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Pedagogical Imagination in Mathematics Teacher Education

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Abstract: After providing a brief summary of what has already been said about pedagogical imagination, data are presented showing how prospective mathematics teachers can become engaged in such imaginations. With reference to this data, the notion of pedagogical imagination is explored further by relating it to dialogue, social justice, mathematics, hope, and sociological imagination. To illustrate these relationships, different episodes from the data are highlighted. Finally, the central role that pedagogical imagination can play in mathematics teacher education is discussed.

Keywords: pedagogical imagination; dialogue; social justice; mathematics; hope; sociological imagination; critical mathematics education

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The notion of imagination does not contain a well-defined semantic nucleus. One can talk about a child having a good imagination when making a drawing, about a criminal demonstrating imagination when inventing an alibi, and about a person showing imagination when cooking a delicious dinner.

To be used in the context of mathematics teacher education, the notion needs careful elaboration. It may help us to articulate what it might mean to do research from a critical perspective. It is important to study what is taking place, say, in a classroom setting, and to ground this research with careful observations. However, when doing critical research, it is equally important to identify alternatives to what is taking place. It is important to research possibilities. When doing so, pedagogical imagination plays a crucial role.

Pedagogical imagination is not only important for educational research; it is also important for developing educational practices. It is important to invite prospective teachers to look beyond the given curriculum and beyond what is presented in textbooks and to involve them in conceptualising teaching-learning processes that are not part of the school mathematics tradition. This way, prospective teachers might become prepared for assuming a role in changing educational routines.

In the following, we *first* provide a summary of what has already been formulated about pedagogical imagination. *Second*, we present data where prospective mathematics teachers were invited to engage in pedagogical imagination. *Third*, with reference to these data, we outline different features of the notion of pedagogical imagination. *Fourth*, we summarise the central role that pedagogical imagination could play in mathematics teacher education.

1. About Pedagogical Imagination

The importance of pedagogical imagination in mathematics education research was recognised in a supervision session that took place in 1996 in Durban. It was two years after Nelson Mandela had been elected president. The Apartheid period had come to an end, and South Africa had turned into a democracy.

Black and Indian PhD students participated in the supervision session; they belonged to the first generation of PhD students in mathematics education in post-apartheid South

Africa. The supervision of this group was organised through a South African-Danish project of cooperation in mathematics education, and Ole Skovsmose acted as the supervisor.

One of the principal issues for the supervision session was the planning of the data production for the PhD studies. There were many things that could be captured by the data. The Apartheid regime had a devastating impact on the educational system, imposing a strict separation between black, Indian, and white students. Although the apartheid period had come to an end, what can be referred to as the "topology of apartheid" continued almost the same. While laws can be changed overnight, material structures cannot. By the topology of apartheid, we refer to, for instance, the location of neighbourhoods, schools, workplaces, hospitals, and roads. As part of the apartheid policy, the centre of Durban (like in most cities in South Africa) was designated as a white area; the neighbourhoods for black people were positioned as isolated enclaves far away from the city centre, and the Indian neighbourhoods were located in between as a kind of buffer zone. The schools were distributed accordingly: the extremely poorly resourced schools were in the black neighbourhoods, the richly resourced schools in the white neighbourhoods, and the badly resourced schools in the Indian neighbourhoods. This topology remained even after the apartheid laws were abandoned.

It is crucial for research in mathematics education in South Africa to document how the topology of apartheid has had, and continues to have, a destructive impact on black and Indian students' possibilities for learning mathematics. However, the PhD students felt strongly that it was *also* crucial to identify alternatives to what had taken place—and what, to a great extent, continued to take place. They found it important to research not only what *did exist* but also what *did not exist* not yet.

But how can we research something that does not exist? What alternatives could one imagine? Would it be possible to research such alternatives before they are actually implemented? Such questions were addressed during the supervision session in 1996, and it was recognised that a *pedagogical imagination* might constitute a crucial component of research that does not only address what actually takes place in the classroom but also tries to conceptualise what could come to take place. One particular issue that the PhD students felt was urgent to explore was the multicultural classroom. But due to the topology of apartheid, no such multicultural classrooms were in sight. So, how does one research learning opportunities that might be established in a multicultural classroom?

Pedagogical imagination is not only relevant to the post-apartheid situation in South Africa; it is relevant to any situation where one wants to investigate alternatives to what is taking place. The general features of research possibilities have been presented in [1,2]. These features can be expressed by means of a triangle (see Figure 1).

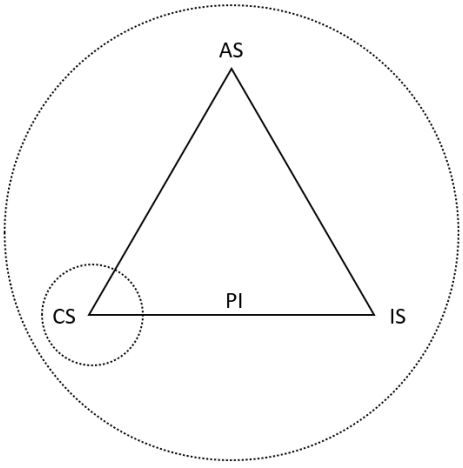

Figure 1. Features of research possibilities.

