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LTR Design of Discrete-time Proportional-Integral Observers

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are taken as $\mu_0 = [0.5 \quad 0.5]^T$ and the model-switching probability matrix as

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.85 & 0.15 \\ 0.30 & 0.70 \end{bmatrix}. \tag{65}$$

A Monte Carlo simulation with 500 experiments was executed, and the results are presented in Fig. 3. In each experiment, the measurement model in effect at each point of time was randomly chosen according to (65). In Fig. 3(a), the root-mean-square-error (RMSE) in the state estimate versus time is presented. Averaging the RMSE's over the time interval gives an average error of 10.75 for the IMM filter, 9.32 for the smoother of Method 1, and 9.42 for Method 2. Fig. 3(b) presents the probability of error in the system-structure detection versus time (i.e., the probability of choosing the incorrect measurement model at each point of time). Averaging the probabilities over the time interval gives an average probability of error of 0.19 for the IMM filter, 0.15 for Method 1, and 0.16 for Method 2. The two smoothers provided noticeably better performances than the IMM filter, while the smoother of Method 1 provided slightly better performance than Method 2. The smoother of Method 1 provided the best overall performance because it considered the most hypotheses.

A simulation example comparing the performances of these algorithms in reconstructing the trajectory of a maneuvering target has also been performed. Detailed results are not presented because of space limitations. Both smoothers provided significantly better performance than the IMM filter in estimating the system state. However, unlike the system-structure simulation results above, the Method 1 smoother provided significantly better mode estimates than the Method 2 smoother. The mode estimates from the Method 2 smoother and the IMM filter were comparable.

V. Summary

Suboptimal approaches to the one-step fixed-lag smoothing problem for Markovian switching systems were examined in this paper. Two algorithms for generating one-step fixed-lag smoothed estimates were presented. In the first algorithm, the models over the two most recent sampling periods were considered. For n models, there are n^2 possible ways of conditioning on models in two sampling periods, and this algorithm evaluated the n^2 hypotheses using n^2 parallel one-step smoothers. In the second algorithm, only the models in the most recent sampling period were considered, and it evaluated n hypotheses using n parallel one-step smoothers. A simulation of a system-structure detection problem was used to compare the performances of the two smoothers and an IMM filter. The results show that the smoother of Method 1 provided the best overall performance. Variants of these one-step smoothing algorithms have been used in conjunction with IMM filtering algorithms to develop techniques for the alignment of asynchronous sensors [6]. Finally, approaches similar to the ones presented in this paper have been applied to the fixed-interval smoothing problem for Markovian switching systems [7].

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LTR Design of Discrete-Time Proportional-Integral Observers

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Abstract—This paper applies the proportional-integral (PI) observer in connection with loop-transfer recovery (LTR) design for discrete-time systems. Both the prediction and the filtering versions of the PI observer are considered. We show that a PI observer makes it possible to obtain time recovery, i.e., exact recovery for $t \to \infty$, under mild conditions. Two systematic LTR design methods, one based on an extension of the linear quadratic Gaussian loop-transfer recovery (LQG/LTR) and the other based on linear matrix inequality (LMI), are derived for the discrete-time PI observer case. Explicit expressions for the recovery error when exact recovery is not achievable for all frequencies are also given.

I. Introduction

Since the appearance of the papers by Doyle and Stein [2], [3] dealing with loop-transfer recovery (LTR), many papers have been written on this topic for both continuous and discrete-time systems. The most notable ones for continuous-time systems are [1], [15], [10], [11], [18], [7], [12], and [13].

Although there are certain similarities between the LTR of continuous and discrete-time systems, there exist also fundamental differences. Without going into the details, it is well known that an arbitrarily specified target loop-transfer function is recoverable if the continuous-time system is minimum phase and left invertible.

However, this is not true for discrete-time systems as discussed in [4], [6], and [5].

For discrete-time systems there are two main types of observers, namely, prediction and filtering (current type) observers. They are used when computation time is either significant or negligible, respectively. The status of the reported results in a discrete-time LTR indicates that the recovery of any arbitrarily specified target loop-transfer function using a filtering observer is possible for the strictly proper square minimum phase systems having only infinite zero of order one. On the other hand, it is impossible in general to have either exact or asymptotic LTR when the plant is a nonminimum phase or a prediction-type observer is used. The fundamental difficulties are due to the fact that sampling usually introduces nonminimum phase zeros, that computation time is sometimes not negligible, and that practical systems contain time delays, and they are often nonstrictly proper. Consequently, recent results [8], [19], [13] were devoted to understanding the behavior of LTR under these conditions.

As pointed out above, it is in general not possible to achieve asymptotic (or exact) recovery for a free target design. However, it is possible to overcome some of these problems by including an integral term in the full-order observer. By using this proportional-integral (PI)-observer in connection with LTR design, it is possible to obtain time recovery, i.e., recovery as $t \to \infty$. The continuous-time case has been thoroughly investigated in [9], where it has been shown that it is possible to obtain time recovery for nonminimum phase systems. In this paper we show explicitly that it is also possible to obtain time recovery in the discrete-time case by using a PI-observer for both minimum phase as well as for nonminimum phase systems.

