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Global practical stabilization of large-scale hydraulic networks*

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Abstract

Proportional feedback control of a large scale hydraulic network which is subject to structural changes is considered. Results regarding global practical stabilization of the non-linear hydraulic network using a set of decentralized proportional control actions are presented. The results show that closed loop stability of the system is maintained when structural changes are introduced to the system.

1 Introduction

An industrial case study involving a system distributed over a network is investigated. The system is a large-scale hydraulic network which underlies a district heating system with an arbitrary number of end-users. The case study considers a new paradigm for constructing district heating systems [1]. The new paradigm is motivated by the possibility of reducing the overall energy consumption of the system while making the network structure more flexible. However, the new system paradigm also calls for a new control architecture, which is able to handle the flexible network structure [1].

The case study is a part of the research program *Plug & Play Process Control* [2] and has been proposed by one of the industrial partners involved in the research program. The goal of the research program is automatic reconfiguration of the control system whenever components, such as sensors or actuators, are added to or

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removed from the system. In the case of the district heating system, the addition (removal) of components could, for instance, be due to the addition (removal) of one or more end-users to (from) the system. Whenever such an addition or removal is made, the structure of the system is changed and the control should accommodate the changes.

The control objective of the system in question is to regulate the pressure drops across the so-called end-user valves in the hydraulic network to a given piecewise constant reference point. This goal shall be obtained in spite of the unknown demand of the end-users. The controllers, which will be considered here, are a set of decentralized proportional controllers, which use only locally available information. This control architecture has been chosen, since it is expected that changes in the system structure can be easily handled [3].

Previous work on a simple system with two end-users has shown that high-gain proportional controllers semi-globally stabilizes the closed loop system towards a set of attractors [4]. The results show that whenever the controller gains are large enough, the basin of attraction contains the set of all possible initial conditions of the system. However, if changes to the structure of the system is introduced, such as the addition or removal of end-users, the results cannot guarantee closed loop stability of the newly obtained system without proper redesign of the controller gains.

The results presented here are threefold. First, the results are applicable for a large-scale hydraulic network, since no assumptions are made regarding the number of end-users in the system. Secondly, the proposed control architecture is decentralized in the sense that the individual controllers use only locally available information. Thirdly, the results show that the closed loop system is globally practically stable with a unique equilibrium point using a set of arbitrary positive controller gains.

Compared to previous results in [4], which are semi-global, the global result here shows that the closed loop system will be stable regardless of the initial conditions. Furthermore, since the result is independent of the number of end-users, the system will also be stable whenever components are added to or removed from the system, since the initial conditions of the newly obtained system are guaranteed to belong to the basin of attraction. This, along with the fact that the control scheme is decentralized, makes structural changes in the system easy to implement.

In Section 2, the models of the individual system components as well as the model of the hydraulic network are presented along with the proposed controllers. The closed loop properties of the system is derived in Section 3. Section 4 provides a proof of an important intermediate proposition, which is used to derive the closed loop stability properties of the system.

1.1 Nomenclature

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, with the scalar product $\langle \mathbf{a}, \mathbf{b} \rangle$ between two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, x_i denotes the i'th element of x. Let $M(n, m; \mathbb{R})$ denote the set of $n \times m$ matrices with real entries, and $M(n; \mathbb{R}) = M(n, n; \mathbb{R})$. For a matrix $\mathbf{A}$, the notation A_{ij} will be used to denote the entry in the i'th row and j'th column of $\mathbf{A}$. For a square matrix $\mathbf{A}$, $\mathbf{A} > 0$ means that $\mathbf{A}$ is positive definite, i.e., $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0 \, \forall \mathbf{x} \neq 0$. For a square matrix $\mathbf{A}$, $\mathbf{A} = \mathrm{diag}(x_i)$ means that $\mathbf{A}$ has x_i as entries on the main diagonal and zero elsewhere. Throughout the following, C^1 denotes a continuously differentiable function (map), and all functions (maps) introduced will be assumed C^1. A continuous function (map) $f : X \to Y$ is said to be: an *injection* if it is into, i.e., for every $a, b \in X$, if $f(a) = f(b)$ then $a = b$; a *surjection* if it is onto, i.e., if for every $\mathbf{y} \in Y$ there exists at least one $\mathbf{x} \in X$ such that $f(\mathbf{x}) = \mathbf{y}$; a *bijection* if it is both an *injection* and a *surjection*; a *homeomorphism* if it is a *bijection* with a continuous inverse f^{-1}; a *diffeomorphism* if it is a *bijection* with a C^1 inverse f^{-1}. A continuous function (map) is said to be *proper* if the inverse image of a compact set is compact. A function $f : \mathbb{R} \to \mathbb{R}$ is called monotonically increasing if it is order preserving, i.e., for all x and y such that $x \leq y$ then $f(x) \leq f(y)$. The open ball with radius r and centred in $\mathbf{x}$ is denoted $B_r(\mathbf{x})$, i.e., $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n | |\mathbf{y} - \mathbf{x}| < r\}$. Likewise, the corresponding closed ball is denoted $\bar{B}_r(\mathbf{x})$, i.e., $\bar{B}_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n | |\mathbf{y} - \mathbf{x}| \leq r\}$.

2 System Model

The system under consideration is a hydraulic network underlying a district heating system. The model has been derived in detail in [3] and will be recalled here but in fewer details.

The hydraulic network consists of a number of connections between two-terminal components, which can be valves, pipes and pumps. The system components are characterized by dual variables, the first of which is the pressure drop Δh across them

$$\Delta h = h_i - h_j, \tag{1}$$

where i, j are nodes in the network; h_i, h_j are the relative pressures at the nodes.

The second variable characterizing the components is the fluid flow q through them. The components have algebraic or dynamic expressions governing the relationships between the two variables.

2.1 Valves

Valves in the network are governed by the following algebraic expression

$$h_i - h_j = \mu(k_v, q), \tag{2}$$

where k_v is the hydraulic resistance of the valve; $\mu(k_v, q)$ is a C^1 and proper function, which for any fixed value of k_v is zero at $q = 0$ and monotonically increasing. Furthermore, $\mu(0, \cdot) = 0$.

