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Linkedness of Cartesian products of complete graphs*

Leif K. Jørgensen 

Department of Mathematical Sciences, Aalborg University, Denmark

Guillermo Pineda-Villavicencio [†] , Julien Ugon [‡] 

Federation University, Ballarat, Australia and

School of Information Technology, Deakin University, Geelong, Australia

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Abstract

This paper is concerned with the linkedness of Cartesian products of complete graphs. A graph with at least $2k$ vertices is k -linked if, for every set of $2k$ distinct vertices organised in arbitrary k pairs of vertices, there are k vertex-disjoint paths joining the vertices in the pairs.

We show that the Cartesian product $K^{d_1+1} \times K^{d_2+1}$ of complete graphs K^{d_1+1} and K^{d_2+1} is $\lfloor (d_1 + d_2)/2 \rfloor$ -linked for $d_1, d_2 \geq 2$, and this is best possible.

This result is connected to graphs of simple polytopes. The Cartesian product $K^{d_1+1} \times K^{d_2+1}$ is the graph of the Cartesian product $T(d_1) \times T(d_2)$ of a d_1 -dimensional simplex $T(d_1)$ and a d_2 -dimensional simplex $T(d_2)$. And the polytope $T(d_1) \times T(d_2)$ is a *simple polytope*, a $(d_1 + d_2)$ -dimensional polytope in which every vertex is incident to exactly $d_1 + d_2$ edges.

While not every d -polytope is $\lfloor d/2 \rfloor$ -linked, it may be conjectured that every simple d -polytope is. Our result implies the veracity of the revised conjecture for Cartesian products of two simplices.

Keywords: k -linked, cyclic polytope, connectivity, dual polytope, linkedness, Cartesian product.

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E-mail addresses: leifkjorgensen@gmail.com (Leif K. Jørgensen), work@guillermo.com.au (Guillermo Pineda-Villavicencio), julien.ugon@deakin.edu.au (Julien Ugon)

1 Introduction

Denote by $V(X)$ the vertex set of a graph. Given sets A, B of vertices in a graph, a path from A to B , called an $A - B$ path, is a (vertex-edge) path $L := u_0 \dots u_n$ in the graph such that $V(L) \cap A = \{u_0\}$ and $V(L) \cap B = \{u_n\}$. We write $a - B$ path instead of $\{a\} - B$ path, and likewise, write $A - b$ path instead of $A - \{b\}$.

Let G be a graph and X a subset of $2k$ distinct vertices of G . The elements of X are called *terminals*. Let $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ be an arbitrary labelling and (un-ordered) pairing of all the vertices in X . We say that Y is *linked* in G if we can find disjoint $s_i - t_i$ paths for $i \in [1, k]$, the interval $1, \dots, k$. The set X is *linked* in G if every such pairing of its vertices is linked in G . Throughout this paper, by a set of disjoint paths, we mean a set of vertex-disjoint paths. If G has at least $2k$ vertices and every set of exactly $2k$ vertices is linked in G , we say that G is k -linked.

This paper studies the linkedness of Cartesian products of complete graphs. Linkedness of Cartesian products has been studied in the past [4]. The *Cartesian product* $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph defined on the pairs (v_1, v_2) with $v_i \in G_i$ and with two pairs (u_1, u_2) and (v_1, v_2) being adjacent if, for some $\ell \in \{1, 2\}$, $u_\ell v_\ell \in E(G_\ell)$ and $u_i = v_i$ for $i \neq \ell$. We prove that the Cartesian product $K^{d_1+1} \times K^{d_2+1}$ of complete graphs K^{d_1+1} and K^{d_2+1} is $\lfloor (d_1 + d_2)/2 \rfloor$ -linked for $d_1, d_2 \geq 0$, and that there are products that are not $\lfloor (d_1 + d_2 + 1)/2 \rfloor$ -linked; hence this result is best possible. Here K^t denotes the complete graph on t vertices.

Our result is connected to questions on the linkedness of a polytope. A (convex) polytope is the convex hull of a finite set X of points in \mathbb{R}^d ; the *convex hull* of X is the smallest convex set containing X . The *dimension* of a polytope in \mathbb{R}^d is one less than the maximum number of affinely independent points in the polytope; a set of points $\vec{p}_1, \dots, \vec{p}_k$ in \mathbb{R}^d is *affinely independent* if the $k - 1$ vectors $\vec{p}_1 - \vec{p}_k, \dots, \vec{p}_{k-1} - \vec{p}_k$ are linearly independent. A polytope of dimension d is referred to as a d -polytope.

The *Cartesian product* $P \times P'$ of a d -polytope $P \subset \mathbb{R}^d$ and a d' -polytope $P' \subset \mathbb{R}^{d'}$ is the Cartesian product of the sets P and P' :

$$P \times P' = \left\{ \begin{pmatrix} p \\ p' \end{pmatrix} \in \mathbb{R}^{d+d'} \mid p \in P, p' \in P' \right\}.$$

The resulting polytope is $(d + d')$ -dimensional. The *graph* $G(P)$ of a polytope P is the undirected graph formed by the vertices and edges of the polytope. It follows that the graph $G(P \times P')$ of the Cartesian product $P \times P'$ is the Cartesian product $G(P) \times G(P')$ of the graphs $G(P)$ and $G(P')$.

