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On formalism and stability of switched systems

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Abstract: In this paper, we formulate a uniform mathematical framework for studying switched systems with piecewise linear partitioned state space and state dependent switching. Based on known results from the theory of differential inclusions, we devise a Lyapunov stability theorem suitable for this class of switched systems. With this, we prove a Lyapunov stability theorem for piecewise linear switched systems by means of a concrete class of Lyapunov functions. Contrary to existing results on the subject, the stability theorems in this paper include Filippov (or relaxed) solutions and allow infinite switching in finite time. Finally, we show that for a class of piecewise linear switched systems, the inertia of the system is not sufficient to determine its stability. A number of examples are provided to illustrate the concepts discussed in this paper.

Keywords: Switched systems; Differential inclusions; Stability; Inertia; Quadratic forms

1 Introduction

The dynamical behavior of many real world systems is subject to instantaneous switches. These systems are part of a rich class of dynamical systems that are commonly known as switched systems or more generally as hybrid systems [1–3]. A number of simple examples of switched systems are given in [2]; an example of a more complex switched system, a supermarket refrigeration process, is described in [4].

In this paper, we study switched systems whose state space is partitioned into subsets, which we call cells. As a result, a local dynamical system is defined on each of the cells, and switching between local dynamical systems takes place whenever a state trajectory travels from one cell to its neighbor.

A crucial notion used when studying dynamic behavior is stability. For switched systems, asymptotic stability is completely characterized by a Lyapunov function [5]. In general, there are no methods for finding such a Lyapunov function, but for a single stable linear system, a quadratic Lyapunov function can be calculated as the solution to a Lyapunov equation. This idea can be generalized to piecewise linear switched systems. As a result, the counterpart of a Lyapunov equation is a linear matrix inequality whose solution is a piecewise quadratic Lyapunov function [6–9]. However, this imposes conservatism as this approach uses the S-procedure [1, 10–11]. Therefore, we ask two intriguing questions. What are the necessary conditions for the existence (or lack of existence) of a piecewise quadratic Lyapunov function for a switched system? Is it possible to formulate the answer in terms of the spectra of the local systems? This was indeed the case in linear system theory, e.g., Theorem 1 in [12]. This paper provides a partial answer to these questions. Specifically, it is shown that in general, a piecewise quadratic Lyapunov function does not restrict the inertia of the local systems. Thus, there is no hope of leaning stability analysis solemnly on the placement of eigenvalues in the complex plane.

The paper is organized in two parts. To some extent, the first part (Sections 3 and 4.1) is a survey. It formulates a uniform mathematical framework for studying the dynamic behavior of switched systems in terms of the theory of differential inclusions. We devise a stability result for a general switched system and specialize it to the class of piecewise linear switched systems. The findings in this part are largely special cases of general results from the theory of differential inclusions [13–15] and of impulse differential inclusions [16]. The exposition is furthermore related to [17], which studies the well-posedness problems of (Carathéodory) solutions for a class of piecewise-linear discontinuous systems, the so-called bimodal systems.

The switched system in this paper generalizes the switched system without control studied in [18]. The stability result in Section 4.1 is related to [6]; however, the current work provides more solutions, since we allow Filippov solutions instead of the less general Carathéodory solutions. In addition, our work allows infinite switching in finite time, which is particularly relevant to the study of Zeno phenomena.

The contribution of this first part is to show how the theory of differential inclusions can be used to formulate stability results for switched systems, e.g., Theorems 1 and 2.

The second part, Section 4.2, describes an application of the stability results obtained in the first part. A switched system composed of linear dynamical systems is chosen for further examination. We show that for a large class of switched systems, one cannot expect to derive stability results based solely on their inertia. This is evidenced by many examples [1, 19], where a system composed of stable systems was shown to be unstable. However, neither necessary nor sufficient conditions for the occurrence of this phenomenon have been characterized so far. The main contribution of this paper, Theorem 3, gives sufficient conditions for the case where the inertia of a switched system is not

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sufficient to derive stability results, and states that a single piecewise quadratic function is Lyapunov for two switched systems with almost arbitrarily different inertia's.

2 Preliminaries

Throughout the paper,

$$E = \mathbb{R}^n$$

denotes the n-dimensional Euclidean space, $(\cdot|\cdot)$ the Euclidean inner product, $|\cdot| = \sqrt{(\cdot|\cdot)}$ the induced norm, and $B_r = \{x \in E \big| |x| \leqslant r, r > 0\}$ the closed r-ball (at $0 \in E$).

2.1 Convex analysis

We recall relevant facts from convex analysis [14,20]. Let $S \subseteq E$ be any subset. The convex hull, $\mathrm{co}(S)$, of S is the smallest convex set containing S, it is given by

$$\mathrm{co}(S) = \{\, x \in E | x = \sum_{i=1}^{n+1} \lambda_i x_i, 1 = \sum_{i=1}^{n+1} \lambda_i, \\ \lambda_i \geqslant 0, x_i \in S, \forall i = 1, \ldots, n+1\}.$$

The convex cone (or conical hull), $\mathrm{cone}(S)$, of S is the smallest convex cone containing S, it is given by

$$\mathrm{cone}(S) = \{\, x \in E | x = \sum_{i=1}^{n+1} \lambda_i x_i, \\ \lambda_i \geqslant 0, x_i \in S, \forall i = 1, \ldots, n+1\}.$$

The affine hull (or affine span), $\mathrm{aff}(S)$, of S is the smallest affine subspace of E containing S, it is given by

$$\mathrm{aff}(S) = \{\, x \in E \Big| x = \sum_{i=1}^{n+1} \lambda_i x_i, 1 = \sum_{i=1}^{k} \lambda_i, \\ \lambda_i \in \mathbb{R}, x_i \in S, \forall i = 1, \ldots, n+1\}.$$

The affine dimension, $\mathrm{afdim}(A)$, of an affine subspace $A \subset E$ is the dimension of the subspace $\{x - y | x, y \in A\}$, and the dimension, $\dim(S)$, of the set S is the affine dimension of $\mathrm{aff}(S)$.

We let $T_S(x)$ denote the contingent cone to S at $x \in S$; see [14] Page 176 or [21] Page 121. Recall that $T_S(x) = E$ if x is in the interior of S, and that $T_S(x) = \mathrm{cl}(\mathrm{cone}(S - x))$ if S is convex; see [14] Page 219 or [21] Page 138.