An alternative way to obtain good recovery at low frequencies is to augment integrators to the plant before the target design is performed [1], [18]. This implies that the target loop is no longer entirely free. In contrast, when the PI-observer approach is used, the integral effect is dictated by the observer structure. Consequently, the target design is completely free.

II. DISCRETE-TIME PI OBSERVER

Consider a finite dimensional, linear, time-invariant discrete system described by a state-space realization (A, B, C)

$$x(t + 1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{1}$$

where $x \in \mathcal{R}^n, u \in \mathcal{R}^m$, and $y \in \mathcal{R}^m$ with $n > m, (A, B)$ stabilizable, (C, A) detectable and C, B full rank. It is further assumed that the system (A, B, C) has no poles or zeros at the origin.

Let the plant be controlled by an observer-based controller having the state feedback

$$u(t) = F\hat{z}(t) + r(t) = w(t) + r(t) \tag{2}$$

where F is the state feedback gain, $\hat{z}$ is the state estimate, and r the reference input. The states are estimated by using a proportional-integral (PI) observer. Analogous to the case of P-observers, it is possible to derive two versions of the PI observer for the discrete-time system (1): a prediction PI observer and a filtering PI observer. The discrete-time, prediction PI observer is equivalent to the continuous-time version [9]. Therefore, we can directly formulate a prediction PI observer as follows:

$$\hat{z}(t + 1) = A\hat{z}(t) + K_P(C\hat{z}(t) - y(t)) + Bu(t) + Bv(t)$$
$$v(t + 1) = v(t) + K_I(C\hat{z}(t) - y(t)) \tag{3}$$

where K_P and K_I are proportional and integral gains, respectively.

To derive a systematic design method, we let the PI observer-based controller be represented by an augmented state system given by

$$z(t + 1) = A_x z(t) + K_x(C_x z(t) - y(t)) + B_x u(t)$$
$$u(t) = F_x z(t) \tag{4}$$

where

$$A_x = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}, \quad B_x = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C_x = [C \quad 0]$$
$$K_x = \begin{bmatrix} K_P \\ K_I \end{bmatrix}, \quad F_x = [F \quad 0]. \tag{5}$$

In a prediction observer the feedback signal $u(t)$ is based on measurements up to time $t - 1$; on the other hand, in a filtering observer $u(t)$ is based on measurements up to time t. The time delay due to calculation of the feedback signal $u(t)$ therefore must be negligible. A filtering PI observer can be derived from the full-order, filtering P observer by including an integral term. The resulting state-space description is equivalent to (3); however, the feedback signal is given by $u(t) = F_f A\hat{z}(t) + F_f K_P(C\hat{z}(t) - y(t))$ where $F = F_f A$. The calculation of F_f requires that A is invertible or that F is a linear-quadratic (LQ) gain. The compact form of the filtering PI observer-based controller is equivalent to (4) and (5) with the matrix F_x given by

$$F_x = [F_f(A + K_P C) \quad 0]. \tag{6}$$

In LTR design, the sensitivity recovery error is defined as

$$E_S(z) = S_I(z) - S_{TFL}(z) \tag{7}$$

where

$$S_{TFL}(z) = (I - F(zI - A)^{-1}B)^{-1}$$
$$S_I(z) = (I - C(z)G(z))^{-1}. \tag{8}$$

Let the applied controller $C(z)$ be a prediction or a filtering PI observer-based controller as described above. We then have the following results which can be proven analogous to the continuous-time case [9].

Lemma 2.1: Let the recovery matrix $M_{PI}(z)$ be given by

$$M_{PI}(z) = F_x(zI - A_x - K_x C_x)^{-1} B_x \tag{9}$$

where A_x, B_x, C_x, K_x, and F_x are given by (5) or F_x by (6) for the filtering observer. Then

$$E_S(z) = S_{TFL}(z)M_{PI}(z). \tag{10}$$

Based on the discrete-time LTR formulation, we now give necessary and sufficient conditions for both exact and time recovery.

Lemma 2.2: Let the sensitivity recovery error be given by (7). Exact LTR is obtained if and only if one of the following equivalent conditions holds:

$$E_S(z) = 0 \tag{11}$$
$$M_{PI}(z) = 0. \tag{12}$$

In some cases the step response of the recovery error E_S tends to zero as $t \to \infty$ which happens exactly when $\lim_{z \to 1} E_S(z) = 0$. We can then define time recovery for discrete-time PI observer-based systems as follows.

Definition 2.1: Let $M_{PI}(z)$ be the recovery matrix. Time recovery is obtained if and only if

$$M_{PI}(1) = 0. \tag{13}$$

Analogous to the continuous-time case, the condition for achieving time recovery with a PI observer can now be derived for the discrete-time case. With respect to the prediction PI observer we have the following result.