A d -simplex $T(d)$ is the convex hull of $d + 1$ affinely independent points in \mathbb{R}^d . The graph of $T(d)$ is the complete graph K^{d+1} . As a consequence, our result implies that the graph of the Cartesian product $T(d_1) \times T(d_2)$ is $\lfloor (d_1 + d_2)/2 \rfloor$ -linked for $d_1, d_2 \geq 0$. Henceforth, if the graph of a polytope is k -linked we say that the polytope is also k -linked.

The first edition of the Handbook of Discrete and Computational Geometry [3, Problem 17.2.6] posed the question of whether or not every d -polytope is $\lfloor d/2 \rfloor$ -linked. This question was answered in the negative by [2]. None of the known counterexamples are *simple d -polytopes*, d -polytopes in which every vertex is incident to exactly d edges. Hence, it may be hypothesised that the conjecture holds for such polytopes.

Conjecture 1.1. *Every simple d -polytope is $\lfloor d/2 \rfloor$ -linked for $d \geq 2$.*

Cartesian products of simplices are simple polytopes, and so our result supports this revised conjecture. Furthermore, Cartesian products of simplices and duals of cyclic polytopes are related; the dual of a cyclic d -polytope with $d+2$ vertices is the Cartesian product of a $\lfloor d/2 \rfloor$ -simplex and a $\lceil d/2 \rceil$ -simplex [6, Example 0.6]. Hence we obtain that the dual of a cyclic d -polytope on $d+2$ vertices is also $\lfloor d/2 \rfloor$ -linked for $d \geq 2$.

Unless otherwise stated, the graph theoretical notation and terminology follows from [1] and the polytope theoretical notation and terminology from [6]. Moreover, when referring to graph-theoretical properties of a polytope such as linkedness and connectivity, we mean properties of its graph.

2 Linkedness of Cartesian products of complex graphs

The contribution of this section is a sharp theorem (Theorem 2.1) that tells the story of the linkedness of Cartesian product of two complete graphs.

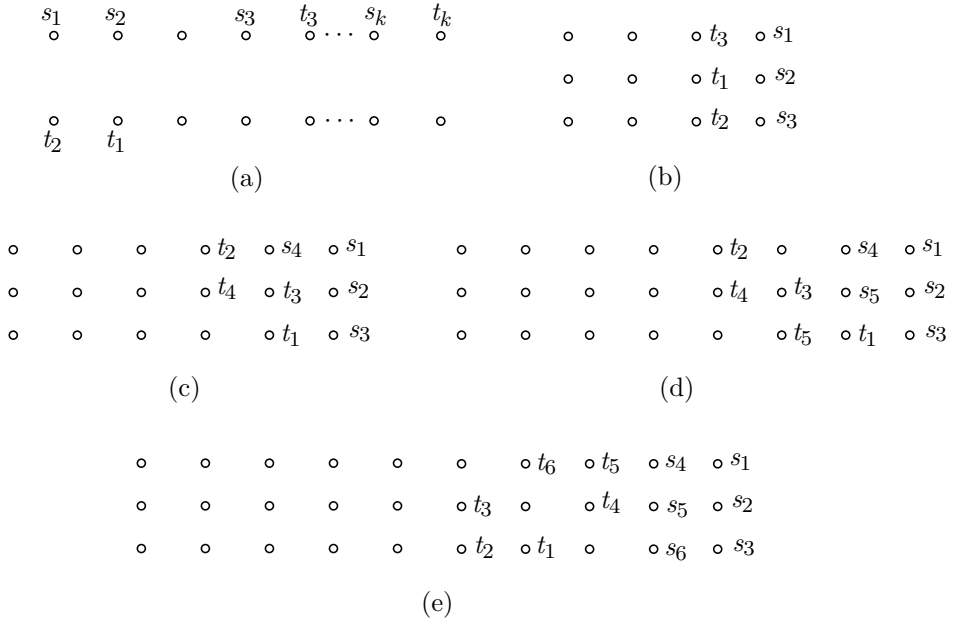


Figure 1: No feasible linkage problems for $K^{d_1+1} \times K^{d_2+1}$, $k = \lfloor (d_1 + d_2 + 1)/2 \rfloor$, $d_1 \leq 2$ and $d_2 > d_1$. (a) The case $d_1 = 1$ and even d_2 with $d_2 > d_1$. (b) The case $d_1 = 2$ and $d_2 = 3$. (c) The case $d_1 = 2$ and $d_2 = 5$. (d) The case $d_1 = 2$ and $d_2 = 7$. (e) The case $d_1 = 2$ and $d_2 = 9$. Each row of each part (a)-(e) is a complete graph whose edges have not been drawn.