For a real valued map $v : E \to \mathbb{R}$, we let $D^+v(x)(u)$ denote the upper contingent derivative of v (at x in the direction u), i.e.,

$$D^+v(x)(u) = \limsup_{h \to 0^+, u' \to u} \frac{v(x + hu) - v(x)}{h}.$$

Recall from [14] Pages 282–286 that

$$D^+v(x)(u) = \limsup_{h \to 0^+} \frac{v(x + hu) - v(x)}{h},$$

if v is locally lipschitzean, and that

$$D^+v(x)(u) = Dv(x)(u)$$

if v is continuously differentiable, i.e., $D^+v(x)(u)$ is just the directional derivative of v at x in the direction u.

2.2 Polyhedral sets

In the sequel, we recall facts related to polyhedral sets [20, 22]. A polyhedral set P (in E) is defined as $P = \{x \in E | Ax \leqslant b, A \in \mathbb{R}^{k \times n}, b \in \mathbb{R}^k\}$, where the inequality is to be understood component wise. The improper faces of P are the subsets $\varnothing$ and P, and the (proper) faces are those $F \subset P$ such that $F = H \cap P$ for some supporting hyperplane H of P. The dimension of a polyhedral set P is $\dim(P)$ as defined in Section 2.1. A polyhedral set P (in E) of dimension $n = \dim(E)$ will also be called a cell, and an $(n-1)$-dimensional face F of a cell P will be called a facet (of P). Recall that a polytope is a bounded polyhedral set, or equivalently the convex hull of finitely many points (hence compact by Corollary 1 in [14] Page 20).

Let I be some index set, and $K = \{P_i\}_{i \in I}$ be a family of polyhedral sets in E. We let $|K| = \bigcup_{i \in I} P_i$ with the subspace topology inherited from E, and call K a (polyhedral) complex if

1) each face of any $P \in K$ is in K,

2) $P \cap P'$ is a face of P and P', for any $P, P' \in K$, and

3) each point of $|K|$ has a neighborhood intersection only finitely many elements of K.

We note that condition 3) is only necessary if I is infinite.

Let E' denote either E or a polytope in E of dimension n. By a (piecewise linear) partition of E', we mean a complex K such that $|K| = E'$.

For a partition K of E' with index set I, we let $I^n = \{i \in I | \dim(P_i) = n\}$ denote the set of indices corresponding to the cells of K, $I^n_x = \{i \in I^n | x \in P_i\}$ denote the set of indices corresponding to the cells containing x, and $K^n = \{P_i\}_{i \in I^n}$ denote the family of cells in K. Note that $|K^n| = |K|$.

3 Switched systems

We define a class of switched systems with a piecewise linear partitioned state space, and state dependent switching.

An n-dimensional switched system $\mathcal{S}$ is a triple $\mathcal{S} = (E', K, F)$ where E' denote either $E = \mathbb{R}^n$ or a polytope in E of dimension n, where K is a (piecewise linear) partition of E' with index set I, and where $F = \{f_i\}_{i \in I^n}$ is a family of smooth functions $f_i : U_i \to E$ with U_i an open neighborhood of P_i.

The switched system $\mathcal{S}$ will be called piecewise linear if $F = \{f_i\}_{i \in I^n}$ is a family of linear operators on E. By the inertia $I(\mathcal{S})$ of such a system, we understand the family $\{I(f_i)\}$ of inertia's. Recall that $I(f_i) = (\pi(f_i), \nu(f_i), \delta(f_i))$, where $\pi(f_i)$ is the number of eigenvalues with positive real part, $\nu(f_i)$ is the number with negative real part, and $\delta(f_i)$ is the number with vanishing real part, all counting multiplicity.

The subspace E' plays the role of the state space and each (vector field) f_i describes the local dynamics of the switched system $\mathcal{S}$. The global dynamics is governed by one of the following differential inclusions

$$x'(t) \in f(x(t)), \tag{1}$$

$$x'(t) \in f^{\mathrm{c}}(x(t)), \tag{2}$$

where the set valued maps f and f^{c} are defined by

$$f : E' \to 2^E;\ x \mapsto \{v \in E | v = f_i(x) \text{ if } x \in P_i\}, \tag{3}$$

$$f^{\mathrm{c}} : E' \to 2^E;\ x \mapsto \mathrm{co}(f(x)), \tag{4}$$

with 2^E the power set of E and $\mathrm{co}(f(x))$ the convex hull of $f(x)$. The choice of whether to use (1) or (2) for describing the dynamics of $\mathcal{S}$ depends on the nature of motion to be modeled by $\mathcal{S}$. For details regarding differential inclusions (respectively, equations), we refer to [14], [13] or [15] (respectively, [23] or [24]). Moreover, a good expository overview can be found in [25].

We are now in a position to introduce the notion of a solution to a switched system. For $T > 0$, let J_T denote either $[0, T]$ or $[0, T)$. By a (Carathéodory) solution at $x \in E'$ to the differential inclusion (1), we understand an absolutely continuous function $J_T \to E'$; $t \mapsto x(t)$ which solves the Cauchy problem

$$x'(t) \in f(x(t)) \text{ a.e., } x(0) = x. \tag{5}$$

Hence, a solution is a.e. differentiable on J_T. A Filippov (or relaxed) solution at $x \in E'$ to (1) is by definition a solution at x to the differential inclusion (2), that is a solution as defined above with f in (5) replaced by f^{c}.

Finally, a classical solution at $x \in E'$ to (1) (or (2)) is a continuously differentiable function $J_T \to E'$; $t \mapsto x(t)$ which solves the Cauchy problem

$$x'(t) \in f(x(t)), \ \ x(0) = x. \tag{6}$$

We adapt the above terminology to the switched system $\mathcal{S}$; e.g., a solution to $\mathcal{S}$ is a solution to the differential inclusion describing the global dynamics of $\mathcal{S}$.

In the following sections, we investigate some general properties of the solutions to the switched system, e.g., existence and stability.