One corner of the triangle, CS, refers to the current situation. It refers to what is actually taking place in the classroom or whatever situation one is researching. It refers to

what can actually be observed by a researcher who does not want to interfere with what he or she is studying. The imagined situation, IS, refers to what one could imagine taking place. This situation can be thought of as an idealised alternative to what is taking place. The arranged situation, AS, refers to what is possible to organise, considering the idealised alternative, IS, and the constraints set by the actual practical and organisational structures. Pedagogical imagination, PI, is a relationship between the current situation, CS, and the imagined one, IS. Pedagogical imagination is the process that brings us to think about possibilities and alternatives to what is currently taking place.

While naturalistic inquiry—such as that described by Lincoln and Guba [3]—and positivist research only focus on what is currently taking place, i.e., the current situation, CS, critical research explores the CS plus idealised alternatives, IS, and what can be managed in practice, AS. Pedagogical imagination is symbolised in Figure 1 by the side of the triangle that connects the CS with the IS. If one does not imagine anything different from the current situation, say, for methodological reasons, then the triangle collapses into one point, the CS. In this case, one is doing naturalistic inquiry or positivistic research. Doing critical research means researching the whole triangle. In Figure 1, the two different foci of research are indicated by two circles: the smaller circle with CS in the centre represents the focus of naturalistic inquiry and positivistic research, while the larger circle encompassing the whole triangle represents the focus of critical research. (In Chapter 16 in [4], pp. 211–221), one can find a discussion of doing critical research and its relevance for critical mathematics education.)

Vithal [5] applies a critical research approach to investigating the possibilities for learning mathematics in a multicultural classroom in post-apartheid South Africa. During the apartheid regime, the school system was strictly segregated, and this segregation continued in practice even after the apartheid system was abolished. To investigate multicultural classrooms meant researching something that did not actually exist but could be imagined as a future possibility. Vithal conducted her research applying the critical approach condensed in Figure 1. She studied what was currently taking place in educational practice, the CS; she engaged in formulating visions about the learning potentials of a multicultural classroom, the IS; and she managed to establish some temporary and heuristic forms of multiculturality where a few black students attended historically Indian schools.

Biotto Filho [6] applied the same critical research approach, although in a different context. His research focused on the learning of mathematics among a group of children from an orphanage in Brazil. It is important to recognise that the pattern for doing critical research can be applied in all kinds of contexts, but what can be conceptualised as idealized alternatives and what can be managed in practice is highly context-dependent. Helliwell and Ng [7] considered the role of pedagogical imagination in innovating in mathematics teacher education to address issues of sustainability. Carrijo [8] addressed the conditions for learning mathematics among immigrant students in Brazil, and in her presentation of possible landscapes of investigation that can facilitate interaction and dialogues between immigrant and non-immigrant students, she applies pedagogical imagination. Chapman [9] worked on mathematics teacher education, did not use the expression "pedagogical imagination", but only "imagination". For Chapman, many of the actions taken by future teachers had the aim of reflecting on concepts and initiatives already completed. In other words, reflect on the past without paying attention to the future. Imagination is a way of including the future in mathematics teacher education, allowing them to think of alternatives. The imagination process proposed by Chapman with 15 prospective mathematics teachers in Canada was based on three steps: imaging, imagining what, and imagining how. Allowing prospective mathematics teachers to reflect more on the future "can help them to be hopeful about the possibility of doing and making things better" (p. 86).

2. A Mathematics Teacher Education Practice

Pedagogical imagination does not only concern research processes but also daily-life classroom practices. For teachers, it is important to consider alternatives to what is actually

taking place in the classroom. It is important to consider possible alternatives to the school mathematics tradition, which assigns special importance to the teachers' exposition of new topics and to the students' work with pre-formulated exercises.

In the following, we draw on data from [10], where prospective teachers worked in groups; they were asked to imagine some mathematics lessons for primary or secondary school students. Lima's study was guided by concern for inclusive education, and one of the conditions set for the prospective teachers' planning was that at least one student with a disability was present in the classes they imagined working with.

Here we concentrate on one of the groups of prospective teachers: Danilo, Denise, Isabel, and Kátia (all pseudonyms). The school they imagined working in was a state public school, which mostly took in students in situations of social vulnerability. The mathematics classes were imagined to use materials that are usually available in school. The classroom was conceptualised with ample physical space. There were computers that could be brought to the classrooms, but there were fewer computers than students. The classrooms were well lit and ventilated, but they were too small for the number of students. The desks were arranged in rows. There was a projector in each room. The school had an internet network for teachers. Each class lasted 50 min. The prospective teachers imagined that the school adopted traditional types of assessments: tests prepared by teachers as well as assessments prepared by the government. This imagination was built from the personal experiences of each of the prospective teachers.

The prospective teachers specified the class that they imagined working with as being a Grade 6 class from elementary school with 30 students aged between 11 and 13 years old. The students were lively, talkative, and curious. They asked questions about everything, even about what colour pen they should use when copying something from the blackboard. In the classroom, there were some small groups that did not separate. The prospective teachers imagined girls sitting in the first few desks and being very participative; they also imagined a group sitting at the back of the classroom not interested in participating.