Theorem 2.1. *The Cartesian product of two complete graphs K^{d_1+1} and K^{d_2+1} is $\lfloor (d_1 + d_2)/2 \rfloor$ -linked for every $d_1, d_2 \geq 0$.*

Remark 2.2. Theorem 2.1 is best possible. There are products $K^{d_1+1} \times K^{d_2+1}$ that are not $\lfloor (d_1 + d_2 + 1)/2 \rfloor$ -linked:

1. $K^2 \times K^{d_2+1}$ for even $d_2 \geq 1$, and

2. $K^3 \times K^{d_2+1}$ for $d_2 = 1, 3, 5, 7, 9$.

For each of these cases, Figure 1 provides a pairing of terminals that cannot be $\lfloor (d_1 + d_2 + 1)/2 \rfloor$ -linked. We conjecture these are the only such cases.

An immediate corollary of Theorem 2.1 is the following.

Corollary 2.3. *The Cartesian product of two simplices $T(d_1)$ and $T(d_2)$ is $\lfloor (d_1 + d_2)/2 \rfloor$ -linked for every $d_1, d_2 \geq 0$.*

The notions of linkage, linkage problem, and valid path will simplify our arguments. A *linkage* in a graph is a subgraph in which every component is a path. Let X be a set of vertices in a graph and let $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ be a pairing of all the vertices of X . A Y -*linkage* $\{L_1, \dots, L_k\}$ is a set of disjoint paths with the path L_i joining the pair $\{s_i, t_i\}$ for $i = 1, \dots, k$. We may also say that Y represents our *linkage problem*, and if Y is linked in G then our linkage problem is *feasible* and *infeasible* otherwise. A path in the graph is called X -*valid* if no inner vertex of the path is in X . Let X be a set of vertices in a graph G . Denote by $G[X]$ the subgraph of G induced by X , the subgraph of G that contains all the edges of G with vertices in X . Write $G - X$ for $G[V(G) \setminus X]$.

Consider a linkage problem $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ on a set X of $2k$ vertices in a graph G . Consider a linkage \mathcal{L} from a subset Z of X to some set Z' disjoint from X and label the vertices of Z' such that the path in \mathcal{L} with end $z_i \in Z$ has its other end $z'_i \in Z'$. Then the linkage \mathcal{L} in G induces a linkage problem Y' in $(G - V(\mathcal{L})) \cup Z'$ where the vertices of $X \setminus Z$ remain and the vertices of Z have been replaced by the vertices of Z' . Slightly abusing terminology, we also call terminals the vertices of Z' . If the problem Y' is feasible in $(G - V(\mathcal{L})) \cup Z'$, so is the problem Y in G .

Since we make heavy use of Menger's theorem [1, Theorem. 3.3.1], we next remind the reader of one of its consequences.

Theorem 2.4 (Menger's theorem). *Let G be a k -connected graph, and let A and B be two subsets of its vertices, each of cardinality at least k . Then there are k disjoint $A - B$ paths in G .*

We fix some notation and terminology for the remaining of the section. Let G denote the graph $K^{d_1+1} \times K^{d_2+1}$. We think of $G = K^{d_1+1} \times K^{d_2+1}$ as a grid with $d_1 + 1$ rows and $d_2 + 1$ columns. In this way, the entry in Row i and Column j can be referred to as $G[i, j]$.

When we write about a row r of subgraph G' of G , we think of r as a subgraph of G' and as the number r so that we can write about the r th row of G' or G ; this ambiguity should cause no confusion. An entry in the grid $K^{d_1+1} \times K^{d_2+1}$ with no terminal is said to be *free*, as is a row or a column of a subgraph of G with no terminal. A row or a column of a subgraph of G with every entry being occupied by a terminal is said to be *full*.

We need the following induced subgraphs of G :

$C_{ab\dots z}$, the subgraph formed by the union of Columns a, b, \dots, z ;

$\bar{C}_{ab\dots z}$, the subgraph obtained by removing Columns a, b, \dots, z ;

$R_{ab\dots z}$, the subgraph formed by the union of Rows a, b, \dots, z ;

$\bar{R}_{ab\dots z}$, the subgraph obtained by removing Rows a, b, \dots, z ;

A_α , the induced subgraph of \bar{C}_{12} obtained by removing its first α rows; and

B_α , the subgraph of C_{12} obtained by removing its first α rows.

For instance, \bar{C}_1 denotes the subgraph of G obtained by removing the first column, C_{12} the subgraph formed by the first two columns of G , and \bar{C}_{12} denotes the subgraph obtained by removing the first two columns of G ; observe \bar{C}_{12} is isomorphic to $K^{d_1+1} \times K^{d_2-1}$. Figure 2 depicts some of the aforementioned subgraphs of $K^{d_1+1} \times K^{d_2+1}$.