3.1 Existence

We address the question of existence of various solutions to (1). Let us start by noting that $f(x) = f^{\mathrm{c}}(x) = f_i(x)$ if x is in the interior of some cell P_i. Hence, on the interior of each cell, the global dynamics is completely described by the local dynamics; here, the theory of ordinary differential equations applies; thus by the Picard-Lindelöf theorem, we conclude that; at any $x \in E'$ which is interior to a cell, there exists a unique classical solution to the differential inclusion (1). However, for a point on a facet non-uniqueness and non-existence can easily occur.

Example 1 Let x be a point on a facet F. Assume that $f(x)$ is a two point set, say $f(x) = \{f_i(x), f_j(x)\}$, and that the intersection

$$f^{\mathrm{c}}(x) \bigcap \mathrm{span}(F - x)$$

contains a relative interior point of $f^{\mathrm{c}}(x)$.

If $f_k(x) \in T_{P_k}(x)$ for $k = i, j$ then there exist two classical solutions at x to (1), and if $f_k(x) \notin T_{P_k}(x)$ for $k = i, j$ then there exists no solution at x to (1). In the case of non-existence, we note that a Filippov solution exists (see Proposition 3), and that for any $x \in F$, there exists a (classical) solution ending at x (see Example 4 for another case of non-existence).

As the above example illustrates, we need to turn our attention to existence (and uniqueness) at points on faces. For ordinary differential equations, the continuity (respectively, Lipschitz continuity) of the vector field guarantees the existence (respectively, existence and uniqueness) of solutions. A similar result holds for differential inclusions. Loosely speaking, if the set valued map is upper semicontinuous (see Proposition 1 for a definition) and has non-empty, closed and convex values then solutions exists Theorem 3 in [14] Page 98.

In our case, f is clearly non-empty and finite (hence compact) valued. Moreover, using that each f_i is continuous we obtain:

Proposition 1 The set-valued map f defined by (3) is upper semicontinuous, i.e., for each $x \in E'$ and any neighborhood U of $f(x)$ there exists a neighborhood V of x such that $f(V) \subset U$.

Unfortunately, if x is on a facet then generically $f(x) = \{f_{i_1}(x), \ldots, f_{i_k}(x)\}$ for some $k > 1$. Hence, $f(x)$ is not convex. However, $f^{\mathrm{c}}(x)$ is a polytope therefore convex and compact, and since the upper semicontinuity of f, established by Proposition 1, carries over to f^{c} Lemma 16 in [13] Page 66, we immediately obtain:

Proposition 2 The set-valued map f^{c} defined by (4) is an upper semicontinuous set valued map with non-empty, convex and compact values.

We are now in a position to prove that at points in the interior of E' (hence at all points if $E' = E$) solutions exist.

Proposition 3 At any interior point x of E' there exists a Filippov solution at x to the differential inclusion (1).

Proof Let $P' = \bigcup\limits_{i \in I_x^n} P_i$ which contains x as an interior point, and $K \subset P'$ be any compact subset with non-empty interior and such that x is an interior point of K. Note that $f^{\mathrm{c}}|P'$ is upper semicontinuous.

Let $K_i = K \cap P_i$, which is compact. By continuity $f_i(K_i)$ is compact; hence, $f(K) = \bigcup\limits_{i \in I_x^n} f_i(K_i)$ is compact (since I_x^n is finite). By Proposition 6 in [14] Page 21, we therefore conclude that $f^{\mathrm{c}}(K)$ is compact.

Let $m(C)$, with C a closed convex subset, denote the element of C with the smallest norm. It then follows that the map $y \mapsto m(f^{\mathrm{c}}(y))$ defined on the interior of K is locally compact, i.e., for each point in the domain of the map there exists an open neighborhood which is mapped into a compact set. Hence, the result follows from Proposition 2, and Theorem 3 in [14] Page 98.

The above proposition does not address the existence of solutions at boundary points of $E' \neq E$, or whether solutions are defined on the whole positive real line J_∞. The later being important when talking about stability. Both questions are related to the tangential condition

$$f^{\mathrm{c}}(x) \bigcap T_{E'}(x) \neq \varnothing, \ \ \forall x \in E', \tag{7}$$

and are answered by the following result.

Proposition 4 For each unbounded P_i, with $i \in I^n$, assume that $f_i(P_i)$ is bounded. Then, at any $x \in E'$, there exists a Filippov solution to (1) defined on $[0, \infty)$

1) if I^n is finite, in the case $E' = E$;

2) iff (7) holds true, in the case $E' \neq E$.

Again this is a direct consequence of existing results. More precisely, use Proposition 1 in [14] Page 60 to conclude that f^{c} is upper hemicontinuous, then the result follows by Proposition 1 and Theorem 1 (b) both in [14] Page 180.

Remark 1 Uniqueness results concerning Filippov solutions may be found in Chapter 2.10 in [13] (see also [14] Page 147).

We end this section with two examples illustrating that

(1) may have solutions whose maximal domain of definition has finite Lebesgue measure even if for each $i \in I^n$ all solutions to the differential equation $x' = f_i(x)$ exist for all time. Hence, we cannot expect a result like Proposition 4 for (Carathéodory) solution. Note also that they illustrate infinite switching in finite time.

Example 2 Consider the switched system

$$\mathcal{S} = (E', K, G),$$

where $E' = \mathbb{R}^3$, $K = \{P_\pm, F\}$ with $P_- \cap P_+ = F$ the x_1x_2-plane, and $G = \{f_\pm\}$ with $f_\pm$ two constant vector fields such that $\mathrm{span}\{f_+(x), f_-(x)\} = F$ for $x \in E'$.

Let $x_0 \in F$, $i \in \{1, 2, \ldots\}$, f_i be f_+ (respectively, f_-) if i is even (respectively, odd), γ_{x_0} denote the classical solution to $x' = f_+(x)$ at x_0, and recursively let γ_{x_i} denote the classical solution to $x' = f_i(x)$ at $x_i = \gamma_{x_{i-1}}(1/2^i)$.

Now, define the curve $\phi_{x_0} : [0, 1) \to F$ by $\phi_{x_0}(t) = \gamma_{x_0}(t)$ if $t \in [0, 1/2]$, and $\phi_{x_0}(t) = \gamma_{x_i}(t - t_i)$ if $t \in [t_i, t_{i+1}]$ with $t_i = \sum\limits_{k=1}^{i} 1/2^k$. Hence, ϕ_{x_0} is a solution to (1) at x_0 which switches (infinitely) between f_+ and f_- at each time instant t_i. Note that ϕ_{x_0} has curve length $1 = \sum\limits_{i=1}^{\infty} 1/2^i$ if $f_\pm(x)$ are unit vectors.