Some students were specifically imagined. Bruno using aggressive language but still enjoying mathematics and participating in the activities. Thiago had a lot of difficulty with mathematics; he was nervous and afraid of making mistakes. Giovana had cerebral palsy with motor and speech difficulties; she wrote slowly and walked with crutches; she attended classes regularly and was very studious. Heitor was autistic and very good at drawing, which had not been encouraged at school; he did not have any friends; sometimes he had the urge to get up and walk. And Josué was blind, with his senses of touch and hearing being very developed; he read in Braille. Like Giovana, he sat at the front of the classroom and he liked to participate in the classroom activities.

For this group of students, the prospective teachers decided to create a sequence of three classes dealing with statistics. They should be introduced to data collection, sampling, mean, median, and the construction of tables and graphs.

We are going to present some episodes. (The original Portuguese version of the dialogue that we present is found in [10]. English translations of some of the dialogues are presented in [11].) The transcriptions we present are based on our translations from the original Portuguese versions. Our analysis of the dialogue is different from those presented in the previous publications, which concentrate on addressing problems related to inclusive education.) from the conversations between the prospective teachers Danilo, Denise, Isabel, and Kátia and the researcher Priscila. The episodes concern the articulation of their pedagogical imaginations. We are going to use these articulations in Section 3, "Features of Pedagogical Imagination", when we elaborate further on the very notion of pedagogical imagination.

2.1. Episode 1

To work on concepts related to statistics, the prospective teachers decided to consider an activity in which students would be working in groups. They planned for each group to receive a box in which each of the students should deposit a piece of paper with their

shoe size written on it. Coloured balls of crumpled paper would be made available so that students could build a visual and tactile graph: for each occurrence of a shoe size, a ball would be glued in the column in the cardboard diagram referring to this number. The dialogue below shows concerns about including the blind student.

> ***Denise**: We have to think about the blind student.*
>
> ***Kátia**: Maybe we could make the axes with glitter paper, because it is very rough. We could also, on the axis that has the shoe sizes, cut out the digit in EVA (Ethylene-vinyl acetate) with glitter that he could feel it.*
>
> ***Danilo**: Yes, we could think of using an EVA with glitter and one without glitter to differentiate.*
>
> ***Kátia**: And so, we can do the same for all groups, because it is more beautiful. Furthermore, we do not differentiate only one. More attractive. More colourful.*
>
> ***Denise**: The good thing about EVA is that it is a cheap material. If it is not available at the school, one can buy it.*
>
> ***Priscila**: Yes. But one thing I kept thinking...You talked about cutting out the digits in EVA for the blind student to read.*
>
> ***Denise**: Wow, guys! But is this how he reads?*
>
> ***Kátia**: I think so. I have seen ready-made EVA digits for sale. I bought some the other day; they are cheap.*
>
> ***Denise**: But Kátia, it is not Braille.*
>
> ***Danilo**: That is what I just thought. It is not Braille. . .*
>
> (Quoted after [10], p. 141–142. Our translation.)

After Priscila interfered, the prospective teachers realised that the blind student did not read the numbers by touching something textured, but rather by using the system of Braille. The prospective teachers searched on the internet about the instrument used for writing in Braille. They found information about the reglete, an instrument similar to a ruler that makes marks on paper. They decided that the texts should be written in Braille with the aid of the reglete.

2.2. Episode 2

The prospective teachers imagined each student would pick a piece of paper from the box, read it aloud, and paste a ball in the corresponding position on the graph. Thus, they would construct a bar graph for the distribution of shoe sizes. After the graphs were made, the imagined teacher would ask which shoe size had the highest occurrence and explain that this number is called the mode. Then the prospective teachers began to reflect on the teaching of the median, as they recognised the need for imagining strategies to include everyone. They considered, for instance, the possibility of asking the students to draw the shoes lined up in ascending order on a sheet of paper.

> ***Denise**: I think drawing can also be done. It could involve the students more... Or maybe not, because only one will draw... That's a difficult decision.*
>
> ***Kátia**: I think asking them to draw could lead to the issue of only one student drawing because we will only give out one sheet per group. Indeed, it would be difficult to think about involving everyone.*
>
> ***Denise**: The concern with drawing is related to the blind, right?*
>
> ***Kátia**: That thinking like this as a whole, it's better not to have the drawing.*
>
> ***Priscila**: Not to have it? But what about the boy who likes to draw?*
>
> ***Kátia**: But the blind one?*
>
> ***Isabel**: Right!*

***Danilo**: How difficult it is!*

***Danilo**: Difficult because we cannot focus on just one and exclude the rest as well.*

(Quoted after [10], p. 145. Our translation.)

