On the Minimal Average Data-Rate that Guarantees a Given Closed Loop Performance Level

da Silva, Eduardo; Østergaard, Jan; Derpich, Milan

Published in:
I F A C Workshop Series

Publication date:
2010

Document Version
Early version, also known as pre-print

On the Minimal Average Data-Rate that Guarantees a Given Closed Loop Performance Level

Eduardo I. Silva, Milan S. Derpich, and Jan Østergaard*

July 7, 2010

Abstract

This paper deals with control system design subject to average data-rate constraints. By focusing on SISO LTI plants, and a class of source coding schemes, we establish lower and upper bounds on the minimal average data-rate needed to achieve a prescribed performance level. We also provide a specific source coding scheme, within the proposed class, that is guaranteed to achieve the desired performance level at average data-rates below our upper bound. Our results are based upon a recently proposed framework to address control problems subject to average data-rate constraints.

1 Introduction

The study of control systems subject to communication constraints has recently received much attention in the control community (see, e.g., the papers in the special issue [1]). Within this framework, a key question relates to the trade-offs between control objectives and communication constraints. This paper focuses on the interplay between average data-rate constraints (in bits per sample) and stationary performance in a class of networked control systems (NCSs).

When stability is the sole control objective, the results of [2] guarantee that, for a noisy LTI plant model and subject to mild conditions on the noise sources statistics, it is possible to find causal coders, decoders and controllers such that the resulting closed loop system is mean square stable, if and only if the average data-rate is greater than the sum of the logarithm of the absolute value of the unstable plant poles. This result establishes a fundamental separation line between what is achievable in NCSs over digital channels and what is not, when the problem of interest is mean square stability (see also the thorough discussion in the survey paper [3]).

When performance bounds subject to average data-rate constraints are sought, there are relatively fewer results available. There exist lower bounds on the mean square norm of the plant state that make explicit the fact that, as the average data-rate approaches the absolute minimum for stability, the performance becomes arbitrarily poor when disturbances are present [2, 3]. This holds irrespective of how the coder, decoder and controller are chosen. Unfortunately, it is unclear whether or not these bounds are tight in general [3].

*E.I. Silva and M.S. Derpich are with the Departamento de Electrónica, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile. J. Østergaard is with the Department of Electronic Systems, Aalborg University, Niels Jernes Vej 12, DK-9220, Aalborg, Denmark. Emails: eduardo.silva@usm.cl, milan.derpich@usm.cl, jo@es.aau.dk

A more general performance-oriented approach has been pursued in [3,4]. In those works, conditions for separation and certainty equivalence have been investigated in the context of quadratic stochastic problems for fully observed plants with data-rate constraints in the feedback path. If the encoder has a specific recursive structure, then certainty equivalence and a *quasi*-separation principle hold [3]. This result is interesting, but [3] does not give a computable characterization of the optimal encoding policies. A similar drawback is shared by the results reported in [4]. In that work, performance related results are expressed in terms of the so-called sequential rate-distortion function, which is very difficult to compute in general. For fully observed Gaussian first order autoregressive systems, [4] provides an expression for the sequential rate-distortion function. However, it is not clear from the results in [4] whether or not the sequential rate-distortion function is operationally tight (see Section IV-C in [4]).

Other works related to the performance of control systems subject to data-rate constraints are reported in [5] and [6]. The first work focuses on noiseless state estimation subject to data-rate constraints under three different criteria. The case most relevant to this work uses an asymptotic (in time) quadratic criterion to measure the state reconstruction error. For such a measure, it is shown in [5] that the bound established in [2] is sufficient to achieve any prescribed asymptotic distortion level. This is achieved, however, at the expense of arbitrarily large estimation errors for any given finite time. This feature of the solution makes the conclusions in [5] too optimistic. On the other hand, [6] considers non-linear stochastic control problems over noisy channels, and a functional (i.e., not explicit) characterization of the optimal control policies is presented.

In this paper, we focus on SISO LTI plants. Our main contribution is a characterization of upper and lower bounds on the minimal average data-rate that allows one to attain a given performance level (as measured by the stationary variance of the plant output). To that end, we focus on a specific class of source coding schemes that contains, as special cases, the coding schemes studied in [7, 8]. We also provide a specific source coding scheme, within the proposed class, that is guaranteed to achieve the desired performance level at average data-rates below our upper bound. Instrumental to our results is the characterization of the minimal signal-to-noise ratio (SNR) that guarantees a prescribed closed loop performance level in a related class of NCSs (cf. [9, 10]).

The remainder of this paper is organized as follows: Section 2 presents the problem of interest in this paper, and defines the class of considered source coding schemes. Section 3 establishes, for the class of source coding schemes introduced in Section 2, a relationship between average data-rates and an internal SNR. Section 4 focuses on the interplay between SNR constraints and closed loop performance. These results are then used in Section 5 to present our main contribution. Section 6 draws conclusions.

Notation: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of strictly positive real numbers, $\mathbb{R}_0^+ \triangleq \mathbb{R}^+ \cup \{0\}$, $\mathbb{N}_0 \triangleq \{0, 1, \cdots\}$; $|x|$ stands for the magnitude (absolute value) of x. $\mathcal{R}_p$ is the set of all proper and real rational SISO transfer functions, $\mathcal{RH}_\infty$ contains all stable transfer functions in $\mathcal{R}_p$, $\mathcal{RH}_2$ contains all the strictly proper transfer functions in $\mathcal{RH}_\infty$, and $\mathcal{U}_\infty$ contains all the biproper and minimum-phase transfer functions in $\mathcal{RH}_\infty$. $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined as usual, and the associated norms are denoted by $||\cdot||_1$ and $||\cdot||_2$, respectively [11].

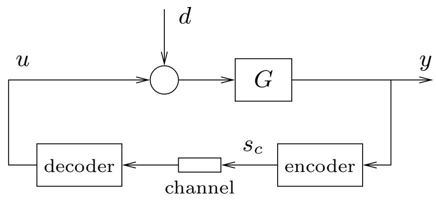

Figure 1: Networked control system.

2 Setup and Problem Definition

2.1 The setup

This paper focuses on the NCS of Figure 1. In that figure, G is a SISO LTI plant, u is the control input, y is the plant output, and d is an input disturbance. The feedback path in Figure 1 comprises an error-free digital channel, and thus quantization becomes mandatory. This task is carried out by an encoder whose output correspond to the binary words s_c. These symbols are then mapped back into real numbers by a decoder. It is clear that the encoder and decoder also embody a controller for the plant. As will become clear as we proceed, a distinction between the controller task and the encoder-decoder task is rather artificial in our setup.