Theorem 2.1: Time recovery is obtained with a prediction PI observer if and only if the largest invariant subspace of $(I - A - K_P C)^{-1} B K_I C$ contained in the controllable subspace of $((I - A - K_P C)^{-1}, (I - A - K_P C)^{-1} B)$ corresponding to $z = 1$ is itself contained in the unobservable subspace of $(F, (I - A - K_P C)^{-1} B K_I C)$.

In connection to Theorem 2.1, the following corollary gives a simple matrix condition which can be checked to determine whether or not time recovery is achievable.

Corollary 2.1: Let the Jordan normal form of the matrix $(I - A - K_P C)^{-1} B K_I C$ be given by

$$T^{-1}((I - A - KC)^{-1} B K_I C)T = \begin{bmatrix} J_0 & 0 \\ 0 & \bar{J} \end{bmatrix} \tag{14}$$

where J_0 contains all the Jordan blocks associated with the eigenvalue $z = 1$ according to the partitionings

$$T = [T_1 \quad T_2], \quad T^{-1} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}. \tag{15}$$

Then time recovery is obtained if and only if

$$FT_1(S_1(I - A - K_P C)^{-1} B, \quad J_0 S_1(I - A - K_P C)^{-1} B \tag{16}$$

$$\cdots, \quad J_0^{n-1} S_1(I - A - K_P C)^{-1} B) = 0. \tag{17}$$

With respect to the filtering PI observer, the only difference is that the target design gain F in Theorem 2.1 and Corollary 2.1 is replaced by $F_f(A + K_P C)$.

Again, the condition on K_I for time recovery will generically be satisfied if $K_I C$ has full row rank. As in the continuous-time case, however, this condition is neither necessary nor sufficient.

III. LQG/LTR DESIGN OF DISCRETE-TIME PI OBSERVERS

In the following subsections we extend the LQG and LQG/LTR design methods of full-order observers for the case of PI-observers.

A. LQG Design

Consider the extended state form of a PI observer-based controller given by (4). To proceed with an LQG design for (1), select weighting matrices Γ and Σ which satisfy

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [L_1 \quad L_2] \geq 0, \quad \Sigma \geq 0 \tag{18}$$

respectively. Solve the algebraic Riccati equation

$$P = A_x P A_x^T - A_x P C_x^T (\Sigma + C_x P C_x^T)^{-1} C_x P A_x^T + \Gamma \tag{19}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}. \tag{20}$$

Then compute K_x by

$$K_x = -A_x P C_x^T (\Sigma + C_x P C_x^T)^{-1} \tag{21}$$

$$= \begin{bmatrix} -A P_{11} C^T - B P_{12}^T C^T \\ -P_{12}^T C^T \end{bmatrix} (\Sigma + C P_{11} C^T)^{-1}. \tag{22}$$

The integral gain K_I has full rank if and only if $C P_{12}$ has full rank. Rewriting (19) as four (effectively three) simultaneous equations leads to

$$0 = -P_{11} + A P_{11} A^T + B P_{12}^T A^T + A P_{12} B^T + B P_{22} B^T$$
$$- A P_{11} C^T D^{-1} C P_{11} A^T - A P_{11} C^T D^{-1} C P_{12} B^T$$
$$- B P_{12}^T C^T D^{-1} C P_{11} A^T - B P_{12}^T C^T D^{-1} C P_{12} B^T + \Gamma_{11} \tag{23}$$

$$0 = -P_{12} + A P_{12} + B P_{22} - A P_{11} C^T D^{-1} C P_{12}$$
$$- B P_{12}^T C^T D^{-1} C P_{12} + \Gamma_{12} \tag{24}$$

$$0 = -P_{12}^T C^T D^{-1} C P_{12} + \Gamma_{22}. \tag{25}$$

From (25) we see that $C P_{12}$ has full rank if and only if $\Gamma_{22} = L_2 L_2^T$ is positive definite. Moreover, Γ_{22} is the only submatrix of Γ which, via P_{12}, influences K_I. As in the continuous time case therefore, LQG design of a PI observer generically yields time recovery.

B. LQG/LTR Design of Full-Order Observers

Derivation of an LQG/LTR design method for discrete-time systems parallels the derivation for continuous-time systems given in [9] with the exception that a design can be obtained with zero weighting on the measurement signals, i.e., $\Sigma = 0$. The solution to the Riccati equation with $\Sigma = 0$ has been given in an explicit form in [14].

Lemma 3.1: Assume that the system (A, B, C) satisfies

$$C A^i B = 0, \quad i = 1, \cdots, l - 2 \tag{26}$$

$$\det [C A^{l-1} B] \neq 0. \tag{27}$$

Then the singular stationary Riccati equation ($\Sigma = 0$) for the system (A, B, C) is given by

$$P = A P A^T + \Gamma - A P C^T (C P C^T)^{-1} C P A^T \tag{28}$$

and with $\Gamma = B B^T$ the observer gain K is given by

$$K = -A^l \hat{B} (C A^{l-1} \hat{B})^{-1} \tag{29}$$

where the system $(A, \hat{B}, C)$ is minimum phase [it is the minimum phase projection of (A, B, C)].