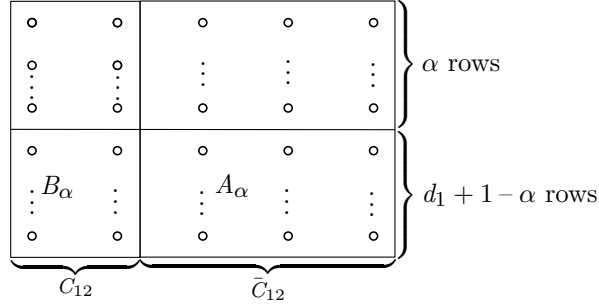


Figure 2: Depiction of the subgraphs B_α , A_α , C_{12} , and \bar{C}_{12} of $K^{d_1+1} \times K^{d_2+1}$.

The connectivity of $K^{d_1+1} \times K^{d_2+1}$ is stated below.

Lemma 2.5 (Špacapan [5, Theorem 1]). *The (vertex)connectivity of $K^{d_1+1} \times K^{d_2+1}$ is precisely $d_1 + d_2$.*

We continue fixing further notation. Henceforth let $k := \lfloor (d_1 + d_2)/2 \rfloor$. And let X be a subset of $2k$ vertices of G and let $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ be a pairing of all the vertices in X .

We first settle the simple cases of $(0, d_2)$ and $(1, d_2)$ for $d_2 \geq 0$.

Proposition 2.6 (Base cases). *For $d_2 \geq 0$ the Cartesian products $K^1 \times K^{d_2+1}$ and $K^2 \times K^{d_2+1}$ are both $\lfloor (1 + d_2)/2 \rfloor$ -linked. This statement is best possible.*

Proof. The lemma is true for the pair $(0, d_2)$ for each $d_2 \geq 0$, since $K^1 \times K^{d_2+1} = K^{d_2+1}$ and K^{d_2+1} is $\lfloor (1 + d_2)/2 \rfloor$ -linked. This is best possible.

The graph $K^2 \times K^{d_2+1}$ is $(1 + d_2)$ -connected by Lemma 2.5. Use Menger's theorem (Theorem 2.4) to bring the $1 + d_2$ terminals to the subgraph \bar{R}_1 through a linkage $\{S_1, \dots, S_k, T_1, \dots, T_k\}$ with $S_i := s_i - \bar{R}_1$ and $T_i := t_i - \bar{R}_1$ for $i \in [1, k]$. Letting $\{\bar{s}_i\} := V(S_i) \cap V(\bar{R}_1)$ and $\{\bar{t}_i\} := V(T_i) \cap V(\bar{R}_1)$, we produce a new linkage problem $Y' := \{\{\bar{s}_1, \bar{t}_1\}, \dots, \{\bar{s}_k, \bar{t}_k\}\}$ in \bar{R}_1 whose feasibility implies that of Y in G . To solve Y' link the pairs of Y' in the subgraph \bar{R}_1 , which is isomorphic to K^{d_2+1} , using the $\lfloor (1 + d_2)/2 \rfloor$ -linkedness of K^{d_2+1} . For even even d_2 , Figure 1(a) shows an infeasible linkage problem with $\lfloor (2 + d_2)/2 \rfloor$ pairs in the graph $K^2 \times K^{d_2+1}$. \square

In what follows we aim to find a Y -linkage $\{L_1, \dots, L_k\}$ in G with L_i joining the pair $\{s_i, t_i\}$ of Y for $i \in [1, k]$. Our proof is by induction on (d_1, d_2) with the base cases settled in Proposition 2.6. If there is a pair of Y , say $\{s_1, t_1\}$, lying in some column or row of G , say in Column 1, we send every terminal $s_i \in C_1$ that is different from s_1 and t_1 and that is not adjacent to t_i to the subgraph \bar{C}_1 , and apply the induction hypothesis on \bar{C}_1 . Otherwise, we may assume every pair of Y lies in two distinct columns or rows, say the pair $\{s_1, t_1\}$ lies in C_{12} ; then we send every terminal $s_i \in C_{12}$ that is different from s_1 and

t_1 and that is not adjacent to t_i to the subgraph \bar{C}_{12} , and apply the induction hypothesis to \bar{C}_{12} . We develop these ideas below.

The definition of k -linkedness gives the following lemma at once; we will use it implicitly hereafter.

Lemma 2.7. *Let $\ell \leq k$. Let X be a set of 2ℓ distinct vertices of a k -linked graph K , let Y be a labelling and pairing of the vertices in X , and let Z be a set of $2k - 2\ell$ vertices in K such that $X \cap Z = \emptyset$. Then there exists a Y -linkage in K that avoids every vertex in Z .*

Besides, basic algebraic manipulation yields the following inequality.

Lemma 2.8. *If $x \geq 2$ and $y \geq 2$ then $x(y - 1) > x + y - 3$.*

Proof. The inequality simplifies to $(x - 1)(y - 2) > -1$. □

We are now ready to put together all the elements of the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $k := \lfloor (d_1 + d_2)/2 \rfloor$. Then $d_1 + d_2 \geq 2k$.