The concern for including everyone made Denise suggest another idea, which everyone approved:

***Denise**: Guys... I'm thinking of changing everything here. I think I'm the "mess aunt". I think it might be a lot of work, or maybe someone might feel uncomfortable, but at the time of the activity, we would have to think better. I'm thinking here of doing it with the student's own shoe.*

***Kátia**: Can you explain?*

***Denise**: One person says the sizes of the shoes and then the other students place their own shoes with the sizes organised in ascending order in a line on the floor as the peer says the numbers. The blind student knows his shoe size. One student can help the other... One can read the shoe size; another can pick up the colleague's shoe and put it on the line... That way, no one is excluded, not even the student with mobility difficulties nor the autistic student ... Everyone may help and be helped.*

***Isabel**: I think they would like that.*

***Danilo**: I really like that idea, Denise.*

***Denise**: Sometimes I think I have some really crazy ideas.*

***Kátia**: It's not crazy—they will love it!*

***Danilo**: Yes, they will get up—move around.*

***Kátia**: They will get excited! I would never have thought of that!*

(Quoted after [10], p. 145. Our translation.)

The concern for including all students meant that the initial idea of making a drawing was replaced by another that prioritised groupwork, touch, and movement.

2.3. Episode 3

The prospective teachers imagined the class working with the concept of median, based on Denise's idea of lining up the students' own shoes. She sets out her idea in the following way:

***Denise**: I think now we ask them to see the shoe sizes. Then they will see that this amount is equal to the number of students. So, we ask, whose shoe is in the middle? If the number of students is odd, visually it will be easy to see, right? Then a student goes there and looks at the size of that shoe in the middle. If it's an even amount, then we'll have to think with them that they'll have to add the two in the middle and divide by two. The blind student can feel the shoes with his hands and pick up what's in the middle...*

Then Kátia shared a preoccupation about a concept that was a prerequisite for this activity, namely the notion of average (arithmetic mean).

***Kátia**: But look here, thinking about it now... If we haven't introduced the concept of arithmetic mean yet... When it's even* [the shoe size], *they'll have to add it up and divide it by two, right? So, they are going to have to do the arithmetic mean.*

***Danilo**: Yes, they can do it together.*

***Kátia**: But don't you think it would be easier if we first worked the arithmetic mean with them? Because then when they get to that point, they will see that if the number is even, the median is the arithmetic mean of the middle ones.*

***Danilo**: I think it's better to do it at this time in the activity. Like when they have a problem and you're going to ask them to fix that problem.* [...]

***Kátia**: I understand. Do you mean that you think it awakens more interest in the student if we have already given them something to apply the method to?*

***Danilo**: Yes!*

***Denise**: Like throwing the problem to them, right? Like asking: and now, in the case of an even number, how we do it? What do you think you should do? I think we must guide students in this discovery, just like Danilo said. Because they will have something... they will have a problem to solve.*

***Isabel**: I prefer it that way too.*

***Kátia**: You are right. I also learned arithmetic mean by a mechanical way. Type: take the number of elements, add and divide by the number of elements. But I really didn't understand why I needed to do this. I just reproduced it. As time passed, I understood.*

(Quoted after [10], p. 134. Our translation.)

Based on these reflections, the prospective teachers decided to introduce the concept of mean when dealing with the median.

2.4. Episode 4

In all interactions, the prospective teachers were referring to Josué as 'the blind student', a fact observed by Danilo, who expressed his discomfort with doing so:

***Danilo**: Hey guys, I'm sorry... we're saying: blind student... blind student... we're talking about the student's characteristics, but he has a name!*

***Kátia**: You're right!*

***Danilo**: Yes... It looks like we are ignoring that they're people.*

***Kátia**: So, what's his name?*

***Danilo**: Josué.*

***Denise**: Let's be careful to call him Josué!*

***Danilo**: I'm sorry, guys, but this was bothering me!*

***Denise**: You're right!*

(Quoted after [10], p. 203. Our translation.)

After the reflection proposed by Danilo, the prospective teachers started to try to refer to Josué by name. This is important for ensuring that they see the student as a person before paying attention to his or her physical condition.

2.5. Episode 5

After discussing the median, the prospective teachers imagined that they would work with data from the entire classroom so that the students could compare the results of the groups and reflect on the sample. The work would begin by pasting the posters with the graphs prepared by each group on the blackboard:

***Danilo**: I think it would be cool if we glued or hung the charts on the board. Then we could ask them to compare the differences on the results of each group and also to think about the graph of the whole classroom.*

***Kátia**: Sure! Because on the board, it will be in a position that everyone will see! But... what about the blind student?* [Apparently, Kátia forgot to refer to Josué by name.]

***Danilo**: Oh, yes, true!*

***Kátia**: I'm thinking that way he won't participate.*

***Priscila**: Let's think about it; what could be cool to do? Do you think you can paste the poster with the graphs on the board?*

***Danilo**: Yes! It would be just letting him get up and touch; it would be his way of seeing.*

***Denise**: We have to be careful to place it at a height that the student can touch and see the information of the other groups.*

***Kátia**: But then he's going to get up and touch one by one?* [Said in a reproachful tone.]

***Denise**: Yes! And as each one will have already been made with subtitles in Braille, he will be able to touch each one. And the autistic student too... And if other people wanted to touch the graphs too... I don't see any problems.*

(Quoted after [10], p. 143. Our translation.)

The suggestion of the students getting up to touch and analyse the graphs would allow everyone to read them, but the disturbance that would cause in the room bothered Kátia. After some conversation, the prospective teachers realised that it was no problem for everyone to get up, and so it was decided.