We introduce the following assumptions on G and d:

Assumption 1 *The plant G is SISO, LTI, strictly proper, free of unstable hidden modes, and has no poles or zeros on the unit circle. The initial state of the plant x_o, and the disturbance d, are such that (x_o, d) is jointly second order Gaussian; d is stationary and has spectral factor $\Omega_d \in \mathcal{U}_\infty$.* ■

In this paper we focus on source coding schemes (i.e., encoder-decoder pairs) of the following type:

Definition 1 *The source coding scheme of Figure 1 is said to be linear if and only if its input y and output u are related via*

$$u = T_q q + T_y y, \tag{1}$$

where q is a second order zero-mean i.i.d. sequence, q is independent of (d, x_o), and $T_q, T_y \in \mathcal{R}_p$ are the transfer functions of LTI systems with deterministic initial states. ■

The class of linear coding schemes extends the class of so-called independent and i.i.d. coding schemes introduced in [7,8]. We acknowledge that it is a restricted class of coding schemes. However, its simplicity allows one to actually address control system design problems subject to average data-rate constraints.

Any linear source coding scheme can be written as shown in Figure 2(a), where E and $\bar{H}$ are proper transfer functions. The design of E and $\bar{H}$ is however non-trivial. Indeed, it is easy to see that optimally designing E and $\bar{H}$ amounts to solving an optimal control problem subject to sparsity constraints [12]. To the best of our knowledge, the only known sufficient conditions (see [12]) that allow one to pose such problems as convex ones are not satisfied in our case.

Motivated by the previous discussion, we introduce the alternative rewriting of a linear source coding scheme shown in Figure 2(b), where $H, E, C, F \in \mathcal{R}_p$ are the transfer functions of filters with

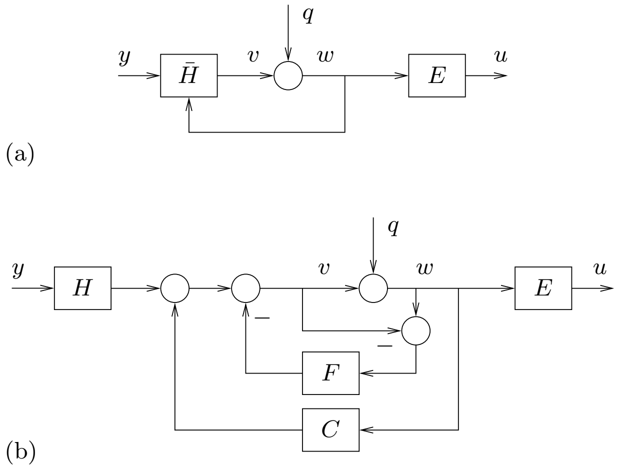

Figure 2: Two alternative representations of a linear source coding scheme.

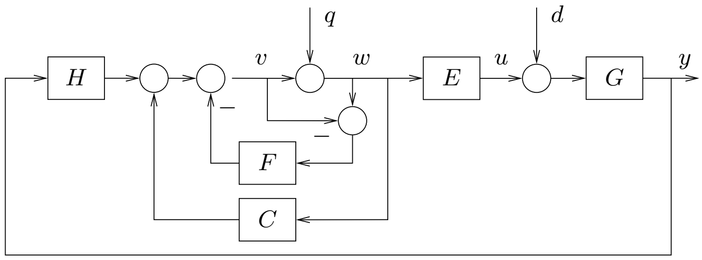

Figure 3: The NCS of Figure 1 when a linear source coding scheme is employed.

deterministic initial states. In the scheme of Figure 2(b), F is redundant. However, as will become clear below, this over-parametrization of linear source coding schemes allows one to construct a convex optimization problem that is equivalent to that of optimally designing $\bar{H}$ and E in Figure 2(a).

When the linear source coding scheme of Figure 2(b) is used in the NCS of Figure 1, the feedback system of Figure 3 arises. For future reference we define[1]

$$\mathcal{S} \triangleq \left\{ (H, C, F, E) \in \mathcal{R}_p^4 : \text{the loop of Figure 3 internally stable and well-posed} \right\}. \tag{2}$$

Remark 1 *In the remainder of this paper, when we refer to a linear source coding scheme, it must be understood that we refer to the specific architecture of Figure 2(b), where q satisfies the conditions in Definition 1.* ■

2.2 Average data-rate constraints

In this section we make a connection between the average data-rate across a linear source coding scheme, and the stationary second order properties of the auxiliary signals w and v in Figure 2. In

[1] Internal stability and well-posedness are defined in the standard way (see, e.g., [13]).

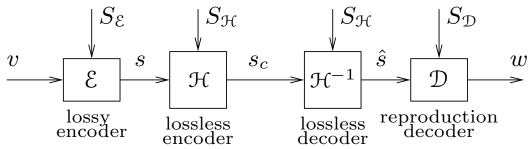

Figure 4: The link between v and w in a linear source coding scheme.

order to do so, we first make the relationship between those signals and the channel symbols s_c in Figure 1 explicit.

Without loss of generality, we assume that the link between v and w in Figure 2 is given by the scheme of Figure 4 [14]. In that figure, $\mathcal{E}$, $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{H}^{-1}$ are causal systems such that[2]

$$\begin{aligned} s(k) &= \mathcal{E}_k(v^k, S_{\mathcal{E}}^k), & s_c(k) &= \mathcal{H}_k\left(s^k, S_{\mathcal{H}}^k\right), \\ \hat{s}(k) &= \mathcal{H}_k^{-1}\left(s_c^k, S_{\mathcal{H}}^k\right), & w(k) &= \mathcal{D}_k(\hat{s}^k, S_{\mathcal{D}}^k), \end{aligned} \tag{3}$$

where $\mathcal{E}_k$, $\mathcal{D}_k$, $\mathcal{H}_k$, $\mathcal{H}_k^{-1}$ are (possibly nonlinear time-varying) deterministic mappings, and $S_{\mathcal{E}}(k)$, $S_{\mathcal{D}}(k)$, $S_{\mathcal{H}}(k)$ correspond to side information that becomes available at time instant k at the encoder side, decoder side, and both at the encoder and decoder sides, respectively. The range of $\mathcal{E}_k$ is assumed to be countable, and that of $\mathcal{H}_k$, to be a countable set of prefix-free binary words [15]. The mappings $\mathcal{H}_k$ and $\mathcal{H}_k^{-1}$ are chosen so as to satisfy

$$\mathcal{H}_k^{-1}(\mathcal{H}_0(s(0), S_{\mathcal{H}}(0)), \cdots, \mathcal{H}_k(s^k, S_{\mathcal{H}}^k), S_{\mathcal{H}}^k) = s(k) \tag{4}$$

for any s^k, $S_{\mathcal{H}}^k$, and any $k \in \mathbb{N}_0$. Condition (4) makes explicit the fact that the blocks $\mathcal{H}$ and $\mathcal{H}^{-1}$ act as a transparent link between the output of $\mathcal{E}$ and the input of $\mathcal{D}$ in Figure 4. Since we assume an error free digital channel, and (4) holds, it follows that $\hat{s} = s$.