Proposition 2.6 gives the result for the pairs $(d_1, 0)$, $(0, d_2)$, $(d_1, 1)$, and $(1, d_2)$ for each $d_1, d_2 \geq 0$. Hence, our bidimensional induction on (d_1, d_2) can start with the assumption of $d_1, d_2 \geq 2$.

We first deal with the case where a pair in Y , say $\{s_1, t_1\}$, lies in some column or some row of G , say in Column 1.

Case 1. A pair in Y , say $\{s_1, t_1\}$, lies in Column 1.

The induction hypothesis ensures that the subgraph \bar{C}_1 is $(k - 1)$ -linked. Hence it suffices to show that all the terminals in C_1 other than s_1, t_1 can be moved to \bar{C}_1 via a linkage; Menger's theorem (Theorem 2.4) guarantees this.

Let U be the set of terminals in C_1 other than s_1 and t_1 , and let W be the set of terminals in \bar{C}_1 . Then $|U| + |W| \leq d_1 + d_2 - 2$, as $|U| + |W| = 2k - 2$ and $2k \leq d_1 + d_2$. Besides, the subgraph $G - (W \cup \{s_1, t_1\})$ is $|U|$ -connected, as G is $(d_1 + d_2)$ -connected (Lemma 2.5). In the case of $d_1, d_2 \geq 2$, Lemma 2.8 yields that \bar{C}_1 has more than $|U \cup W|$ vertices:

$$|\bar{C}_1| = (d_1 + 1)d_2 > d_1 + 1 + d_2 + 1 - 3 > d_1 + d_2 - 2 = |U| + |W|.$$

Use Menger's theorem (Theorem 2.4) to bring the $|U|$ terminals in C_1 to the subgraph \bar{C}_1 through a linkage Y_U . For every path L in Y_U , if $s_i \in L$, let $\{\bar{s}_i\} := V(L) \cap V(\bar{C}_1)$ and if $t_i \in L$ let $\{\bar{t}_i\} := V(L) \cap V(\bar{C}_1)$. For $s_i \in W$ (respectively $t_i \in W$) let $\bar{s}_i = s_i$ (respectively $\bar{t}_i = t_i$). This produces a new linkage problem $Y' := \{\{\bar{s}_2, \bar{t}_2\}, \dots, \{\bar{s}_k, \bar{t}_k\}\}$ in \bar{C}_1 whose feasibility implies that of Y in G , since s_1 and t_1 are adjacent in C_1 . The $(k - 1)$ -linkedness of \bar{C}_1 now settles the case.

By symmetry, we can assume that every pair $\{s_i, t_i\}$ in Y lies in two different columns or rows and that s_i, t_i are not adjacent. Without loss of generality, assume that

$$s_1 \text{ is in Column 1 and } t_1 \text{ is in Column 2 of } C_{12}. \quad (*)$$

The induction hypothesis also ensures that both \bar{C}_{12} and \bar{R}_{12} are $(k - 1)$ -linked. We consider two further cases based on the number of terminals in C_{12} or R_{12} .

Case 2. The subgraph C_{12} contains precisely $d_1 + 2 - \alpha$ terminals, including $\{s_1, t_1\}$, where $0 \leq \alpha \leq d_1$.

Excluding $\{s_1, t_1\}$, there are at most d_1 terminals in C_{12} , and there are $d_1 + 1$ internally-disjoint $s_1 - t_1$ paths in C_{12} of length at most three: two length-two paths and $d_1 - 1$ length-three paths. One of these $s_1 - t_1$ paths, say L_1 , avoids every other terminal in C_{12} .

Without loss of generality, assume that Row 1 in C_{12} is part of the path L_1 ; that is,

$$\{G[1, 1], G[1, 2]\} \subseteq V(L_1). \quad (**)$$

In the subcase $\alpha = d_1$, every pair in $Y \setminus \{s_1, t_1\}$ is in \bar{C}_{12} , and the induction hypothesis on \bar{C}_{12} settles the subcase.

Suppose that $\alpha = d_1 - 1$, say C_{12} contains $\{s_1, t_1, s_2\}$. Then $s_2 \in B_1$ and $t_2 \in \bar{C}_{12}$. We may assume s_1, s_2 are in Column 1 and t_1 is in Column 2. We show there is an X -valid $s_2 - A_1$ path L'_2 such that the vertex $\bar{s}_2 \in V(L'_2) \cap V(A_1)$ is either t_2 or a nonterminal.

Through each entry of Column 1 of B_1 , there are $d_2 - 1$ paths from s_2 to A_1 of length at most two (one for each column in A_1). Moreover, there are at least $d_1 - 1$ free entries in Column 1 of B_1 . Therefore, to ensure the existence of L'_2 , we need to show that at least one of these $(d_1 - 1)(d_2 - 1)$ paths from s_2 to A_1 either contains t_2 or a nonterminal in A_1 . Indeed, according to Lemma 2.8, the inequality

$$(d_1 - 1)(d_2 - 1) > d_1 - 1 + d_2 - 3 \geq |X \setminus \{s_1, t_1, s_2, t_2\}|$$

holds for $d_1, d_2 \geq 2$. Hence we get the existence of L'_2 . As a result, the solution of the new problem $Y' := \{\{\bar{s}_2, t_2\}, \{s_3, t_3\}, \dots, \{s_k, t_k\}\}$ in \bar{C}_{12} induces a solution of the problem Y in G . And the solution of Y' follows from the $(k - 1)$ -linkedness of \bar{C}_{12} .