2.6. Episode 6

To continue the analysis of statistical data in the whole classroom, each group should receive a printed table to record the collected information. The group with Josué would have its table presented in a textured way. The prospective teachers were concerned about making sure that Josué could come to do things like the others:

***Danilo**: You can put his hand on the chart so he can see the number of balls too.*

***Isabel**: He can count one of the columns and say the result aloud to the others.*

***Denise**: Yes, he can do that.*

***Priscila**: Since you're thinking about Josué, will he be able to do the shoes part?*

***Kátia**: He will! Maybe he will need his peers' help, but he will.*

***Danilo**: I think he will, but Giovana could have problems.*

***Denise**: She has motor difficulties, but I think if she's patient, she'll be able to put the shoe there.*

***Danilo**: And she can get help from other peers as well. This is important!*

(Quoted after [10], p. 151. Our translation.)

Imagining inclusive classes led the prospective teachers to talk about the importance of students helping each other, and they identified an important idea of inclusive education, namely that everyone can help and be helped.

2.7. Episode 7

The prospective teachers imagined that the discussion of the comparison of data from the groups would close the activities related to statistics. However, Kátia felt they were missing the teacher making a formalization, a kind of closure of the content:

***Kátia**: Can we come up with something to formalise on the blackboard?*

[...]

***Danilo**: Guys, this thing about formalising there... I'm thinking... when we do an activity like that, full of things to build, full of graphs, full of steps... I at least think it would be interesting for us as teachers to ask students to write what they thought, what they learned... like a self-assessment. They formalise the content in their own way so that we could know how they thought, if they understood the content... For us to evaluate them to know if they learned... Before we formalise something for them.*

***Kátia**: I think this is cool, but at this stage of schooling, 6th grade, if you ask them to write, some will write what the mode, the arithmetic mean, and the median are... But others will talk about the paper balls. So, if we did a simple questionnaire asking: what did they understand? What is mode? What is arithmetic mean? What is median?*

> ***Denise****: I think that this questionnaire . . . at the beginning, they will be nervous. And then, if they don't succeed, their self-esteem goes downhill.*
>
> ***Kátia****: Yeah... could be.*
>
> ***Denise****: And I thought Danilo's idea was wonderful. They will have to tie the ideas together. I think we can help... direct. Like if they say: Teacher, I don't know what to write! We will say: Ah, write down what you have learned! What did you do? This is nice so they don't get lost and also know what to answer. And we do not interfere in the result.*
>
> (Quoted after [10], p. 154. Our translation.)

This dialogue led the prospective teachers to think about the need for formalization for a final word from the teacher.

2.8. Episode 8

When in the end the prospective teachers were invited by Priscila to evaluate the experience of participating in the pedagogical imagination process, Denise and Kátia made the following observations:

> ***Denise****: In the part where we thought about activities, it was a matter of breaking down barriers in the mind, really. There was something I did not think about, but Danilo mentioned it, or Kátia, or Isabel. Then I thought: Wow! How could I not think of that? And the importance of sharing things too. When you put several heads together, you know?*
>
> ***Kátia****: Yes, talking with colleagues here, at various moments I deconstructed many things that were in my mind.*
>
> (Quoted after [10], p. 209. Our translation.)

These comments indicate that the prospective teachers realised the importance of dialogical interaction in the process of formulating a pedagogical imagination.

3. Features of Pedagogical Imagination

We acknowledge that the notion of pedagogical imagination is not well defined, and we let it preserve its open nature. Still, we want to elaborate on the complexity of the notion by relating it to other equally open notions, namely *dialogue*, *social justice*, *mathematics*, *hope, and sociological imagination*. In order to provide this elaboration, we are going to refer to the different episodes presented in Section 2. We are going to use the episodes to illuminate aspects of pedagogical imagination rather than analyse the episodes as such.

3.1. Pedagogical Imagination and Dialogue

The importance of dialogue for educational processes has been pointed out by Freire [12]. Alrø and Skovsmose [13] try to bring some specification to the notion by identifying eight dialogic acts, namely *getting in contact*, *locating*, *identifying*, *advocating*, *thinking aloud*, *reformulating*, *challenging*, and *evaluating*. A dialogue can also be characterised "negatively" in terms of non-dialogical communicative acts. Faustino and Skovsmose [14] have identified eight such acts: ignoring, disqualifying, and lecturing being some of them. A dialogue can then be characterised as a communicative process where plenty of dialogic acts occur and few non-dialogic acts.

Dialogue is important for constructing and articulating pedagogical imaginations. Dialogic interactions might help to distance pedagogical imaginations from personal and private fantasies and turn them into collective ambitions.