2.2 Pipes

Pipes in the network are governed by the dynamic equation

$$\mathcal{J}\dot{q} = (h_i - h_j) - \lambda(k_p, q), \tag{3}$$

where $\mathcal{J}$ and k_p are parameters of the pipe; $\lambda(k_p, q)$ is a function with the same properties as $\mu(k_v, q)$.

2.3 Pumps

A (typically centrifugal) pump is a component which delivers a desired pressure difference Δh regardless of the value of the fluid flow through it. Thus, the pumps in the network are governed by the following expression

$$h_i - h_j = -\Delta h_p, \tag{4}$$

where Δh_p is a non-negative control input.

2.4 Component Model

A generalised component model can be made using the following expression

$$\Delta h = \mathcal{J}\dot{q} + \lambda(k_p, q) + \mu(k_v, q) - \Delta h_p \tag{5}$$

where $\mathcal{J}, k_p$ are non-zero for pipe components and zero for other components; k_v is non-zero for valve components and zero for other components; Δh_p is non-zero for pump components and zero for other components.

The values of the parameters k_p and k_v are typically unknown, but they will be assumed to be piecewise constant functions of time ranging over a compact set of non-negative values. Likewise, the functions $\mu(k_v, q)$ and $\lambda(k_p, q)$ are not precisely known, only their properties of being C^1, monotone and proper are guaranteed. The varying heating demand of the end-users, which is the main source of disturbances in the system, is modelled by a (end-user) valve with variable hydraulic resistance. In the network model, a distinction is to be made between end-user valves and the rest of the valves in the network. Two types of pumps are present in the network; the end-user pumps, which are mainly used to meet the demand at the end-users, and booster pumps which are used to meet constraints on the relative pressures in the network [5].

2.5 Network Model

The network model has been derived using standard circuit theory [3]. The hydraulic network consists of m components and n end-users ($m > n$). The network is associated with a graph $\mathcal{G}$ which has nodes coinciding with the terminals of the network components. The edges of the network are the components themselves. By the use of graph theory, a set of n independent flow variables q_i have been identified. These flow variables have the property that their values can be set independently from other flows in the network. The independent flow variables coincide with the flows through the chords of the graph [3]. To each chord in the graph, a fundamental (flow) loop is associated, and along this loop Kirchhoffs voltage law holds. This means that the following equality holds

$$\mathbf{B}\boldsymbol{\Delta}\mathbf{h} = \mathbf{0}, \tag{6}$$

where $\mathbf{B} \in M(n, m; \mathbb{R})$ is called the fundamental loop matrix; $\boldsymbol{\Delta}\mathbf{h}$ is a vector consisting of the pressure drops across the components in the network.

The entries of the fundamental loop matrix $\mathbf{B}$ are $-1, 1$ or 0, dependent on the network topology. For the case study in question, the hydraulic network underlies a district heating system. Because of this, the following statements can be made regarding the network.

Assumption 2.1. *[3] Each end-user valve is in series with a pipe and a pump, as seen in Fig. 1. Furthermore, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.*

Assumption 2.2. *[3] There exists one and only one component called the heat source. It corresponds to a valve[1] of the network, and it lies in all the fundamental loops.*

Proposition 2.1. *[3] Any hydraulic network satisfying Assumption 2.1 admits the representation*

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \mathbf{B}^T\mathbf{q}) + \mathbf{u} \tag{7}$$

$$y_i = \mu_i(k_{vi}, q_i)\ ,\ i = 1, \ldots, n, \tag{8}$$

[1]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.

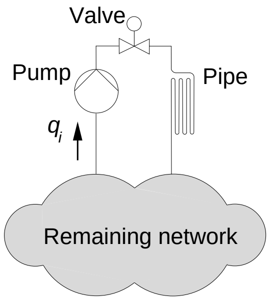

Figure 1: The series connection associated with each end-user [3].

where $\mathbf{q} \in \mathbb{R}^n$ is the vector of independent flows; $\mathbf{u} \in \mathbb{R}^n$ is a vector of independent inputs consisting of a linear combination of the delivered pump pressures; y_i is the measured pressure drop across the ith end-user valve; $\mathbf{J} > 0 \in M(n;\mathbb{R})$; $\mathbf{K}_p, \mathbf{K}_v$ are vectors of system parameters; $\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \cdot)$ is a C^1 vector field; $\mu_i(k_{vi}, \cdot)$ is the fundamental law of the ith end-user valve. In (8), it is assumed that the first n components coincide with the end-user valves.

Under Assumption 2.1 and Assumption 2.2, it is possible to select the orientation of the components in the network such that the entries of the fundamental loop matrix $\mathbf{B}$ are equal to 1 or 0, where B_{ij} is 1 if component j belongs to fundamental flow loop i and 0 otherwise.

Defining the vector of flows through the components in the system as $\mathbf{x} = \mathbf{B}^T\mathbf{q} \in \mathbb{R}^m$, the vector field $\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \cdot)$ can be written as [3]

$$\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \mathbf{x}) = -\mathbf{B}(\boldsymbol{\lambda}(\mathbf{K}_p, \mathbf{x}) + \boldsymbol{\mu}(\mathbf{K}_v, \mathbf{x})), \tag{9}$$

$\forall \mathbf{x} \in \mathbb{R}^m$,

where $\boldsymbol{\lambda}(\mathbf{K}_p, \cdot) = [\lambda_1(k_{p1}, x_1), \ldots, \lambda_m(k_{pm}, x_m)]^T$; $\boldsymbol{\mu}(\mathbf{K}_v, \cdot) = [\mu_1(k_{v1}, x_1), \ldots, \mu_m(k_{vm}, x_m)]^T$, and k_{pi} is non-zero for pipe components and k_{vi} is non-zero for valve components.