The connection between the minimum phase system $(A, \hat{B}, C)$ and the nonminimum phase system (A, B, C) is given by

$$G(z) = \hat{G}(z) \hat{B}_a(z) = C(zI - A)^{-1} \hat{B} \hat{B}_a(z) \tag{30}$$

where $\hat{B}_a(z)$ is stable, has zeros coinciding with the nonminimum phase zeros of $G(z)$, and satisfies $\hat{B}_a(z^{-1})^T \hat{B}_a(z) = I$. The transfer function $\hat{G}(z)$ is minimum phase and is termed the minimum phase counterpart of $G(z)$. One method for calculating $\hat{G}(z)$ and $\hat{B}_a(z)$ can be found in [19].

Now, using the recovery matrix for the full-order prediction observer

$$M_P(z) = F(zI - A - K_P C)^{-1} B \tag{31}$$

and the gain given in Lemma 3.1 with $\Gamma = L L^T$ and $\Sigma = 0$, we have the following result.

Theorem 3.1: Let the full-order prediction observer gain K be

$$K = -A^l \hat{L}(CA^{l-1}\hat{L})^{-1}.$$

Then the recovery matrix $M_P(z)$ is given by

$$M_P(z) = F(zI - A)^{-1}[B - z^{-l}A^l\hat{L}(C(zI - A)^{-1}\hat{L})^{-1} \\ \cdot C(zI - A)^{-1}B]$$

where $(A, \hat{L}, C)$ is the minimum phase projection of (A, L, C).

Proof: Rewriting the recovery matrix as

$$\begin{aligned} M_P(z) &= F(zI - A - KC)^{-1}B \\ &= F(zI - A)^{-1}(I - KC(zI - A)^{-1})^{-1}B \\ &= F(zI - A)^{-1}[I + KC((zI - A)^{-1} - KC)^{-1}]B \\ &= F(zI - A)^{-1}[I + K(I - C(zI - A)^{-1}K)^{-1} \\ &\quad \cdot C(zI - A)^{-1}]B \\ &= F\Phi(z)[I + K(I - C\Phi(z)K)^{-1}C\Phi(z)]B \end{aligned}$$

and substituting $K = -A^l\hat{L}(CA^{l-1}\hat{L})^{-1}$ in the above equation, we get

$$\begin{aligned} M_P(z) &= F\Phi(z)[I - A^l\hat{L}(CA^{l-1}\hat{L})^{-1}(I + C\Phi(z)A^l \\ &\quad \cdot \hat{L}(CA^{l-1}\hat{L})^{-1})^{-1}C\Phi(z)]B \\ &= F\Phi(z)[I - A^l\hat{L}(CA^{l-1}\hat{L} + C\Phi(z)A^l\hat{L})^{-1} \\ &\quad \cdot C\Phi(z)]B \\ &= F\Phi(z)[I - A^lL(zC\Phi(z)A^{l-1}L)^{-1}C\Phi(z)]B. \end{aligned}$$

Now, using $C\Phi(z)A\hat{L} = zC\Phi(z)\hat{L}$ the recovery matrix becomes

$$\begin{aligned} M_P(z) &= F\Phi(z)[I - A^l\hat{L}(z^lC\Phi(z)\hat{L})^{-1}C\Phi(z)]B \\ &= F\Phi(z)[B - z^{-l}A^l\hat{L}(C\Phi(z)\hat{L})^{-1}C\Phi(z)B] \end{aligned}$$

which completes the proof of Theorem 3.1. $\square$

When a filtering observer is applied instead, the recovery matrix is given by

$$M_F(z) = F_f(A + K_PC)(zI - A - K_PC)^{-1}B \quad (32)$$

and with the observer gain given in Lemma 3.1, we have directly

$$M_F(z) = F(zI - A)^{-1}[B - z^{-l+1}A^{l-1} \\ \cdot \hat{L}(C(zI - A)^{-1}\hat{L})^{-1}C(zI - A)^{-1}B]. \quad (33)$$

It is now very easy to derive explicit expressions for the recovery matrices for the P-observer. When CB does not have full rank but satisfies the conditions given in Lemma 3.1, we obtain the expressions for the recovery matrices given in the following lemma.