Example 3 Consider the switched system

$$\mathcal{S} = (E', K, F),$$

where $E' = \mathbb{R}^2$, $K^n = \{P_i\}_{i \in I^n}$ with $I^n = \{1, 2, 3, 4\}$ and P_i the ith quadrant, and $F = \{f_i\}$ with f_1, f_2, f_3 and f_4 the constant vector fields $(-2, 1)$, $(-1, -1)$, $(1, -1)$ and $(1, 1)$, respectively.

It follows that the unique solution ϕ_{x_0} to (1) at $x_0 = (2, 0)$ is defined on $[0, 8]$, that it 'spirals' towards the origin ($\lim\limits_{t \to 8} \phi_{x_0}(t) = 0$), and that it switches infinitely from f_i to f_{i+1} (of course $f_{4+1} = f_1$). That is, at each time instant $t_{j+1} = 4 \sum\limits_{i=0}^{j} 1/2^i$ (respectively, state instant $x_{j+1} = (1/2^j, 0)$), with $j \in \{0, 1, 2, \ldots\}$, the system switches from f_4 to f_1.

Note that at $x = 0$ no solution exists. Hence, no solution to $\mathcal{S}$ can be extended to J_∞. However, each Filippov solution exists on J_∞ by Proposition 4. Indeed, each solution can be extended by means of the (unique trivial) Filippov solution at $x = 0$; $x(t) = 0$ for all $t \in J_\infty$.

3.2 Stability

Having established criteria for the existence of solutions defined on J_∞ we can now move on to introduce the notion of stability.

Let g denote either f or f^{c} defined by (3) and (4), respectively. We consider the differential inclusion

$$x'(t) \in g(x(t)), \tag{8}$$

and recall that a point $x_* \in E'$ is called an equilibrium (of (8)) if $0 \in g(x_*)$, and that an equilibrium x_* is stable (respectively, weakly stable) if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_*| < \delta \Rightarrow |x(t) - x_*| < \epsilon, \ \ \forall t \in [0, \infty),$$

for each (respectively, some) solution to the Cauchy problem

$$x'(t) \in g(x(t)) \ \text{ a.e.}, \ \ x(0) = x. \tag{9}$$

An equilibrium point x_* is called asymptotically stable (respectively, weakly asymptotically stable) if it is stable (respectively, weakly stable) and $x(t) \to x_*$ for $t \to \infty$.

Note that for an equilibrium point x_* to be weakly stable, it is necessary that there exists a globally viable neighborhood (under g) of x_*, i.e., a neighborhood U of x_* such that for each $x \in U$ the Cauchy problem $x'(t) = g(x(t))$, $x(0) = x$ has a solution $J_\infty \to E'$; $t \mapsto x(t)$ with $x(t) \in U$ for all $t \in J_\infty$.

An equilibrium point which is not weakly stable is called unstable. e.g., if $x_* = 0$ and f_i is linear then x_* is unstable if there exists $x \in P_i$ with $\lambda x = f_i(x)$ and $\mathrm{Re}(\lambda) > 0$.

The above terminology will be used in connection with the switched system $\mathcal{S}$ (whose global dynamics is governed by g), e.g., x_* is said to be a weakly stable equilibrium for $\mathcal{S}$ if it is so for (8).

Example 4 By convexifying in Example 3, we obtain that $x_* = 0$ is an asymptotically stable equilibrium point. Note, however, that before the convexification, $x_* = 0$ was not even an equilibrium point since no solution exists at $x_* = 0$.

In the case $g = f^{\mathrm{c}}$, we have the following stability result which is a direct consequence of Therorem 8.2 in [15]. It should be seen as a switched system version of the Lyapunov stability theorem. We remark that this result rely crucially on the properties of f^{c} given in Proposition 2.

Theorem 1 Assume that $0 \in f^{\mathrm{c}}(0)$. If there exists $r > 0$ and a continuous positive (respectively, negative) definite function $v : E \to \mathbb{R}$ (respectively, $w : E \to \mathbb{R}$) such that for each $x \in B_r$,

$$D^+v(x)(u) \leqslant w(x), \tag{10}$$

for all $u \in f^{\mathrm{c}}(x)$. Then, the equilibrium point 0 (of $x' \in f^{\mathrm{c}}(x)$) is asymptotically stable. Moreover, the equilibrium point 0 is stable if w is negative semidefinite.

Note that Theorem 1 in particular guarantees that there exists a positive invariant neighborhood of 0, i.e., a neighborhood U of 0 such that for each $x \in U$ all solutions to the Cauchy problem $x'(t) = f^{\mathrm{c}}(x(t))$, $x = x(0)$ exist on J_∞ and belong to U for all $t \in J_\infty$.

4 Piecewise linear switched systems

We fix a piecewise linear switched system

$$\mathcal{S} = (E', K, F)$$

and let (2) describe the overall dynamics of $\mathcal{S}$ (mainly due to Theorem 1). It will be assumed that 0 is an interior point of E', and that it is on a facet (this is to avoid trivialities). Note that $0 \in f^{\mathrm{c}}(0)$ so 0 is an equilibrium.

In the sequel, we will use a family of quadratic forms to construct a continuous positive definite function v and then show that there exists a continuous negative definite function w such that (10) holds true. Based on this construction, we show that 0 is an asymptotically stable equilibrium.

We end this section by showing that the inertia of a piecewise linear switched system is not sufficient to determine its stability. The result is motivated by examples in references [1, 19] evidencing unstable switched systems composed of stable linear systems.

In the sequel, we use standard notation and terminology from the theory of quadratic forms; our main references are [27] and [28].

4.1 Quadratic functions and stability

Inspired by the standard Lyapunov stability theorem, we will now prove a Lyapunov like stability result for piecewise linear switched systems. The idea behind the proof is as follows; for each subsystem f_i, find a quadratic form positive on P_i and decreasing along solutions in P_i.

Let $\{\Phi_i\}_{i\in I^n}$ be a family of quadratic forms on E, and let $\{\phi_i\}_{i\in I^n}$ be the corresponding family of (unique) symmetric bilinear forms, i.e.,

$$\phi_i(x,y) = \frac{1}{2}(\Phi_i(x+y) - \Phi_i(x) - \Phi_i(y)).$$

Each Φ_i should be thought as a candidate quadratic Lyapunov function for the local dynamical system $x' = f_i(x)$ on P_i.