Henceforth assume that $\alpha \leq d_1 - 2$. To finalise Case 2, we require a couple of claims.

Claim 2.9. *Suppose that there are at most $d_1 + 2 - \alpha$ terminals in $B_{\alpha+1} = K^{d_1-\alpha} \times K^2$. Then there is an injection from the set of rows of $B_{\alpha+1}$ that contain two terminals x_1, x_2 such that $\{x_1, x_2\} \cap \{s_1, t_1\} = \emptyset$ to the set of rows of $B_{\alpha+1}$ that contain no terminal other than possibly s_1 and t_1 .*

Proof. This follows from a simple counting argument. The number of rows in $B_{\alpha+1}$ is $d_1 - \alpha$. Let m denote the number of rows of $B_{\alpha+1}$ that contain two terminals x_1, x_2 such that $\{x_1, x_2\} \cap \{s_1, t_1\} = \emptyset$ and let $n := |(X \cap V(B_{\alpha+1})) \setminus \{s_1, t_1\}|$; that is, n counts the total number of terminals in $B_{\alpha+1}$ other than s_1 and t_1 . It follows that the number of rows of $B_{\alpha+1}$ that contain precisely one terminal $x \notin \{s_1, t_1\}$ is $n - 2m$; either s_1 or t_1 may be in these rows. As a result, the number of rows of $B_{\alpha+1}$ that contain no terminal other than $\{s_1, t_1\}$ is $d_1 - \alpha - m - (n - 2m)$. Combining $n \leq d_1 - \alpha$ with all these numbers, we get that

$$d_1 - \alpha - m - (n - 2m) = d_1 - \alpha - n + m \geq d_1 - \alpha - (d_1 - \alpha) + m = m.$$

The claim is proved. \square

Claim 2.10. *Suppose that there are at most $d_1 + 2 - \alpha$ terminals in $B_{\alpha+1} = K^{d_1-\alpha} \times K^2$. If every row in the subgraph $A_{\alpha+1} = K^{d_1-\alpha} \times K^{d_2-1}$ of \bar{C}_{12} has a free entry, then, for every terminal $x \notin \{s_1, t_1\}$ in $B_{\alpha+1}$, there is an X -valid $x - A_{\alpha+1}$ path L to a free entry in $A_{\alpha+1}$; and all these X -valid paths are disjoint.*

Proof. If a row of $B_{\alpha+1}$ contains exactly one terminal $x \notin \{s_1, t_1\}$, then send x to a free entry in the same row of $A_{\alpha+1}$. Let x_1 and x_2 be two terminals in $B_{\alpha+1}$ that satisfy

$\{x_1, x_2\} \cap \{s_1, t_1\} = \emptyset$ and occupy a row r_f of $B_{\alpha+1}$. From Claim 2.9 ensues the existence of a row r_e of $B_{\alpha+1}$ that contain no terminal other than possibly s_1 and t_1 ; in short, there is at least a free entry in r_e .

Consider a pair (r_f, r_e) of rows granted by Claim 2.9. Send either x_1 or x_2 , say x_1 , to the free entry in the row r_e of $A_{\alpha+1}$ passing through the corresponding free entry in the row r_e of $B_{\alpha+1}$, and send x_2 to a free entry in the row r_f of $A_{\alpha+1}$. The proof of the claim is now complete. \square

Now suppose that $\alpha = 0$ or $2 \leq \alpha \leq d_1 - 2$. In this subcase, the subgraph \bar{C}_{12} contains at most α full rows: if $\alpha + 1$ rows were full in \bar{C}_{12} then there would be at least $(\alpha + 1)(d_2 - 1)$ terminals in \bar{C}_{12} but $(\alpha + 1)(d_2 - 1) > d_2 - 2 + \alpha$ (Lemma 2.8). Even when the path L_1 uses the first row of C_{12} by (**), there is no loss of generality by assuming that the full rows of \bar{C}_{12} are among the first $\alpha + 1$ rows of \bar{C}_{12} . It follows that every row of $A_{\alpha+1}$ has a free entry.

