999), 128–145.
- [8] Andrew R. Teel, *A nonlinear small gain theorem for the analysis of control systems with saturation*, IEEE Trans. Automat. Control **41** (1996), no. 9, 1256–1270. MR MR1409471 (98b:93051)
- [9] H. Wang, A. A. Mian, D. Wang, and H. Duan, *Robust multi-mode flight control design for an unmanned helicopter based on multi-loop structure*, International Journal of Control, Automation, and Systems **7** (2009), 723–730.
- [10] Xilin Yang, Hemanshu Pota, and Matt Garratt, *Design of a Gust-Attenuation Controller for Landing Operations of Unmanned Autonomous Helicopters*, 18th IEEE International Conference on Control Applications, July 2009.