Lemma 3.2: Let the system (A, B, C) be nonminimum phase, and let the optimal LQG/LTR gain be given by (29). Then the recovery matrices for the full order observers are given by

$$\begin{aligned} M_P(z) &= F(zI - A)^{-1}(B - z^{-l}A^l\hat{B}\hat{B}_a(z)) \\ M_F(z) &= F(zI - A)^{-1}(B - z^{-l+1}A^{l-1}\hat{B}\hat{B}_a(z)). \end{aligned} \quad (34)$$

Proof: Lemma 3.2 follows directly from Theorem 3.1 by using $L = \hat{B}$. $\square$

C. LQG/LTR Design of PI Observers

The LQG/LTR design of a full-order P observer can be realized by using $\Gamma = BB^T$ and $\Sigma = 0$, as shown in the above section. Similarly, if we let $\Gamma = B_xB_x^T$ and $\Sigma = 0$ in the PI observer design, we have $K_I = 0$ due to the fact that $\Gamma_{22} = 0$. Instead, let us use a modified matrix $\Gamma = L_xL_x^T$, where $L_x^T = [B^T \quad L_2^T]$. It is assumed that L_2 is selected such that $I - L_2$ has the eigenvalues inside the unit circle. This is a technical assumption which will simplify the equations for the recovery matrices. However, there is no need to introduce nonminimum phase zeros into the system by the selection of the free parameter L_2. With this choice of Γ, we will get a nonzero K_I in general for a nonzero L_2.

Let $\hat{L}_x$ satisfy the minimum phase condition on $(A_x, \hat{L}_x, C_x)$ with $\hat{L}_x^T = [\hat{B}_l^T \quad L_2^T]$, whereby it is also required that

$$\text{rank}\begin{bmatrix} zI - A_x & \hat{L}_x \\ C_x & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} zI - A & -B & \hat{B}_l \\ 0 & zI - I & L_2 \\ C & 0 & 0 \end{bmatrix} \\ = n + 2m, \quad \forall z: \rho(z) \geq 1. \quad (35)$$

Then one can compute $\hat{L}_x$ by applying the method proposed in [19].

Using L_x and (29), the following PI-observer gain is derived for the nonminimum phase case:

$$\begin{aligned} K_x &= -A_x^l\hat{L}_x(C_xA^{l-1}\hat{L}_x)^{-1} \quad (36) \\ &= \begin{bmatrix} -A^l\hat{B}_l - A^{l-1}BL_2 - \cdots - BL_2 \\ -L_2 \end{bmatrix}(CA^{l-1}\hat{B}_l)^{-1}. \quad (37) \end{aligned}$$

It is now reasonable to state the following result.

Theorem 3.2: Let the matrix L_2 be selected such that $I - L_2$ has the eigenvalues inside the unit circle. The recovery matrix $M_{PI}(z)$ for the prediction PI-observer take the following form when the optimal LQG/LTR gain in (37) is used:

$$\begin{aligned} M_{PI}(z) &= F(zI - A)^{-1}[B - z^{-l}(A^l\hat{B}_l + A^{l-1}BL_2 + \cdots \\ &\quad + BL_2 + (zI - I)^{-1}BL_2)\hat{B}_a(z)(zI - I) \\ &\quad \cdot (zI - I + L_2)^{-1}] \quad (38) \end{aligned}$$

where $\hat{B}_a(z)$ satisfies

$$C_x\Phi_xL_x = C_x\Phi_x\hat{L}_x\hat{B}_a(z).$$

Proof: Substitution of A_x, B_x, C_x, F_x, and $\hat{L}_x$ in the general recovery matrix of Theorem 3.1 yields

$$M_{PI}(z) = F_x\Phi_x(z)[B_x - z^{-l}A_x^l\hat{L}_x(C_x\Phi_x(z)\hat{L}_x)^{-1} \\ \cdot C_x\Phi_x(z)B_x].$$

Moreover, the different terms in the above equation can be expressed by the following results:

$$\begin{aligned} F_x\Phi_x(z) &= [F\Phi(z) \quad (zI - I)^{-1}F\Phi(z)B] \\ F_x\Phi_x(z)B_x &= F\Phi(z)B \\ C_x\Phi_x(z)B_x &= C\Phi(z)B \\ -A_x^l\hat{L} &= \begin{bmatrix} -A^l\hat{B}_l - A^{l-1}BL_2 - \cdots - BL_2 \\ -L_2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} -z^{-l}&F_x\Phi_x(z)A_x^l\hat{L} \\ &= z^{-l}F\Phi(z)[A^l\hat{B}_l + A^{l-1}BL_2 + \cdots + BL_2 \\ &\quad + (zI - I)^{-1}BL_2] \end{aligned}$$