It is clear that, when one uses the scheme of Figure 4 as the link between v and w in a linear source coding scheme, one needs to focus on mappings $\mathcal{E}_k, \mathcal{D}_k, \mathcal{H}_k, \mathcal{H}_k^{-1}$ such that the process

$$q \triangleq w - v \tag{5}$$

satisfies the conditions in Definition 1. We also note that, since $\hat{s} = s$, the feedback link between w and the encoder side in Figure 2 does not require a physical channel. (To make w available at the encoder side, it suffices to replicate $\mathcal{D}$ at the encoder side.)

We denote the expected length of the symbol $s_c(i)$ by $R(i)$, and define the average data-rate across the considered source coding scheme as

$$\mathcal{R} \triangleq \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} R(i). \tag{6}$$

We will work under the following assumptions:

Assumption 2

[2] X^k stands for $X(0), \cdots, X(k)$.

(a) $S_{\mathcal{D}}$ *is independent of* (x_o, d).

(b) The mapping $\mathcal{D}_i$ *is such that,* $\forall i \in \mathbb{N}_0$*, there exists another deterministic mapping* g_i *such that* $\hat{s}^i = g_i(w^i, S^i_{\mathcal{D}})$ *(i.e.,* $\mathcal{D}$ *is invertible upon knowledge of* $\mathcal{S}_{\mathcal{D}}$*).* ■

Assumption 2(a) is motivated by the sensible requirement that the block $\mathcal{D}$ in Figure 4 uses only past and present symbols, and side information not related to the message being sent, to construct its current output. On the other hand, if, for some mappings $\mathcal{E}_k$ and $\mathcal{D}_k$, Assumption 2(b) does not hold, then one can define another set of mappings that achieve a coding noise q with the same statistics as in the original situation, but at the expense of a lower average data-rate $\mathcal{R}$.[3] Accordingly, if one aims at minimizing $\mathcal{R}$, then one can focus, without loss of optimality, on mappings $\mathcal{E}_k$ and $\mathcal{D}_k$ satisfying Assumption 2(b).

2.3 Problem definition

The main focus of this paper is on the minimal average data-rate that allows one to attain a given performance level, as measured by the stationary variance of the plant output y. With the definitions introduced above, we are now in a position to formally define the problem of interest as follows:

Problem 1 *Consider the NCS of Figure 1, suppose that Assumption 1 holds, that the source coding scheme is linear, and that the scheme of Figure 4 is used as the link between* v *and* w*. Denote by* D_{inf} *the minimum stationary variance of* y *that is achievable in the NCS of Figure 3 when* $q = 0$*. Find, for a given performance level* $D \in (D_{\text{inf}}, \infty)$*,*

$$\mathcal{R}_D \triangleq \inf_{\sigma_y^2 \leq D} \mathcal{R}, \tag{7}$$

where σ_y^2 *is the stationary variance of* y*, and the optimization is carried out with respect to:*

- *All causal blocks* $\mathcal{E}, \mathcal{D}, \mathcal{H}, \mathcal{H}^{-1}$ *described by* (3)-(4)*, such that the coding noise* q *(see* (5)*) is as in Definition 1, and* $\mathcal{D}$ *satisfies Assumption 2(b).*
- *All side information processes* $S_{\mathcal{E}}$*,* $S_{\mathcal{D}}$*,* $S_{\mathcal{H}}$ *satisfying Assumption 2(a).*
- *All filters* $(H, C, F, E) \in \mathcal{S}$ *(see* (2)*).* ■

Remark 2 *The definition of independent source coding scheme guarantees that, provided* $(H, C, F, E) \in \mathcal{S}$ *and the auxiliary noise* q *satisfies the conditions in Definition 1, the NCS considered in Problem 1 is mean square stable.* ■

In order to solve Problem 1, we will first establish a lower bound on the average data-rate across a linear source coding scheme in terms of stationary second order properties of the auxiliary signals w and v (see Figure 4). The existence of this bound motivates Section 4, where we study the optimal design of linear source coding schemes subject to SNR constraints. These results are then used in Section 5 to give both upper and lower bounds on $\mathcal{R}_D$.

[3] The proof of this claim can be found in [8].

3 Bounding the Average Data-Rate Across Linear Source Coding Schemes

This section extends the results of Section V-A in [8] to show that, when a linear source coding scheme is used in the NCS of Figure 1, the minimum average data-rate across it (subject to a performance constraint) is bounded from below by a simple function of the minimum ratio between the stationary variances of v and q (subject to the same performance constraint). To do so, we start by noting that the following holds:

Theorem 1 *Consider the NCS of Figure 1 where the source coding scheme is linear, and the link between v and w is given by the scheme of Figure 4. If $(H, C, F, E) \in \mathcal{S}$ and Assumptions 1 and 2 hold, then*

$$\mathcal{R} \geq I_\infty(v \to w) \geq \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega, \tag{8}$$

where $I_\infty(v \to w)$ denotes the directed mutual information rate [16, 17] between v and w, S_w is the stationary power spectral density of w, and σ_q^2 is the variance of q. Moreover, equality holds in the last inequality if and only if q is Gaussian.