The conversations we have presented in the eight episodes are rich in dialogic acts, while non-dialogic acts seldom appear. The conversation in Episode 2 concerned the possibility of students making drawings. Denise thought aloud: *I think drawing can also be done.* However, Kátia perceived difficulties and made a challenge: *I think asking them to draw could lead to the issue of only one student drawing because we will only give out one sheet per*

group. Indeed, it would be difficult to think about involving everyone. Denise located a different challenge: *The concern with drawing is related to the blind, right?* She identified a different possibility: *One person says the shoe sizes, and then the other students place their own shoes with the sizes organised in ascending order in a line on the floor as the peer says the numbers.* She advocated for this by adding: *The blind student knows his shoe size. One student can help the other* [. . .] *That way, no one is excluded, not even the student with mobility difficulties nor the autistic student . . . Everyone may help and be helped.* Isabel followed up with an evaluation: *I think they would like that.* Danilo made his evaluation by stating: *I really like that idea, Denise.*

In Episode 3, Kátia asked: But look here, thinking about it now... If we haven't introduced the concept of arithmetic mean yet... When it's even [the shoe size], they'll have to add it up and divide it by two, right? So, they are going to have to do the arithmetic mean. She believed that for students to learn a new concept (median), it was necessary for them to master a concept that she considered a prerequisite, namely the average. She did so even if, as in the present case, it only involved a simple calculation. Through dialogue recalling previous school experiences, the prospective teachers realised that letting students try to learn a concept based on the need to solve a problem might be more meaningful for them.

In Episode 4, the prospective teachers reflected on the importance of taking care of the way in which they referred to a blind student: he should be referred to by name and not by his disability. They recognised that calling a person by name was an attitude of respect. In Episodes 5 and 6, the dialogue allowed the prospective teachers to think of strategies so that everyone could participate in the activities. From the dialogue in Episode 7, they came to consider the need and feasibility of formalizations on the part of the teacher. In Episode 8, Denise precisely defines the role of dialogue in articulating a pedagogical imagination: *There was something I did not think about, but Danilo mentioned it, or Kátia, or Isabel. Then I thought: Wow! How could I not think of that? And also the importance of sharing things too. When you put several heads together, you know?*

3.2. Pedagogical Imagination and Social Justice

We do not assume that a mathematics education for social justice needs to start out with a specific definition of social justice. We do not see mathematics education for social justice as being an education that "informs" students, prospective teachers, or anybody else about what social justice means. Instead, we see education for social justice as education that engages the participants in expressing what they find to be just and unjust. It is education that brings to the forefront the participants' concerns and ideas. One need not expect the existence of broad agreement about what justice is; the crucial point is that possible divergences are expressed and provide points of departure for conversations and dialogues. Conceptions of social justice are social constructions; they can take place in mathematics classrooms, both in schools and at universities. Dialogues are crucial for such constructions. (For a discussion of social justice as a social construction that can take place in the classroom, see [4].

Visions about social justice are formed through pedagogical imagination. Logically speaking, pedagogical imagination might be rooted in any set of educational and political visions. One could, for instance, claim that the management approach in education is guided by a pedagogical imagination directed towards a neo-liberal horizon. However, we do not intend to operate with an all-embracing and free-flowing concept of pedagogical imagination. We pay particular attention to conceptions of social justice and to ways of pointing out social injustices. Such injustices could concern economic inequalities, structural poverty, exploitation of workers, racism, sexism, any type of homophobia, and social exclusion. The notion of pedagogical imagination that we want to operate with tries to conceptualise educational practices that contest any such cases of social injustice.

Pedagogical imagination can encompass grand visions about social justice. However, it can also be concrete and concern particular cases. It was part of the whole contextualization of the prospective teachers' pedagogical imagination that students with different disabilities

would be present in the classroom. Josué was a blind student. How do we meet his needs in the mathematics classroom? We are dealing with a particular issue of social justice. In Episode 1, Kátia made a proposal: *Maybe we could make the axes with glitter paper because it is very rough. We could also, on the axis that has the shoe sizes, cut out the digit in EVA with glitter that he could feel it.* Danilo followed up on this idea, and Kátia concluded: *And so, we do the same for all groups because it is more beautiful. Furthermore, we do not differentiate only one. More attractive. More colourful.* For the prospective teachers, it was important that Josué, like the others, could identify the axes of the coordinate systems in which graphs were depicted. Understanding that a blind person reads with their hands was essential for thinking about activities that would include Josué.

When Danilo, in Episode 4, exclaimed that the student had a name and they should use it, it concerned a specific feature of social justice. Not calling Josué by name but referring to him as "the blind person", is a way of dehumanizing him. It means seeing the disability before the person. The quest to call him by name is a humanizing act. Freire [12] highlights that humanization is a vocation of human nature. However, for Freire dehumanization is present both in individuals who have their humanity stolen and in those who steal it from others. The search for humanization is possible because dehumanization is not a given destiny, but the result of an unjust social order. In this sense, the search for the humanization of Josué was also a search for social justice.

The process of pedagogical imagination led the prospective teachers to conceive a dialogical classroom in which students interacted, respected, and helped each other. Denise's comments in Episode 2 explain this: *That way, no one is excluded, not even the student with mobility difficulties nor the autistic student. Everyone may help and be helped. Everyone may help and be helped.* A statement in the same direction was made by Danilo when, in Episode 6, the group was concerned about the autonomy of Giovana, a student with motor difficulties, in carrying out an activity. The prospective teachers found that stimulating collaboration among students was something positive, as Danilo pointed out: *And she can get help from other peers as well. This is important!*

3.3. Pedagogical Imagination and Mathematics

In a school context, mathematics might be considered to be fixed in the form of a pre-defined curriculum. A curriculum might be carefully specified in terms of topics to be covered and in terms of the chosen textbook. What can be referred to as the school mathematics tradition sets a definite agenda for what is taking place in the classroom. An important task for the teacher is to make an exposition of a new topic as presented in the textbook and to clarify students' doubts and uncertainties. The students' task is to solve exercises as they are formulated in the textbook. Finally, it is the teacher's task to check if the students have solved the exercises correctly. According to the school mathematics tradition, a principal criterion for students having understood mathematics is for them to be able to solve the relevant exercises correctly.