$$\begin{aligned} (C_x\Phi_x\hat{L}_x)^{-1}&C_x\Phi_xB_x \\ &= (C_x\Phi_x\hat{L}_x)^{-1}C_x\Phi_xL_x(C_x\Phi_xL_x)^{-1}C_x\Phi_xB_x \\ &= \hat{B}_a(z)(C_x\Phi_xL_x)^{-1}C_x\Phi_xB_x \\ &= \hat{B}_a(z)(C\Phi B + (zI - I)^{-1}C\Phi BL_2)^{-1}C\Phi B \\ &= \hat{B}_a(z)(zI - I)(zI - I + L_2)^{-1}. \quad \square \end{aligned}$$
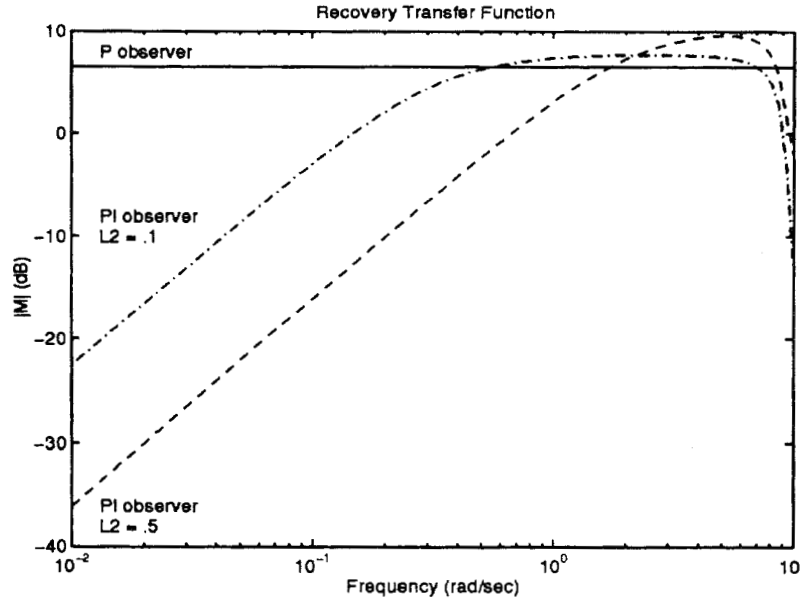
Fig. 1.   The recovery matrix for the filtering P and PI observer.

Note that the assumption on L_2 has been used in connection with the calculation of the last equation. Thus, using these equations, the recovery matrices $M_{PI}(z)$, $M_P(z)$ can be derived directly. Note that we get the recovery matrices for the P-observer from Theorem 3.2 by using $L_2 = 0$. As a direct consequence of Theorem 3.2, we achieve time recovery. This is reflected in the following result.

Theorem 3.3: Let the recovery matrix for the prediction PI-observer be given as in Theorem 3.2. Then for $z = 1$, the recovery matrix satisfies

$$M_{PI}(1) = 0.$$

Proof: Since the recovery matrix $M_{PI}(z)$ can be expressed by

$$
\begin{aligned}
M_{PI}(z) &= F(zI - A)^{-1}[B - z^{-l}(A^l \hat{B}_l + A^{l-1}BL_2 + \cdots + BL_2 \\
&\quad + (zI - I)^{-1}BL_2)(C_x \Phi_x \hat{L}_x)^{-1}C_x \Phi_x B_x] \\
&= F(zI - A)^{-1}[B - z^{-l}(A^l \hat{B}_l + A^{l-1}BL_2 + \cdots + BL_2 \\
&\quad + (zI - I)^{-1}BL_2)(zI - I)((zI - I)C\Phi \hat{B}_l \\
&\quad + C\Phi BL_2)^{-1} \cdot C\Phi B]
\end{aligned}
$$

we simply evaluate it at $z = 1$ and the result follows, i.e.,

$$
\begin{aligned}
M_{PI}(1) &= F(I - A)^{-1}[B - BL_2(C\Phi BL_2)^{-1}C\Phi B] \\
&= F(I - A)^{-1}[B - B] \\
&= 0. \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \square
\end{aligned}
$$

Equivalently, we can calculate the recovery matrix for the filtering PI observer which is given by

$$
\begin{aligned}
M_{PI,F}(z) &= F(zI - A)^{-1}[B - z^{1-l}(A^{l-1}\hat{B}_l + A^{l-2}BL_2 \\
&\quad + \cdots + BL_2 + (zI - I)^{-1}BL_2)\hat{B}_a(z)(zI - I) \\
&\quad \cdot (zI - I + L_2)^{-1}].
\end{aligned}
\tag{39}
$$

Here, it is also possible to get time recovery.

In the minimum phase case, we have $\hat{B}_a(z) = I$ and $\hat{B}_l = B$. Thus, the equations for the recovery matrices take the following forms:

$$
\begin{aligned}
M_{PI}(z) &= F(zI - A)^{-1}[B - z^{-l}(A^l B + A^{l-1}BL_2 + \cdots \\
&\quad + BL_2 + (zI - I)^{-1}BL_2)(zI - I) \\
&\quad \cdot (zI - I + L_2)^{-1}]
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
M_{PI,F}(z) &= F(zI - A)^{-1}[B - z^{1-l}(A^{l-1}B + A^{l-2}BL_2 \\
&\quad + \cdots + BL_2 + (zI - I)^{-1}BL_2)(zI - I) \\
&\quad \cdot (zI - I + L_2)^{-1}].
\end{aligned}
\tag{41}
$$