Proof: The first inequality follows immediately from Theorem 1 in [8]. The second inequality follows from Part 3 of Lemma 5.2 in [7], and from the proof of Theorem 3 in [8] (see also Theorem 4.6 in [17]). ■

Theorem 1 gives a lower bound on the average data-rate across a linear source coding scheme in terms of a simple function of the spectrum of w and the equivalent noise variance σ_q^2. This key result, upon which the remainder of this paper is based upon, can be further simplified. Note that Jensen's inequality yields

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega \leq \frac{1}{2} \ln (1 + \gamma), \quad \gamma \triangleq \frac{\sigma_v^2}{\sigma_q^2}, \tag{9}$$

where σ_v^2 is the stationary variance of v, and γ is the SNR of the linear source coding scheme. By finding the minimal SNR γ subject to a performance constraint (e.g., an upper bound on the stationary variance of the plant output y), one is also calculating an upper bound on the minimal value of the right hand side of (8), subject to the same performance constraint. If, in addition, the optimal solution to the former SNR minimization problem is such that the gap between the left and right hand sides of the inequality in (9) is arbitrarily small (or can be made so without compromising optimality), then, by virtue of Theorem 1, one would immediately get a lower bound on $\mathcal{R}_D$ by solving the much simpler SNR minimization problem. The following result guarantees that this is actually the case:

Lemma 1 *Consider the NCS of Figure 1, where the source coding scheme is linear and has a fixed noise source q. Suppose that Assumption 1 holds and define $\phi \triangleq \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega$. If the choice $(H, C, F, E) = (H_0, C_0, F_0, E_0) \in \mathcal{S}$ is such that $\sigma_y^2 = \sigma_{y,0}^2$ and $\phi = \phi_0$, then, for any arbitrarily small $\eta > 0$, there exist a choice of filters, $(H, C, F, E) = (H_1, C_1, F_1, E_1) \in \mathcal{S}$ such that $\sigma_y^2 = \sigma_{y,0}^2$ $\phi = \phi_0$ and, in addition, $\frac{1}{2}\ln(1 + \frac{\sigma_v^2}{\sigma_q^2}) = \phi_0 + \eta$.*

Proof: The proof of this result goes along the lines of the proof of Theorem 4 in [8]. ■

4 Optimal Performance Subject to SNR Constraints

Motivated by the discussion preceding Lemma 1, we will now focus on the following problem:

Problem 2 *Consider the NCS of Figure 1 where the source coding scheme is linear, and suppose that Assumption 1 holds. Define* $\gamma_{\inf} \triangleq (\prod_{i=1}^{n_p} |p_i|^2) - 1$*, where* p_i *is the* i^{th} *unstable pole of* G*. Find, for a given* $\Gamma \in (\gamma_{\inf}, \infty)$*,*

$$\left[\sigma_y^2\right]_\Gamma \triangleq \inf_{\substack{(H,C,F,E)\in\mathcal{S},\sigma_q^2\in\mathbb{R}^+ \\ \gamma\leq\Gamma}} \sigma_y^2, \tag{10}$$

where all the symbols are as defined before. ■

Remark 3 *It follows from Theorem 17 in [10] that* $\gamma_{\inf}$ *corresponds to the minimal SNR compatible with mean square stability in the NCS of Figure 3.* ■

We note that Problem 2 is concerned with the best achievable performance, as measured by σ_y^2, that is achievable when an upper bound Γ is placed on the SNR γ. As shown in Lemma 4 below, this problem is equivalent to the problem of finding the minimal achievable SNR γ subject to an upper bound on σ_y^2. Our momentary change of focus is only due to technical reasons (see Remark 4 below).

The next lemma states necessary conditions for the 4-tuple (H, C, F, E) to belong to $\mathcal{S}$:

Lemma 2 *Consider the NCS of Figure 1 where the source coding scheme is linear, and suppose that Assumption 1 holds. Define*

$$S \triangleq \frac{1}{1 - C - HEG}, \tag{11}$$

and consider a coprime factorization of G *over* $\mathcal{RH}_\infty$*, i.e., consider* X*,* Y*,* N*,* $D \in \mathcal{RH}_\infty$*, with* X*,* D *biproper, such that* $G = ND^{-1}$ *and* $XD - YN = 1$*. If* $(H, C, F, E) \in \mathcal{S}$*, then* $F \in \mathcal{RH}_2$*,* $HSGE = NY - NDQ$ *for some* $Q \in \mathcal{RH}_\infty$*, and*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln |(1 - F)S| d\omega \geq \sum_{i=1}^{n_p} \ln |p_i|, \tag{12}$$

where p_i *is the* i^{th} *unstable pole of* G*.*

Proof: It is clear from Figure 3 that F is in open loop. Thus F must be stable to ensure the internal stability of the loop. On the other hand, q models a possibly non-linear and discontinuous mapping between v and w and, accordingly, F must be strictly proper for the architecture to be well-posed (the same conclusion applies to C). Therefore, F must belong to $\mathcal{RH}_2$.

In Figure 3, denote the open loop transfer function from y to u by K, i.e., $K \triangleq HE(1 - C)^{-1}$. If $(H, C, F, E) \in \mathcal{S}$, then K must be an admissible one degree-of-freedom controller for G.[4] Thus $S_K \triangleq (1 - GK)^{-1}$ must contain as non-minimum phase (NMP) zeros all the unstable poles of H, E, G and

[4] That is, the closed loop that arises around G when $u = Ky$ is internally stable and well-posed.

$(1-C)^{-1}$ (see Lemma 3.1.3 in [18]). Denote all the unstable poles of H, E and G by $\bar{p}_i$, $i \in \{1, \cdots, n_{\bar{p}}\}$. Since $S = S_K(1-C)^{-1}$ we have from the Bode Integral Theorem [18] that, for any $F \in \mathcal{RH}_2$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|(1-F)S| d\omega \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|S| d\omega = \sum_{i=1}^{n_{\bar{p}}} \ln|\bar{p}_i| \geq \sum_{i=1}^{n_p} \ln|p_i|. \tag{13}$$

To complete the proof, we note that $HSGE$ is the closed loop transfer function from a disturbance at the input of H to the output y. By invoking the Youla-Kucera parametrization [19], we thus conclude that $HSGE$ must be as claimed. ■

By virtue of Lemma 2, we are now ready to present a lower bound on the best achievable performance $\left[\sigma_y^2\right]_\Gamma$. To that end, we will make use of the following mild additional assumption:

Assumption 3 *In Problem 2, $EH \neq 0$ at the optimum.* ■

If EH were identically zero at the optimum, then optimal performance would be achieved by leaving G in open loop. The cases where this happens are clearly of no interest in a networked control setting.