Using the family $\{\Phi_i\}$, we define the set valued map

$$v : E \to 2^{\mathbb{R}};\ x \mapsto \{a \in \mathbb{R} | a = \Phi_i(x) \text{ if } x \in P_i\}. \tag{11}$$

Clearly v should be thought of as a switched system version of a candidate quadratic Lyapunov function. Note that if $\Phi_i(x) = \Phi_j(x)$ for all $x \in P_i \cap P_j$ and $i, j \in I^n$ then v is real single valued ($v : E \to \mathbb{R}$) and locally lipschitzean.

Now, for each $i \in I^n$ define the quadratic form Ψ_i on E by

$$\Psi_i(x) = \phi_i(x, f_i(x));$$

hence, the corresponding symmetric bilinear form ψ_i is

$$2\psi_i(x,y) = \phi_i(x, f_i(y)) + \phi_i(f_i(x), y). \tag{12}$$

We note that $D\Phi_i(x)(f_i(x)) = 2\phi_i(x, f_i(x)) = 2\Psi_i(x)$; hence, Ψ_i is the derivative of Φ_i along the (classical) solutions of $x' = f_i(x)$, i.e., equation (12) is the standard Lyapunov equation.

Let L^n be the set of ordered pairs (i,j) in $I^n \times I^n$ such that $P_i \cap P_j \neq \varnothing$. Similar to the above we define for each $(i,j) \in L^n$ the quadratic form Ψ_{ij} on E by

$$\Psi_{ij}(x) = \phi_i(x, f_j(x)).$$

In order to prove our stability result, Theorem 2 below, we need the following technicality, where we, here and in the sequel, let $S_* \subset E$ denote the set $S - \{0\}$, and let $L_0^n = I_0^n \times I_0^n$.

Lemma 1 Assume that

I) $\Psi_i(x) < 0$ for all $x \in P_{i*}$ and each $i \in I_0^n$, and

II) $\Psi_{ij}(x) < 0$ for all $x \in (P_i \cap P_j)_*$ and each $(i,j) \in L_0^n$.

Then, there exists a continuous negative definite function $w : E \to \mathbb{R}$ such that

III) $w(x) \geqslant \Psi_i(x)$ for all $x \in P_{i*}$ and each $i \in I_0^n$, and

IV) $w(x) \geqslant \Psi_{ij}(x)$ for all $x \in (P_i \cap P_j)_*$ and each $(i,j) \in L_0^n$.

Proof For each $i \in I_0^n$, we claim that there exists $\lambda_i < 0$ such that $\lambda_i(x|x) \geqslant \Psi_i(x)$ for all $x \in P_{i*}$. Because if not $\lambda_i(x|x) < \Psi_i(x)$ for all $\lambda_i < 0$ and some $x \in P_{i*}$; hence, $0 = \lim\limits_{\lambda_i \to 0} \lambda_i(x|x) \leqslant \Psi_i(x)$ which contradicts the assumption. Now, note that $0 > \lambda = \min\limits_{i \in I_0^n} \lambda_i$ since I_0^n is finite. Hence, the continuous negative definite function $x \mapsto \lambda(x|x)$ satisfy III).

In exactly the same manner, we may obtain a continuous negative definite function $x \mapsto \beta(x|x)$ satisfy IV). Hence, the map $w(x) = \alpha(x|x)$, with $\alpha = \min\{\lambda, \beta\}$ can be used.

We are now ready to prove a Lyapunov like stability result for piecewise linear switched systems.

Theorem 2 Let $\mathcal{S}$, Φ_i, Ψ_i and Ψ_{ij} be defined as above. If

$$\Psi_i(x) < 0,\ \ \forall x \in P_{i*}, \tag{13}$$

$$\Phi_i(x) > 0,\ \ \forall x \in P_{i*}, \tag{14}$$

for all $i \in I_0^n$,

$$\Psi_{ij}(x) < 0,\ \ \forall x \in (P_i \bigcap P_j)_*, \tag{15}$$

for all $(i,j) \in L_0^n$, and

$$\Phi_i(x) = \Phi_j(x),\ \ \forall x \in P_i \bigcap P_j, \tag{16}$$

for all $i, j \in I_0^n$. Then, the equilibrium point 0 of $\mathcal{S}$ is asymptotically stable.

Proof We will use Theorem 1, i.e., we need to construct v and w. Therefore, let v be as in (11); hence, by (16) and (14), we conclude that v is a real valued function which is locally lipschitzean (hence continuous) and positive definite. Moreover, by applying $\{\Psi_i\}_{i\in I_0^n}$ and $\{\Psi_{ij}\}_{(i,j)\in L_0^n}$ to Lemma 1, we obtain, by (13) and (15), the continuous negative definite function w.

Hence in order to complete the proof, we need to show that (10) holds true. Therefore, let $r > 0$ be small and such that $B_r \subset E'$.

If $x \in B_r$ is in the interior of some cell say P_i, then $f^{\mathrm{c}}(x) = f_i(x)$ and

$$D^+v(x)(f_i(x)) = D\Phi_i(x)(f_i(x)) = 2\Psi_i(x) \leqslant 2w(x);$$

hence, (10) holds true in this case.

Now, let $x \in B_r$ be a point on a facet. For each $u \in f^{\mathrm{c}}(x)$ there exists $i \in I_x^n$ and $h' > 0$ small such that $(x + hu) \in P_i$ for all $h \in [0, h']$, hence

$$\begin{aligned} D^+v(x)(u) &= D\Phi_i(x)(u) \\ &= D\Phi_i(x)\big(\sum_{j\in I_x^n} \lambda_j f_j(x)\big) \\ &= \sum_{j\in I_x^n} \lambda_j D\Phi_i(x)(f_j(x)) \\ &= \sum_{j\in I_x^n} \lambda_j 2\Psi_{ij}(x) \\ &\leqslant \sum_{j\in I_x^n} \lambda_j 2w(x) = 2w(x), \end{aligned}$$

for any $u \in f^{\mathrm{c}}(x)$.