D. LMI-Based Solutions for LQG/LTR Design

The above Riccati equation in Lemma 3.1 has also been considered in [17]. The result derived in [17] is based on an LMI (linear matrix inequality) formulation of the Riccati equation. Let us consider a linear matrix function $G(Q)$

$$
G(Q) = \begin{bmatrix} AQA^T - Q + BB^T & AQC^T \\ CQA^T & CQC^T \end{bmatrix}.
\tag{42}
$$

Let Q be the largest real symmetric solution of the LMI

$$G(Q) \geq 0.
\tag{43}$$

It can be shown that whenever (C, A) is detectable, the largest real symmetric solution Q to (43) exists and is unique. Moreover, the solution Q to (43) is the solution to the Riccati equation in Lemma 3.1. Knowing the solution Q to (43), we define the following relation between $G(Q)$ and B_Q, D_Q:

$$
G(Q) = \begin{bmatrix} B_Q \\ D_Q \end{bmatrix} \times [B_Q^T \quad D_Q]
\tag{44}
$$

where the system characterized by the quadruple (A, B_Q, C, D_Q) is the minimum phase projection of (A, B, C), including also the infinite zero structure. The direct matrix D_Q is nonsingular, so the system (A, B_Q, C, D_Q) has no infinite zeros at all. Furthermore, the observer gain can be expressed in terms of the matrices B_Q and D_Q as
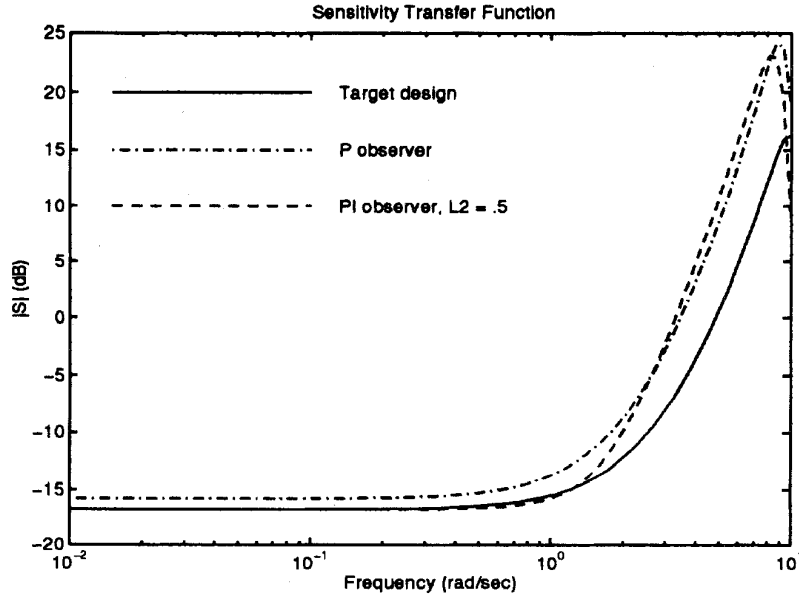
$$K = -B_Q D_Q^{-1}.
\tag{45}$$

Fig. 2. The sensitivity transfer for the target loop and the full loop.

Based on the above LMI result for the Riccati equation, we can derive the following factorization of the system (A, B, C):

$$G(z) = G_{m,Q}(z)\hat{B}_{a,Q}(z)$$
$$= (C(zI - A)^{-1}B_Q + D_Q)\hat{B}_{a,Q}(z) \qquad (46)$$

where $\hat{B}_{a,Q}(z)$ is its all pass factor. Note that this factorization takes the infinite zero structure into account.

By using the two equations for the Kalman filter gain, we get directly the connection between the matrices in (30) and (46)

$$B_Q = A^l \hat{B}, \quad D_Q = CA^{l-1}\hat{B}, \quad \hat{B}_{a,Q}(z) = z^{-l}\hat{B}_a(z). \quad (47)$$

The connection between the two all-pass factors follows directly by the observation that the system (A, B, C) has l zeros at infinity.

If we use the Kalman gain (45) in the recovery matrix for the prediction observer, we obtain directly

$$M_P(z) = F(zI - A)^{-1}(B - B_Q\hat{B}_{a,Q}(z)). \qquad (48)$$

Now, let us consider the LMI solution for the PI-observer by considering the extended system (A_x, L_x, C_x) given by

$$G(Q) = \begin{bmatrix} A_x^T Q A_x - Q + L_x L_x^T & A_x Q C_x^T \\ C_x Q A_x^T & C_x Q C_x^T \end{bmatrix} \geq 0 \qquad (49)$$

where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \qquad (50)$$

is the largest real symmetric solution of (49). Knowing the solution Q, we get the following relation between $G(Q)$ and L_Q, D_Q:

$$G(Q) = \begin{bmatrix} L_Q \\ D_Q \end{bmatrix} \times [L_Q^T \quad D_Q^T]. \qquad (51)$$

The PI-observer gain is then given by

$$K_I = -L_Q D_Q^{-1} = -\begin{bmatrix} B_Q \\ L_2 \end{bmatrix} D_Q^{-1}. \qquad (52)$$

Using this gain in the recovery matrix $M_{PI}(z)$, we have the following result.