Theorem 2 *Consider the NCS of Figure 1 where the source coding scheme is linear, and suppose that Assumptions 1 and 3 hold. Then,*

$$\left[\sigma_y^2\right]_\Gamma \geq \inf_{\substack{f \in \mathcal{M} \\ Q \in \mathcal{RH}_\infty}} J_\Gamma(f, Q) \triangleq J_{\Gamma,\inf}, \tag{14}$$

where[5]

$$\mathcal{M} \triangleq \left\{ f : \mathbb{D} \to \mathbb{R}_0^+ : ||f||_2^2 < \Gamma + 1, \text{ and } \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f d\omega \geq \sum_{i=1}^{n_p} \ln|p_i| \right\}, \tag{15}$$

and

$$J_\Gamma(f, Q) \triangleq ||W_Q + \Omega_x||_2^2 + \frac{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |W_Q| f d\omega\right)^2}{\Gamma + 1 - ||f||_2^2}, \tag{16}$$

where $W_Q \triangleq NY\Omega_x - ND\Omega_x Q$, N, Y, D and p_i are as in Lemma 2, $\Omega_x \triangleq \xi G \Omega_d$ and $\xi \triangleq \left(\prod_{i=1}^{n_p} \frac{z-p_i}{1-zp_i}\right) \times \left(\prod_{j=1}^{n_c} \frac{1-zc_i}{z-c_i}\right)$, where c_i denotes the i^{th} NMP zero of G. Moreover, the optimization problem on the right hand side of (14) *is convex (i.e., $J_\Gamma(\cdot,\cdot)$ is a convex function of its arguments, and $\mathcal{M} \times \mathcal{RH}_\infty$ is a convex set).*

Proof: Under our assumptions we have that, for any $(H, C, F, E) \in \mathcal{S}$ and $\sigma_q^2 \in \mathbb{R}^+$, σ_y^2 and γ exist and are given by

$$\sigma_y^2 = ||(1-C)SG\Omega_d||_2^2 + \sigma_q^2 ||GES(1-F)||_2^2, \tag{17}$$

$$\gamma = \frac{||HSG\Omega_d||_2^2}{\sigma_q^2} + ||(1-F)S||_2^2 - 1, \tag{18}$$

[5] $\mathbb{D}$ denotes the unit circle.

and also that $||(1-F)S||_2^2 \leq \gamma + 1$.

A simple contradiction-based argument shows that, since Assumption 3 holds, the SNR constraint in (10) is active at the optimum. Hence, at the optimum,

$$||(1-F)S||_2^2 < \Gamma + 1 \tag{19}$$

and

$$\sigma_y^2 = ||(1-C)SG\Omega_d||_2^2 + \frac{||HSG\Omega_d||_2^2 \, ||GES(1-F)||_2^2}{\Gamma + 1 - ||(1-F)S||_2^2} \tag{20}$$

$$\geq ||((HGES+1)G\Omega_d||_2^2 + \frac{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |HSGE| \, |S(1-F)| \, |\Omega_d G| \, d\omega\right)^2}{\Gamma + 1 - ||(1-F)S||_2^2}, \tag{21}$$

where the first equality follows upon solving (18) for σ_q^2 with $\gamma = \Gamma$, and the inequality follows from the Cauchy-Schwartz inequality and the fact that $HSGE+1 = (1-C)S$. If one defines $f \triangleq |(1-F)S|$ and notes that $|\xi| = 1$, then the first part of the result follows immediately from (21), (19) and Lemma 2.

To complete the proof we need to show that the optimization problem on the right hand side of (14) is convex. Since W_Q is an afine function of Q, and both $\mathcal{RH}_\infty$ and $\mathcal{M}$ are convex, it follows from Lemma 4 in [20] that $J_\Gamma(\cdot,\cdot)$ is a convex function of its arguments. The proof is thus completed. ■

Theorem 2 shows that one can calculate a lower bound on the best achievable performance subject to an SNR constraint by solving a convex optimization problem. Convexity guarantees, among other things, that the problem of finding the bound $J_{\Gamma,\inf}$, and also $Q_\delta \in \mathcal{RH}_\infty$ and $f_\delta \in \mathcal{M}$ such that, for any $\delta > 0$,

$$J_\Gamma(f_\delta, Q_\delta) \leq J_{\Gamma,\inf} + \delta, \tag{22}$$

can be addressed numerically using standard algorithms [21].

Remark 4 *As an alternative to Problem 2, one can also consider the problem of finding the minimal SNR subject to a performance constraint. By doing so, and by proceeding as in the proof of Theorem 2, one arrives at an auxiliary optimization problem whose convexity we have not been able to prove yet.* ■

We will next show that a solution of the optimization problem on the right hand side of (14) actually yields a solution to Problem 2. To that end, we start by noting that the following holds:

Lemma 3 *Consider $Q_\delta \in \mathcal{RH}_\infty$ and $f_\delta \in \mathcal{M}$ satisfying $J_\Gamma(f_\delta, Q_\delta) \leq J_{\Gamma,\inf} + \delta$.*

1. *For any $\epsilon_1 > 0$, there exist $\hat{F}_{\delta,\epsilon_1} \in \mathcal{RH}_2$ such that $\prod_{i=1}^{n_p} \frac{1-zp_i}{z-p_i}(1-\hat{F}_{\delta,\epsilon_1}) \in \mathcal{U}_\infty$ and*

$$\left|\left| f_\delta - |1-\hat{F}_{\delta,\epsilon_1}| \right|\right|_2^2 \leq \epsilon_1. \tag{23}$$

2. Consider X, N, Y and D as in Lemma 2, and define[6]

$$K_\delta \triangleq (X - Q_\delta N)^{-1}(Y - Q_\delta D), \tag{24}$$

$$S_\delta \triangleq \frac{z^n}{\text{den}\{K_\delta\}(1 - GK_\delta)}, \tag{25}$$

$$F_{\delta,\epsilon_1} \triangleq 1 - \frac{1 - \hat{F}_{\delta,\epsilon_1}}{S_\delta}. \tag{26}$$

There exists $E_{\delta,\epsilon_2} \in \mathcal{RH}_\infty$ such that, for any $\epsilon_2 > 0$,

$$\left|\left| E_{\delta,\epsilon_2} - \sqrt{\frac{\text{num}\{K_\delta\}\Omega_d}{z^n(1 - F_{\delta,\epsilon_1})}} \right|\right|_2^2 \leq \epsilon_2. \tag{27}$$

Proof:

1. Equation (23) follows by mimicking the proof of Lemma 1 in [22], and by using the definition of $\mathcal{M}$.

2. The Youla-Kucera parametrization guarantees that K_δ is an admissible one degree-of-freedom controller for G. Thus, $(1 - GK_\delta)^{-1}$ has as zeros all the unstable poles of both G and K_δ. Accordingly, S_δ is stable, $S_\delta(\infty) = 1$, and S_δ has as NMP zeros the unstable plant poles only. Given the definition of $\hat{F}_{\delta,\epsilon_1}$, we thus conclude that $F_{\delta,\epsilon_1} \in \mathcal{RH}_2$ and that $(1 - F_{\delta,\epsilon_1})^{-1}$ is also stable. Since, by assumption, Ω_d is stable, we have that $\sqrt{\text{num}\{K_\delta\}\Omega_d(z^n(1 - F_{\delta,\epsilon_1}))^{-1}} \in \mathcal{L}_2$ is analytic outside and on the unit circle and can thus be approximated to any degree of accuracy $\epsilon_2 > 0$ by a stable and proper E_{δ,ϵ_2}. ■

With the aid of Lemma 3 we can state the main result of this section:

Theorem 3 *Consider the NCS of Figure 1 where the source coding scheme is linear, and recall the notation introduced in Lemma 3. If Assumptions 1 and 3 hold, then $\left[\sigma_y^2\right]_\Gamma = J_{\Gamma,\text{inf}}$. Moreover, for any $\epsilon > 0$, there exists $\epsilon_1, \epsilon_2, \delta > 0$ such that choosing $Q_\delta \in \mathcal{RH}_\infty$ and $f_\delta \in \mathcal{M}$ satisfying $J_\Gamma(f_\delta, Q_\delta) \leq J_{\Gamma,\text{inf}} + \delta$, F_{δ,ϵ_1} as in* (26) *with $\hat{F}_{\delta,\epsilon_1}$ satisfying the conditions in Part 1 of Lemma 3, and $E_{\delta,\epsilon_2} \in \mathcal{RH}_\infty$ satisfying* (27)*, guarantees that the choice of parameters*

$$F = F_{\delta,\epsilon_1}, \quad E = E_{\delta,\epsilon_2}, \quad H = \frac{1 - F}{\Omega_d}E, \tag{28}$$

$$C = 1 - \frac{\text{den}\{K_\delta\}}{z^n}, \tag{29}$$

$$\sigma_q^2 = \frac{||HS_\delta G\Omega_d||_2^2}{\Gamma + 1 - ||(1 - F)S_\delta||_2^2}, \tag{30}$$

defines a 4*-tuple $(H, C, F, E) \in \mathcal{S}$, $\sigma_q^2 \in \mathbb{R}_0^+$, and yields $\sigma_y^2 \leq J_{\Gamma,\text{inf}} + \epsilon$ whilst achieving $\gamma = \Gamma$.*

[6]For any $X \in \mathcal{R}_p$, num $\{X\}$ and den $\{X\}$ denote the corresponding numerator and *monic denominator* polynomials (after all cancellations, if any, have been performed).

Proof: Our first claim follows immediately from the second, which is proved below.

Since $H, E \in \mathcal{RH}_\infty$ and $F, C \in \mathcal{RH}_2$ (recall the definition of den $\{\cdot\}$), it follows that the proposed choice of parameters defines a well-posed feedback loop. We next show that the proposed choice of parameters achieves internal stability. To do so, we first note that the open loop transfer function from y to u is given by $HE(1-C)^{-1} = K_\delta$. From the proof of Lemma 3, we know that K_δ is an admissible one degree-of-freedom controller for G. As such, there exist no unstable pole-zero cancellations between K_δ and G [18]. Moreover, it is clear by construction (see (28) and (29)) that there are no unstable pole-zero cancellation among any two transfer functions in the set $\{H, E, (1-C)^{-1}\}$. The above facts imply that the closed loop is internally stable.

We next show that $\sigma_q^2 \in \mathbb{R}_0^+$. Equation (23) implies

$$\left| ||f_\delta||_2^2 - ||1 - F_{\delta,\epsilon_1}||_2^2 \right| \leq \epsilon_1. \tag{31}$$

Thus, since $f_\delta \in \mathcal{M}$, it follows that there exists a sufficiently small $\epsilon_1 > 0$ such that $||(1-F)S_\delta||_2^2 = ||1 - \hat{F}_{\delta,\epsilon_1}||_2^2 < \Gamma + 1$, where $\hat{F}_{\delta,\epsilon_1}$ satisfies the conditions in Part 1 of Lemma 3. On the other hand, $HS_\delta G\Omega_d$ is stable since K_δ is admissible for G and $H\Omega_d$ is stable. As a consequence of the above, σ_q^2 in (30) is finite and positive, as required.

The fact that our choice of parameters guarantees $\gamma = \Gamma$ follows upon using (30) in (18), and noting that $S = S_\delta$.

To complete the proof, it remains to show that the proposed set of parameters is such that $\sigma_y^2 \leq J_{\Gamma,\inf} + \epsilon$ for any $\epsilon > 0$. By definition of E_{δ,ϵ_2} (see (27)), the choice for E in (28) is such that there exists $\Delta E \in \mathcal{L}_2$ such that

$$E = \sqrt{\frac{\text{num}\,\{K_\delta\}\,\Omega_d}{z^n(1-F)}} + \Delta E, \quad ||\Delta E||_2^2 \leq \epsilon_2. \tag{32}$$

Thus,

$$H = \sqrt{\frac{\text{num}\,\{K_\delta\}\,(1-F)}{z^n \Omega_d}} + \frac{(1-F)}{\Omega_d}\Delta E \tag{33}$$

and the numerator of the second term on the right hand side of (20) can be written as

$$\begin{aligned} ||HSG\Omega_d||_2^2\, ||GES(1-F)||_2^2 &= \left|\left| SG\sqrt{\frac{\text{num}\,\{K_\delta\}\,(1-F)\Omega_d}{z^n}} + (1-F)SG\Delta E \right|\right|_2^4 \\ &\leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\text{num}\,\{K_\delta\}\, SG(\Omega_d G)}{z^n} \right| |(1-F)S|\, d\omega \right)^2 + \alpha(\epsilon_2) \\ &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |W_{Q_\delta}| \left| 1 - \hat{F}_{\delta,\epsilon_1} \right| d\omega \right)^2 + \alpha(\epsilon_2), \end{aligned} \tag{34}$$

where α, satisfying $0 \leq \alpha(\epsilon_2) < \infty$, $\forall \epsilon_2 \in \mathbb{R}^+$, is given by

$$\alpha(\epsilon_2) \triangleq M^2\epsilon_2^2 + 2\left|\left|SG\sqrt{\frac{\text{num}\left\{K_\delta\right\}(1-F)\Omega_d}{z^n}}\right|\right|_2^2 M\epsilon_2,$$
$$M \triangleq \max_{w\in[-\pi,\pi]}\left|SG\sqrt{\frac{\text{num}\left\{K_\delta\right\}(1-F)\Omega_d}{z^n}}\right|^2.$$