The triple $(E', K, \{\Phi_i\}_{i\in I^n})$ or v given by (11) will be called a piecewise quadratic function if it satisfies (16), and a (candidate) piecewise quadratic Lyapunov function for $\mathcal{S}$ if it moreover satisfies (13) and (14). Hence, from the proof we may restate Theorem 2 as: the equilibrium point 0 of a piecewise linear switched system $\mathcal{S}$ is asymptotically stable, if there exists a piecewise quadratic Lyapunov function for $\mathcal{S}$.

As indicated by the proof of Theorem 2, we remark that the assumption involving (15) can be relaxed (this will not be pursued further here but will be addressed in future work). However, this assumption cannot be removed completely as the next example shows.

Example 5 Consider the piecewise linear switched system $\mathcal{S} = \{E', K, F\}$, where $E' = \mathbb{R}^2$, where $K^2 = \{P_1, \ldots, P_6\}$ with the partition illustrated on the left Fig. 1, and where $F = \{f_1, \ldots, f_6\}$ with

$$f_1(x) = f_4(x) = (-3x_1 - 2x_2, 5x_1 + 2x_2),$$
$$f_2(x) = f_5(x) = (-3x_1 + 5x_2, -2x_1 + 2x_2),$$
$$f_3(x) = f_6(x) = (-x_1, -x_2).$$

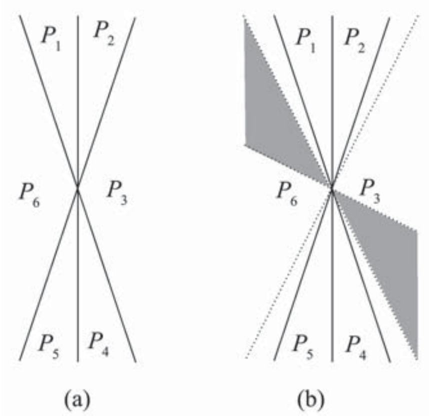

Fig. 1 (a) A partition of E' by means of the sets $\{x|x_1 = 0\}$ and $\{x|x_2 = \pm 3x_1\}$ illustrated by solid lines. (b) The partition with the dotted lines corresponding to the sets $\{x|x_2 = \pm 2x_1\}$ and $\{x|x_2 = -1/2x_1\}$.

With the quadratic forms $\Phi_1, \ldots, \Phi_6$ defined by

$$\Phi_1(x) = \Phi_4(x) = 4x_1^2 + x_2^2 + 4x_1x_2,$$
$$\Phi_2(x) = \Phi_5(x) = 4x_1^2 + x_2^2 - 4x_1x_2,$$
$$\Phi_3(x) = \Phi_6(x) = x_1^2,$$

it follows that

1) Φ_1, Φ_2 and Φ_3 are positive semidefinite (with $\ker(\Phi_1) = \{x|x_2 = -2x_1\}$, $\ker(\Phi_2) = \{x|x_2 = 2x_1\}$ and $\ker(\Phi_3) = \{x|x_1 = 0\}$, see the right Fig. 1), hence (14) is satisfied.

2) $\Psi_3(x) = \Psi_6(x) = -x_1^2$ hence these are negative semidefinite (with $\ker(\Phi_3) = \{x|x_1 = 0\}$) so (13) holds in this case.

3) $\Psi_1(x) = \Psi_4(x) = -4x_1^2 - 4x_2^2 - 10x_1x_2$ and $\Psi_2 = \Psi_5 = 4\Psi_1$; hence, these are negative in the interior of the two larger cones bounded by $\{x|x_2 = -2x_1\}$ and $\{x|x_2 = -1/2x_1\}$, the nonshaded area in the right Fig. 1; therefore, (13) also holds in this case.

Hence, all assumptions of Theorem 2 but (15) are satisfied, since e.g., $\Psi_{12}(x) = -32x_1^2 + 24x_2^2 + 32x_1x_2$ is positive on $\{x|x_1 = 0, x_2 \neq 0\}$.

Now, at any $x \in P_{1*} \cap P_{2*}$ or $x \in P_{4*} \cap P_{5*}$, there exist three solutions of which two convergences to 0; whereas, for the last one we have $|x(t)| \to \infty$ as $t \to \infty$. Hence, 0 is a weakly stable equilibrium point since all other solutions converge to 0.

Note that in the above example, 0 is an asymptotically stable equilibrium point for $\mathcal{S}$ if we used f, rather than f^{c}, to describe the global dynamics. This result should be compared with [6] where only (Carathéodory) solutions to f are considered when studying stability.

4.2 Spectral analysis and stability

We now turn our attention to the assumptions (13) and (14) of Theorem 2. It is well known that in the case of just one subsystem (i.e., $f_i = f = f^{\mathrm{c}}$), these assumptions restrict the inertia of f. For a piecewise linear switched system, it is no longer the case as we will prove in Theorem 3. However, before doing so we illustrate that in some (special) cases the assumptions (13) do indeed restrict the inertia of the system.

Assume that there exists $j \in I^n$ such that Φ_j is nondegenerate; hence, for each $i \in I^n$ there exist two unique linear operators h_{ji} and g_{ji} on E such that

$$\psi_i(x, y) = \phi_j(h_{ji}(x), y) = \phi_j(x, h_{ji}(y)),$$
$$\phi_i(x, y) = \phi_j(g_{ji}(x), y) = \phi_j(x, g_{ji}(y)).$$

Note that each of the very last equalities above argues that $h_{ji} = h_{ji}^*$ and $g_{ji} = g_{ji}^*$ denoting that h_{ji} and g_{ji} are self-dual with respect to ϕ_j (see Chapter II.5 in [27]).

Now with $k_{ji} = g_{ji} \circ f_i$, we have

$$\begin{aligned}&\phi_j(x, k_{ji}(y)) + \phi_j(k_{ji}(x), y)\\ &= \phi_i(x, f_i(y)) + \phi_i(f_i(x), y)\\ &= 2\psi_i(x, y) = 2\phi_j(h_{ji}(x), y),\end{aligned}$$

and therefore,

$$\phi_j(x, k_{ji}(y)) = \phi_j((2h_{ji} - k_{ji})(x), y).$$

As a consequence

$$2h_{ji} = k_{ji} + k_{ji}^* = g_{ji} \circ f_i + f_i^* \circ g_{ji},$$

and in particular $2h_{jj} = f_j + f_j^*$ (here k_{ji}^* and f_j^* denote the dual, with respect to ϕ_j, of k_{ji} and f_j respectively).