Theorem 3.4: Let the matrix L_2 be selected such that $I - L_2$ has the eigenvalues inside the unit circle. The recovery matrix $M_{PI}(z)$ for the prediction PI-observer takes the following form when the gain in (52) is used:

$$M_{PI}(z) = F(zI - A)^{-1}[B - (\overline{B}_Q + (zI - I)^{-1}BL_2)$$
$$\cdot \hat{B}_{a,Q}(z)(zI - I)(zI - I + L_2)^{-1}] \qquad (53)$$

where $\hat{B}_{a,Q}(z)$ satisfies

$$C_x \Phi_x L_x = (D_Q + C_x \Phi_x L_Q)\hat{B}_{a,Q}(z).$$

Proof: See the proof of Theorem 3.2.

Note that the equation for the recovery matrix based on the LMI solution has the same structure as the recovery matrix given in Theorem 3.2. Therefore, time recovery is still obtained.

IV. EXAMPLE

Consider a nonminimum phase single-input–single output (SISO) system represented by

$$A = \begin{bmatrix} 1.1036 & 1 & 0 \\ -0.4060 & 0 & 1 \\ 0.0498 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0803 \\ 0.1544 \\ 0.0179 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0], \quad D = 0.$$

The discrete-time system is nonminimum phase with two zeros at -1.7989 and -0.1239. Note also that CB has full rank. As target design we use [13]

$$F = [7.1222 \quad 7.5293 \quad 2.7373].$$

Applying the optimal LQG/LTR gains for the P and PI observers considered in the above section results in the recovery matrices shown in Fig. 1. The PI-observer has been designed for different values of the gain L_2. As expected, the PI observer achieves time recovery, i.e., the recovery matrix has small gain at low frequencies. It is also important to point out that one can shape the gain of the recovery matrix at low frequencies by the selection of L_2. In Fig. 2, the associated sensitivity transfer functions are shown. Again, the controller based on the PI observer recovers the target sensitivity
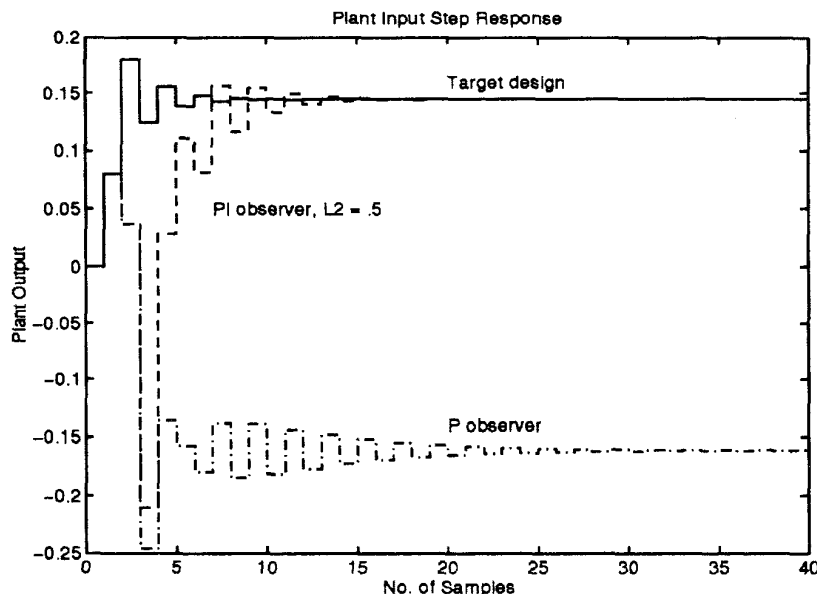
Fig. 3. The step responses for the closed-loop systems.

function at low frequencies. In Fig. 3, the step responses are shown for the target closed loop and the observer-based closed-loop transfer functions. The advantage of the PI observer, as compared to P observer, is clearly shown in this plot as well. Moreover, the step response corresponding to PI observer-based implementation reaches the steady-state value very fast.

V. CONCLUSION

This paper presented two versions of the discrete-time PI observer: a prediction and a filtering PI-observer. Both LQG and LQG/LTR design methods were derived for each observer type with special attention to the time recovery effect of the PI observer. Necessary and sufficient conditions for achieving LTR and time recovery in PI observer-based systems are given.

Moreover, explicit expressions have been derived for the recovery matrices for both the P and the PI-observer in light of optimal LQG/LTR design. These explicit forms are derived for both minimum phase as well as for nonminimum phase systems. As a direct consequence of these derivations, we established that it is always possible to obtain time recovery when PI-observer is applied. Furthermore, the LQG/LTR design method does not have to be employed for achieving time recovery.

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