The matrix $\mathbf{J}$ in (7) is given by

$$\mathbf{J} = \mathbf{B}\boldsymbol{\mathcal{J}}\mathbf{B}^T \tag{10}$$

where $\boldsymbol{\mathcal{J}} = \text{diag}(\mathcal{J}_1, \ldots, \mathcal{J}_m)$ and $\mathcal{J}_i$ is non-zero for pipe components.

The input $\mathbf{u}$ to the system deserves a few comments as well. Define the vectors $\boldsymbol{\Delta}\mathbf{h}_{pe}$ and $\boldsymbol{\Delta}\mathbf{h}_{pb}$ as the vectors of pump pressures delivered by respectively the end-user pumps and the booster pumps. Then $\mathbf{u}$ is given as

$$\mathbf{u} = \boldsymbol{\Delta}\mathbf{h}_{pe} + \mathbf{F}\boldsymbol{\Delta}\mathbf{h}_{pb} \tag{11}$$

where $\mathbf{F} \in M(n, k; \mathbb{R})$ is the sub-matrix of $\mathbf{B}$ which maps the booster pumps to the fundamental flow loops; k is the number of booster pumps in the network.

A sketch of a simple district heating system with a heat source and two apartment buildings is illustrated in Fig. 2. The corresponding hydraulic network is illustrated in Fig. 3. The two end-users are represented by the series connections $\{c_{12}, c_{13}, c_{14}\}$ and $\{c_5, c_6, c_7\}$. The heat source is represented by the valve $\{c_{10}\}$ which models the pressure losses in the secondary side of the heat exchanger of the heat source.

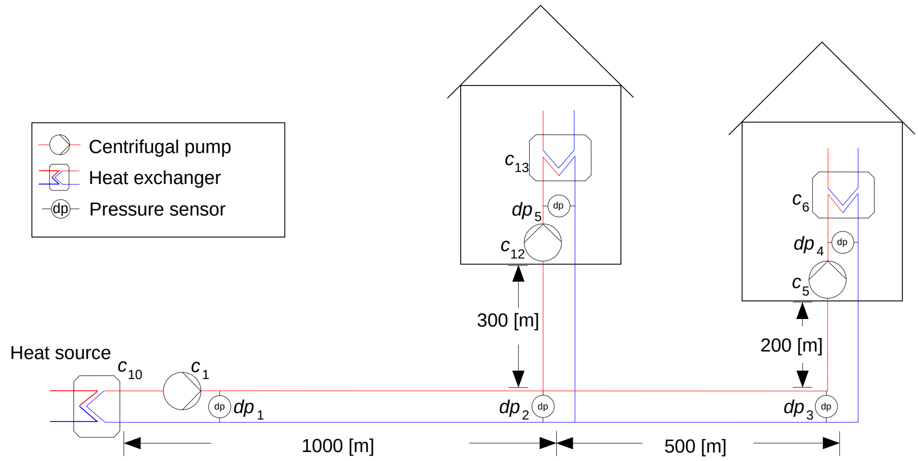

Figure 2: A sketch of a small district heating system.

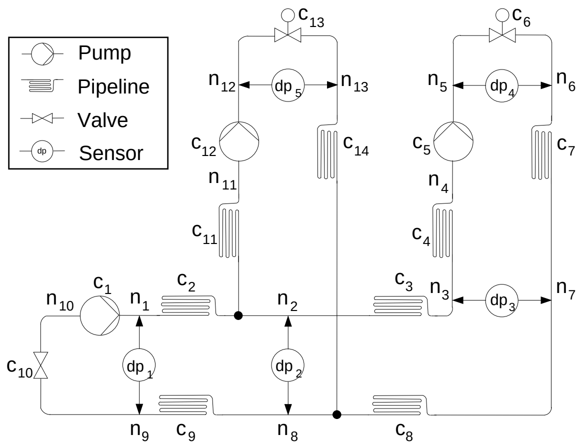

Figure 3: The hydraulic network diagram.

It is desired to regulate the pressure y_i across the ith end-user valve to a given reference value r_i with the use of a feedback controller using locally available information only. The desired reference value of the pressure across the end-user valve is assumed to be a piecewise constant function of time, and it ranges in a known set $[r_m, r_M]$. Thus, the vector $\mathbf{r} = (r_1, \ldots, r_n)$ of reference values takes values in a known compact set $\mathcal{R}$

$$\mathcal{R} = \{\mathbf{r} \in \mathbb{R}^n | r_m \leq r_i \leq r_M\}. \tag{12}$$

For the purpose of practical output regulation, a set of decentralized proportional controllers will be the focus of the work presented here. The controllers considered will be of the form

$$u_i = -\gamma_i(y_i - r_i)\ ,\ i = 1, \ldots, n, \tag{13}$$

where $\gamma_i > 0$ is the controller gain at end-user i.

The pressure control for the ith end-user valve use only the pressure measurement obtained at said valve. Thus, the controllers are decentralized in the sense that the individual controller use only locally available information.

3 Stability Properties of Closed Loop System

In this section, the main result regarding the closed loop stability properties of the feedback control system introduced in the previous section will be presented.

To simplify the notation, $\mathbf{f}^K(\cdot)$ will be used to denote $\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \cdot)$. Likewise, $\boldsymbol{\lambda}^K(\cdot)$ and $\boldsymbol{\mu}^K(\cdot)$ will be used to denote $\boldsymbol{\lambda}(\mathbf{K}_p, \cdot)$ and $\boldsymbol{\mu}(\mathbf{K}_v, \cdot)$. The closed loop system defined by (7), (8) and (13) is given by

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{f}^K(\mathbf{B}^T\mathbf{q}) - \boldsymbol{\Gamma}(\mathbf{y}(\mathbf{q}) - \mathbf{r}). \tag{14}$$

where $\boldsymbol{\Gamma} = \mathrm{diag}(\gamma_i)$.