The next result tells us that if the eigenspace of $f_j + f_j^*$ has full dimension n, then (in particular) P_j contains no eigenvector of $f_j + f_j^*$ corresponding to a positive eigenvalue. Moreover, it illustrates what restrictions, the assumptions (13) and (14) can impose on the system.

Proposition 5 Assume that Φ_j is nondegenerate for some $j \in I^n$ and that h_{jj} has n linear independent eigenvectors. If

$$\Psi_j(x) < 0, \ \forall x \in P_{j*}, \tag{17}$$
$$\Phi_j(x) > 0, \ \forall x \in P_{j*}, \tag{18}$$

then $P_{j*} \cap E^+(h_{jj}) = \varnothing$ where $E^+(h_{jj})$ denotes the positive eigenspace corresponding to h_{jj}.

Proof For simplicity, let $h = h_{jj}$, $\Phi = \Phi_j$, $\Psi = \Psi_j$ and $P = P_j$. Let $\alpha_1, \ldots, \alpha_k$ be the distinct eigenvalues of h and write E as the direct sum $E = E_1 \oplus \ldots \oplus E_k$ with E_u ($u = 1, \ldots, k$) the eigenspace corresponding to the eigenvalue α_u of h. For convenience, re-index (if necessary) such that

$$E^+(h) = E_1 \oplus \ldots \oplus E_{k'},$$
$$E^-(h) = E_{k'+1} \oplus \ldots \oplus E_{k''},$$
$$\ker(h) = E_{k''+1} \oplus \ldots \oplus E_k.$$

Note that $\psi(x, y) = \alpha_u \phi(x, y)$ for $x, y \in E_u$, and that $\phi(x, y) = 0$ (and hence $\psi(x, y) = 0$) for $x \in E_u, y \in E_{u'}$, and $u \neq u'$. Hence, E can be written as the orthogonal (with respect to Φ) direct sum $E = E_1 \hat{\oplus} \ldots \hat{\oplus} E_k = E^+(h) \hat{\oplus} E^-(h) \hat{\oplus} \ker(h)$, so in particular,

$$\Phi(x) = \sum_{u=1}^{k'} \Phi(x_u) = \sum_{u=1}^{k'} \frac{1}{\alpha_u} \Psi(x_u), \tag{19}$$

with $x = x_1 \hat{\oplus} \ldots \hat{\oplus} x_{k'} \in E^+(h)$. Now, if there exists $x \in P_* \cap E^+(h)$ then by (19) and (17), we conclude that $\Phi(x) < 0$ which contradict (18). This proves the result.

Corollary 1 Assume that Φ_j is nondegenerate for some $j \in I^n$ and let $i \in I^n$ be such that $P_j \cap P_i = F$ for some facet $F \in K$. Assume that h_{ji} has n linear independent eigenvectors. If

$$\Psi_i(x) < 0, \ \forall x \in P_{i*}; \ \Phi_j(x) > 0, \ \forall x \in P_{j*},$$

then $F_* \cap E^+(h_{ji}) = \varnothing$ where $E^+(h_{ji})$ denotes the positive eigenspace corresponding to h_{ji}.

Proof Use the proof of Proposition 5 with $h = h_{ji}$.

Hence, if a piecewise linear switched system is proven stable via a piecewise quadratic Lyapunov function where it is known that the assumptions of Corollary 1 or Proposition 5 holds, then the inertia of $h_{ji} = 1/2(g_{ji} \circ f_i + f_i^* \circ g_{ji})$ is restricted according to the conclusions of either Corollary 1 or Proposition 5. The following example illustrates this.

Example 6 Consider the piecewise linear switched system $\mathcal{S} = \{E, K, G\}$ where $K^n = \{P_1, P_2\}$ and $P_1 \cap P_2 = F$ with $0 \in F$. Let $u = 1$ or $u = 2$ and assume that Φ_1 is nondegenerate, h_{1u} has n linear independent eigenvectors, and that

$$\Phi_1(x) > 0, \ \forall x \in P_{1*},$$
$$\Psi_u(x) < 0, \ \forall x \in P_{u*}.$$

Due to the configuration of the partition, it follows immediately that Φ_1 is positive definite, and that Ψ_u is negative definite. Moreover, using the results above we conclude that $E^+(h_{11}) = \{0\}$ or $E^+(h_{12}) \cap F = \{0\}$.

Now, let $\mathcal{B}$ be an orthonormal basis with respect to (the inner product) ϕ_1. Then,

$$2h_{11} = A_1 + A_1^t, \ \ 2h_{12} = Q_2A_2 + A_2^tQ_2,$$

where Q_2 and A_u are the matrices with respect to $\mathcal{B}$ corresponding to Ψ_2 and f_u, respectively. Note that given any basis $\mathcal{B}'$, we may produce the above equations by an orthogonal coordinate change. So for a switched system to satisfy the set up in this example, it is necessary that all eigenvalue of $A_1 + A_1^t$ are nonpositive or all except possibly one eigenvalue of $Q_2A_2 + A_2^tQ_2$ are nonpositive.

The above example shows that, in some (simple) cases, the existences of a piecewise quadratic Lyapunov function for a piecewise linear switched system restricts the inertia of the system. However, in general, this is not true as Theorem 3 below shows. In references [1, 19], a particular instance of an unstable switched system consisting of two stable linear systems is presented. This has inspired our research, in which, we show that for a large class of piecewise linear switched systems, there is no hope of obtaining stability results based purely on their inertia. For this purpose, we will introduce a notion of Φ-boundedness. For $v \in E$, let $v^\perp$ denote the hyper plane $\{x \in E | (x|v) = 0\}$.

A cell P will be called Φ-bounded with respect to a quadratic form Φ if Φ is nondegenerate, if $\Phi(x) > 0$ for all $x \in P_*$, and if

$$v^\perp \bigcap P_* = \varnothing, \tag{20}$$

for at least one eigenvector v corresponding to the (unique) linear operator l given by $\phi(x, y) = (l(x)|y)$. Hence, if P is a Φ-bounded polyhedral set then P_* is contained in precisely one of the open half-spaces defined by $v^\perp$. As a result, P is bounded in this very special way. For this reason, we have called this notion 'Φ-bounded'.