Pedagogical imagination signifies a readiness to conceptualise alternative educational possibilities. It is important that the mathematics teacher is ready to consider other ideas and themes than those defined by the curriculum. Working with landscapes of investigation is one such possibility (see, for instance, [15]). It is important as well that the teacher recognise ways of organising the teaching-learning processes different from those engraved in the school mathematics tradition. We see a readiness to construct educational alternatives as an important component of the mathematics teacher's professionalism; as a consequence, pedagogical imagination becomes an important ingredient in any mathematics teacher education programme.

The imagined classroom activities deviated from the traditional model of mathematics teaching; the prospective teachers looked for possibilities to engage their students in an investigative process. In Episode 3, the prospective teachers discussed the importance of teaching the concept of mean so that students could also find the median of a distribution with an even number of elements. Kátia's initial discomfort was dealt with dialogically,

where, for instance, Danilo highlighted that it was better for the students to learn when they were involved in solving a problem—a fact that was corroborated by Denise. The discussion led Kátia to realise that, in fact, students' investigations could contribute to non-systematic and more meaningful learning processes.

In Episode 7, the need for a formalization made by the teacher was put on the agenda. Until then, the imagined classroom activities concerned the students' activities, a fact that worried Kátia. She asked: *Can we come up with something to formalise on the blackboard?* The prospective teachers considered the need for formalization, which, however, Danilo found might go in the opposite direction of what they had proposed up to then. He proposed that a formalization could be completed by the students: *I at least think it would be interesting for us as teachers to ask students to write what they thought, what they learned... like a self-assessment . . . Before we formalise something for them.*

The collective pedagogical imagination allowed Kátia to move away from a comfort zone. Unexpected things might happen when students get up to touch the posters (Episode 5), when the explanation of the concept of average comes before the students are faced with the calculation of the median (Episode 3), and when the students are recommended to sketch definitions for concepts learned before the teacher presents formal clarification of the applied notions (Episode 7). These imagined situations seemed to make Kátia ready to enter a risk zone.

According to Penteado [16], the practice of many teachers is located in a comfort zone where, most of the time, they can predict and control what is going to take place. The word "comfort" indicates, for instance, that the teachers do not risk facing mathematics questions that they cannot answer. Moving into a risk zone means that the teacher might lose control and predictability with respect to the students' activities and questions. However, moving into a risk zone also generates educational possibilities. Pedagogical imagination led Kátia to consider entering a risk zone and revealed that openness to risks generates possibilities.

3.4. Pedagogical Imagination and Hope

Freire [17] pays particular attention to the notion of hope. To him, hope is an integral part of a struggle for a better society. In Freire's words: "I do not understand human existence and the struggle needed to improve it, apart from hope and dream. Hope is an ontological need" (p. 8). Freire does not assume a Marxist-like determinism, according to which a classless society will emerge due to some pre-identified economic laws. Freire points out that the struggle for improving the world cannot be reduced to some "calculated acts alone", or to a "purely scientific approach" (p. 9). He highlights that, without a minimum of hope, "we cannot so much as start the struggle" (p. 9).

In the monumental work *The Principle of Hope* [18], first published in German in three volumes in 1954, 1955, and 1959, Bloch provides a careful discussion of the concept of hope. Bloch was a Marxist, but not in any orthodox way. In line with Freire, he claims that socio-political changes do not take place according to some economic laws that can be identified in advance. Political actions need to be fueled by hope. Bloch points out that hope concerns "dreams of a better life" ([18], Volume I, p. 11), and it needs to be added that he is talking about dreams of a better life *on earth*. In the *Principles of Hope*, Bloch uses the expression "concrete-utopian horizon" in order to unite utopian visions with real-life political actions ([18], Volume I, p. 146). To operate within a concrete-utopian horizon means to articulate visions about what could be desirable. However, such ideas need not be wild speculations; they can be directed towards the horizon of real-life possibilities. They can be formulated within a "concrete-utopian horizon". This is an important feature of pedagogical imagination.

When the prospective teachers started their project work, they were presented with a range of conditions that needed to be considered. It was made explicit that the class they were going to consider would include students with different disabilities. It was stipulated that it would be a public school, which in a Brazilian context indicates that resources might be limited. It was also stipulated that the curriculum must be observed. By

adding such conditions, the prospective pedagogical imagination became formed within a "concrete-utopian horizon", exemplified when Kátia in Episode 1 proposed the use of a material and promptly recognised that it was cheap and therefore possible to use.