Subsequently, a more specific class of functions will be used in the expressions of $\mu(k_v, \cdot)$ and $\lambda(k_p, \cdot)$. This more specific class is motivated by the presence of turbulent[2] flows in the system [3]. The class of functions, which will be considered, is the following

$$\mu_i(k_{vi}, x_i) = k_{vi}|x_i|x_i \tag{15}$$

$$\lambda_i(k_{pi}, x_i) = k_{pi}|x_i|x_i \tag{16}$$

An important intermediate result, which will be used for establishing the stability properties of the closed loop system, is presented below.

Define the map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$\mathbf{F}(\mathbf{z}) = \mathbf{y}(\mathbf{z}) - \boldsymbol{\Gamma}^{-1}\mathbf{f}^K(\mathbf{B}^T\mathbf{z}). \tag{17}$$

Proposition 3.1. *For the class of functions defined in (15) and (16), the map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ defined in (17) is a homeomorphism.*

[2]Since the motivation for considering the new paradigm is reducing the diameters of the pipes used in the network, the likelihood for turbulent flows increases.

The proof of Proposition 3.1 has been left out of this section to maintain the flow of the exposition, but can be found in Section 4.

As a consequence of Proposition 3.1, for any vector $\mathbf{r} \in \mathcal{R}$ of output reference values, there exists a unique vector of flows $\mathbf{q}^* \in \mathbb{R}^n$ such that

$$\mathbf{q}^* = \mathbf{F}^{-1}(\mathbf{r}), \tag{18}$$

which means that $\mathbf{r}$ can be expressed in terms of $\mathbf{q}^*$ as

$$\mathbf{r} = \mathbf{y}(\mathbf{q}^*) - \mathbf{\Gamma}^{-1}\mathbf{f}^K(\mathbf{B}^T\mathbf{q}^*). \tag{19}$$

Using the identity in (19), the expression of the closed loop system in (14) can be replaced by

$$\mathbf{J}\dot{\mathbf{q}} = \tilde{\mathbf{f}}^K(\mathbf{q}, \mathbf{q}^*) - \mathbf{\Gamma}(\mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{q}^*)), \tag{20}$$

where $\tilde{\mathbf{f}}^K(\mathbf{q}, \mathbf{q}^*) = \mathbf{f}^K(\mathbf{B}^T\mathbf{q}) - \mathbf{f}^K(\mathbf{B}^T\mathbf{q}^*)$. Recall that the vector $\mathbf{q}^* \in \mathbb{R}^n$ is some unknown but unique vector of flows, which is constant for every constant vector $\mathbf{r}$ of reference values.

Proposition 3.2. *The point $\mathbf{q}^*$ defined by (18) is a globally asymptotically stable equilibrium point of the closed loop system defined by (7), (8) and (13).*

Proof of Proposition 3.2. Define the variable $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}^*$, and the function $V(\tilde{\mathbf{q}})$ as

$$V(\tilde{\mathbf{q}}) = \frac{1}{2}\langle \tilde{\mathbf{q}}, \mathbf{J}\tilde{\mathbf{q}} \rangle, \tag{21}$$

which has the properties

- $V(\mathbf{0}) = 0$
- $V(\tilde{\mathbf{q}}) > 0$, $\forall \tilde{\mathbf{q}} \neq 0$
- $\lim_{||\tilde{\mathbf{q}}|| \to \infty} V(\tilde{\mathbf{q}}) = \infty$.

The time derivative of $V(\tilde{\mathbf{q}})$ is given by

$$\frac{d}{dt}V(\tilde{\mathbf{q}}) = \langle \tilde{\mathbf{q}}, \mathbf{J}\dot{\mathbf{q}} \rangle \Leftrightarrow \tag{22}$$

$$\frac{d}{dt}V(\tilde{\mathbf{q}}) = \langle \mathbf{q} - \mathbf{q}^*, \tilde{\mathbf{f}}^K(\mathbf{q}, \mathbf{q}^*) - \mathbf{\Gamma}[\mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{q}^*)] \rangle. \tag{23}$$

The functions $\lambda_i(k_{pi}, x_i)$ have the properties that they are monotonically increasing and zero for $x_i = 0$, consequently it applies that

$$-(x_i - x_i^*)\left[\lambda_i(k_{pi}, x_i) - \lambda_i(k_{pi}, x_i^*)\right] < 0, \forall x_i \neq x_i^* \Rightarrow \tag{24}$$

$$-\Big\langle \mathbf{x}-\mathbf{x}^*, \boldsymbol{\lambda}^K(\mathbf{x})-\boldsymbol{\lambda}^K(\mathbf{x}^*)\Big\rangle < 0,\ \forall \mathbf{x}\neq\mathbf{x}^*. \tag{25}$$

The map $\boldsymbol{\mu}^K(\mathbf{x})$ has the same properties as $\boldsymbol{\lambda}^K(\mathbf{x})$, i.e., it consists of monotonically increasing functions which are zero for $x_i = 0$. Due to these properties, the fact that $\mathbf{x} = \mathbf{B}^T\mathbf{q}$ and the identity in (9), the following inequality holds

$$\langle \mathbf{q}-\mathbf{q}^*, \mathbf{f}^K(\mathbf{B}^T\mathbf{q})-\mathbf{f}^K(\mathbf{B}^T\mathbf{q}^*)\rangle < 0,\ \forall \mathbf{q}\neq\mathbf{q}^*. \tag{26}$$

Furthermore, since $y_i(q_i)$ is a monotonically increasing function which is zero at $q_i = 0$, the inequality below is true

$$\langle \mathbf{q}-\mathbf{q}^*, \mathbf{y}(\mathbf{q})-\mathbf{y}(\mathbf{q}^*)\rangle > 0,\ \forall \mathbf{q}\neq\mathbf{q}^*. \tag{27}$$

Using (26) and (27) in (23) and observing that $\boldsymbol{\Gamma} > 0$, the following inequality is obtained

$$\frac{d}{dt}V(\tilde{\mathbf{q}}) < 0,\ \forall \tilde{\mathbf{q}}\neq 0. \tag{28}$$