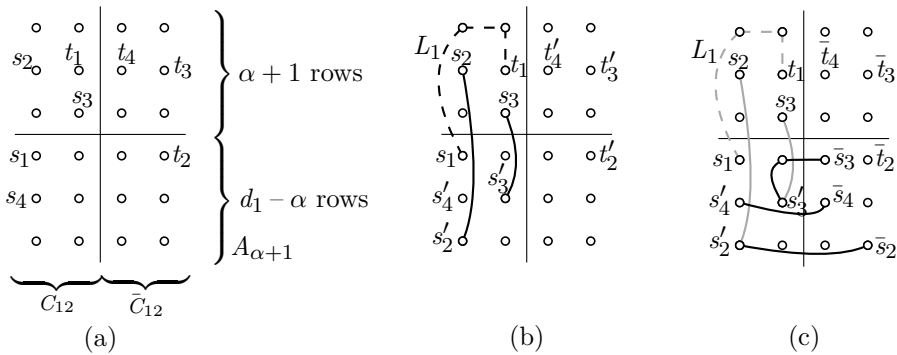


Figure 3: Auxiliary figure for Case 2 (a) This shows a scenario where $d_1 = 5$, $d_2 = 3$, and $\alpha = 2$. (b) The path $L_1 = s_1 - t_1$ in dashed line, the paths that send the terminals in $B_1 \setminus B_3$ other than s_1 and t_1 to B_3 , and the resulting new linkage $Y' = \{\{s_2', t_2'\}, \{s_3', t_3'\}, \{s_4', t_4'\}\}$ in $\bar{C}_{12} \cup B_{\alpha+1}$. (c) The paths that send the terminals in B_3 to A_3 , and the resulting new linkage $Y'' = \{\{\bar{s}_2, \bar{t}_2\}, \{\bar{s}_3, \bar{t}_3\}, \{\bar{s}_4, \bar{t}_4\}\}$ in \bar{C}_{12} .

Next we show how to send to $B_{\alpha+1}$ the terminals other than s_1 and t_1 that are in the rows 2 to $\alpha + 1$ of C_{12} ; that is, the terminals other than s_1 and t_1 that are in $B_1 \setminus B_{\alpha+1}$. For $\alpha = 0$, $B_1 \setminus B_{\alpha+1} = \emptyset$ and there is nothing to do. We now focus on the subcase $2 \leq \alpha \leq d_1 - 2$. Let n_1 and n_2 denote the number of terminals in $B_1 \setminus B_{\alpha+1}$ and $B_{\alpha+1}$, respectively. Then the following inequalities hold

$$\begin{aligned} n_1 + n_2 &\leq d_1 + 2 - \alpha \leq d_1 \quad (\text{since } 2 \leq \alpha), \\ n_1 + n_2 &\leq d_1 + 2 - \alpha \leq 2d_1 - 2\alpha = |V(B_{\alpha+1})| \quad (\text{since } \alpha \leq d_1 - 2). \end{aligned}$$

From the second inequality, it follows that there are at least n_1 free vertices in $B_{\alpha+1}$. Since B_1 is d_1 -connected by Lemma 2.5, Menger's theorem gives n_1 disjoint paths in B_1 from the terminals in $B_1 \setminus B_{\alpha+1}$ to n_1 free entries in $B_{\alpha+1}$, avoiding the n_2 terminals in $B_{\alpha+1}$. For a terminal s_i in $B_1 \setminus B_{\alpha+1}$, let L_i' be the path from s_i to $B_{\alpha+1}$ and let $s_i' := V(L_i') \cap B_{\alpha+1}$. Define t_i' similarly for a terminal t_i in $B_1 \setminus B_{\alpha+1}$. Furthermore, for s_i (respectively, t_i) in

$B_{\alpha+1} \cup \bar{C}_{12}$, let $s'_i := s_i$ (respectively, $t'_i := t_i$). This produces a new linkage problem $Y' := \{\{s'_2, t'_2\}, \dots, \{s'_k, t'_k\}\}$ in $\bar{C}_{12} \cup B_{\alpha+1}$. See Figure 3(b).

There are at most $d_1 + 2 - \alpha$ terminals in $B_{\alpha+1} = K^{d_1-\alpha} \times K^2$, and every row in $A_{\alpha+1} = K^{d_1-\alpha} \times K^{d_2-1}$ has a free entry. Hence, Claim 2.10 applies, and there is a linkage formed by X -valid paths from the terminals in $B_{\alpha+1}$, other than s_1 and t_1 , to free entries in $A_{\alpha+1}$. For every such path L''_i , if $s'_i \in V(L''_i) \cap V(B_{\alpha+1})$, let $\{\bar{s}_i\} := V(L''_i) \cap V(A_{\alpha+1})$, and if $t'_i \in V(L''_i) \cap V(B_{\alpha+1})$, let $\{\bar{t}_i\} := V(L''_i) \cap V(A_{\alpha+1})$. Besides, for $s'_i \in \bar{C}_{12}$ (respectively $t'_i \in \bar{C}_{12}$), let $\bar{s}_i = s'_i$ (respectively, $\bar{t}_i = t'_i$). This produces a new linkage problem $Y'' := \{\{\bar{s}_2, \bar{t}_2\}, \dots, \{\bar{s}_k, \bar{t}_k\}\}$ in \bar{C}_{12} whose feasibility implies that of Y' , and therefore that of Y in G , by completing each linkage problem with the path L_1 . See Figure 3(c).