In (34), the first equality follows upon using (32) and (33), the inequality follows from the Cauchy-Schwartz inequality, and the last equality follows from the fact that our choice of parameters guarantees that $S = S_\delta$ and thus $\text{num}\left\{K_\delta\right\} SG(\Omega_d G)z^{-n} = GK_\delta(1-GK_\delta) = W_{Q_\delta}$ (use the definition of W_{Q_δ} and K_δ), and from our choice for F and the definition of F_{δ,ϵ_1}. By using the above facts and the identity $HSGE + 1 = (1-C)S$ we can write (20) as

$$\sigma_y^2 \leq ||W_{Q_\delta} + \Omega_x||_2^2 + \frac{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|W_{Q_\delta}|\left|1-\hat{F}_{\delta,\epsilon_1}\right|d\omega\right)^2 + \alpha(\epsilon_2)}{\Gamma + 1 - \left|\left|1-\hat{F}_{\delta,\epsilon_1}\right|\right|_2^2}. \tag{35}$$

Now, note that (23) implies $\left|\left||1-\hat{F}_{\delta,\epsilon_1}| - f\right|\right|_1 \leq \sqrt{\epsilon_1}$. The latter inequality and (31) allow one to rewrite, for sufficiently small ϵ_1, (35) as

$$\sigma_y^2 \leq ||W_{Q_\delta} + \Omega_x||_2^2 + \frac{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|W_{Q_\delta}|f_\delta d\omega\right)^2 + \beta(\epsilon_1) + \alpha(\epsilon_2)}{\Gamma + 1 - ||f_\delta||_2^2 - \epsilon_1}, \tag{36}$$

where β, satisfying $0 \leq \beta(\epsilon_1) < \infty$, $\forall \epsilon_1 \in \mathbb{R}^+$, is given by

$$\beta(\epsilon_1) \triangleq \left(\max_{w\in[-\pi,\pi]}|W_Q|\right)^2\epsilon_1 + 2\sqrt{\epsilon_1}\left(\max_{w\in[-\pi,\pi]}|W_Q|\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}|W_Q|f_\delta d\omega. \tag{37}$$

The result follows immediately from (36). (Note that ϵ_2 depends upon ϵ_1 and, hence, has to be chosen, once ϵ_1 is chosen.) ■

Theorem 2 provides a characterization of the solution of Problem 2. In the next section, we will use this result to characterize bounds on $\mathcal{R}_D$.

5 Bounds on the Minimal Average Data-rate needed to Achieve a Given Performance Level

This section presents a characterization of the solution of Problem 1. In particular, we present both upper and lower bounds on $\mathcal{R}_D$, and also a specific implementation of a linear source coding scheme that achieves an average data-rate that is guaranteed to be below our upper bound.

We start by stating the following auxiliary result:

Lemma 4 *Consider the NCS of Figure 1 where the source coding scheme is linear, and suppose that Assumptions 1 and 3 hold. Define*

$$\gamma_D \triangleq \inf_{\substack{(H,C,F,E)\in\mathcal{S},\sigma_q^2\in\mathbb{R}^+ \\ \sigma_y^2\leq D}} \gamma, \tag{38}$$

where all symbols are as defined before. Then, $\gamma_{[\sigma_y^2]_\Gamma} = \Gamma$.

Proof: By solving Problem 1 one finds a set of parameters such that $\sigma_y^2 = [\sigma_y^2]_\Gamma$ and $\gamma = \Gamma$ (recall that Assumption 3 guarantees that the SNR constraint is active at the optimum). It is therefore immediate to see that, by definition, $\gamma_{[\sigma_y^2]_\Gamma} > \Gamma$ is impossible. Assume now that Q_o and f_o are the parameters that minimize $J_{\gamma_{[\sigma_y^2]_\Gamma}}(f, Q)$. Then, since the SNR constraint in Problem 2 is active at the optimum, we have that

$$\gamma_{[\sigma_y^2]_\Gamma} < \Gamma \Rightarrow J_{\gamma_{[\sigma_y^2]_\Gamma}}(Q_o, f_o) > J_\Gamma(Q_o, f_o), \tag{39}$$

which contradicts the optimality of Q_o and f_o. We thus see that $\gamma_{[\sigma_y^2]_\Gamma} = \Gamma$. ■

Lemma 4 shows that Problem 2 is equivalent to the problem of finding γ_D in (38). That is, if some choice of the parameters (H, C, F, E) and σ_q^2 solve Problem 2 and thus achieve a performance level $\sigma_y^2 = [\sigma_y^2]_\Gamma$ and an SNR $\gamma = \Gamma$, then the same parameters optimize the SNR γ subject to the constraint $\sigma_y^2 \leq [\sigma_y^2]_\Gamma$.

We are now in a position to present the main result of this paper:

Theorem 4 *Consider the setup and assumptions of Problem 1. If, in addition, Assumption 3 holds and* $[\sigma_y^2]_\Gamma = D^*$, *then* $\mathcal{R}_{D^*} \geq \frac{1}{2}\ln(1+\Gamma)$. *Moreover, there exists a linear source coding scheme such that* $\mathcal{R} < \frac{1}{2}\ln(1+\Gamma) + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2$, *while satisfying* $\sigma_y^2 = D^* + \epsilon$ *for any* $\epsilon > 0$.

Proof: Our first claim is immediate from Theorems 1 and 3, and Lemmas 1 and 4.

We now prove our second claim. Use Theorem 3 to find filters $(H, C, F, E) = (H_o, C_o, F_o, E_o)$ and a noise variance $\sigma_q^2 = \sigma_o^2$ such that $\gamma = \Gamma$ and $\sigma_y^2 = D^* + \epsilon$ for the desired value of $\epsilon > 0$. Consider an *entropy coded dithered quantizer* (ECDQ) [23] as the link between v and w, i.e., pick blocks $\mathcal{E}$, $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{H}^{-1}$ in Figure 4 such that

$$s(k) = \mathcal{Q}(v(k) + d_h(k)), \qquad s_c(k) = \mathcal{H}_k(s(k), d_h(k)),$$
$$\hat{s}(k) = \mathcal{H}_k^{-1}(s_c(k), d_h(k)), \qquad w(k) = \hat{s}(k) - d_h(k),$$

where d_h is a dither signal available at both the encoder and decoder sides (accordingly, $S_\mathcal{E}(k) = S_\mathcal{D}(k) = S_\mathcal{H}(k) = d_h(k)$), $\mathcal{Q} : \mathbb{R} \to \{i\Delta; i \in \mathbb{Z}\}$ corresponds to a uniform quantizer with step size $\Delta \in \mathbb{R}^+$, $\mathcal{H}_k$ corresponds to the mapping describing an entropy-coder (also called loss-less encoder [15, Ch.5]) whose output symbol is chosen using the conditional distribution of $s(k)$, given $d_h(k)$, and $\mathcal{H}_k^{-1}$ corresponds to the mapping describing the entropy-decoder that is complementary to the entropy-coder at the encoder side. If d_h is an i.i.d. sequence, independent of (x_o, d), and uniformly distributed on $(-\Delta/2, \Delta/2)$, with $\Delta = \sqrt{12\sigma_o^2}$, then our claim follows from Corollary 3 in [8]. ■