As a consequence of the properties of $V(\tilde{\mathbf{q}})$ and (28), the point $\tilde{\mathbf{q}} = 0$ is a globally asymptotically stable equilibrium point of the closed loop system (see for instance [6], Theorem 4.2). Considering the change of coordinates $\tilde{\mathbf{q}} = \mathbf{q}-\mathbf{q}^*$ it is concluded that $\mathbf{q} = \mathbf{q}^*$ is a globally asymptotically stable equilibrium point of the closed loop system. □

Proposition 3.2 shows that for every constant vector $\mathbf{r}$ and gain $\gamma_i > 0$, there exists a unique constant vector $\mathbf{q}^*$ such that $\mathbf{q}^*$ is a globally asymptotically stable equilibrium point of the closed loop system. Note that only the properties of the functions $\mu(k_v, q)$ and $\lambda(k_p, q)$ being monotonically increasing and zero at $q = 0$ are used in the proof of Proposition 3.2. This means that the control system is robust towards uncertainties in the system parameters.

With the flows in the system converging to $\mathbf{q}^*$, the output of the system will converge to the value $\mathbf{y}^* = \mathbf{y}(\mathbf{q}^*)$. Using (19), the following relation is given between the vector $\mathbf{r}$ of reference values and $\mathbf{q}^*$

$$\mathbf{r}-\mathbf{y}(\mathbf{q}^*) = -\boldsymbol{\Gamma}^{-1}\mathbf{f}^K(\mathbf{B}^T\mathbf{q}^*). \tag{29}$$

Using the definition of $\boldsymbol{\Gamma}$, the i'th component is

$$r_i - y_i(q_i^*) = -\frac{1}{\gamma_i} f_i^K(\mathbf{B}^T\mathbf{q}^*). \tag{30}$$

Letting $\gamma_i \to \infty$, the right hand side of (30) will converge to zero. From this it can be seen that the use of large gains in the controller will let the output regulation error become small.

Since the system is globally asymptotically stable at $\mathbf{q}^*$, the system state will converge to $\mathbf{q}^*$ regardless of the initial conditions. Furthermore, the stability property is independent of the number n of end-users. This has the consequence that flow loops along with their respective controllers can be added to or removed from the system without the need for redesigning the controller gains in order for the system to be stable. However, controller gains may have to be redesigned for the purpose of fulfilling some specifications on the regulation error. From 30, it can be seen that each individual controller can adjust its own gain freely.

4 Properties of $\mathbf{F}(\mathbf{q}^*)$

This section provides a proof of Proposition 3.1, which has been used in deriving the closed loop properties of the system.

For the specific class of $\mu(k_v, \cdot)$ and $\lambda(k_p, \cdot)$ defined in (15) and (16), the output map (8) can be rewritten as

$$\mathbf{y} = (k_{v1}|q_1|q_1, \ldots, k_{vn}|q_n|q_n)^T, \tag{31}$$

which in turn can be rewritten as

$$\mathbf{y} = \mathbf{H}(\mathbf{q})\mathbf{q}, \tag{32}$$

where $\mathbf{H}(\mathbf{q}) \in M(n;\mathbb{R})$ is given by

$$\mathbf{H}(\mathbf{q}) = \mathrm{diag}(k_{vi}|q_i|), \tag{33}$$

$i = 1, \ldots, n$.

Likewise, by substituting back $\mathbf{x}$ with $\mathbf{B}^T\mathbf{q}$, the expression for $\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \cdot)$ in (9) can be rewritten as

$$\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \mathbf{B}^T\mathbf{q}) = -\mathbf{B}\mathbf{N}(\mathbf{B}^T\mathbf{q})\mathbf{B}^T\mathbf{q}, \tag{34}$$

where $\mathbf{N}(\mathbf{B}^T\mathbf{q}) \in M(m;\mathbb{R})$ is given by

$$\mathbf{N}(\mathbf{B}^T\mathbf{q}) = \mathrm{diag}((k_{vj} + k_{pj})|\mathbf{B}_j^T\mathbf{q}|), \tag{35}$$

$j = 1, \ldots, m$, where k_{vj} is non-zero for valve components and k_{pj} is non-zero for pipe components; $\mathbf{B}_j$ is the jth column of $\mathbf{B}$.

Define the function $\bar{\mathbf{F}} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\bar{\mathbf{F}}(\mathbf{z}) = \mathbf{\Gamma}\mathbf{y}(\mathbf{z}) - \mathbf{f}^K(\mathbf{z}). \tag{36}$$

For the specified class of $\mu(k_v, q)$ and $\lambda(k_p, q)$, $\bar{\mathbf{F}}(\mathbf{z})$ can be written as

$$\bar{\mathbf{F}}(\mathbf{z}) = \mathbf{\Gamma}\mathbf{H}(\mathbf{z})\mathbf{z} + \mathbf{B}\mathbf{N}(\mathbf{B}^T\mathbf{z})\mathbf{B}^T\mathbf{z}, \tag{37}$$

From the above, it can be established that $\bar{\mathbf{F}}(\mathbf{z})$ scales in the sense stated below

$$\bar{\mathbf{F}}(\lambda\mathbf{z}) = \lambda|\lambda|\bar{\mathbf{F}}(\mathbf{z}), \tag{38}$$

where $\lambda \in \mathbb{R}$.

Furthermore, note that $g(\lambda) = \lambda|\lambda|$ is bijective, i.e. for every $\kappa \in \mathbb{R}$ there exists a unique $\lambda \in \mathbb{R}$ such that

$$\kappa = \lambda|\lambda|. \tag{39}$$

The properties (38) and (39) are instrumental in the proof of Proposition 3.1.