Now we have a new linkage problem Y'' in \bar{C}_{12} with $(k-1)$ pairs. The solution of Y'' in \bar{C}_{12} implies a solution of the linkage problem Y in G . To link the pairs of Y'' use the $(k-1)$ -linkedness of \bar{C}_{12} .

Finally assume that $\alpha = 1$. Then there are exactly $d_1 + 1$ terminals in C_{12} and at most $d_2 - 1$ terminals in \bar{C}_{12} . In a first scenario suppose that either both entries in $B_1 \setminus B_2$ are nonterminals or each terminal other than s_1 and t_1 in $B_1 \setminus B_2$ is adjacent to a nonterminal in B_2 . Then we can send these terminals in $B_1 \setminus B_2$ to B_2 . In the second scenario, suppose that there is a terminal s_i ($i \neq 1$) in $B_1 \setminus B_2$ whose neighbours in B_2 are all terminals. Then the column of s_i in B_1 would contain exactly d_1 terminals, including s_i . We send s_i to a free entry in A_1 , in the same row as s_i (the first row of A_1): if this free entry didn't exist, then s_i would be adjacent to the $d_2 - 1$ terminals in A_1 and the $d_1 - 1$ terminals in B_2 . Since there are $d_1 + d_2$ terminals in total, it would follow that s_i is adjacent to t_i . This contradiction shows that we can send s_i to a free entry in A_1 .

In both scenarios, it remains to send the terminals other than s_1 and t_1 in $B_2 = K^{d_1-1} \times K^2$ to $A_2 = K^{d_1-1} \times K^{d_2-1}$. To do so, we reason as in the subcase $2 \leq \alpha \leq d_1 - 2$. It follows that there are at most $d_1 + 2 - 1$ terminals in B_2 , and that every row in A_2 has a free entry. Claim 2.10 applies again and gives a linkage formed by X -valid paths from the terminals in B_2 , other than s_1, t_1 , to free entries in A_2 .

With all the terminals other than s_1 and t_1 in \bar{C}_{12} , therein we have a new linkage problem Y' with $k-1$ pairs whose solution in \bar{C}_{12} implies a solution of the linkage problem Y in G . To solve Y' in \bar{C}_{12} use the $(k-1)$ -linkedness of \bar{C}_{12} .

By symmetry, we also have the result if there are at most $d_2 + 2$ terminals in R_{12} , including $\{s_1, t_1\}$.

Case 1. The subgraph C_{12} contains at least $d_1 + 3$ terminals, including $\{s_1, t_1\}$.

This case reduces to the previous case. If C_{12} contains at least $d_1 + 3$ terminals then R_{12} contains at most $d_2 - 3 + 4 = d_2 + 1$ terminals, since there are four entries shared by C_{12} and R_{12} . Because we make no distinction between columns and rows, this case is already covered. This completes the proof of the theorem. \square

3 Duals of cyclic polytopes

There is a close connection between duals of cyclic d -polytopes with $d + 2$ vertices and Cartesian products of complete graphs.

The *moment curve* in \mathbb{R}^d is defined by $x(t) := (t, t^2, \dots, t^d)$ for $t \in \mathbb{R}$, and the convex hull of any $n > d$ points on it gives a *cyclic polytope* $C(n, d)$. The *combinatorics* of a cyclic

polytope, the face lattice of the polytope faces partially ordered by inclusion, is independent of the points chosen on the moment curve. Hence we talk of the cyclic d -polytope on n vertices [6, Example 0.6].

For a polytope P that contains the origin in its interior, the *dual polytope* P^* is defined as

$$P^* = \{y \in \mathbb{R}^d \mid x \cdot y \leq 1 \text{ for all } x \text{ in } P\}.$$

If P does not contain the origin, we translate the polytope so that it does. Translating the polytope P changes the geometry of P^* but not its face lattice. The face lattice of P^* is the inclusion reversed face lattice of P . In particular, the vertices of P^* correspond to the facets of P , and the edges of P^* correspond to the $(d-2)$ -faces of P . The *dual graph* of a polytope P is the graph of the dual polytope, or equivalently, the graph on the set of facets of P where two facets are adjacent in the dual graph if they share a $(d-2)$ -face.

Duals of cyclic d -polytopes are simple d -polytopes. It is also the case that the dual of a cyclic d -polytope with $d+2$ vertices can be expressed as $T(\lfloor d/2 \rfloor) \times T(\lceil d/2 \rceil)$ ([6, Example 0.6]). From this observation and Theorem 2.1 the next corollary follows at once.

Corollary 3.1. *Duals of cyclic polytopes with $d+2$ vertices are $\lfloor d/2 \rfloor$ -linked for every $d \geq 2$.*

ORCID iDs

Leif K. Jørgensen  <https://orcid.org/0000-0003-4922-3937>

Guillermo Pineda-Villavicencio  <https://orcid.org/0000-0002-2904-6657>

Julien Ugon  <https://orcid.org/0000-0001-5290-8051>

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