One could have invited the prospective teachers to apply their pedagogical imagination freely. They could be invited to consider any possible mathematics content and any form of educational environment. A pedagogical imagination can be elaborated as "free speculations", and this might invite a more fundamental critique of what is actually taking place in the mathematics classroom. Specific conditioning imposes limitations on the pedagogical imagination, but such conditioning might also help to intensify an imagination and ensure that it takes place within a "concrete-utopian horizon". A pedagogical imagination brings us beyond what is currently taking place, directing us towards realistic alternatives.

It was specified that the prospective teachers had to consider a classroom that included students with different disabilities. This made prospective teachers consider mathematics classes from an inclusive perspective. At no time was the non-participation of Josué, Giovana, or Heitor considered. Classes were imagined to be for everyone.

In Episode 1, the initial planning was changed in order to include everyone, and the students were lining up their shoes. However, it is important to be aware of the fact that whenever a mathematics activity operates with numbers referring to the students—like height, weight, body mass, or shoe size—new problems might occur. By putting things in numbers, some students might be pointed out as being, say, the tallest, the smallest, and the heaviest. Being pointed out as having the smallest or biggest pair of shoes might be a problem to some: "Making differences among students public in terms of numbers might provoke bullying and ruin the self-esteem of some students" ([4], p. 141). (See [19]) for similar remarks concerning the use of the Body Mass Index in the mathematics classroom.) In Episode 2, the danger of bullying was somehow indicated by Denise's remark: *maybe someone might feel uncomfortable*. This remark was not further unfolded in the dialogue. However, a pedagogical imagination is temporary and preliminary; it is always in need of being further elaborated upon; it is rooted in a dialogue that is always in need of being continued.

As part of the pedagogical imagination, the prospective teachers considered that all students were important actors and responsible for their mathematics learning. There was no discrimination. The pedagogical imagination made it possible for them to approach a pedagogy of hope, as formulated by Freire [17]. The prospective teachers shared the hope associated with an inclusive education, in which the classroom turns into a favourable place for encounters between differences. The process of pedagogical imagination invited the prospective teachers to express their hopes with respect to the mathematics classroom but also with respect to the organisation of the school and society in general.

3.5. Pedagogical Imagination and Sociological Imagination

In 1959, Charles Wright Mills published the book *Sociological Imagination* [20]. It was during a time when the positivism paradigm dominated sociology. According to this paradigm, the overall aim of sociology is to provide extensive and reliable descriptions of social facts. The descriptions should be objective in the sense that they should not reflect the perspectives of the researchers, and they should be neutral in the sense that they do not encompass political or ethical priorities. According to logical positivism, all scientific disciplines should contribute to the unity of science by providing such descriptions. The natural sciences—in particular, physics—were considered paradigmatic role models for sociology too.

Wright Mills did not agree. According to him, sociology should not only try to describe social realities; it should also try to present possible alternatives to such realities. This led Wright Mills to present sociological imagination as an integral part of sociological studies.

The notion of pedagogical imagination is closely related to the notion of sociological imagination. The purpose of articulating pedagogical imagination is to move beyond the descriptive paradigm in educational research. The aim is not only to describe educational

realities as they might appear in schools and classrooms but also to formulate visions about alternative educational possibilities. The aim is not only to research what is, but also what could become. Such ideas bring the notion of pedagogical imagination to the forefront of what can be referred to as critical research. As indicated in Figure 1, positivistic research (and naturalistic research in general) focuses on the current situation, CS, while critical research also addresses what could be imagined, IS, and what could be managed, AS.

Pedagogical imagination is not only part of critical research; it also concerns daily-life classroom practices. Pedagogical imagination can be formulated by teachers when they consider possible alternatives to what is usually taking place in their classroom practices. Teachers can consider if there are new topics that they could address in their teaching. They can consider the students' possible reactions and also the parents' possible reactions to controversial issues. Teachers can consider different possibilities for covering the curriculum and alternative ways for preparing students for tests and exams. To address such issues means to engage in pedagogical imagination.

We find it important to engage prospective teachers in formulating pedagogical imaginations. Pedagogical imagination is part of teachers' professional expertise, and it is important to prepare prospective teachers for this.

4. Pedagogical Imagination and Mathematics Teacher Education

Pedagogical imagination emerges from a set of collective constructions. One can assume that pedagogical imagination can take the form of free-floating personal speculations, but this is not the notion we are operating with. We see pedagogic imagination as developing through processes of *dialogue*. We relate pedagogical imagination to visions about *social justice*. Pedagogical imagination also concerns *mathematics*. The mathematics curriculum might appear to be a given, and so might many of the accompanying classroom routines. But the mathematics curriculum and the classroom routines can be recognised as being not necessities but contingencies. Pedagogical imagination is an expression of *hope*. Hope can concern social and political development, but simultaneously, it may concern a different life in the classroom. The notion of pedagogical imagination is related to the notion of *sociological imagination*. However, while sociological imagination first of all concerns ways of doing research, pedagogical imagination concerns ways of doing education research *as well as* ways of making educational innovation.

It is important that mathematics teacher education does not coagulate as preparation for adapting to the given social and educational order. It is important to prepare mathematics teachers for moving beyond what is normally taken as given. As a consequence, *we see pedagogical imagination as a crucial component of mathematics teacher education*.

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