Proof of Proposition 3.1. As a consequence of (27) and the fact that $\mathbf{\Gamma} > 0$, the following inequality is satisfied

$$\langle \mathbf{z} - \mathbf{z}^*, \mathbf{\Gamma}[\mathbf{y}(\mathbf{z}) - \mathbf{y}(\mathbf{z}^*)] \rangle > 0, \, \forall \mathbf{z} \neq \mathbf{z}^*. \tag{40}$$

Likewise, from (26) the following inequality is obtained

$$-\left\langle \mathbf{z} - \mathbf{z}^*, \mathbf{f}^K(\mathbf{B}^T\mathbf{z}) - \mathbf{f}^K(\mathbf{B}^T\mathbf{z}^*) \right\rangle > 0, \, \forall \mathbf{z} \neq \mathbf{z}^*. \tag{41}$$

A combination of these two inequalities gives

$$\left\langle \mathbf{z} - \mathbf{z}^*, \bar{\mathbf{F}}(\mathbf{z}) - \bar{\mathbf{F}}(\mathbf{z}^*) \right\rangle > 0, \, \forall \mathbf{z} \neq \mathbf{z}^*. \tag{42}$$

Definition 4.1. *[7]. Let $\mathbf{f} : X \to Y$, $X \subset \mathbb{R}^n$, $Y = \mathbb{R}^n$. Let the following inner product be denoted by*

$$\langle \mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2), \mathbf{x}_1 - \mathbf{x}_2 \rangle \equiv \alpha(\mathbf{x}_1, \mathbf{x}_2).$$

Then $\mathbf{f}$ is said to be increasing on X, or simply an increasing function if and only if

$$\alpha(\mathbf{x}_1, \mathbf{x}_2) > 0, \, \forall \mathbf{x}_1, \mathbf{x}_2 \in X \text{ and } \mathbf{x}_1 \neq \mathbf{x}_2.$$

From (42) and Definition 4.1, it can be seen that $\bar{\mathbf{F}}(\mathbf{z})$ is an increasing function for every point $\mathbf{z} \in \mathbb{R}^n$.

Lemma 4.1. *[7]. Let $\mathbf{f} : U \to \mathbb{R}^n$, where U is an open convex subset of $\mathbb{R}^n$.*

(a) If $\mathbf{f}$ is increasing on U, then $\mathbf{f}$ is injective on U.

(b) If $\mathbf{f}$ is continuous and increasing on U, then $\mathbf{f}$ is a homeomorphism on U[3] and its inverse function $\mathbf{f}^{-1} : \mathbf{f}(U) \to U$ is also increasing on $\mathbf{f}(U)$.

Since $\bar{\mathbf{F}}(\mathbf{z})$ is continuous and increasing for every point $\mathbf{z} \in \mathbb{R}^n$, it follows from Lemma 4.1 that $\bar{\mathbf{F}}(\mathbf{z})$ is a local homeomorphism.

[3] $\mathbf{f}$ is a homeomorphism on U if and only if $\mathbf{f} : U \to V$ is a homeomorphism, where $V = \mathbf{f}(U)$

Proposition 4.1. *For the specified class of $\mu(k_v, q)$ and $\lambda(k_p, q)$ defined in (15) and (16), the map $\bar{\mathbf{F}} : \mathbb{R}^n \to \mathbb{R}^n$ defined in (17) is proper.*

Proof of Proposition 4.1. In the proof, the following lemma will be used.

Lemma 4.2. *[8]. Let $\mathbf{f}$ be a continuous map from $\mathbb{R}^n$ into $\mathbb{R}^n$, then $\mathbf{f}$ is a proper map if and only if:*

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{f}(\mathbf{x})| = \infty$$

Thus, if $\bar{\mathbf{F}}(\cdot)$ is proper it should fulfil

$$\lim_{|\mathbf{z}|\to\infty} |\bar{\mathbf{F}}(\mathbf{z})| = \infty. \tag{43}$$

Suppose by contradiction, that some sequence $\{\mathbf{z}_n\}_{n\in\mathbb{N}}$ exists, where

$$\lim_{n\to\infty} |\mathbf{z}_n| = \infty \tag{44}$$

and

$$\bar{\mathbf{F}}(\mathbf{z}_n) \in B_r(0),\ \forall n \in \mathbb{N}, \tag{45}$$

for some $r \in \mathbb{R}$.

Since $\bar{\mathbf{F}}(\cdot)$ is a local homeomorphism, there exits some open set $\mathcal{U} \subset \mathbb{R}^n$ containing 0 and an open set $\mathcal{V} \subset \mathbb{R}^n$, such that $\bar{\mathbf{F}} : \mathcal{U} \to \mathcal{V}$ is a homeomorphism. Furthermore, it is known that $0 \in \mathcal{V}$ since $\bar{\mathbf{F}}(0) = 0$.

Because of the scaling property (38) of $\bar{\mathbf{F}}(\cdot)$, there exists some $\hat{\mathbf{z}}_n \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $r_{\mathcal{V}} \in \mathbb{R}$, such that

$$\lambda\hat{\mathbf{z}}_n = \mathbf{z}_n, \tag{46}$$

$$\bar{\mathbf{F}}(\mathbf{z}_n) = \bar{\mathbf{F}}(\lambda\hat{\mathbf{z}}_n) = \lambda|\lambda|\bar{\mathbf{F}}(\hat{\mathbf{z}}_n) \tag{47}$$

and

$$\bar{\mathbf{F}}(\hat{\mathbf{z}}_n) \in \bar{B}_{r_{\mathcal{V}}}(0) \subset \mathcal{V}, \tag{48}$$

where $\hat{\mathbf{z}}_n$ is unique for a specific choice of λ.

However, this indicates that

$$\hat{\mathbf{z}}_n \in \bar{K} \subset \mathcal{U} \tag{49}$$

where $\bar{K} = \bar{\mathbf{F}}^{-1}(\bar{B}_{r_{\mathcal{V}}}(0))$ is some compact and thus bounded set.