Theorem 4 establishes a lower bound on the minimal average data-rate that is needed to achieve a given performance level, when linear source coding schemes are employed to control SISO LTI plants. This lower bound is not tight, but the worst case gap is given by $\frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2$ nats per sample (i.e., ≈ 1.254 bits per sample). The first term of this gap corresponds to the divergence of the ECDQ quantization noise q from Gaussianity, and appears due to the fact that ECDQs generate uniform and not Gaussian coding noise. The second term arises because practical entropy-coders (see proof of Theorem 4) are not perfectly efficient [15, Chapter 5]. (A detailed discussion of these facts can be found in [7,8].) Since we constrain ourselves to a simple class of source coding schemes, this gap seems inescapable but, in our view, it is a fair price to be paid given the simplicity of our approach.

A key aspect of our results is that they are built upon the solution of an SNR optimization problem. This is a key feature of our work, and allows one to easily provide average data-rate guarantees in feedback loops, by using standard control system design techniques.

Remark 5 *If in Problem 1 one removes the performance constraint $\sigma_e^2 \leq D$, then the problem reduces to the calculation of the minimal average data-rate that is compatible with mean square stability, say $\mathcal{R}_{\text{MSS}}$. By using Theorem 17 in [10], and proceeding as in the proof of Theorem 4, it follows that, for the setup and assumptions of the latter theorem,*

$$\sum_{i=1}^{n_p} \ln|p_i| \leq \mathcal{R}_{\text{MSS}} < \sum_{i=1}^{n_p} \ln|p_i| + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2,$$

where p_i is i^{th} unstable pole of G. That is, independent source coding schemes can achieve MSS at average data-rates that are at most $\frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2$ nats per sample away from the absolute lower bound established in [2]. ■

6 Conclusions

This paper has studied control systems closed over noiseless digital channels. By focusing on a class of source coding schemes, we have established lower and upper bounds on the minimal average data-rate needed to achieve a prescribed performance level. Instrumental to our result was the characterization of the minimal SNR that guarantees a given closed loop performance in a related LTI control system.

Future work should focus on situations that include causal but otherwise unrestricted source coding schemes. Extensions to the MIMO case are also under study by the authors.

References

[1] P. Antsaklis and J. Baillieul, "Special issue on technology of networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 5–8, 2007.

[2] G. Nair and R. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 413–436, 2004.

[3] G. Nair, F. Fagnani, S. Zampieri, and R. Evans, "Feedback control under data rate constraints: An overview," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 108–137, 2007.

[4] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1549–1561, 2004.

[5] S. Yuksel and T. Basar, "Minimum rate coding for LTI systems over noiseless channels," *IEEE Transactions on Automatic Control*, vol. 51, no. 12, pp. 1878–1887, 2006.

[6] A. Mahajan and D. Teneketzis, "Optimal Performance of Networked Control Systems with Non-classical Information Structures," *SIAM Journal on Control and Optimization*, vol. 48, no. 3, pp. 1377–1404, 2009.

[7] E. Silva, "A unified framework for the analysis and design of networked control systems," Ph.D. dissertation, School of Electrical Eng. and Comp. Sci., The University of Newcastle, Australia, 2009.

[8] E. Silva, M. Derpich, and J. Østergaard, "A framework for control system design subject to average data-rate constraints," *Conditionally accepted in IEEE Transactions on Automatic Control (Tech. Rep. available at* `http://altair.elo.utfsm.cl/public_elo`*)*, 2010.

[9] J. Braslavsky, R. Middleton, and J. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1391–1403, 2007.

[10] E. Silva, G. Goodwin, and D. Quevedo, "Control system design subject to SNR constraints," *Automatica*, vol. 46, no. 2, pp. 428–436, 2010.

[11] D. Luenberger, *Optimization by Vector Space Methods.* London: John Wiley and Sons, Inc., 1969.

[12] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 274–286, 2006.

[13] K. Zhou, J. Doyle, and K. Glover, *Robust and optimal control.* Prentice Hall, 1996.

[14] R. Gray and D. Neuhoff, "Quantization," *IEEE Transactions on Information Theory*, vol. 44, no. 6, pp. 2325–2383, 1998.

[15] T. Cover and J. Thomas, *Elements of Information Theory*, 2nd ed. John Wiley and Sons, Inc., 2006.

[16] J. Massey, "Causality, feedback and directed information," in *Proc. of the International Symposium on Information Theory and its Applications.*, Hawaii, USA, 1990.

[17] N. Elia, "When Bode meets Shannon: Control oriented feedback communication schemes," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1477–1488, 2004.

[18] M. Seron, J. Braslavsky, and G. Goodwin, *Fundamental Limitations in Filtering and Control.* Springer, 1997.

[19] M. Vidyasagar, *Control Systems Synthesis: A Factorization Approach.* Cambridge, USA: MIT Press, 1985.

[20] M. Derpich and J. Østergaard, "Improved upper bounds to the causal quadratic rate-distortion function for Gaussian stationary sources," *Submitted to IEEE Transactions on Information Theory (available at* ***http://arxiv.org/abs/1001.4181****)*, 2009.

[21] S. Boyd and L. Vandenberghe, *Convex Optimization.* Cambridge University Press, 2004.

[22] M. Derpich, E. Silva, D. Quevedo, and G. Goodwin, "On optimal perfect reconstruction feedback quantizers," *IEEE Transactions on Signal Processing*, vol. 56, no. 8, pp. 3871–3890, 2008.

[23] R. Zamir and M. Feder, "On universal quantization by randomized uniform/lattice quantizers," *IEEE Transactions on Information Theory*, vol. 38, no. 2, pp. 428–436, 1992.