Before moving on to the above-mentioned result, we note that a straight forward calculation shows that (20) is equivalent to:

either $(x_i|v) > 0$ for each $i = 1, \dots, m$
or $(x_i|v) < 0$ for each $i = 1, \dots, m$,

with $\{x_1, \dots, x_m\}$ a set of generators for P. Hence, the assumption of Φ-boundedness is easy to verify.

Recall that for a linear operator l on E, we write $\nu(l)$ to denote the number of eigenvalues of l with negative real part, counting multiplicity.

Theorem 3 Let $\mathcal{S} = (E', K, F)$ be a piecewise linear switched system, and $(E', K, \{\Phi_i\}_{i \in I^n})$ a piecewise quadratic Lyapunov function for $\mathcal{S}$. If there exists $i \in I^n$ such that P_i is Φ_i-bounded with $0 \in P_i$, then

- for any $0 \leqslant j \leqslant n-1$ there is a linear operator τ on E such that $\nu(\tau) = j$ or $\nu(\tau) = j+1$;
- $(E', K, \{\Phi_i\}_{i \in I^n})$ is also a piecewise quadratic Lyapunov function for the switched system obtained from $\mathcal{S}$ by replacing f_i with τ.

Proof For simplicity, write $\Phi = \Phi_i$ and $P = P_i$. Let $v_1, \dots, v_n$ be the eigenvectors of the linear operator l given by $\phi(x, y) = (l(x)|y)$, and recall that $\{v_1, \dots, v_n\}$ is an orthogonal basis for E with respect to both $(\cdot|\cdot)$ and ϕ. Without loss of generality, we assume that $|v_i| = 1$ for $i = 1, \dots, n$.

By the Φ-bounded condition, we may assume that v_1 satisfy (20), where we have re-indexed if necessary. Note that $\{v_1, \dots, v_n\} \cap P$ is either $\varnothing$ or $\{v_1\}$. Moreover, by rescaling (if necessary), we may assume that the unit ball contain no vertices of P except 0.

Now, consider the function

$$g : D \to \mathbb{R}; \ x = \sum_{i=1}^n \alpha_i v_i \mapsto \sum_{i=2}^n \alpha_i^2,$$

where D denotes the intersection of P and the boundary of the unit ball.

Claim Let $\bar{\alpha} = \max\limits_{x \in D} g(x)$ then $0 \leqslant \bar{\alpha} < 1$: Clearly $0 \leqslant \bar{\alpha}$, and since $1 = |x|^2 = \sum\limits_{i=1}^n \alpha_i^2$, we also have $1 \geqslant \sum\limits_{i=2}^n \alpha_i^2$ hence $\bar{\alpha} \leqslant 1$. Now assume that $\bar{\alpha} = 1$. Then, there is an $x = \sum\limits_{i=1}^n \alpha_i v_i \in D$ with $\alpha_1 = 0$; thus $(x|v_1) = 0$ contradicting the Φ-bounded assumption since $x \in P$. Hence, $\bar{\alpha} < 1$.

Let λ_i $(i = 1, \dots, n)$ be the eigenvalue corresponding to v_i, and for $0 \leqslant j \leqslant n-2$ define a symmetric bilinear form $\psi = \psi_j$ on the basis $\{v_1, \dots, v_n\}$ by

$$\psi(v_1, v_1) = -\frac{1}{(1-\bar{\alpha})^2},$$
$$\begin{aligned} \psi(v_u, v_u) &= \operatorname{sign} \phi(v_u, v_u) \\ &= \operatorname{sign} \lambda_u, \ \ u = 2, \dots, n-j, \end{aligned} \tag{21}$$
$$\begin{aligned} \psi(v_w, v_w) &= -\operatorname{sign} \phi(v_w, v_w) \\ &= -\operatorname{sign} \lambda_w, \ \ w = n-j+1, \dots, n, \end{aligned}$$
$$\psi(v_s, v_t) = 0, \ \ s, t = 1, \dots, n, \ s \neq t. \tag{22}$$

Furthermore, define $\psi = \psi_{n-1}$ as above but with (21) removed. In any case, we extend ψ to E by linearity.

Similarly, for $0 \leqslant j \leqslant n-2$, define the linear operator $\tau = \tau_j$ on the basis $\{v_1, \dots, v_n\}$ by

$$\tau(v_i) = \begin{cases} -\dfrac{1}{(1-\bar{\alpha})^2 \lambda_1} v_1, \\ \dfrac{\operatorname{sign} \lambda_i}{\lambda_i} v_i, & \text{for } i \in \{2, \dots, n-j\}, \\ -\dfrac{\operatorname{sign} \lambda_i}{\lambda_i} v_i, & \text{for } i \in \{n-j+1, \dots, n\}, \end{cases} \tag{23}$$

and let $\tau = \tau_{n-1}$ be defined as above but with (23) removed. In any case, we extend τ to E by linearity.

By construction, we have $\Psi(x) = \phi(x, \tau(x))$ for all $x \in E$, and either $\nu(\tau) = j$ or $\nu(\tau) = j+1$ depending on $\operatorname{sign} \lambda_1$. Hence, the proof is complete if we show that $\Psi(x) < 0$ for all $x \in P - \{0\}$. However, since the unit ball contain no vertices of P (except 0), we only need to prove that $\Psi(x) < 0$ for all $x \in D$. Hence, if $x \in D$, then

$$\begin{aligned} \Psi(x) &= \Psi(\sum_{i=1}^{n} \alpha_i v_i) = \sum_{i=1}^{n} \alpha_i^2 \Psi(v_i) \\ &\leqslant -\frac{\alpha_1^2}{(1-\bar{\alpha})^2} + \sum_{i=2}^{n} \alpha_i^2 \leqslant -\frac{(1-\bar{\alpha})^2}{(1-\bar{\alpha})^2} + \bar{\alpha} \\ &< 0, \end{aligned} \tag{24}$$

where (24) follows from (22).

5 Conclusions

We have used the theory of differential inclusions to formulate stability results for switched systems, namely Theorems 1 and 2, which allow Filippov solutions and infinite switching in finite time. Moreover, a sufficient condition has been proven in Theorem 3 under which the inertia of a switched system is not sufficient to derive stability results. The condition is easily tested as it amounts to verifying simple inequalities.

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