This is a contradiction since

$$\lim_{n\to\infty} |\hat{\mathbf{z}}_n| = |\frac{1}{\lambda}| \lim_{n\to\infty} |\mathbf{z}_n| = \infty \tag{50}$$

□

Theorem 4.1. *[8]. Let* $\mathbf{f}$ *be a map from* $\mathbb{R}^n$ *into* $\mathbb{R}^n$*, then* $\mathbf{f}$ *is a homeomorphism of* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$ *if and only if* $\mathbf{f}$ *is:*

1) a local homeomorphism

2) a proper map

From Theorem 4.1 it follows that $\bar{\mathbf{F}}(\mathbf{z})$ is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Now, consider the linear transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ where $\mathbf{T}(\mathbf{v}) = \mathbf{\Gamma}^{-1}\mathbf{v}$. Since $\mathbf{\Gamma}^{-1}$ is non-singular, the transformation $\mathbf{T}(\cdot)$ is a diffeomorphism. Thus, the composition

$$(\mathbf{T} \circ \bar{\mathbf{F}})(\mathbf{z}) = \mathbf{y}(\mathbf{z}) - \mathbf{\Gamma}^{-1}\mathbf{f}^K(\mathbf{B}^T\mathbf{z}), \tag{51}$$

is a homeomorphism.

□

Since $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism it is bijective and has a continuous inverse $\mathbf{F}^{-1}$

$$\mathbf{F}(\mathbf{F}^{-1}(\mathbf{r})) = \mathbf{r}. \tag{52}$$

5 Numerical Results

The proposed proportional controllers have been tested using numerical simulations. The results of the simulations are shown in Fig. 4 and Fig. 5. The simulated system consists of two end-users corresponding to the hydraulic network illustrated in Fig. 3. The parameters used in the system are: $J_{11} = 0.3697, J_{12} = J_{21} = 0.0559, J_{22} = 0.2738$; $k_{p2} = k_{p9} = 0.0024$; $k_{p3} = k_{p8} = 0.0012$; $k_{p4} = k_{p7} = 0.0014$; $k_{p11} = k_{p14} = 0.0021$; $k_{v6} = k_{v13} = 0.01$; $k_{v10} = 0.0013$. Furthermore, the functions $\mu(k_v, \cdot)$ and $\lambda(k_p, \cdot)$ used in the simulation are the ones introduced in Section 4.

First, a scenario where the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$ is removed from and later re-inserted into the system has been simulated. This is simulated by increasing the hydraulic resistance k_{v13} of c_{13} to a large value and thereby reducing q_2 to close to zero. The results are shown in Fig. 4. Explicitly, the end-user connection is removed at time 100 s and re-inserted at time 200 s. All system parameters are maintained at the same values throughout the simulation, and the controller gain $\gamma_1 = \gamma_2 = 2$ has been used. The reference value for the pressure across the end user valves is indicated by the solid line at 0.5 Bar in the two plots in the middle.

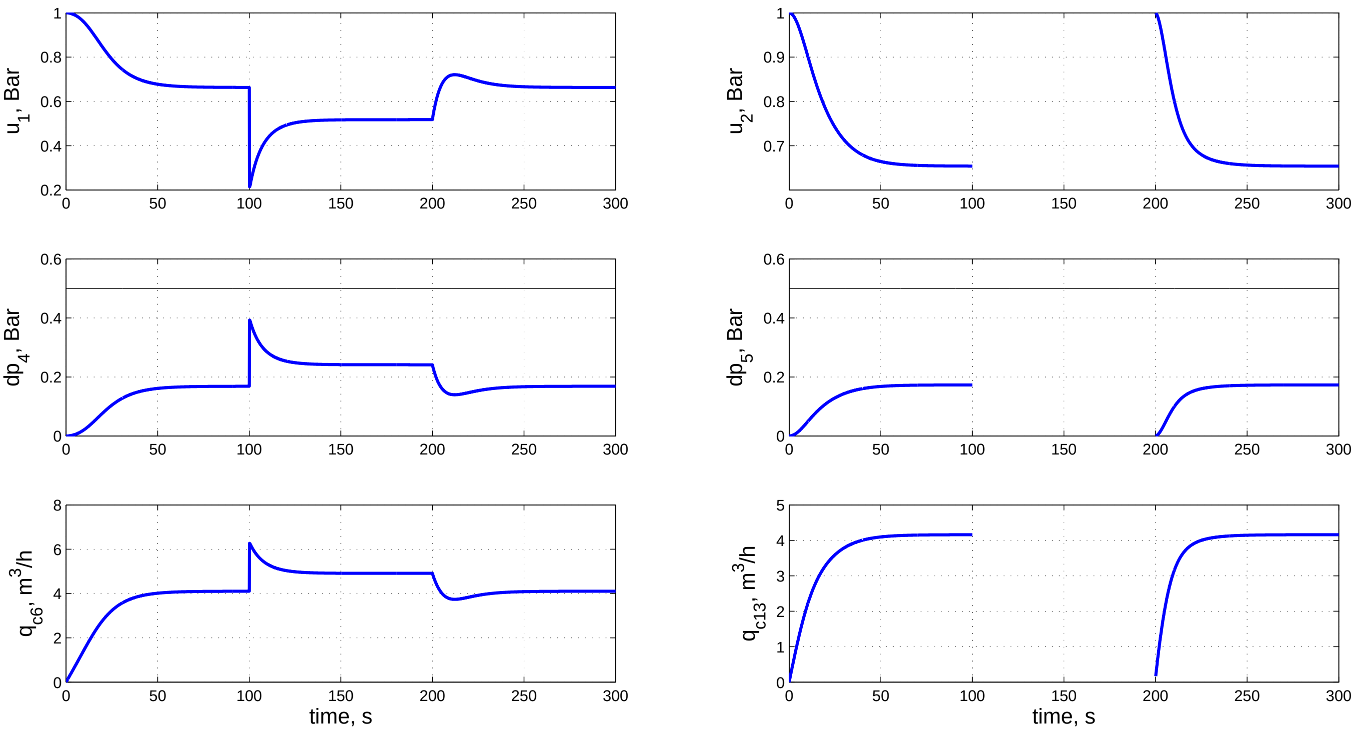

Figure 4: The figure shows the result of a numerical simulation of the system in Fig. 3. The figure shows control inputs u_1 and u_2, the controlled variable dp_4 and dp_5, and the flow through valve c_6 and c_{13} obtained with the proportional feedback control. At time 100 s, the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$ is removed from the system. At time 200 s the end-user connection is re-inserted into the system.

In Fig. 4, it can be seen that a steady state, with an equilibrium point $\mathbf{q}^* = (q^*_{c6}, q^*_{c13}) \approx (4.1, 4.2)$ Bar, has been reached at time 100 s. Later, when the above mentioned end-user connection is re-inserted, the same equilibrium point $\mathbf{q}^*$ has been reached again at time 300 s. Since the same system parameters are used throughout the simulation, it is expected that the same equilibrium point will be reached since the relation between the reference value and the equilibrium point is the homeomorphism given by the expression in (19). Furthermore, when only one end-user is present, it can be seen that a steady state with an equilibrium point $q^* = q^*_{c6} \approx 4.9$ Bar is reached between time 100 s and 200 s.

Secondly, a scenario has been simulated where steps in the hydraulic resistance k_{v6}, k_{v13} of the end-user valves c_6, c_{13} are made. This corresponds to a varying demand for heating at the end-users. The steps are between the values 0.01 and 0.11. The results of the simulation are seen in Fig. 5. Again, $\gamma_1 = \gamma_2 = 2$ and the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$ is removed from and later re-inserted into the system. Specifically, it is removed between time 300 s and time 600 s.

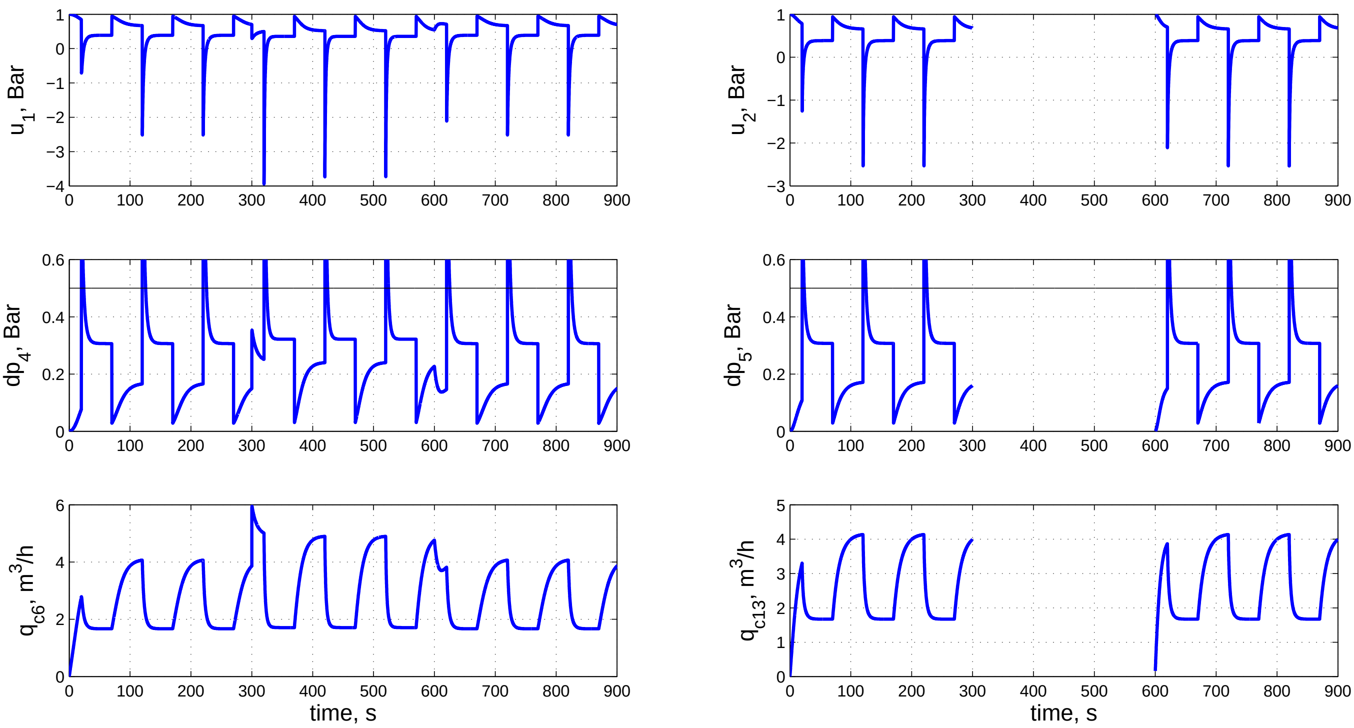

Figure 5: The figure shows the result of a numerical simulation of the system in Fig. 3. Throughout the simulation, steps between values 0.01 and 0.11 are made in the hydraulic resistance (k_{v6}, k_{v13}) of the end-user valves c_6, c_{13}. At time 300 s, the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$ is removed from the system. At time 600 s the end-user connection is re-inserted into the system.

In Fig. 5, it can be seen that the system remains stable when a step is made in the hydraulic resistance of the end-user valves.

6 Conclusion

An industrial case study involving a large-scale hydraulic network underlying a district heating system was investigated. The system under investigation is subject to structural changes. A set of decentralized proportional controllers for practical output regulation were proposed. The results show that the closed loop system is globally practically stable with a unique equilibrium point. The decentralized architecture of the control design and the fact that the closed loop system is globally stable make it easy to implement structural changes in the system, while maintaining closed loop stability. The results were supported by numerical simulations of a simple two end-user system.

Some natural future extensions of the work presented here will be restricting the control actions to only positive values and the incorporation of integral control actions. Since the (centrifugal) pumps used in the network are able to deliver only positive pressures, it should be examined if the stability properties of the system are kept when this restriction is taken into consideration. The incorporation of integral control actions would be interesting with respect to accommodating for the output regulation error which is present with the proportional control actions.

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